Fracture mechanics

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The early work of Griffith and others on crack propagation has in recent years been extended and systematized to form the science of Fracture Mechanics. In dealing with stationary cracks its principal theoretical tools are the stress intensity factor which characterizes the elastic field near a crack tip and the energy release rate associated with the extension of a crack. The behaviour of fast-moving cracks presents some intriguing theoretical problems which are only beginning to be solved.

1. Introduction

The scientific study of fracture began with Griffith's\(^1\) work on cracks in brittle solids like glass and its extension by Orowan\(^2\) to metals and other more or less ductile materials. More recently, largely as a result of the work of Irwin,\(^3\) their ideas have been further developed and formalized into the science of Fracture Mechanics, considered by its exponents to be an independent discipline, and now supported by two journals (the *International Journal of Fracture Mechanics* and *Engineering Fracture Mechanics, an International Journal*), a seven-volume treatise,\(^4\) and numerous conferences.

The main task of fracture mechanics is a conservative one, to learn enough about cracks to prevent them from appearing or growing, or at least to be in a position to predict their behaviour. On the other hand, the phenomenon of brittle fracture is, or has been, used constructively in various technical processes, ranging from flint-knapping\(^5\) through blasting and comminution\(^6\) (the grinding of powders) to the shattering of aircraft canopies\(^7\) to allow the pilot to eject. The ideas of fracture mechanics have also been used in geology, in particular in the study of earthquakes; so far this is largely an observational branch of the subject.

The fracture of materials must ultimately be explained in terms of their atomic, or at least microscopic constitution. This topic was the subject of a recent article by Hull\(^8\) in this journal. Here we shall be concerned with some topics drawn from the macroscopic side of fracture mechanics which uses concepts drawn from the mechanics of continua to set up the formal framework within which the detailed physical processes of fracture must act, and also to

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provide practical criteria for estimating fracture toughness, the resistance of a material to the propagation of cracks.

The following is an outline of the contents of the present article. Section 2 is concerned with a problem fundamental to fracture mechanics, the determination of the stress field which appears around a crack when the body containing it is subjected to external forces. For most purposes it is enough to know the elastic field near the tip of a crack. This field can be completely specified by three constants, the stress intensity factors.

Section 3 is concerned with energetics. Griffith emphasized the importance of the energy flow associated with crack propagation. In modern fracture mechanics it is handled with the help of the energy release rate or crack extension force, a quantity which measures the rate at which the total energy of the system, made up of the cracked body and the external loading mechanism, varies with the position of a crack tip.

The stress intensity factor and the energy release rate are the two characteristic concepts of modern fracture mechanics and they provide an adequate basis for the discussion of the onset of fracture and the slow propagation of cracks. However, a crack which is actually running may reach a speed which is a sizeable fraction of the velocity of elastic waves in the material, and then the static theory is inadequate. Section 4 is concerned with the theory of moving cracks.

2. Stresses around cracks: the stress intensity factor

When a specimen containing a crack is acted on by external forces a complicated system of stresses is induced in its interior. The stresses are particularly large near the ends of the crack and for most of the purposes of fracture mechanics it is only necessary to know the form of the stress field near the crack tips. In order to give some body to the general discussion, and because stress analysis bulks so large in fracture mechanics, it seems worthwhile outlining the elements of the mathematical theory and carrying it far enough to find the elastic field at a crack tip. (A more detailed, but straightforward account may be found in reference 9.) Only a little of the formal apparatus of the theory of elasticity is required. In fact all we shall really need are the equations of equilibrium

\[
\begin{align*}
\nabla^2 u + \alpha \partial e/\partial x &= 0, \\
\nabla^2 v + \alpha \partial e/\partial y &= 0 \\
\nabla^2 w + \alpha \partial e/\partial z &= 0
\end{align*}
\]

with \( \alpha = 1/(1-2v) \) \hfill (2.1)

written in terms of the displacement vector \( \mathbf{u} \) with Cartesian components \( u, v, w \) and the relation between the stress components \( \sigma_{xx}, \sigma_{xy}, \ldots \) and the displacement, typified by

\[
\begin{align*}
\sigma_{xx} &= \lambda e + \mu \partial u/\partial x, \\
\sigma_{xy} &= \mu (\partial u/\partial y + \partial v/\partial x)
\end{align*}
\]

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Here
\[ e = \text{div} \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial w}{\partial z} \]
(2.2)
is the dilatation (the fractional change of volume on deformation), \( \mu \) is the shear modulus, \( \nu \) Poisson's ratio and \( \lambda \) stands for \( 2\mu\nu/(1-2\nu) \).

It is first necessary to introduce some standard nomenclature. Fig. 1(a) represents a body containing a crack. In Fig. 1(b) the crack has been opened up by a tension \( T \) transverse to its length, producing a state of mode I deformation. In Fig. 1(c) the shear stress \( S \) has made the faces of the crack slide over one another parallel to the plane of the figure and we have mode II deformation. In Fig. 1(d) a shear stress \( \tau \) (represented by the conventional signs for arrow heads and tails) is applied so as to make the crack faces slide over each other perpendicular to the plane of the figure producing a state of mode III deformation. If the left-hand end of the crack had reached all the way to the surface, the resulting configuration would be roughly what one gets when a telephone directory is torn in half, hence the alternative name tearing mode for mode III. For more complicated loading, even if it is non-uniform, the elastic field near the tip of a crack will still be the sum of simple mode I, II and III deformations.

