Proceedings of DETC'97
1997 ASME Design Engineering Technical Conferences
September 14-17, 1997, Sacramento, California

DETC97/VIB-4141

ON WAVELET-ANALYZED SEISMIC RESPONSE OF SDOF SYSTEMS

Biswajit Basu
Graduate Student
Department of Civil Engineering
Indian Institute of Technology, Kanpur
Kanpur-208016 INDIA
Email: bbasu@iitk.ernet.in

Dr. Vinay K. Gupta [1]
Associate Professor
Department of Civil Engineering
Indian Institute of Technology, Kanpur
Kanpur-208016 INDIA
Email: vinaykg@iitk.ernet.in

ABSTRACT

This paper presents the technique using wavelet analysis to solve the response of a single-degree-of-freedom (SDOF) system subjected to transient excitations such as earthquakes. A brief review of the wavelet transform and its discretization, time-frequency representation of the ground motion and the input-output relation for a SDOF system is presented. A new orthogonal basis function has been used for describing the temporal and frequency characteristics of the input and the response in case of a recorded accelerogram during 1971 San Fernando earthquake. Also, the input-output relation in the wavelet domain has been illustrated through comparison with the time domain simulation results.

Key words: Non-stationarity, wavelets, Littlewood-Paley basis, seismic response.

INTRODUCTION

The problem of analyzing dynamical systems subjected to transient excitations has been of great interest to the engineering community due to its common occurrence in several practical situations, e.g., in the case of structures subjected to seismic excitations. It has been established in vibration studies that a system responds in a transient manner before reaching its steady state when it is subjected to a periodic excitation. This is true even for a stationary stochastic excitation (Caughey and Stumpf (1961)). However, for the earthquake excitations, the response is very different from that in the steady-state situation. Due to the dispersion of the seismic waves, the earthquake excitations have non-stationary frequency characteristics. A rigorous treatment for such situations becomes complicated both in analytical and computational terms. Hence, researchers have tried to extend the stationary description of ground motions to such cases in an approximate manner and to provide workable solutions (e.g., see Cornell (1964), Amin and Ang (1968), Shinozuka and Sato (1967), Iyengar and Iyengar (1969), Ruiz and Penzien (1971), Boore (1983), Quek et al. (1990)). Several researchers have also used those models to obtain the responses of single-degree-of-freedom (SDOF) and multi-degree-of-freedom (MDOF) systems (e.g., see Gasparini (1979), Gasparini and Debchaudhury (1980), Muscolino (1988), Borino et al. (1988), Iwan and Hou (1989), Senthilnathan and Lutes (1991)) in an approximate manner. However, most of these efforts have not considered the temporal variations in the frequency characteristics of the input excitation which may be crucial in calculating accurate responses.

Wavelet analysis provides a convenient way to tackle the problems associated with ground motion non-stationarity. Whereas the Fourier transform provides only an 'average' information regarding the frequency content of the process, the wavelet analysis accounts for the temporal variations in its frequency content. Wavelet basis functions used for such an analysis are locally decaying functions with reasonably good time-frequency localization properties. A large amount of research has been carried out in the past few years on this subject (e.g., see Goupillaud et al. (1984), Grossman and Morlet (1984), Lemarié (1988), Mallat (1989), Mallat and Zhong (1990), Meyer (1992), Daubechies (1992), Chui (1992)). Newland (1993) developed the Discrete Wavelet Transform (DWT) and the

[1] Address all correspondence to this author.

Fast Wavelet Transform (FWT) algorithms, and Newland (1994a,b) successfully applied the wavelet analysis to analyze vibration problems. This has greatly enhanced the possibility of wavelet transform application to engineering problems with sufficient computational ease. Recently, Basu and Gupta (1996) have derived the input-output relationship in wavelet domain for a SDOF system oscillator subjected to seismic excitation.

This paper reviews the input-output relationship formulation of Basu and Gupta (1996) based on the use of Mexican hat basis function. Then, a new orthogonal basis function which is a modified version of the Harmonic Wavelet basis by Newland (1993) and the Littlewood-Paley basis has been used and the time-frequency characteristics of the input and the output in a SDOF system have been studied. For the purpose of illustration, a numerical study has been carried out in case of the recorded 1971 San Fernando ground motion, S16E component, at Pacoima Dam site.

