Chapter 14

THERMOELASTICITY

Most materials tend to expand if their temperature rises and, to a first approximation, the expansion is proportional to the temperature change. If the expansion is unrestrained, all dimensions will expand equally — i.e. there will be a uniform dilatation described by

$$e_{xx} = e_{yy} = e_{zz} = \alpha T \tag{14.1}$$

$$e_{xy} = e_{yz} = e_{zx} = 0\ , \tag{14.2}$$

where α is the *coefficient of linear thermal expansion.* Notice that no shear strains are induced in unrestrained thermal expansion, so that a body which is heated to a uniformly higher temperature will get larger, but will retain the same shape.

Thermal strains are additive to the elastic strains due to local stresses, so that Hooke's law is modified to the form

$$e_{xx} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E} - \frac{\nu\sigma_{zz}}{E} + \alpha T \tag{14.3}$$

$$e_{xy} = \frac{\sigma_{xy}(1+\nu)}{E}\ . \tag{14.4}$$

14.1 The governing equation

The Airy stress function can be used for two-dimensional thermoelasticity, but the governing equation will generally include additional terms associated with the temperature field. Repeating the derivation of §4.4.3, but using (14.3) in place of (1.47), we find that the compatibility condition demands that

$$\frac{\partial^2\sigma_{xx}}{\partial y^2} - \nu\frac{\partial^2\sigma_{yy}}{\partial y^2} + E\alpha\frac{\partial^2 T}{\partial y^2} - 2(1+\nu)\frac{\partial^2\sigma_{xy}}{\partial x\partial y} + \frac{\partial^2\sigma_{yy}}{\partial x^2} - \nu\frac{\partial^2\sigma_{xx}}{\partial x^2} + E\alpha\frac{\partial^2 T}{\partial x^2} = 0 \tag{14.5}$$

and after substituting for the stress components from (4.1) and rearranging, we obtain

$$\nabla^4\phi = -E\alpha\nabla^2 T\ , \tag{14.6}$$

for plane stress.

The corresponding plane strain equations can be obtained by a similar procedure, noting that the restraint of the transverse strain e_{zz} (14.1) will induce a stress $\sigma_{zz} = -E\alpha T$ and hence additional in-plane strains $\nu\alpha T$. Equation (14.6) is therefore modified to

$$\nabla^4\phi = -\frac{E\alpha}{(1-\nu)}\nabla^2 T \; , \tag{14.7}$$

for plane strain and we can supplement the plane stress to plane strain conversions (3.18) with the relation

$$\alpha = \alpha'(1+\nu') \; . \tag{14.8}$$

Equations (14.6, 14.7) are similar in form to that obtained in the presence of body forces (7.8) and can be treated in the same way. Thus, we can seek any particular solution of (14.6) and then satisfy the boundary conditions of the problem by superposing a more general biharmonic function, since the biharmonic equation is the complementary or homogeneous equation corresponding to (14.6, 14.7).

14.1.1 Example

As an example, we consider the case of the thin circular disk, $r < a$, with traction-free edges, raised to the temperature

$$T = T_0 y^2 = T_0 r^2 \sin^2\theta \; , \tag{14.9}$$

where T_0 is a constant.

Substituting this temperature distribution into equation (14.6), we obtain

$$\nabla^4\phi = -2E\alpha T_0 \tag{14.10}$$

and a simple particular solution is

$$\phi_0 = -\frac{E\alpha T_0 r^4}{32} \; . \tag{14.11}$$

The stresses corresponding to ϕ_0 are

$$\sigma_{rr} = -\frac{E\alpha T_0 r^2}{8} \;\; ; \;\; \sigma_{\theta\theta} = -\frac{3E\alpha T_0 r^2}{8} \;\; ; \;\; \sigma_{r\theta} = 0 \tag{14.12}$$

and the boundary $r = a$ can be made traction-free by superposing a uniform hydrostatic tension $E\alpha T_0 a^2/8$, resulting in the final stress field[1]

$$\sigma_{rr} = \frac{E\alpha T_0(a^2 - r^2)}{8} \;\; ; \;\; \sigma_{\theta\theta} = \frac{E\alpha T_0(a^2 - 3r^2)}{8} \;\; ; \;\; \sigma_{r\theta} = 0 \; . \tag{14.13}$$

[1] It is interesting to note that the stress field in this case is axisymmetric, even though the temperature field (14.9) is not.

14.2 Heat conduction

The temperature field might be a given quantity — for example, it might be measured using thermocouples or radiation methods — but more often it has to be calculated from thermal boundary conditions as a separate boundary-value problem. Most materials approximately satisfy the Fourier heat conduction law, according to which the heat flux per unit area $\boldsymbol{q}$ is linearly proportional to the local temperature gradient, i.e.

$$\boldsymbol{q} = -K\boldsymbol{\nabla}T \; , \tag{14.14}$$

where K is the thermal conductivity of the material. The conductivity is usually assumed to be constant, though for real materials it depends upon temperature. However, the resulting non-linearity is only important when the range of temperatures under consideration is large.