For the moment we shall only be concerned with two-dimensional states of strain in which the elastic field is independent of the \( z \)-coordinate. Modes I and II, for which \( w \) is zero and the displacement \((u, v)\) is in the plane of the figure,
are examples of plane strain. Mode III, with the displacement \((0, 0, w)\) perpendicular to the plane of the figure, is an example of anti-plane strain deformation.

Because for it the theory is particularly simple we begin with mode III. Since \(u\) and \(v\) are zero and \(w\) is independent of \(z\), the dilatation (2.2) vanishes and the equations (2.1) reduce to one, namely

\[
\nabla^2 w = 0,
\]

which says that the surviving displacement component is a harmonic function of \(x, y\). The only non-zero stress components are

\[
\sigma_{xx} = \mu \frac{\partial w}{\partial x}, \quad \sigma_{yy} = \mu \frac{\partial w}{\partial y}.
\]

![Diagram](image)

Fig. 2

To deal with the elastic field near a crack tip we set up the Cartesian and polar coordinates shown in Fig. 2. The displacement can then be expanded in terms of the basic set of harmonic functions

\[
r^n \sin n\theta, \quad r^n \cos n\theta
\]

(2.3)

Obviously \(w\) will be odd in \(\theta\), and so

\[
w = \sum_n a_n r^n \sin n\theta
\]

(2.4)

We cannot expect \(n\) to be integral, for then \(w\) would be continuous across the crack, and its faces would not suffer a relative displacement. In fact it is easy to show that if the crack faces are to be free of stress, so that \(\sigma_{xy} = 0\) for \(\theta = \pm \pi\), then \(n\) must be plus or minus half an odd integer. We can exclude negative values of \(n\) by an energy argument. Take the term with \(n = -\frac{1}{2}\). Then \(w\) behaves like \(r^{-\frac{1}{2}}\), the stresses like \(r^{-\frac{3}{2}}\) and the energy per unit volume, proportional to \((\text{stress})^2 / (\text{elastic modulus})\), behaves like \(r^{-3}\). By integration one finds that the elastic energy per unit length of a cylinder of outer and inner radii \(R, r_0\) centred on the crack tip is proportional to \(r_0^{-1} - R^{-1}\). Letting \(r_0\) tend to zero we find that the total energy in any region embracing the tip is infinite. This is not necessarily mathematically or even physically improper for a permanently existing singularity (a classical point electron for example) but in the present case we start with an energy-free unstressed body and ‘induce’ a stress field
around the crack in it by applying forces to the material. Since these forces would have to do an infinite amount of work to provide the energy round the tip we must exclude \( n = -\frac{1}{4} \), and likewise any more negative value.

Close enough to the tip the term with \( n = \frac{1}{4} \) will be dominant and, with a rather clumsy notation to be explained in a moment, the displacement is

\[
w = \frac{2 K_{\text{III}}}{\mu (2\pi)^{1/4}} r^{1/4} \sin \frac{1}{4}\theta
\]

and the stresses are

\[
\sigma_{zz} = \frac{K_{\text{III}}}{(2\pi)^{1/4}} r^{-1/4} \cos \frac{1}{4}\theta, \quad \sigma_{zz} = -\frac{K_{\text{III}}}{(2\pi)^{1/4}} r^{-1/4} \sin \frac{1}{4}\theta
\]

In particular the stress immediately ahead of the tip (\( \theta = 0 \)) is

\[
\sigma_{zz} = \frac{K_{\text{III}}}{(2\pi)^{1/4}} r^{-1/4}
\]

and the relative displacement of the faces just behind it (\( \theta = \pm \pi \)) is

\[
\Delta w = \frac{4 K_{\text{III}}}{\mu (2\pi)^{1/4}} r^{1/4}
\]

The quantity \( K_{\text{III}} \) is by definition the mode III stress intensity factor. It is defined only in relation to the stress across the plane of the crack just ahead of the tip, being \((2\pi)^{1/4}\) times the coefficient of the \( r^{-1/4} \) singularity, as in (2.7), but as (2.5) and (2.6) make clear a knowledge of 

of its value also enables one to reconstruct the entire elastic field around the tip for small enough \( r \). The factor \((2\pi)^{1/4}\) is conventional but convenient. In a less common definition of the stress intensity factor it is left out.

If the crack is subjected to mode I or mode II loading (Figs. 1b and 1c) instead of mode III the elastic field near the tip is similar though more complicated. It is now governed by the first two of equations (2.1). If we apply \( \partial / \partial x \) to the first and \( \partial / \partial y \) to the second and add we get \( \nabla^2 \epsilon = 0 \), so that the dilatation is harmonic. If we then apply \( \nabla^2 \) to the two original equations we get \( \nabla^2 \nabla^2 \epsilon = 0 \), \( \nabla^2 \nabla^2 \nu = 0 \), so that \( \epsilon \) and \( \nu \) are not harmonic, but rather so-called biharmonic functions.