WAVELET REPRESENTATION AND TIME-FREQUENCY CHARACTERISTICS

Let us consider a function, $f(t)$ which may represent a ground motion or the response of a dynamical system to this motion, such that

$$\int_{-\infty}^{\infty} |f(t)|^2 \mathrm{d}t < \infty \quad . \tag{1}$$

The wavelet transform pair for $f(t)$ can be written as (see, e.g., Daubechies (1992))

$$W_\psi f(a,b) = \langle f, \psi_{a,b} \rangle = \frac{1}{|a|^{1/2}} \int_{-\infty}^{\infty} f(t)\psi\left(\frac{t-b}{a}\right) \mathrm{d}t \tag{2}$$

and

$$f(t) = \frac{1}{2\pi C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{a^2} W_\psi f(a,b)\psi\left(\frac{t-b}{a}\right) \mathrm{d}a\,\mathrm{d}b \tag{3}$$

where, $W_\psi f(a,b)$ is the wavelet transform of $f(t)$ with respect to the wavelet basis function $\psi(.)$. Further, C_ψ is a scalar defined by

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} \mathrm{d}\omega < \infty \ , \tag{4}$$

where,

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(t) e^{-i\omega t} \mathrm{d}t \tag{5}$$

is the Fourier transform of the wavelet basis function. In the expressions given by Eq. (2) and (3), the variables a and b are the scaling and translation parameters respectively. The variable a scales the wavelet basis function such that this convolutes (in Eq. (2)) with the function $f(t)$ in different temporal windows, and the variable b acts as a shift operator by centering the window at several locations. Interpreting in frequency domain, the variation in a gives basis function with different frequency contents. The possibility whether those frequencies are present at a particular stretch of the signal is analyzed by centering the basis function with the help of variable b.

For numerical calculations, it is convenient to discretize Eqs. (2) and (3) by assuming $a_j = \sigma^j$ and $b_j = (j-1)\Delta b$ such that

$$\Delta b_j = [(b_{j+1} - b_j) + (b_j - b_{j-1})]/2 = \Delta b \tag{6}$$

and

$$\Delta a_j = [(a_{j+1} - a_j) + (a_j - a_{j-1})]/2 = \frac{a_j}{2}\left(\sigma - \frac{1}{\sigma}\right) \ . \tag{7}$$

On using Eqs. (6) and (7), the discretized form of Eq. (3) becomes

$$f(t) = \sum_i \sum_j \frac{K\Delta b}{a_j} W_\psi f(a_j, b_i)\psi_{a_j,b_i}(t) \tag{8}$$

with

$$K = \frac{1}{4\pi C_\psi}\left(\sigma - \frac{1}{\sigma}\right) \ . \tag{9}$$

The above discretization is similar to that used by Basu and Gupta (1996) close to the scheme of Alkemade (1993).

To discuss the time-frequency characteristics of $f(t)$, let us use the following expression for the inner product of f (see Daubechies (1992))

$$\langle f, f \rangle = \int_{-\infty}^{\infty} f^2(t)\mathrm{d}t = \frac{1}{2\pi C_\psi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{[W_\psi f(a,b)]^2}{a^2} \mathrm{d}a\,\mathrm{d}b \, . \tag{10}$$

On discretizing the integral as above, it is possible to obtain the instantaneous mean square response as (Basu and Gupta (1996))

$$f^2|_{t=b_i} = K \sum_j \frac{1}{a_j} [W_\psi f(a_j, b_i)]^2 \ . \quad (11)$$

Eq. (11) leads to the integral square response or the the growth in the temporal energy upto time $t = b_i$ as

$$E(t)|_{t=b_i} = \int_0^{t=b_i} f^2(\tau) \mathrm{d}\tau = K \sum_j \sum_{l=0}^{i} \frac{[W_\psi f(a_j, b_l)]^2}{a_j} \ . \quad (12)$$