We next apply the principle of conservation of energy to a small cube of material. Equation (14.14) governs the flow of heat across each face of the cube and there may also be heat generated, Q per unit volume, within the cube due to some mechanism such as electrical resistive heating or nuclear reaction etc. If the sum of the heat flowing into the cube and that generated within it is positive, the temperature will rise at a rate which depends upon the *thermal capacity* of the material. Combining these arguments we find that the temperature T must satisfy the equation[2]

$$\rho c\frac{\partial T}{\partial t} = K\nabla^2 T + Q \; , \tag{14.15}$$

where ρ, c are respectively the density and specific heat of the material, so that the product ρc is the amount of heat needed to increase the temperature of a unit volume of material by one degree.

In equation (14.15), the first term on the right hand side is the net heat flow into the element per unit volume and the second term, Q is the rate of heat generated per unit volume. The algebraic sum of these terms gives the heat available for raising the temperature of the cube.

It is convenient to divide both sides of the equation by K, giving the more usual form of the heat conduction equation

$$\nabla^2 T = \frac{1}{\kappa}\frac{\partial T}{\partial t} - \frac{Q}{K} \; , \tag{14.16}$$

where

$$\kappa = \frac{K}{\rho c} \tag{14.17}$$

[2]More detail about the derivation of this equation and other information about the linear theory of heat conduction can be found in the classical text H.S.Carslaw and J.C.Jaeger, *Conduction of Heat in Solids,* 2nd.ed., Clarendon Press, Oxford (1959).

is the *thermal diffusivity* of the material. Thermal diffusivity has the dimensions area/time and its magnitude gives some indication of the rate at which a thermal disturbance will propagate through the body.

We can substitute (14.16) into (14.6) obtaining

$$\nabla^4\phi = -E\alpha\left(\frac{1}{\kappa}\frac{\partial T}{\partial t} - \frac{Q}{K}\right) \tag{14.18}$$

for plane stress.

14.3 Steady-state problems

Equation (14.18) shows that if the temperature is independent of time and there is no internal source of heat in the body ($Q = 0$), ϕ will be biharmonic. It therefore follows that in the steady-state problem without heat generation, the stress field is unaffected by the temperature distribution. In particular, if the boundaries of the body are traction-free, a steady-state (and hence harmonic) temperature field will not induce any thermal stresses.

Furthermore, if there are internal heat sources — i.e. if $Q \neq 0$ — the stress field can be determined directly from equation (14.18), using Q, without the necessity of first solving a boundary value problem for the temperature T. This also implies that the thermal boundary conditions in two-dimensional steady-state problems have no effect on the thermoelastic stress field.

All of these deductions rest on the assumption that the elastic problem is defined in terms of *tractions*. If some of the boundary conditions are stated in terms of displacements, the resulting thermal distortion will induce boundary tractions and the stress field will be affected, though it will still be the same as that which would have been produced by the same tractions if they had been applied under isothermal conditions.

Recalling the arguments of §2.2.1, we conclude that similar considerations apply to the multiply-connected body, for which there exists an implied displacement boundary condition. In other words, multiply-connected bodies will generally develop non-zero thermal stresses even under steady-state conditions with no boundary tractions. However, the resulting stress field is essentially that associated with the presence of a dislocation in the hole and hence can be characterized by relatively few parameters[3].

Appropriate conditions for multiply-connected bodies can be explicitly imposed, but in general it is simpler to obviate the need for such a condition by reverting to a displacement function representation. We shall therefore postpone discussion of thermoelastic problems for multiply-connected bodies until Chapter 20 where such a formulation is introduced.

[3]See for example J.Dundurs, Distortion of a body caused by free thermal expansion, *Mech. Res. Comm.*, Vol. 1 (1974), 121-124.

14.3.1 Dundurs' Theorem

If the conditions discussed in the last section are satisfied and the temperature field therefore induces no thermal stress, the strains will be given by equations (14.1, 14.2). It then follows that

$$\frac{\partial^2 u_y}{\partial x^2} = -\frac{\partial^2 u_x}{\partial x \partial y} , \tag{14.19}$$

because of (14.2)

$$= -\frac{\partial e_{xx}}{\partial y} = -\alpha \frac{\partial T}{\partial y} = \frac{\alpha q_y}{K} , \tag{14.20}$$

from (14.1, 14.14). In view of (14.8), the corresponding result for plane strain can be written

$$\frac{\partial^2 u_y}{\partial x^2} = \frac{\alpha(1+\nu)q_y}{K} . \tag{14.21}$$

In this equation, the constant of proportionality $\alpha(1+\nu)/K$ is known as the *thermal distortivity* of the material and is denoted by the symbol δ.

Equations (14.20, 14.21) state that the curvature of an initially straight line segment in the x-direction $(\partial^2 u_y/\partial x^2)$ is proportional to the local heat flux across that line segment. This result was first proved by Dundurs[4] and is referred to as *Dundurs' Theorem*. It is very useful as a guide to determining the effect of thermal distortion on a structure. Figure 14.1 shows some simple bodies with various thermal boundary conditions and the resulting steady-state thermal distortion.