However, we can stay within the more familiar realm of harmonic functions by a trick. Because \( \epsilon \) is harmonic one may write \( \partial \epsilon / \partial x = \nabla^2 (\frac{1}{2} x \epsilon) \), \( \partial \epsilon / \partial y = \nabla^2 (\frac{1}{2} y \epsilon) \), as is easily checked. Consequently the first two of (2.1) can be thrown into the forms

\[
\nabla^2 [u + \frac{1}{2} x \epsilon] = 0, \quad \nabla^2 [v + \frac{1}{2} y \epsilon] = 0
\]

which state that the bracketed expressions are harmonic functions, \( U \) and \( V \) say, so that we have
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\[ u = U - \frac{1}{2}axe, \quad v = V - \frac{1}{2}aye. \]  

(2.8)

For a mode I crack tip the harmonic \( U, V, e \) can be built up from the set of functions (2.3) which served for mode III. Obviously \( U \) and \( e \) will contain cosines and \( V \) sines, and corresponding to a term in \( r^n \) for \( U, V \) there will be one in \( r^{n-1} \) for \( e \). The demand that the tensile stress \( \sigma_{xy} \) shall vanish on the crack faces and that the total energy near the tip shall be finite requires, as in mode III, that in \( U \) and \( V \) each \( n \) shall be half a positive odd integer. Hence close to the tip, where the terms with \( n = \frac{1}{2} \) are dominant, the displacement takes the form (2.8) with

\[ e = Ar^{-\frac{1}{2}} \cos\frac{1}{2}\theta, \quad U = Br^{\frac{1}{2}} \cos\frac{1}{2}\theta, \quad V = Cr^{\frac{1}{2}} \sin\frac{1}{2}\theta \]

The constants \( B, C \) can be related to \( A \) by imposing two conditions. One is that the shear stress \( \sigma_{xy} \) shall vanish on the crack faces. The other, less obvious, is that the dilatation \( e \) formed from the two expressions (2.8) shall agree with the \( e \) already incorporated in them. If the reader cares to carry out this programme he should find that, with a suitable choice for the remaining constant \( A \), the elastic field near the tip takes the form

\[
\sigma_{yy} = \frac{K_I}{(2\pi)^{\frac{1}{2}}} r^{-\frac{1}{2}} \frac{1}{4} \left( 5 \cos\frac{1}{2}\theta - \cos\frac{5}{2}\theta \right)
\]

\[
\sigma_{xx} + \sigma_{yy} = \frac{K_I}{(2\pi)^{\frac{1}{2}}} r^{-\frac{1}{2}} 2 \cos\frac{1}{2}\theta
\]

\[
\sigma_{xy} = \frac{K_I}{(2\pi)^{\frac{1}{2}}} r^{-\frac{1}{2}} \frac{1}{4} \left( - \sin\frac{1}{2}\theta + \sin\frac{5}{2}\theta \right)
\]

\[
u = \frac{K_I}{(2\pi)^{\frac{1}{2}}} r^{\frac{1}{2}} \frac{4(1-v^2)}{E} \left[ 1 - \frac{\cos^2\frac{1}{2}\theta}{2(1-v)} \right] \sin\frac{1}{2}\theta
\]

(2.9)

To conform with engineering usage we have expressed the displacements in terms of Young's modulus \( E = 2(1+v)\mu \).

Two important quantities contained in (2.9) are the stress across the plane of the crack just ahead of the tip and the relative displacement of the crack faces just behind the tip:

\[
\sigma_{yy} = \frac{K_I}{(2\pi)^{\frac{1}{2}}} r^{-\frac{1}{2}}, \quad \Delta u = \frac{K_I}{(2\pi)^{\frac{1}{2}}} r^{\frac{1}{2}} \frac{8(1-v^2)}{E}
\]

(2.10)

To deal with mode II it is only necessary to interchange sine and cosine in (2.8) and find the new \( B, C \) in the same way. We shall only quote the analogue of (2.10):

\[
\sigma_{xy} = \frac{K_{II}}{(2\pi)^{\frac{1}{2}}} r^{-\frac{1}{2}}, \quad \Delta u = \frac{K_{II}}{(2\pi)^{\frac{1}{2}}} r^{\frac{1}{2}} \frac{8(1-v^2)}{E}
\]
The quantities $K_1$, $K_{II}$ are the stress intensity factors for mode I and II loading. As in the case of mode III they are in the first instance defined only in terms of the stress immediately ahead of the crack, but they actually determine the complete elastic field near the tip. Generally speaking the tip field will be a superposition of the three modes; it is then completely specified by $K_1$, $K_{II}$ and $K_{III}$.

Although this analysis determines the form of the tip field it does not tell us its magnitude, which is fixed by the stress intensity factors. Their values depend on the details of the external load acting on the specimen, and also on the geometry, on whether, for example, the crack line terminates at the other end in another tip or at the surface and generally on the relation of the crack to external surfaces. Much effort has gone into the calculation of stress intensity factors. Paris & Sih\textsuperscript{11} have given a compendium of results. Here we shall merely indicate some of the simpler ones.

Suppose that the crack in Fig. 1 extends along the $x$-axis from $x = -a$ to $x = a$ and that it is subjected not to one of the simple types of stress indicated in the figure but to a more complicated system of loading which produces stresses $\sigma^A_{ij}(x,y)$ in the absence of the crack. The stress intensity factor at the right-hand tip can be found in terms of the values of $\sigma^A_{ij}$ along the crack from the formula

$$K = \frac{1}{\pi a^2} \int_{-a}^{a} \left( \frac{x+a}{a-x} \right)^\frac{3}{2} \sigma^A(x, 0) \, dx$$  \hspace{1cm} (2.11)

To get $K_1$, $K_{II}$, $K_{III}$, $\sigma^A$ must be put equal to $\sigma_{yy}^A$, $\sigma_{xx}^A$, $\sigma_{zy}^A$ successively.