Similarly, the integral square response or the growth in the temporal energy upto time $t = b_i$ for the frequency band corresponding to the dilation parameter $a_j = \sigma^j$ is obtained as

$$E(t)|_{t=b_i}^{j} = K \sum_{l=0}^{i} \frac{[W_\psi f(a_j, b_l)]^2}{a_j} \ . \quad (13)$$

INPUT-OUTPUT RELATIONSHIP

We now consider a linear SDOF oscillator with natural frequency, ω_n, and viscous damping co-efficient, ζ, and subjected to ground acceleration, $\ddot{u}_g$, at its base. The equation of motion for this system is

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = -\ddot{u}_g \quad (14)$$

where, x, $\dot{x}$ and $\ddot{x}$ respectively denote the relative displacement, relative velocity and relative acceleration of the oscillator with respect to the ground.

On obtaining wavelet transform of both sides of Eq. (14) with respect to the wavelet basis, $\psi_{a,b}(t)$, and on applying integration by parts, it is possible to write (Basu and Gupta (1996))

$$\frac{\partial^2}{\partial b^2} W_\psi x(a,b) + 2\zeta\omega_n \frac{\partial}{\partial b} W_\psi x(a,b) + \omega_n^2 W_\psi x(a,b) = W_\psi u(a,b); \quad a, b \in R \ . \quad (15)$$

This is the differential equation relating the wavelet coefficients, $W_\psi x(a,b)$, of the response with the wavelet coefficients, $W_\psi u(a,b)$, of the ground acceleration with respect to the same basis function. It may be noted that both Eqs. (14) and (15) have the same form except that Eq. (15) is a partial differential equation in b. By discretizing, Eq. (15) can be converted, for a particular value of a_j, into an ordinary differential equation identical to Eq. (14). The response process in this differential equation however has a time-invariant frequency content, even though the process in Eq. (14) is non-stationary in nature. Further, the wavelet domain relationship given by Eq. (15) has the advantage over the time-domain relationship in Eq. (14) as it provides the time-dependent frequency content information in the response directly through the wavelet co-efficients, $W_\psi x(a,b)$. This equation also provides us a clear picture of how the non-stationarity in input propagates through the dynamical system. Thus, Eq. (15) affords the development of a more realistic non-stationary stochastic formulation, provided the ground motion processes are characterized via wavelet co-efficients.

It may be noted that even though Eqs. (14) and (15) have been obtained for the SDOF systems, those are also applicable to the linear MDOF structural systems subjected to base acceleration. In such systems, any response quantity of interest may be estimated by a linear combination of the responses of a set of SDOF systems. In some cases, e.g., those of the base-excited multistoried buildings, these equations are directly applicable as the fundamental mode often dominates the total response.

WAVELET BASIS

The wavelet basis function used by Basu and Gupta (1996) was Mexican hat which had a reasonable time-frequency localization property but was non-orthogonal. A new orthogonal basis has been used in this paper. Following guidelines were followed for constructing the wavelet function: (i) the basis should be orthogonal, (ii) the Fourier transform of the basis function should be defined on finite support so as to avoid overlap in frequency content between different bands corresponding to different a_j's, and (iii) the fluctuations in the Fourier transform amplitudes with frequency should be captured while the basis function should be temporally decaying to represent local variations. Hence, the wavelet function in frequency domain has been characterized by the Fourier transform

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2(p-1)\pi}} \ , \quad \pi \le |\omega| \le p\pi$$
$$= 0 \ , \quad \text{otherwise} \ . \quad (16)$$

It may be noted that for $p = 2$, this is the Littlewood-Paley basis which may not be suitable to characterize the fluctuations in the Fourier amplitudes of the realistic ground motions. In fact, the value of p should be less than 2, and based on experiences from several recorded motions, a value of $p = 2^{1/4}$ has been found to be suitable. Further, even though it is very common to use a dyadic wavelet ($\sigma = 2$), any arbitrary choice of σ may not satisfy the required criteria as above. Interestingly, assigning $\sigma = p$ satisfies these criteria and also, on taking the inverse Fourier transform of Eq. (16), we obtain the basic or the mother wavelet function

$$\psi(t) = \frac{1}{\pi\sqrt{(\sigma-1)}} \frac{\sin\sigma\pi t - \sin\pi t}{t} \quad . \tag{17}$$

It may be further noted that the Harmonic wavelet by Newland (1993) is a complex version of the above wavelet with the Fourier amplitudes defined only over the positive frequency axis for $p = 2$.