Notice that straight boundaries that are unheated remain straight, those that are heated become convex outwards, whilst those that are cooled become concave. The angles between the edges are unaffected by the distortion, because there is no shear strain. Since the thermal field is in the steady-state and there are no heat sources, the algebraic sum of the heat input around the boundary must be zero. Thus, although the boundary is locally rotated by the cumulative heat input from an appropriate starting point, this does not lead to incompatibility at the end of the circuit.

Many three-dimensional structures such as box-sections, tanks, rectangular hoppers etc., are fabricated from plate elements. If the unrestrained thermal distortions of these elements are considered separately, the incompatibilities of displacement developed at the junctions between elements permits the thermal stress problem to be described in terms of dislocations, the physical effects of which are more readily visualized.

[4] J.Dundurs *loc. cit.*

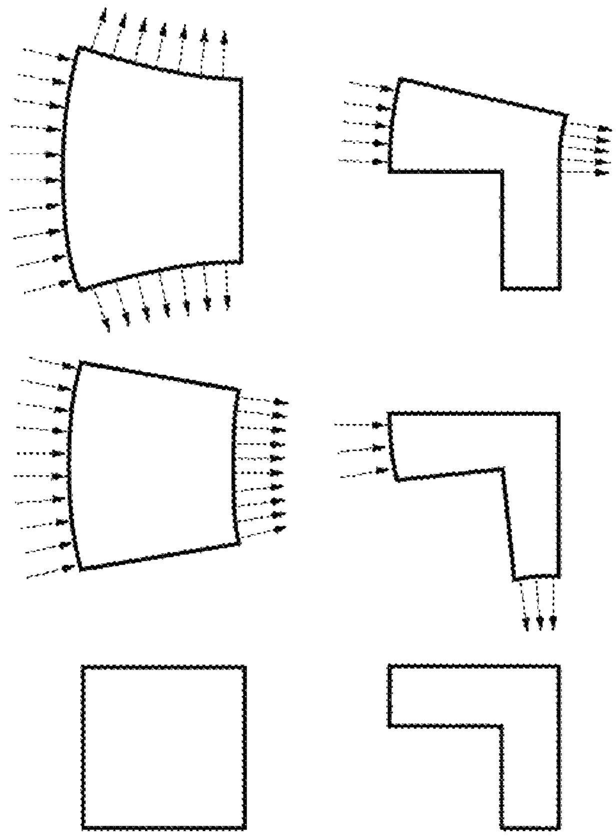

Figure 14.1: Distortion due to thermal expansion.

Dundurs' Theorem can also be used to obtain some useful simplifications in two-dimensional contact and crack problems involving thermal distortion[5].

PROBLEMS

1. A direct electric current I flows along a conductor of rectangular cross-section $-4a<x<4a$, $-a<y<a$, all the surfaces of which are traction free. The conductor is made of copper of electrical resistivity ρ, thermal conductivity K, Young's modulus E, Poisson's ratio ν and coefficient of thermal expansion α. Assuming the current density to be uniform and neglecting electromagnetic effects, estimate the thermal stresses in the conductor when the temperature has reached a steady state.

[5]For more details, see J.R.Barber, Some implications of Dundurs' Theorem for thermoelastic contact and crack problems, *J. Strain Analysis,* Vol. 22 (1980), 229-232.

2. A fuel element in a nuclear reactor can be regarded as a solid cylinder of radius a. During operation, heat is generated at a rate $Q_0(1+Ar^2/a^2)$ per unit volume, where r is the distance from the axis of the cylinder and A is a constant.

Assuming that the element is immersed in a fluid at pressure p and that *axial* expansion is prevented, find the radial and circumferential thermal stresses produced in the steady state.

3. The instantaneous temperature distribution in the thin plate $-a<x<a$, $-b<y<b$ is defined by

$$T(x,y) = T_0\left(\frac{x^2}{a^2}-1\right) ,$$

where T_0 is a positive constant. Find the magnitude and location of (i) the maximum tensile stress and (ii) the maximum shear stress in the plate if the edges $x=\pm a, y=\pm b$ are traction-free and $a\gg b$.

4. The half plane $y>0$ is subject to periodic heating at the surface $y=0$, such that the surface temperature is

$$T(0,t) = T_0\cos(\omega t) .$$

Show that the temperature field

$$T(y,t) = T_0 e^{-\lambda y}\cos(\omega t - \lambda y)$$

satisfies the heat conduction equation (14.16) with no internal heat generation, provided that

$$\lambda = \sqrt{\frac{\omega}{2k}} .$$

Find the corresponding thermal stress field as a function of y, t if the surface of the half plane is traction-free.

Using appropriate material properties, estimate the maximum tensile stress generated in a large rock due to diurnal temperature variation, with a maximum daytime temperature of 30°C and minimum nighttime temperature of 10°C.

5. The layer $0<y<h$ rests on a frictionless rigid foundation at $y=0$ and the surface $y=h$ is traction-free. The foundation is a thermal insulator and the free surface is subjected to the steady state heat input

$$q_y = q_0\cos(mx) .$$

Use Dundurs' theorem to show that the layer will not separate from the foundation and find the amplitude of the sinusoidal perturbation in the free surface due to thermal distortion.