For the simple types of loading in the figure (2.11) gives

$$K_1 = T(\pi a)^{\frac{3}{2}}, \quad K_{II} = S(\pi a)^{\frac{3}{2}}, \quad K_{III} = \tau(\pi a)^{\frac{3}{2}}$$  \hspace{1cm} (2.12)

If internal forces are applied to the crack faces equal and opposite to those produced by the load in the absence of the crack the faces will be pulled together again and the stress will be just $\sigma^A_{ij}(x,y)$ everywhere. This means that the internal forces have cancelled the tip singularities due to the load, so that (2.11) can also be used for internal loading. A little puzzling over signs shows that if there is a variable pressure $p(x)$ inside the crack, tending to open it when positive, then the $K_i$ it induces is given by (2.11) with $\sigma^A(x,0) = p(x)$. If there is a real gas pressure $p$ in the crack (hydrogen in steel) the result will merely be the first of (2.12) with $T = p$, but the idea of a variable internal pressure (or rather tension) will be useful later.

All we have said so far relates to two-dimensional situations, but the idea of a stress intensity factor carries over into three dimensions. Fig. 3 shows a flat disc-shaped crack. It is assumed that referred to the coordinates indicated the stresses are locally the same as those of a tangential two-dimensional crack suitably loaded, so that the elastic field near the crack border can be specified.
by three parameters \( K_1, K_{II}, K_{III} \) which vary round the periphery. This assumption is confirmed in all cases for which an analytical solution has been obtained; the most elaborate of these is a flat elliptical crack arbitrarily loaded.

The theory of cracks in plates, of particular importance to fracture mechanics, is at present in an unsatisfactory condition. In a state of plane strain there is a tensile stress \( \sigma_{zz} \) perpendicular to the \( x,y \) plane, equal to \( \nu(\sigma_{xx} + \sigma_{yy}) \), which prevents the material expanding or contracting parallel to the \( z \)-axis. Thus if we cut a slab out of the cracked solid of Fig. 1(b) to get a plate traversed by a crack, retaining the original loading round the edges, the relaxation of \( \sigma_{zz} \) on the freshly formed faces of the plate will throw the material into a complicated three-dimensional state of stress. From what has just been said about the disc-shaped crack we might expect that for a crack in a plate under mode I loading the field close enough to the tip would be given by (2.9) with \( K_1 = K_f(z) \) now some function of \( z \), though we may suspect that 'close enough' must be interpreted more and more strictly as the faces of the plate are approached. No exact solution has so far been published,* and this is an embarrassment since much of fracture toughness testing is carried out on various forms of cracked plate. In the absence of the exact solution a so-called 'plane stress' approximation is commonly used. It is of doubtful validity and in any case only claims to give the average of the field across the thickness of the plate. In particular the averages corresponding with (2.10) are

\[
\bar{\sigma}_{yy} = \frac{K_1}{(2\pi)^{\frac{1}{2}}} r^{-\frac{1}{2}}, \quad \Delta \bar{\sigma} = \frac{K_1}{(2\pi)^{\frac{1}{2}}} r^{\frac{1}{2}} \cdot \frac{8}{E}
\]

where the average \( \bar{K}_1 \) is the same as the plane strain \( K_f \) for the same type of loading. Because of the suppression of the factor \((1 - \nu^2)\) (2.13) is inconsistent with (2.10) with variable \( K_1 \). We return to the matter in the next section.

The importance of the stress intensity factor in fracture mechanics stems from

* Cruse & VanBuren\textsuperscript{114} have recently given some interesting numerical results.
the fact that if at the tip of a certain crack \( K_1, K_{II}, K_{III} \) have the same values as they do at the tip of a second crack in identical material then the elastic fields are identical at the two tips despite possible differences of geometry and loading. For brevity we shall suppose that \( K_{II} \) and \( K_{III} \) are both zero. Then if one crack is found to begin to extend and initiate fracture when \( K_1 \) reaches a critical value \( K_{IC} \) we should expect the second crack to do the same and we have a fracture criterion

\[
K_1 = K_{IC}
\]  

(2.14)

where \( K_{IC} \) is a function which may be tabulated for materials of different chemical composition, crystal structure, grain size, heat treatment and so forth.

The above argument is based on the linear theory of elasticity used to calculate \( K_1 \). However, the idea of a critical \( K_{IC} \) is still useful when there are departures from linearity or plastic flow near the tip. Imagine that we have a material in which the forces responsible for non-linearity or plasticity can be switched on or off at will. With the forces switched off, load two crack tips to the same value of \( K_1 \). Now switch the forces on. As these forces are identical and find themselves acting in identical environments (the identical linear elastic fields) they will, with a certain limitation, produce identical changes in the elastic field, so that physical conditions are still identical at the tips, and they should both begin to extend at the same value of \( K_1 \), as calculated from the linear theory. The limitation is that the changes produced by the forces should be confined to a region small compared with the length of the crack and its distance from the surface of the specimen.