NUMERICAL ILLUSTRATIONS

Fig. 1 shows the S16E component of the recorded motion at the Pacoima Dam site for the 1971 San Fernando earthquake and the temporal variations in the wavelet co-efficients (for j = –17, –12, –7, –2 and 4) with the variable, b. The wavelet co-efficients are obtained according to Eq. (2) for i varying from 1 to 1024 over the integers. The value of Δb has been chosen to be 0.02. The basis function as in Eq. (17) has been used for obtaining the wavelet co-efficients. It may be noted that the wavelet co-efficient functions corresponding to j = –17, –12, –7, –2 and 4 respectively form narrow-band stationary processes with the frequency bands, 59.78–71.09 rad/sec, 25.13–29.89 rad/sec, 10.57–12.57 rad/sec, 4.44–5.28 rad/sec, and 1.57–1.87 rad/sec, and that the original ground motion can be thought of as composed of these processes. Further, each of these processes is a amplitude modulated stationary process and the instantaneous mean-square value of the parent process can be obtained as the sum of the instantaneous mean-square value in each case. These observations can be seen clearly in Fig. 1. It may also be seen that most of the energy is contributed from the frequency bands corresponding to j = –12 and –7 (the wavelet co-efficient squared times the factor $1/\sigma^j$ represents the mean-square energy). This is consistent with the presence of peak at around 0.2 to 0.4 sec period in the pseudo spectral acceleration (PSA) ($= \omega_n^2 |x(t)|_{\max}$) spectrum of the example ground motion for 5% damping (see Fig. 2). The time-domain analysis here has been carried out by using a semi-analytical numerical scheme of Nigam and Jennings (1969) with the time-step of 0.02 sec.

To illustrate the input-output relationship presented in this paper, the pseudo root-mean-square acceleration

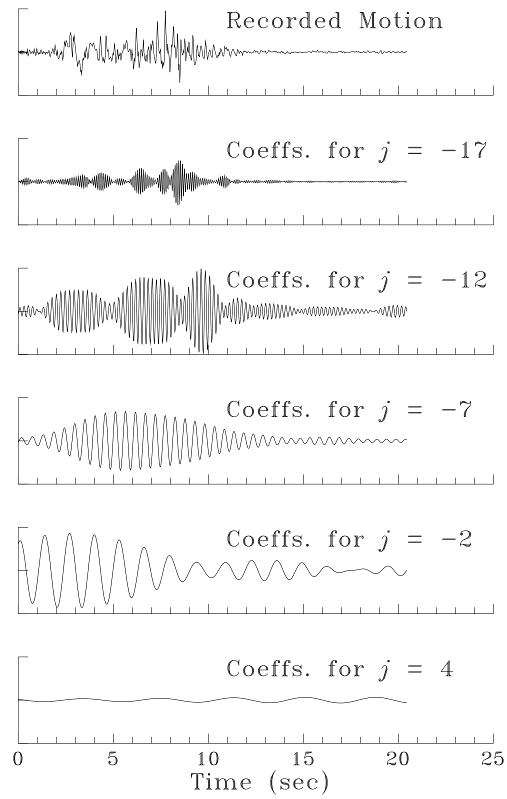

Figure 1. TIME-HISTORY AND TEMPORAL VARIATIONS OF WAVELET CO-EFFICIENTS FOR THE 1971 SAN FERNANDO EARTHQUAKE MOTION.

(PRMSA) spectrum of the example ground motion for 5% damping as obtained from the wavelet-domain analysis has been compared with that obtained via time-domain analysis (see Fig. 3). PRMSA is defined as the natural frequency squared times the root-mean-square (r.m.s.) value of the relative displacement response of the SDOF oscillator. It may be seen that except for the high-period zone, the two results are generally in good agreement. The discrepancies in the high-period zone will be understood when we will discuss the time-frequency characteristics in more detail.

