Why Complex Numbers?

T. Muthukumar
tmk@iitk.ac.in

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Complex numbers are introduced in high school mathematics today. A common question that springs up in our mind is:

> Why do we need complex numbers? We all know $x^2 > 0$ for all non-zero real numbers. Then why bother to seek a "number" x such that $x^2 = -1$?

First note that it is very clear that there is no real number satisfying $x^2 = -1$. Up to some point in the history when people encountered such an equation, they ignored it as being absurd. It was Gerolamo Cardano (1545) who pursued $\sqrt{-1}$ as an "imaginary" number and Rafael Bombelli (1572) developed as a number system. This article is an attempt to recall the motivation behind their interest in a "number" that was not "real".

The "imaginary" numbers were introduced as a mathematical tool to make life simple in the real world. In contrast to quadratic equations, it is *impossible* to have a formula to compute real roots of certain cubic equation without solving for $x^2 = -1$. Thus, it was observed that even to compute real roots of certain cubic equations one has to go outside the realm of real numbers.

It was known for, at least, 2300 years that the formula to compute roots of the quadratic equation $ax^2 + bx + c = 0$, for given $a, b, c \in \mathbb{R}$ with $a \neq 0$ is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

A straight forward way of deriving this formula is

$$ax^2 + bx + c \quad = \quad 0$$

$$
\begin{aligned}
x^2 + \frac{b}{a}x &= -\frac{c}{a} \\
x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 &= -\frac{c}{a} + \left(\frac{b}{2a}\right)^2 \\
\left(x + \frac{b}{2a}\right)^2 &= \frac{b^2 - 4ac}{4a^2} \\
x + \frac{b}{2a} &= \pm\frac{\sqrt{b^2 - 4ac}}{2a} \\
x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}
$$

The case $b^2 - 4ac < 0$ always corresponds to non-existence of roots. Geometrically, in this case, the graph of the quadratic function never intersects x-axis. This situation sits well with our understanding that it is absurd to consider square root of negative numbers.

Let us derive the formula for roots of quadratic equation in an alternate way. This alternate approach will help us in deriving a formula for cubic equation. Note that if $b = 0$ in the quadratic equation, then $ax^2 + c = 0$ and

$$x = \sqrt{\frac{-c}{a}}.$$

Let us seek for a ξ such that replacing x in $ax^2 + bx + c$ with $y - \xi$ reduces the equation to have the form $Ay^2 + C$. Set $x = y - \xi$. Then

$$
\begin{aligned}
ax^2 + bx + c &= a(y - \xi)^2 + b(y - \xi) + c \\
&= ay^2 - 2a\xi y + a\xi^2 + by - b\xi + c \\
&= ay^2 + (b - 2a\xi)y + a\xi^2 - b\xi + c.
\end{aligned}
$$

We shall choose ξ such that the coefficient of y is zero. Therefore, we choose $\xi = \frac{b}{2a}$. Thus,

$$ay^2 + \frac{b^2}{4a} - \frac{b^2}{2a} + c = ay^2 - \frac{b^2}{4a} + c.$$

The roots of $ay^2 - \frac{b^2}{4a} + c = 0$ are

$$y = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$$

and the roots of $ax^2 + bx + c = 0$ are

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}.$$

Let us employ the above approach to find a formula for roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$. We seek for a ξ such that replacing x in $ax^3 + bx^2 + cx + d$ with $y - \xi$ reduces the equation to have the form $Ay^3 + Cy + D$. Then

$$\begin{aligned} ax^3 + bx^2 + cx + d &= a(y-\xi)^3 + b(y-\xi)^2 + c(y-\xi) + d \\ &= ay^3 - 3a\xi y^2 + 3a\xi^2 y - \xi^3 + by^2 - 2b\xi y + b\xi^2 \\ &\quad + cy - c\xi + d \\ &= ay^3 + (b - 3a\xi)y^2 + (3a\xi^2 - 2b\xi + c)y + b\xi^2 \\ &\quad - \xi^3 - c\xi + d. \end{aligned}$$

Demanding ξ to be such that the coefficients of both y^2 and y vanish is too restrictive because in that case we must have $b - 3a\xi = 3a\xi^2 - 2b\xi + c = 0$. This would imply that $\xi = b/(3a)$ and $b^2 = 3ac$ which is too restrictive for a general cubic equation.

Let us eliminate the coefficient of y^2, by choosing $\xi = b/(3a)$. Then the cubic equation in y has the form

$$ay^3 + \left(\frac{3ac - b^2}{3a}\right) y + \left(\frac{b}{3a}\right)^3 (3a - 1) + \frac{3ad - bc}{3a} = 0.$$

Now, set

$$p := \frac{3ac - b^2}{3a}$$

and

$$q := \left(\frac{b}{3a}\right)^3 (3a - 1) + \frac{3ad - bc}{3a}$$

to make the equation appear as $ay^3 + py + q = 0$. To reduce this cubic equation to a quadratic equation in a new variable we use the *Vieta's substitution* which says define a new variable z such that

$$y = z - \frac{p}{3az}.$$

The Vieta's substituion can be motivated but we shall not digress in to this domain. Then the equation $ay^3 + py + q = 0$ transforms to a quadratic equation

$$27a^4(z^3)^2 + 27qa^3z^3 - p^3 = 0.$$

The roots of this equation are

$$z^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27a^2}}}{2a}.$$

At first glance, it seems like z has six solutions, but three of them will coincide. Thus, finding z leads to finding y and, hence, to x. Thus far, we have only derived a formula for roots of cubic equation. We are yet to address the question of complex numbers.

As a simple example, consider the cubic function $x^3 - 3x$. The roots of this equation can be easily computed by rewriting $x^3 - x = x(x^2 - 3) = x(x + \sqrt{3})(x - \sqrt{3})$ and hence has exactly three roots $0, \sqrt{3}, -\sqrt{3}$.

Let us try to compute the roots of $x^3 - x$ using the formula derived above. Note that $x^3 - x$ is already in the form with no x^2 term. Thus, $a = 1, p = -3, q = 0$ and $z^3 = \pm\sqrt{-1}$. The cubic equation has real roots but z is not the realm of real numbers. $\sqrt{-1}$ makes no sense, since square of any non-zero real number is positive. This is the motivation for complex numbers! We set $i := \sqrt{-1}$ and, hence, $i^2 = -1$. We introduce this notation to avoid using the property of square root, $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Negative numbers will not inherit this property because $\sqrt{-1}\sqrt{-1} = -1 \neq 1 = \sqrt{(-1)(-1)}$. With this setting, $z^3 = \pm i$. Now recall what you learnt in your course on complex numbers: the cube roots of i are $z = -i, \frac{\sqrt{3}+i}{2}, \frac{-\sqrt{3}+i}{2}$. Using this in the formula

$$x = z + \frac{1}{z}$$

we get $x = 0, \sqrt{3}, -\sqrt{3}$. The cube roots of $-i$ are $z = i, \frac{\sqrt{3}-i}{2}, \frac{-\sqrt{3}-i}{2}$ which yield the same roots. Expanding to complex number system helped us in solving a real cubic equation with only real roots! This little bold step helped us understand/expand in many ways. Complex numbers and complex functions has found application in engineering and other sciences, especially, via analytic functions and analytic continuations.