Explanations of (2.14) which say that the stress or strain or energy density reach a critical value near the crack tip do not really add anything to the assumption that like causes lead to like effects, but the Russian school of Barenblatt\(^{12} \) gives a more elaborate interpretation. It is assumed that the crack tip is, so to speak, pulled apart by the applied forces, and held together by cohesive forces acting between the faces just inside the tips. For equilibrium it is required that \( \Delta \nu \) shall behave smoothly like \( r^{3/2} \) near the tip, in place of the usual abrupt \( r^{1/2} \). The condition for this is that the \( K_1 \) produced by the combined action of the external load and the cohesive forces shall be zero. In fact if \( K_1 = 0 \) the \( r^{1/2} \) term in the stresses vanishes, and with it the \( r^{1/2} \) term in \( \Delta \nu \), and the second term in (2.4) (or rather its analogue for mode I) dominates. We have seen that, for the crack of Fig. 1, the \( K \) value can be calculated from the same formula for both internal and external forces, and the same is true for more general geometries. Hence Barenblatt's criterion can be written in the form

\[
K_1(\text{app}) + K_1(\text{coh}) = 0
\]  

(2.15)

It is assumed by him that for small enough loads the cohesive forces are able to adjust themselves so that (2.15) is satisfied, but that ultimately they are unable
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to do so any longer, so that a stress-singularity develops and the crack begins to extend. It is further assumed that just before this point is reached the shape of the tip, specified by $A r$, is a function characteristic of the material, which implies that the cohesive force is a definite function of distance back from the tip. (These assumptions are equivalent to our argument above that with a definite applied $K_1$ prescribed non-linear forces produce a unique configuration near the tip.) The upshot of all this is that $K_1(\text{coh})$ in (2.15) is a constant characteristic of the material, negative since the cohesive forces tend to close the crack. If we call it $-K_{IC}$ (2.15) gives a fracture criterion $K_1(\text{app}) = K_{IC}$ of the usual type. Barenblatt calls $(\frac{1}{2\pi})^{1/2}K_{IC}$ the modulus of cohesion.

Other interpretations of the fracture process are based, not on the stress intensity factor, but on a second basic concept of fracture mechanics, the crack extension force or energy release rate, which we introduce in the next section.

3. Energy relations: the crack extension force

If the right-hand tip of the crack in Fig. 1 moves to the right the elastic energy $E_{el}$ of the specimen and the potential energy $E_{pot}$ of the loading mechanism responsible for the applied stress $T$ both change. In his theory of brittle fracture Griffith pointed out that for the crack to be able to extend, the decrease of the total energy $E_{tot} = E_{el} + E_{pot}$ must be equal to or exceed the surface energy of the freshly formed crack surfaces. Suppose that the total energy increases by $\delta E_{tot}$ when a length $\delta l$ of the crack border advances by $\delta a$. Then we write

$$-\delta E_{tot} = G \delta a \delta l$$

(3.1)

where, by definition, $G$ is the energy release rate or crack extension force. As the alternative name suggests we may regard (3.1) as the work done by a fictitious force $G$ per unit length acting along the edge of the crack.

If the applied tension $T$ in Fig. 1(b) is produced by pulling the upper and lower surfaces of the block apart and then clamping them (hard loading) the elastic energy obviously decreases when the crack lengthens, while there is no change in the potential energy of the loading mechanism. On the other hand, if $T$ is produced by, say, a weight (dead loading) the lengthening of the crack makes the specimen more flexible, and the weight descends, losing potential energy, and at the same time doing work on the specimen which thus increases its elastic energy. For intermediate types of loading the elastic energy may either increase or decrease. The energy release rate may be calculated without distinguishing between these different cases with the help of the following set of imaginary operations performed at the crack tip.

Fig. 4 shows the stress and displacement near the tip of a crack. Allow the crack tip to advance by $\delta a$, but apply to the freshly-formed surfaces forces chosen so as to prevent them separating. The original elastic field remains
unchanged. From (2.10) it follows that if these forces are gradually relaxed the stress across the x-axis between 0 and \( \delta a \) falls from

\[
\sigma_{yy} = \frac{K_1}{(2\pi)^{\frac{1}{2}}} x^{-\frac{1}{2}}
\]

to zero, and the crack opening \( \Delta v \) increases from zero to the value

\[
\Delta v = \frac{K_1}{(2\pi)^{\frac{1}{2}}} \frac{8(1-v^2)}{E} (-x + \delta a)^{\frac{1}{2}}
\]

represented by the dotted curve. The work extracted by the surface forces as they relax is thus

\[
\frac{1}{2} \int_0^{\delta a} \sigma_{yy} \Delta v \, dx = \frac{1}{2} \frac{K_1^2}{2\pi} \frac{8(1-v^2)}{E} \int_0^{\delta a} x^{-\frac{1}{2}}(\delta a - x)^{\frac{1}{2}} \, dx
\]

(3.2)

per unit length perpendicular to the plane of the figure. The integral has the value \( \frac{1}{4} \pi \delta a \), and (3.2) is, by definition, \( G_1 \delta a \), so that we have

\[
G_1 = \frac{K_1^2}{E} (1-v^2)
\]

(3.3)

Entirely similar calculations for modes II and III give

\[
G_{II} = \frac{K_{II}^2}{E} (1-v^2), \quad G_{III} = \frac{K_{III}^2}{2\mu}
\]

(3.4)

The formula

\[
\bar{G}_{I,II} = \frac{G_{I,II}}{E}
\]

(3.5)

is commonly used for the average of \( G \) across a crack in a plate. It is obtained by inserting (2.13) in the first member of (3.2). As well as ignoring the uncertain nature of the underlying plane stress solution it in effect assumes that the
average of a product is the product of the averages. Later in this section we shall outline an argument which suggests that, nevertheless, (3.5) is correct, though it does not explain precisely how this comes about.

Since \( G \) is a definite function of \( K \) and the elastic constants we can transcribe the fracture criterion \( K_i = K_{ic} \) into the entirely equivalent one

\[
G_i = G_{ic}
\]

where \( G_i = K_i^2(1-v^2)/E \) and \( G_{ic} = K_{ic}^2(1-v^2)/E \) and similarly for other modes. However, because of its energetic interpretation \( G_{ic} \) has a more concrete meaning than does \( K_{ic} \). For a brittle material we may, following Griffith, equate \( G_{ic} \) with \( 2\gamma \), the surface energy of the free faces of the crack. For a ductile material we may write, with Orowan, \( G_{ic} = 2(\gamma + \gamma_{pl}) \) where \( \gamma_{pl} \) is a fictitious surface energy which accounts for the work done in plastic deformation near the tip.