To see how the different bands of energy contribute to the output mean-square energy, the cumulative contributions to the PRMSA upto j = –12, –7, –2 and 4 have been compared in Fig. 4 for 5% damping oscillators. It may be seen that a specific band of frequencies contributes only locally to the temporal mean square energy of the response. For example, the band of frequencies corresponding to j values upto –12 solely contributes to the oscillators with natural periods between 0.1 to 0.2 sec. Similarly, for oscillators

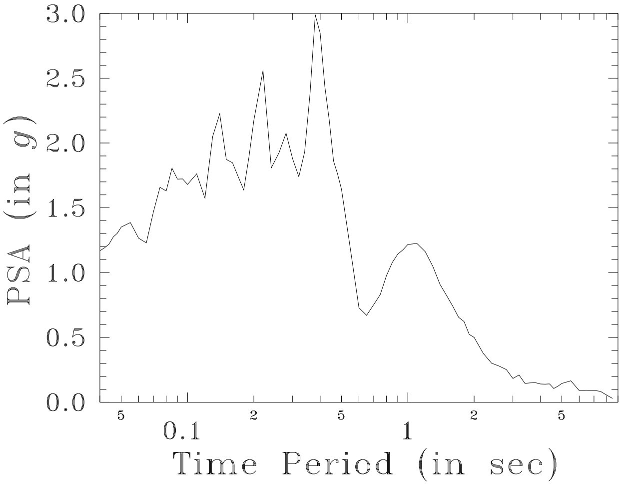

Figure 2. 5% DAMPING PSA SPECTRUM FOR THE 1971 SAN FERNANDO EARTHQUAKE MOTION.

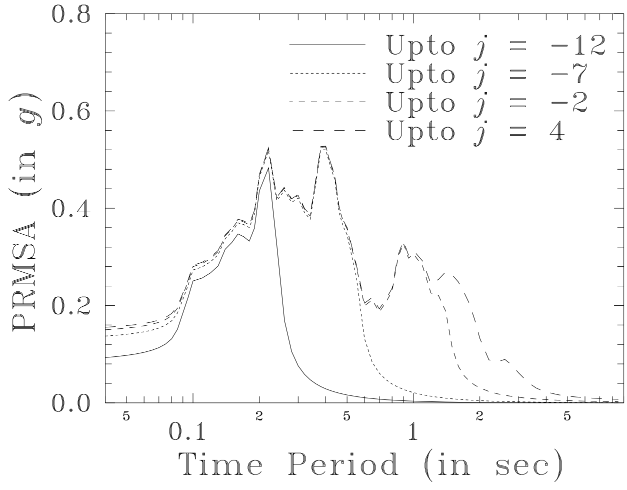

Figure 4. CONTRIBUTIONS OF DIFFERENT ENERGY BANDS TO THE R.M.S. SPECTRA.

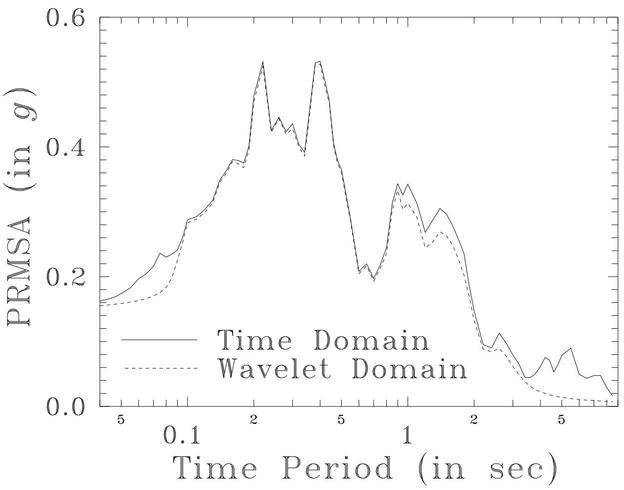

Figure 3. COMPARISON OF THE R.M.S. SPECTRA FROM TIME DOMAIN AND WAVELET DOMAIN ANALYSES.

