‘Convergence’ of Some Divergent Series!

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19 Jun 2014

The topic of this article, the idea of attaching a finite value to divergent series, is no longer a purely mathematical exercise. These finite values of divergent series have found application in string theory and quantum field theory (Casimir effect).

The finite sum of real/complex numbers is always finite. The infinite sum of real/complex numbers can be either finite or infinite. For instance,

$$\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$$

is finite and

$$\sum_{k=1}^{\infty} k = 1 + 2 + 3 + \ldots = +\infty.$$

If an infinite sum has finite value it is said to *converge*, otherwise it is said to *diverge*. Divergence do not always mean it grows to $\pm\infty$. For instance,

$$\sum_{k=1}^{\infty} k, \sum_{k=1}^{\infty} \frac{1}{k} \text{ and } \sum_{k=0}^{\infty} (-1)^k$$

are diverging, while

$$\sum_{k=1}^{\infty} \frac{1}{k^s} \quad s \in \mathbb{C} \text{ and } \Re(s) > 1,$$

is converging. The convergence of a series is defined by the convergence of its partial sum. For instance,

$$\sum_{k=1}^{\infty} k$$

diverges because its partial sum

$$s_n := \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

diverges, as $n \to \infty$. The geometric series

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \tag{1}$$

for all $|z| < 1$, because its partial sum

$$s_n := \frac{z^n - 1}{z - 1} \xrightarrow{n\to\infty} \frac{-1}{z-1}.$$

Note that the map $T(z) = \frac{1}{1-z}$ is well-defined between $T : \mathbb{C} \setminus \{1\} \to \mathbb{C}$. In fact, T is a holomorphic (complex differentiable) functions and, hence, analytic. The analytic function T exists for all complex numbers except $z = 1$ and, inside the unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$, T is same as the power series in (1). Thus, T may be seen as an *analytic continuation* of

$$\sum_{k=0}^{\infty} z^k$$

outside the unit disk, except at $z = 1$. A famous *unique continuation* result from complex analysis says that the analytic continuation is unique. Thus, one may consider the value of $T(z)$, outside the unit disk, as an 'extension' of the divergent series

$$\sum_{k=0}^{\infty} z^k.$$

In this sense, setting $z = 2$ in $T(z)$, the divergent series may be seen as taking the finite value

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \ldots = -1.$$

Similarly, setting $z = -1$ in $T(z)$, the oscillating divergent series may be seen as taking the finite value

$$\sum_{k=0}^{\infty} (-1)^k = \frac{1}{1-(-1)} = \frac{1}{2}.$$

This sum coincides with the Cesàro sum which is a special kind of convergence for series which do not diverge to $\pm\infty$. For instance,

$$\sum_{k=1}^{\infty} k$$

diverges in Cesàro sum too. The Cesàro sum of a series

$$\sum_{k=1}^{\infty} a_k$$

is defined as the

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=1}^{i} a_k \right).$$

Observe that the sequence

$$\frac{1}{n} \sum_{i=1}^{n} \left(\sum_{k=0}^{i-1} (-1)^k \right) \longrightarrow \frac{1}{2}.$$

Since T is not defined on 1, we have not assigned any finite value to the divergent series

$$1 + 1 + 1 + \ldots.$$

The *Riemann zeta function*

$$\zeta(z) := \sum_{k=1}^{\infty} \frac{1}{k^z}$$

converges for $z \in \mathbb{C}$ with $\Re(z) > 1$. Note that for $z \in \mathbb{C}$, $k^z = e^{z \ln(k)}$, where ln is the real logarithm. If $z = \sigma + it$ then $k^z = k^\sigma e^{it \ln(k)}$ and $|k^z| = k^\sigma$ since $|e^{it \ln(n)}| = 1$. Therefore, the infinite series converges for all $z \in \mathbb{C}$ such that $\Re(z) > 1$ (i.e. $\sigma > 1$) because it is known that $\sum_{k=1}^{\infty} k^{-\sigma}$ converges for all $\sigma > 1$ and diverges for all $\sigma \leq 1$.

