# Bernoulli Numbers and Polynomials

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The sum of first $n$ natural numbers $1, 2, 3, \ldots, n$ is

$$S_1(n) := \sum_{m=1}^{n} m = \frac{n(n+1)}{2} = \frac{n^2}{2} + \frac{n}{2}.$$

This formula can be derived by noting that

$$\begin{aligned} S_1(n) &= 1 + 2 + \ldots + n \\ S_1(n) &= n + (n-1) + \ldots + 1. \end{aligned}$$

Therefore, summing term-by-term,

$$2S_1(n) = \underbrace{(n+1) + \ldots + (n+1)}_{n-\text{times}} = n(n+1).$$

An alternate way of obtaining the above sum is by using the following two identity:

(i) $(m+1)^2 - m^2 = 2m + 1$ and, hence,

$$2\sum_{m=1}^{n} m = \sum_{m=1}^{n} \left[(m+1)^2 - m^2\right] - n.$$

(ii)

$$\begin{aligned} \sum_{m=1}^{n} \left[(m+1)^2 - m^2\right] &= [2^2 - 1^2] + [3^2 - 2^2] + \ldots + \\ &\quad + [n^2 - (n-1)^2] + [(n+1)^2 - n^2] \\ &= (n+1)^2 - 1. \end{aligned}$$

Thus,

$$S_1(n) = \frac{(n+1)^2 - (n+1)}{2} = \frac{n(n+1)}{2}.$$

More generally, the sum of $k$-th power of first $n$ natural numbers is denoted as

$$S_k(n) := 1^k + 2^k + \ldots + n^k.$$

Since $a^0 = 1$, for any $a$, we have $S_0(n) = n$. For $k \in \mathbb{N}$, one can compute $S_k(n)$ using the identities:

(i)

$$(m+1)^{k+1} - m^{k+1} = \sum_{i=0}^{k} \binom{k+1}{i} m^i$$

and, hence,

$$(k+1)\sum_{m=1}^{n} m^k = \sum_{m=1}^{n} \left[(m+1)^{k+1} - m^{k+1}\right] - \sum_{i=0}^{k-1}\sum_{m=1}^{n} \binom{k+1}{i} m^i.$$

(ii) $\sum_{m=1}^{n} \left[(m+1)^{k+1} - m^{k+1}\right] = (n+1)^{k+1} - 1.$

Thus,

$$\begin{aligned}
(k+1)\sum_{m=1}^{n} m^k &= (n+1)^{k+1} - 1 - \sum_{i=0}^{k-1} \binom{k+1}{i} \sum_{m=1}^{n} m^i \\
S_k(n) &= \frac{(n+1)^{k+1}}{k+1} - \frac{(n+1)}{k+1} - \frac{1}{k+1}\sum_{i=1}^{k-1} \binom{k+1}{i} S_i(n).
\end{aligned}$$

The formula obtained in RHS is a $(k+1)$-degree polynomial of $n$. Using the above formula, one can compute

$$\begin{aligned}
S_2(n) &= \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}, \\
S_3(n) &= \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}, \\
S_4(n) &= \frac{n^5}{5} + \frac{n^4}{2} + \frac{n^3}{3} - \frac{n}{30}, \\
S_5(n) &= \frac{n^6}{6} + \frac{n^5}{2} + \frac{5n^4}{12} - \frac{n^2}{12}, \\
S_6(n) &= \frac{n^7}{7} + \frac{n^6}{2} + \frac{n^5}{2} - \frac{n^3}{6} + \frac{n}{42}, \\
S_7(n) &= \frac{n^8}{8} + \frac{n^7}{2} + \frac{7n^6}{12} - \frac{7n^4}{24} + \frac{n^2}{12}, \\
\ldots
\end{aligned}$$

James/Jacques/Jakob Bernoulli observed that the sum of first $n$ whole numbers raised to the $k$-th power can be concisely written as,

$$S_k(n) = \frac{n^{k+1}}{k+1} + \frac{1}{2}n^k + \frac{1}{12}kn^{k-1} + 0 \times n^{k-2} + \ldots.$$