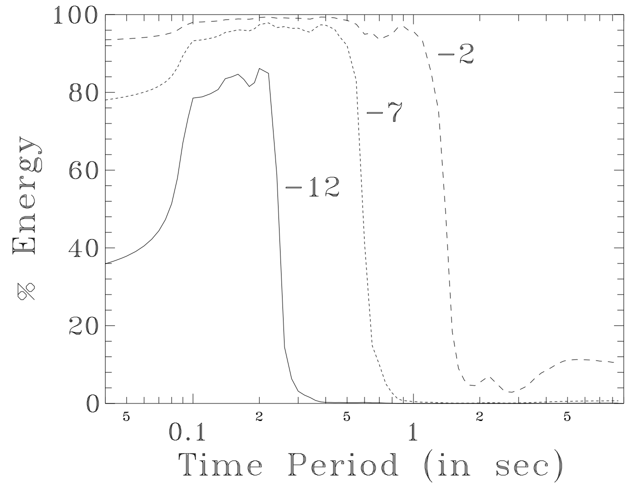

Figure 5. PERCENTAGE CONTRIBUTIONS OF DIFFERENT ENERGY BANDS TO THE TOTAL RESPONSE ENERGY.

with periods between 0.2 to 0.5 sec, the frequency bands corresponding to j values between –12 and –7 mainly contribute. The only exception is the low period range where almost all the bands contribute. This is because the oscillators in this range are stiffer as compared to all the bands, thus resulting in pseudo-static response in each band. Fig. 5 again shows the observed effects of localized contribution of certain bands of frequencies to the response in form of cumulative percentage energy contributions. Thus, it can be concluded that a response spectrum is separable and depending on the oscillator period, only a few bands may have to be considered in computing the response.

Fig. 6 shows the plots of the normalized integrated square of the response (i.e., $\int_0^t x^2(\tau)\mathrm{d}\tau / \int_0^T x^2(\tau)\mathrm{d}\tau$, T being the total duration) with time, t for different oscillators of period 0.1, 0.3 and 1.0 sec and damping 5% of the critical. These curves represent the growth in the temporal energy and are compared with those obtained from the wavelet co-efficients by using Eq. (12). Close agreement between the two sets of curves indicates that the wavelet co-efficients can adequately be used to represent the growth in the temporal energy. There is slight deviation for the oscillator with 1.0 sec natural period as also seen in case of the PRMSA values in the high period range. To further investigate this discrepency, the growth in temporal energy has been plotted in Fig. 7 for three different bands of frequencies cor-

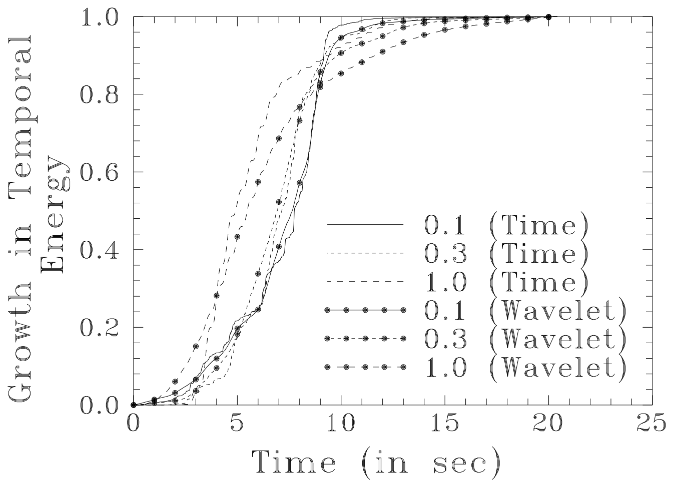

Figure 6. COMPARISON OF THE GROWTH IN TEMPORAL ENERGY IN TIME DOMAIN AND WAVELET DOMAIN FOR OSCILLATORS WITH 0.1, 0.3 AND 1.0 SEC PERIODS.