The Riemann zeta function is a special case of the *Dirichlet series*

$$D(z) := \sum_{k=1}^{\infty} \frac{a_k}{k^z}$$

with $a_k = 1$ for all k. Note that for $z = 2$, we get the *Basel series*

$$\zeta(2) := \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

and $\zeta(3) = 1.202056903159594\ldots$ called the *Apéry's constant.* In fact, the following result of Euler gives the value of Riemann zeta function for all even positive integers.

Theorem 1 (Euler)**.** *For all $k \in \mathbb{N}$,*

$$\zeta(2k) = (-1)^{k+1} \frac{(2\pi)^{2k}}{2(2k)!} B_{2k}$$

where B_{2k} is the $2k$-th Bernoulli number.

If $\zeta(z)$ is well-defined for $\Re(z) > 1$ then $2^{1-z}\zeta(z)$ is also well-defined for $\Re(z) > 1$. Therefore,

$$\zeta(z)(1 - 2^{1-z}) = \sum_{k=1}^{\infty} (-1)^{k-1} n^{-z} =: \eta(z).$$

Observe that $\eta(z)$ is also a Dirichlet series. It can be shown that $(1-2^{1-z})\eta(z)$ is analytic for all $\Re(z) > 0$ and $\Re(z) \neq 1$. Thus, we have analytically continued $\zeta(z)$ for all $\Re(z) > 0$ and $\Re(z) \neq 1$. There is a little work to be done on the zeroes of $1-2^{1-z}$ but is fixable. In the strip $0 < \Re(z) < 1$, called the *critical strip*, the Riemann zeta function satisfies the relation

$$\zeta(z) = 2^z (\pi)^{z-1} \sin\left(\frac{z\pi}{2}\right) \Gamma(1-z)\zeta(1-z).$$

This relation is used to extend the Riemann zeta function to z with non-positive real part, thus, extending to all complex number $z \neq 1$. Setting $z = -1$ in the relation yields

$$\zeta(-1) = \frac{1}{2\pi^2} \sin\left(\frac{-\pi}{2}\right) \Gamma(2)\zeta(2) = \frac{1}{2\pi^2}(-1)\frac{\pi^2}{6} = \frac{-1}{12}.$$

Since $\zeta(-1)$ is extension of the series

$$\sum_{k=1}^{\infty} \frac{1}{k^z}$$

one may think of $\zeta(-1)$ as the finite value corresponding to $1+2+3+\ldots$. The value of $\zeta(0)$ is obtained limiting process because it involves $\zeta(1)$ which is not defined. However, there is an equivalent formulation of Riemann zeta function for all $z \in \mathbb{C} \setminus \{1\}$ as

$$\zeta(z) = \frac{1}{1-2^{1-z}} \sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{i=0}^{k} (-1)^i \binom{k}{i} (i+1)^{-z}$$

with a simple pole and residue 1 at $z = 1$.

Recall that the analytic continuation T of the geometric series is not defined for $z = 1$ and, hence, we could not assign a finite value to

$$T(1) = 1 + 1 + 1 + \ldots.$$

But, setting $z = 0$ above,

$$T(1) = \zeta(0) = -\sum_{k=0}^{\infty} \frac{1}{2^{k+1}} \sum_{i=0}^{k} (-1)^i \binom{k}{i} = -\sum_{k=0}^{\infty} \frac{\delta_{0k}}{2^{k+1}} = -\frac{1}{2}$$

where δ_{0k} is the Kronecker delta defined as

$$\delta_{0k} = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases}$$

The harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

corresponds to $\zeta(1)$ which is not defined. The closest one can conclude about $\zeta(1)$ is that

$$\lim_{n\to\infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \ln(n) \right) = \gamma$$

where γ is the Euler-Mascheroni constant which has the value $\gamma = 0.57721566\ldots$.