In its role as an effective force on the tip of a crack the energy release rate is closely related to the force on a defect (dislocation, impurity atom, lattice vacancy and so forth) as the term is understood in the theory of lattice defects. This force can be calculated from the formula

\[
F_x = \int_S \left( W n_x - \frac{\partial u}{\partial x} \cdot T \right) dS
\]

(3.7)

with similar expressions for the \( y \) and \( z \) components. Here \( S \) is a closed surface which envelops the defect we are interested in, and no others, \( T \ dS \) is the force on an element \( dS \) of \( S \), \( (n_x, n_y, n_z) \) are the components of the normal to \( dS \), \( W \) is the elastic energy density and \( u \) is the elastic displacement. With the symbols suitably interpreted, (3.7) is valid not only for infinitesimal but also for finite deformation and an arbitrary stress strain law; \( T \ dS \) is the force on an element which had area \( dS \) and normal \( (n_x, n_y, n_z) \) before deformation, and \( W \) is the energy per unit undeformed volume. An important property of the integral (3.7) is that its value is unchanged when \( S \) is deformed, provided it does not embrace any other defect.

The physical significance of \( F_x \) is that the defect is displaced by \( \delta a \) parallel to the \( x \)-axis \( F_x \delta a \) gives the decrease in the sum of the elastic energy of the material and the potential energy of the loading mechanism, or in other words \( F_x \) is the 'energy release rate' for the defect. One can show that a crack tip qualifies as a singularity in its own right, to which (3.7) may be applied. In a two-dimensional situation we have simply

\[
G = F_x
\]

(supposing that the crack is parallel to the \( x \)-axis) where in (3.7) \( S \) is any circuit embracing the tip. The resulting integral is identical with Rice's path-independent integral for \( G \). If we need the local \( G \) value at some point on a straight or
curved crack we must take for $S$ a small closed surface, a sphere, say, which, so
to speak, takes a bite out of the edge of the crack. If $S$ intercepts a length $\delta l$
of the crack border then $F_x$ is equal to $G \delta l$, provided the axes are oriented as in
Fig. 3. There are various ways of making good these assertions; we shall
outline one later in connection with the corresponding dynamical problem.
The fact that (3.7) is valid for finite deformation and that it is surface or
path-independent make it a useful tool. When the elastic field is known only
near the tip one takes for $S$ an infinitesimal circuit. On the other hand, when
there are non-linearities near the tip but we believe that the linear solution is
adequate far from the tip we take $S$ to be large. We then know that the $G$ so
calculated will be the same as the $G$ which would have been obtained from a
small circuit and the unknown tip field. If for this case a fracture criterion of the
usual form $G_1 = G_{lc}$ is found to be adequate an alternative criterion of the
form $K_1 = K_{lc}$ will also be adequate, where $K_1$ is related to $G_1$ by the linear
formula (3.3), even though this formula is inapplicable near the tip. This
provides a justification for our earlier assertion that a stress intensity factor based
on the linear theory is still significant when the linear theory is inadequate near
the tip.
The same kind of argument can be used to justify the plate formula (3.5).
The integral of $G$ along the crack is given by (3.7) with $S$, say, a large circular
cylinder with the crack as axis closed by the two circles in which it intersects
the faces of the plate. These circular areas actually make no contribution. One
can show that the plane stress solution is adequate remote from the tip, the
averages given by the theory now being interpreted as constant values across
the plate. After inserting the plane stress solution into (3.7) we can contract $S$
on to the tip and use (2.13) to reproduce (3.5). This last step is allowable be-
cause the plane stress solution, though not an exact solution of any problem
for the actual plate, is in fact the exact solution for a fictitious, but possible,
anisotropic plate, and this is enough to make (3.7) independent of $S$.

4. Fast cracks
Crack tips may propagate through a material with speeds which are comparable
with the velocity of elastic waves in it, and then the dynamic equations of
elasticity must be used. Various mathematical solutions have been obtained for
cracks whose tips move with constant velocity, but since it is found that cracks
do not commonly move uniformly what is really wanted is the solution for a
crack which is moving arbitrarily.
Ideally one would like to be able to predict the motion of a crack in a given
material under given loading. A programme for doing this would go somewhat
as follows. (i) Calculate the elastic field near an arbitrarily moving tip. (ii) From
it find the energy release rate $G$. (iii) As in the static case equate $G$ to a quantity
$G_c$ which gives the rate at which energy is absorbed near the tip by non-elastic processes. (iv) Solve the resulting equation of motion. Some progress has already been made along these lines. So far solutions for unsteady crack motion have only been found for anti-plane elastic fields$^{16,17}$ and so we shall have to limit ourselves to mode III, with the hope that for the more important modes I and II things will not be grossly different.