responding to $j = -12$, -7 and -2 in case of the 0.3 sec oscillator. These results are obtained both by the wavelet domain calculations as outlined in this paper and by taking the wavelet transform of the response time-history of the oscillator. It is seen that the matching between the two sets of results is perfect for $j = -12$ and -7, while for $j = -2$, the curves start deviating from each other. This suggests that the differential relation as in Eq. (15) becomes inaccurate with the increasing value of j. Actually, with increasing j,

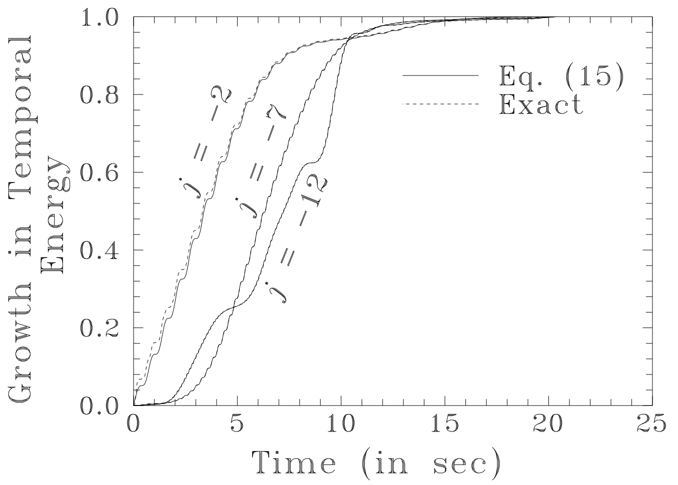

Figure 7. COMPARISON OF THE GROWTH IN TEMPORAL ENERGY USING EXACT ANALYSIS AND EQ. (15) FOR DIFFERENT j VALUES.

the wavelet functions become more and more dilated, and in case of finite duration signals, the edge effects in the integration by parts cause inaccuracy in the calculations. Since the contributions to the response in the high period range are from the higher values of j as seen in Fig. 5, there result slight discrepancies in the responses of the long period oscillators. It may also be seen in Fig. 7 that the the growth of temporal energy follows different patterns for different band of frequencies. This clearly implies that the amplitude modulating functions of strong ground motions are in fact frequency-dependent, not frequency-independent as commonly assumed in non-stationary modeling of ground motions.

CONCLUSIONS

An application of wavelet-based technique in case of the seismic response of a linear SDOF oscillator has been reviewed in this paper. A new wavelet basis function has been considered, and the input-output relation in wavelet-domain has been validated by comparison with the time-domain simulation results. It has been seen that the wavelet co-efficients adequately describe the time-frquency characteristics of the input as well as of the response. It has also been observed that the new basis decomposes any aperiodic (broad-banded) process into narrow-banded processes in a statistical sense. Further, it has been shown that the practice of modeling the earthquake ground motions via the (frequency-independent) amplitude modulating functions may be an oversimplification.

ACKNOWLEDGMENT

This research investigation was supported by the Department of Atomic Energy, Government of India under grant number 11/16/94-G. The authors are grateful for this support.

REFERENCES

Alkemade, J.A.H., 1993, “The finite wavelet transform with an application to seismic processing,” Wavelets: An Elementary Treatment of Theory and Applications edited by T.H. Koornwinder, World Scientific, New Jersey, pp. 183–208.

Amin, M. and A.H.-S. Ang, 1968, “Nonstationary stochastic model of earthquake motions,” *J. Eng. Mech. Div., Proc. ASCE,* Vol. 94, No. 2, pp. 559–583.

Basu, B. and V. K. Gupta, 1996 “Stochastic Seismic Response of SDOF Systems Through Wavelets,” (submitted for publication).

Boore, D.M., 1983, “Stochastic simulation of high-frequency ground motions based on seismological models of the radiated spectra," *Bull. Seism. Soc. Amer.,* Vol. 73, No. 6, pp. 1865–1894.

Borino, G., M. Di Paola, and G. Muscolino, 1988, "Nonstationary spectral moments of base excited MDOF systems," *Earthq. Eng. Struct. Dyn.,* Vol. 16, pp. 745–756.

Caughey, T.K. and H.J. Stumpf, 1961, " Transient response of a dynamic system under random excitation," *J. Appl. Mech., Trans. ASME,* Vol. 28(E), pp. 563–566.