Note that the coefficients $1, 1/2, 1/12, 0, \ldots$ are independent of $k$. Jakob Bernoulli rewrote the above expression as

$$\begin{aligned}
(k+1)S_k(n) &= n^{k+1} - \frac{1}{2}(k+1)n^k + \frac{1}{6}\frac{k(k+1)}{2}n^{k-1} + 0 - \\
&\quad - \frac{1}{30}\frac{(k-2)(k-1)k(k+1)}{4!} + \ldots
\end{aligned}$$

and, hence,

$$S_k(n) = \frac{1}{k+1}\sum_{i=0}^{k}\binom{k+1}{i}B_i n^{k+1-i}, \qquad (1)$$

where $B_i$ are the $i$-th *Bernoulli* numbers and the formula is called *Bernoulli* formula. An easier way to grasp the above formula for $S_k(n)$ is to rewrite[1] it as

$$S_k(n) = \frac{(n+B)^{k+1} - B_{k+1}}{k+1}$$

[1] Umbral Calculus

where $B$ is a notation used to identify the $i$-th power of $B$ with the $i$-th Bernoulli number $B_i$ and

$$(n+B)^{k+1} := \sum_{i=0}^{k+1} \binom{k+1}{i} B_i n^{k+1-i}.$$

This notation also motivates the definition of *Bernoulli polynomial* of degree $k$ as

$$B_k(t) := \sum_{i=0}^{k} \binom{k}{i} B_i t^{k-i}$$

where $B_i$ are the Bernoulli numbers. In terms of Bernoulli polynomials, the $k$-th Bernoulli number $B_k = B_k(0)$.

Two quick observation can be made from (1).

(i) There is no constant term in $S_k(n)$ because $i$ does not take $k+1$.

(ii) The $k$-th Bernoulli number, $B_k$, is the coefficient of $n$ in $S_k(n)$. For instance, $B_0$ is coefficient of $n$ in $S_0(n) = n$ and, hence, $B_0 = 1$. Similarly, $B_1 = 1/2, B_2 = 1/6, B_3 = 0, B_4 = -1/30, B_5 = 0, B_6 = 1/42, B_7 = 0, \ldots$.

The beauty about the sequence of Bernoulli numbers is that one can compute them *a priori* and use it to calculate $S_k(n)$, i.e., given $n \in \mathbb{N}$ and $k \in \mathbb{N} \cup \{0\}$ it is enough to know $B_i$, for all $0 \leq i \leq k$, to compute $S_k(n)$. We already computed $B_0 = 1$. Using $n = 1$ in the Bernoulli formula (1), we get

$$1 = \frac{1}{k+1} \sum_{i=0}^{k} \binom{k+1}{i} B_i \tag{2}$$

and, this implies that the $k$-th Bernoulli number, for $k > 0$, is defined as

$$B_k = 1 - \frac{1}{k+1} \sum_{i=0}^{k-1} \binom{k+1}{i} B_i.$$

Recall that $B_3, B_5, B_7$ vanish. In fact, it turns out that $B_k = 0$ for all odd $k > 1$. The odd indexed Bernoulli numbers vanish because they have no $n$-term. Since there are no constant terms in $S_k(n)$, the vanishing of Bernoulli numbers is equivalent to the fact that $n^2$ is a factor of $S_k(n)$. Some properties of Bernoulli numbers:

(i) For odd $k > 1$, $B_k = 0$.

(ii) For even $k$, $B_k \neq 0$.

(iii) $B_k \in \mathbb{Q}$ for all $k \in \mathbb{N} \cup \{0\}$.

(iv) $B_0 = 1$ is the only non-zero integer.

(v) $B_{4j}$ is negative rational and $B_{4j-2}$ is positive rational, for all $j \in \mathbb{N}$.

(vi) $|B_6 = 1/42| < |B_{2k}|$, for all $k \in \mathbb{N}$.