For a non-static state of anti-plane strain the equilibrium equation $\nabla^2 w = 0$ is replaced by the wave-equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} - \frac{1}{c^2} \frac{\partial^2 w}{\partial t^2} = 0$$

(4.1)

where $c = (\mu/\rho)^{\frac{1}{2}}$ is the velocity of transverse waves ($\rho$ is the density of the medium). We begin with the simplest possible case. Initially the crack is at rest and has the displacement (2.5) not only near the tip but everywhere, that is, all but the first of the terms in the expansion (2.4) are zero. This displacement could, for example, be maintained in a large cylinder slit along the negative $x$-axis by forces proportional to $\sin \frac{1}{2} \theta$ distributed round the circumference. We re-write (2.5) in the form

$$w = B W_0(x, y)$$

(4.2)

where $B$ is an abbreviation for the cumbersome coefficient $2K_{III}/\mu(2\pi)^{\frac{3}{2}}$, and

$$W_0(x, y) = r^4 \sin \frac{1}{2} \theta = \left\{ \frac{1}{2} r - \frac{1}{2} y \right\}^{\frac{3}{2}}$$

(4.3)

is a basic solution which will play an important part in what follows. We now suppose that at time $t=0$ the tip of the crack starts to move along the $x$-axis according to the arbitrary law

$$x = \xi(t)$$

(4.4)

and ask what the resulting elastic field will be.

The answer is provided by a rather pretty theorem of Bateman's.$^{18}$ To explain it we need to introduce the retarded time $\tau$. Suppose that a point moving according to (4.4) is continually emitting signals which propagate with velocity $c$. A signal received by an observer at $x, y$ at time $t$ must have been emitted at an earlier time, $\tau$ say, when the emitter was at $\xi(\tau)$. The retarded time $\tau$ is determined as a function of $x, y, t$ by solving the implicit equation

$$c(t-\tau) = \left\{ (x - \xi(\tau))^2 + y^2 \right\}^{\frac{3}{2}}$$

(4.5)

which says that the time of transmission is equal to the distance between the points of emission and reception, divided by the signal velocity. Bateman's theorem states that if $\nabla^2 w(x, y) = 0$ then $w[x - \xi(\tau), y]$ is a solution of the wave-equation (4.1), provided $w(x, y)$ is homogeneous of degree $\frac{1}{2}$. A function
W(x, y) is said to be homogeneous of degree 1 if $w(λx, λy) = λ^1 w(x, y)$. The theorem may be verified, though with some difficulty, by substituting in (4.1) and using (4.5). The basic solution (4.2) meets the conditions of the theorem and yields the solution

$$w = BW_0 \left[ x - ξ(t), y \right]$$

(4.6)

It is easy to verify that (4.6) has the right properties. In particular the stress $σ_{xx}$ is zero on the x-axis to the left of $x = ξ(t)$ and there is a stress singularity of the right type near $x = ξ(t)$. Outside a circle of radius $ct$ about the original position of the tip $τ$ is negative, $ξ(τ) = 0$ and, as would be expected, the static solution remains undisturbed.

If the tip moves uniformly, so that

$$ξ = 0, t < 0; \quad ξ = vt, t > 0$$

(4.5) can be solved for $τ$, and after some algebra the displacement is found to be

$$w = B A(v) W_0 \left[ \frac{x - vt}{β(v)}, y \right]$$

(4.7)

inside the circle $r = ct$, with

$$A(v) = \left( \frac{1 - v/c}{1 + v/c} \right)^{1/2}, \quad β(v) = \left( 1 - \frac{v^2}{c^2} \right)^{1/2}$$

(4.8)

Equation (4.7) differs from (4.2) by the substitution (‘Lorentz transformation’)

$$x → \frac{x - vt}{β(v)}, \quad y → y$$

and the multiplier $A(v)$.

For arbitrary tip motion (4.5) can be solved for $τ$ with the help of Lagrange’s expansion\(^{19}\) to give the displacement

$$w = B A(ξ) W_0 \left[ \frac{x - ξ(t)}{β(ξ)}, y \right] + O(R^{\frac{1}{2}})$$

(4.9)

where $R$ is the distance from the current position of the tip and $ξ$ is the current velocity. Comparison with (4.7) shows that close to the tip the elastic field is the same as that of a uniformly-moving fictitious crack which happens to have the same position and velocity as the real crack at the instant of observation.

All this refers to the case where the disturbance spreading from the moving tip is, so to speak, eating into the particularly simple static field (4.2). When this is not so the static field is not homogeneous of degree $\frac{1}{2}$ and Bateman’s theorem can only be applied indirectly.\(^{17}\) An arbitrary tip motion can be analysed into a succession of elementary jumps. It is fairly clear that if the tip
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jumps from $\xi$ to $\delta \xi$ at time $t'$ the resulting disturbance will be a universal function of $x-\xi, y$ and $t-t'$, a Green's function for crack motion in fact, multiplied by $\delta \xi$ and a weight factor depending on the (unknown) conditions near the tip. The Green's function can be found by allowing the jump to occur in the simple stress field to which Bateman's theorem does apply. An integral over the jumps will give the total disturbance, and the unknown variable weight factor in the integral can be found from the condition that the disturbance shall annul the original static stress $\sigma_{xy}$ over the segment of the $x$-axis so far swept over by the tip. The final result is that (4.9) still holds close to the tip but that now $B$ depends on the current position $\xi$ of the tip. The actual expression for it is

$$B(\xi) = \frac{2}{\pi \mu} \int_0^\xi \sigma_{xy}(x', 0)(\xi-x')^{-\frac{1}{2}} \, dx'$$

(4.9a)

a weighted average of the static stress $\sigma_{xy}(x, 0)$ which originally existed ahead of the crack.

For a crack of finite length (4.9) is ultimately upset by the arrival at the moving tip of the original disturbance reflected back from the other tip. This is not very restrictive; for example the crack will have already trebled its original length by the time the first reflection overtakes it if its average tip velocity up to this time has the reasonable value $\frac{1}{2}c$.