Chui, C.K., 1992, " An Introduction to Wavelets," Academic Press, Inc., San Diego, California.

Cornell, C.A., 1964, " Stochastic process models in structural engineering," Tech. Report No. 34, Dept. of Civil Eng., Stanford University, California.

Daubechies, I., 1992, " Ten Lectures on Wavelets," Society for Industrial & Applied Mathematics, Philadelphia, Pennsylvania.

Gasparini, D.A., 1979, " Response of MDOF systems to nonstationary random excitation," *J. Eng. Mech. Div., Proc. ASCE,* Vol. 105, No. 1, pp. 13–27.

Gasparini, D. and A. DebChaudhury, 1980, " Dynamic response to nonstationary nonwhite excitation," *J. Eng. Mech. Div., Proc. ASCE,* Vol. 106, No. 6, pp. 1233–1248.

Gaupillaud, P., A. Grossmann, and J. Morlet, 1984, "Cycle-octave and related transforms in seismic signal analysis," *Geoexploration,* Vol. 23, pp. 85–102.

Grossmann, A. and J. Morlet, 1984, "Decomposition of Hardy functions into square integrable wavelets of constant shape," *SIAM J. Math. Anal.,* Vol. 15, pp. 723–736.

Iwan, W.D. and Z.K. Hou, 1989, "Explicit solutions for the response of simple systems subjected to nonstationary random excitation," *Struct. Safety*, Vol. 6, pp. 77–86.

Iyengar, R.N. and K.T.S. Iyengar, 1969, " A nonstationary random process model for earthquake accelerograms," *Bull. Seism. Soc. Amer.,* Vol. 59, No. 3, pp. 1163–1188.

Lemarié, P.G., 1988, "Une nouvelle base d'ondelettes de L^2 ($\mathcal{R}^n$)," *J. de Math. Pures et Appl.,* Vol. 67, pp. 227–236.

Mallat, S., 1989, "Multiresolution approximation and wavelets," *Trans. Amer. Math. Soc.,* Vol. 315, pp. 69–88.

Mallat, S. and S. Zhong, 1990, "Signal characterization from multiscale edges," *Proc. Tenth Int. Conf. on Pattern Recognition,* IEEE Comput. Soc. Press, Atlantic City, pp. 891–896.

Meyer, Y., 1992, "Wavelets and Operators," Cambridge University Press, New York.

Muscolino, G., 1988, "Nonstationary envelope in random vibration theory," *J. Eng. Mech. (ASCE),* Vol. 114, No. 8, pp. 1396–1413.

Newland, D.E., 1993, "An Introduction to Random Vibrations," Spectral and Wavelet Analysis, Longman, U.K.

Newland, D.E., 1994a, "Wavelet analysis of vibration, Part 1: Theory," *J. Vib. Acoust., Trans. ASME,* Vol. 116, pp. 409–416.

Newland, D.E., 1994b, "Wavelet analysis of vibration, Part 2: Wavelet maps," *J. Vib. Acoust., Trans. ASME,* Vol. 116, pp. 417–425.

Nigam, N.C. and P.C. Jennings, 1969, "Calculation of response spectra from strong-motion earthquake records," *Bull. Seism. Soc. Amer.,* Vol. 59, No. 2, pp. 909–922.

Quek, S.-T., Y.-P. Teo, and T. Balendra, 1990, "Nonstationary structural response with evolutionary spectra using seismological input model," *Earthq. Eng. Struct. Dyn.,* Vol. 19, pp. 275–288.

Ruiz, P. and J. Penzien, 1971, "Stochastic seismic response of structures," *J. Struct. Div., Proc. ASCE,* Vol. 97, No. 4, pp. 441-456.

Senthilnathan, A. and L.D. Lutes, 1991, "Nonstationary maximum response statistics for linear structures," *J. Eng. Mech. (ASCE),* Vol. 117, No. 2, pp. 294–311.

Shinozuka, M. and Y. Sato, 1967, "Simulation of nonstationary random process," *J. Eng. Mech. Div., Proc. ASCE,* Vol. 93, No. 1, pp. 11–40.