L. Euler gave a nice generating function for the Bernoulli numbers. He seeked a function $f(x)$ such that $f^{(k)}(0) = B_k$ where $f^{(k)}$ denotes the $k$-th derivative of $f$ with the convention that $f^{(0)} = f$. If such an $f$ exists then it admits the Taylor series expansion, around 0,

$$f(x) = \sum_{k=0}^{\infty} f^{(k)}(0) \frac{x^k}{k!}.$$

Therefore, for such a function

$$f(x) = \sum_{k=0}^{\infty} B_k \frac{x^k}{k!}.$$

Recall the Taylor series of $e^x$,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

defined for all $x \in \mathbb{R}$. Consider the product (discrete convolution/Cauchy product) of the infinite series

$$\begin{aligned} f(x)e^x &= \left( \sum_{k=0}^{\infty} B_k \frac{x^k}{k!} \right) \left( \sum_{k=0}^{\infty} \frac{x^k}{k!} \right) \\ &= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} B_i \frac{x^i}{i!} \frac{x^{k-i}}{(k-i)!} \right) \\ &= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} \frac{B_i}{i!(k-i)!} \right) x^k \\ &= \sum_{k=0}^{\infty} \left( \sum_{i=0}^{k} \binom{k}{i} B_i \right) \frac{x^k}{k!}. \end{aligned}$$

Let

$$c_k := \sum_{i=0}^{k} \binom{k}{i} B_i = \sum_{i=0}^{k-1} \binom{k}{i} B_i + B_k = k + B_k.$$

Then

$$f(x)e^x = \sum_{k=0}^{\infty} (k + B_k) \frac{x^k}{k!} = \sum_{k=1}^{\infty} \frac{x^k}{(k-1)!} + f(x) = xe^x + f(x).$$

Thus, the $f$ we seek satisfies

$$f(x) = \frac{xe^x}{e^x - 1}$$

and is called the *generating function*. Since $e^x > 0$ for all $x \in \mathbb{R}$, we may rewrite $f(x)$ as

$$f(x) = \frac{x}{1 - e^{-x}}. \tag{3}$$

The entire exercise of seeking $f$ can be generalised to complex numbers and

$$f(z) = \frac{z}{1 - e^{-z}} \quad \forall z \in \mathbb{C} \text{ with } 0 \leq \Im(z) < 2\pi.$$

A word of caution that idenitities (2) and (3) are different from the standard formulae because we have derived them for second Bernoulli numbers, viz., with $B_1 = 1/2$. The standard convention is to work with first Bernoulli numbers, viz., with $B_1 = -1/2$. The first Bernoulli numbers can be obtained by following the approach of summing the $k$-th powers of first $n - 1$ natural numbers, for any given $n$.

The Bernoulli numbers with appeared while computing $S_k(n)$ is appears in many crucial places.

(a) In the expansion of $\tan z$.

$$\tan z = \sum_{k=1}^{\infty} (-1)^{k-1} \frac{2^{2k}(2^{2k} - 1)B_{2k}}{2n!} z^{2k-1}.$$

for all $|z| < \pi/2$.

(b) In computing the sum of Riemann-zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

for positive even integers $s$. The case $s = 2$ is the famous *Basel problem* computed by Euler to be $\pi^2/6$.

**Theorem 1** (Euler)**.** *For all $k \in \mathbb{N}$,*

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}.$$

Further, the relation $\zeta(-2k) = -\frac{B_{2k+1}}{2k+1}$ gives the trivial zeroes of the Riemann zeta function.

(c) The Bernoulli numbers were also used as an attempt to prove Fermat's last theorem (already discussed in a previous article/blog).

**Definition 1.** *An odd prime number $p$ is called* regular *if $p$ does not divide the numerator of $B_k$, for all even $k \leq p-3$. Any odd prime which is not regular is called* irregular*.*

The odd primes $3, 5, 7, \ldots, 31$ are all regular primes. The first irregular prime is 37. It is an open question: are there infinitely many regular primes? However, it is known that there are infinitely many irregular primes.

**Theorem 2** (Kummer, 1850)**.** *If $p$ is a regular odd prime then the equation*

$$a^p + b^p = c^p$$

*has no solution in $\mathbb{N}$.*