To find the energy releases rate for a moving crack tip we draw a circuit $S$ around the tip (Fig. 5). (The discussion is no longer confined to mode III.) If $T \cdot dS$ is the force acting on the surface element $dS$ with outwardly-directed normal the integral

$$I = \int_S \mathbf{u} \cdot T \, dS$$

(4.10)

is the rate at which the material outside $S$ does work on the material inside $S$, that is, $I$ is the rate of flow of energy (elastic plus kinetic) into $S$. (More formally $I$ is the integral of the normal component of the energy flow vector.) Of the inflow $I$ a part, which by definition is $vG$, flows out again at the tip, while the
rest goes towards increasing the elastic and kinetic energy within $S$. To get $vG$ we must subtract from (4.10) the rate of increase of the energy contained within $S$. Suppose for the moment that the elastic field is carried along rigidly with the tip, so that we may write

$$u = u(x - vt, y)$$ (4.11)

Then in time $dt$ the energy within $S$ is decreased by the energy contained in the crescent-shaped region 1 and increased by the energy in the crescent-shaped region 2. The diagram makes clear that the total rate of change of energy is

$$- \int vE n_x dS$$ (4.12)

where $E$ is the sum of the elastic and kinetic energy densities and $n_x$ is the $x$-component of the outward normal to $dS$. Subtracting (4.12) from (4.10) we get

$$vG = \int_S (vE n_x + u \cdot T) dS$$ (4.13)

But if (4.11) is true we have $\dot{u} = - v \partial u / \partial x$ and (4.13) becomes

$$G = \int_S \left( E n_x - \frac{\partial u}{\partial x} \cdot T \right) dS$$ (4.14)

One can easily show that if (4.11) is satisfied the integrals (4.13) and (4.14) are path-independent, in the sense explained in connection with (3.7), and so we are spared the embarrassing feature of some expressions which have been proposed, that they give different values according as $S$ is, say, a circle or a square.

Of course (4.11) is not strictly satisfied by the elastic field of a moving crack. Nevertheless we can arrange for it to be satisfied to any degree of accuracy by going close enough to the tip, or at least, to be more cautious, this is true for every solution so far published; (4.9) is an example. Consequently (4.4) can be used to calculate $G$ generally, on the understanding that $S$ is to be contracted on to the tip until the integral reaches a constant limiting value. Equation (4.14) can also be used to find the local $G$ value at a point on a crack border in the same way as (3.7), but the appropriate closed surface $S$ must be supposed to shrink on to the crack border.

If the crack is moving slowly enough for the kinetic energy to be ignored (4.4) becomes the static expression (3.7). The integral is now, in fact, path-independent whether (4.11) is satisfied or not and so, as stated in Section 3, (4.14) can be used with a finite circuit $S$ even though it was derived only for an infinitesimal one. In the static case there are, however, more satisfactory ways of deriving (4.14) which start out with a finite circuit.
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For the moving anti-plane tip field described by (4.9) and (4.9a) the formula (4.14) gives

\[ G(\xi, \dot{\xi}) = \frac{1}{4} \pi \mu \left( \frac{1 - \dot{\xi}/c}{1 + \dot{\xi}/c} \right)^4 B^2(\xi) \]  

(4.15)

The velocity factor in (4.15) falls to zero at \( c \). The analogues of (4.15) for modes I and II is as yet unknown but the indications are that they are similar to (4.15) but with a velocity factor which falls to zero at the Rayleigh velocity \( c_R \), about 0.9c.

The expression (4.15) or its unknown analogues represents the rate at which energy flows into the tip in obedience to the equations which govern the elastic field. To find the way in which the tip actually moves we must equate (4.15) to some quantity \( G_c \) which specifies how this energy is absorbed near the tip by physical processes not taken account of by the theory of elasticity. The simplest assumption is that \( G_c \) has a constant value representing a real or effective surface energy. Since \( B(\xi) \) increases with the length of the crack and the velocity factor decreases as the velocity increases, reaching zero at \( \xi = c \) or \( \xi = c_R \), the tip would then accelerate up to a limiting velocity \( c \) or \( c_R \). This is in disagreement with experiment; limiting velocities in the neighbourhood of 0.6c are commonly quoted. To reproduce this sort of behaviour we have to suppose that \( G_c \) is not constant but increases as the crack lengthens. The tip field and its rate of change are fixed if \( K = K_{III} \) and \( \dot{\xi} \) are given, and we may perhaps expect \( G_c \) to depend mainly on these quantities. This leads to an equation of motion for the tip of the form

\[ G(\xi, \dot{\xi}) = G_c(K, \dot{\xi}) \]

where the left-hand side is determined through (4.15) by the applied stresses and the law of tip advance (4.4) while the right-hand side represents a material function. Note that \( G(\xi, \dot{\xi}) \) does not depend on the acceleration of the tip, so that if we regard the tip as a ‘particle’ it is one which exhibits no inertia. Küppers²¹ drew the same conclusion for mode I from experimental evidence.

In the foregoing we have introduced the basic ideas of fracture mechanics and looked at some of the more interesting theoretical problems, solved and unsolved, connected with them. This may perhaps have conveyed the incorrect impression that fracture mechanics is a somewhat rarified branch of science. In fact, fracture toughness tests, largely based on the ideas we have discussed, are now regularly incorporated into material specifications for high-duty materials, and they have done much to improve the reliability and efficiency of engineering design.

References