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THE USE OF SOAP FILMS IN SOLVING TORSION PROBLEMS*

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INTRODUCTION

The equations which represent the torsion of an elastic bar of any uniform cross-section are of exactly the same form as those which represent the displacement of a soap film, due to a slight pressure acting on its surface, the film being stretched over a hole in a flat plate of the same shape as the cross-section of the bar. The theory of this relationship is briefly outlined, and it is shown that advantage may be taken of the analogy to find the stresses and torsional stiffness of a twisted bar or shaft of any cross-section whatever, by making appropriate measurements of soap films. The method is technically useful because there is no restriction on the shape of sections with which it is capable of dealing, whereas the number of cases in which the equations can be solved analytically is extremely limited.

The apparatus used for measuring films is described and illustrated, and examples of its use are given. These include simple geometrical figures, for which the results of the soap-film method may be checked by calculation, and also two instances of technically important sections which are not amenable to mathematical treatment. In the first of these, the magnitude of the stress in an internal corner, and its dependence on the radius of the fillet, are investigated, while in the second the stresses and torque of a twisted aeroplane wing spar, of I section, are discussed, and comparisons between the results of the method and those of some direct torsion experiments are given.

Finally, a number of general theorems relating to, and approximate formulae for, the stiffness and strengths of shafts and beams, are obtained with the help of the soap-film analogy. It is shown, by comparison with other results, that it is possible to deduce thus, in nearly every case, figures for those torsional data usually required in practice, which are within a small percentage of the exact values. The superiority of these formulae over those now in use appears to be due to the introduction, it is believed for the first time, of the length of the perimeter of the cross-section as a factor. This was suggested almost immediately by the soap-film analogy, and is an instance of the value of the latter as a means of forming a clear idea of the nature of the torsion problem.

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General Considerations

In the old theory of the torsion of shafts or beams of uniform cross-section, which was originated by Coulomb, it was assumed that sections of the bar, initially plane and at right angles to the axis of torsion, remained so when the bar was twisted, and that the only strains set up were those due to the relative rotation of adjacent sections about the axis.

In his classical memoir on the mathematical theory of torsion, Saint-Venant showed that the assumptions made by Coulomb were valid only in the case of circular shafts, either solid or having concentric circular holes. In every other instance the initially plane section is distorted into a curved surface, and the stresses and strains set up in the bar cannot be calculated until the shape of this curved surface has been found.

A complete discussion of the theory of torsion put forward by Saint-Venant would be out of place in the present Paper. It is fully dealt with in books on the mathematical theory of elasticity, among which the treatise of Professor Love* may be mentioned. It is necessary to remark, however, that he showed how to reduce the problem to that of finding a function of the co-ordinates of points on the cross-section, which satisfies a certain partial differential equation. There is, however, no known general analytical method of finding this function for any assigned cross-section, and therefore the torsion problem cannot be solved mathematically for the great majority of technically important sections.

A simple method of determining these stresses would be of the very greatest assistance in general engineering work, and even more so in the many fresh problems which have to be dealt with in aeronautical calculations. In the very complex sections which occur in this work, such as those of airscrew blades, and the many forms of spars and struts, etc., used, it is of the highest importance that correct knowledge should be available, and therefore the authors have carried out work at the Royal Aircraft Factory, Farnborough, with a view to solving the problem by means of a simple experimental method. The following is a very brief description of the method which has been developed.

A hole is cut, in a thin plate, of the section required to be investigated, and a circular hole of a predetermined diameter is cut alongside it. The plate is placed in a box and soap films are stretched across the holes. The films are blown out slightly by reducing the air pressure on one side of them. By making suitable measurements of the shape of the resulting film surfaces, as will be explained later, it is possible to find the stresses in a bar of the given section, in terms of the stresses in a circular bar of the same diameter as the circular hole, when the two bars are twisted through the same angle per unit length. It is equally easy, by means of other measurements, to find the ratio of the torques which must be applied to the two bars in order to produce the same twist in each. It will readily be seen that by this means the most complicated sections can be dealt with.

* A. E. H. Love, *Mathematical Theory of Elasticity*, 2nd ed., chap. XIV.

The experimental work is described in the body of the paper, while the mathematical theory of the method is discussed in an Appendix.

Experimental Methods

It is seen from the mathematical discussion given in the Appendix (p. 20), that, in order that full advantage may be taken of the information on torsion which soap films are capable of furnishing, apparatus is required with which three kinds of measurements can be made, namely:

(A) Measurements of the inclination of the film to the plane of the plate at any point, for the determination of stresses.

(B) Determination of the contour lines of the film.

(C) Comparison of the displaced volumes of the test film and circular standard for finding the corresponding torque ratio.

The available means of measurement will now be enumerated under these three heads.

(A) For this purpose optical reflection methods naturally suggest themselves. In the apparatus used by the authors, the image of an electric-lamp filament is viewed in the film in such a way that the reflected ray is coincident with the incident one, so that their common direction gives the inclination of the normal to the surface of the film. This experiment may conveniently be referred to as the measurement of angles by auto-collimation.

(B) For mapping contour lines, a steel needle point, moistened with soap solution, is arranged to move about over the plate carrying the film, its distance therefrom being adjustable by means of a micrometer screw. The point is made to approach the film till the distortion of the image in the latter shows that contact has occurred. This position is remarkably definite, so much so, indeed, that it is possible, with ordinary care, to limit the error in the measurement of the normal co-ordinate to $\pm 0{\cdot}001$ in. This method of mapping contours will be referred to as the 'spherometer' method. Another method, which was suggested to the authors by Mr Vernon Boys, F.R.S., though not so convenient as the one already described, is, nevertheless, useful in affording a ready means of exhibiting the shape of the contour lines to the eye. If a film be left undisturbed for, say, 15 min., owing to drainage and consequent thinning of the film, a black spot appears at the highest point and gradually increases in size till, after the lapse of several hours, it may include the whole surface of the film. Its edge is quite sharply defined and is horizontal. Hence, if the plate has been levelled up beforehand, the edge of the black spot coincides at any moment with a contour line of the film.

(C) The most obvious way of measuring the displaced volume of the films is to blow them up by running a known volume of water, or, preferably, soap solution, into the apparatus from a pipette or burette. The volume of the circular film may be calculated from the observed value of the inclination at its boundary, since its surface is a portion of a sphere, and hence the volume of the other film may be obtained by difference. The most accurate results are obtained by giving the film a slight initial displacement before running in the known volume of liquid, and measuring the difference of the inclinations at the boundary of the circular film.

Another method, which requires a certain amount of practice, but which has the advantage of great simplicity, is to blow up the two films, observe the angle at the edge of the circular one, and then carefully place a flat plate, moistened with soap solution, on the test film, so as to cover it completely, until the flat plate is in contact with the test-plate. The total volume is then contained in the circular film, and it can be determined in the ordinary way by again observing the inclination. Hence the volume of the test film may be found.

Description of Apparatus

In the apparatus used by the authors, the films are formed on holes cut to the required shapes in flat aluminium plates, of no. 18 s.w.g. thickness. The plates are held in a horizontal position during the experiment, and the edges of the holes are chamfered off on the underside, to an angle of about 45°, in order to fix the plane of the boundary. The soap solution used is that recommended by Mr Boys, namely, pure sodium or potassium oleate, glycerine and distilled water. It may be obtained ready for use from Messrs Griffin, Kingsway, London.

The photograph (plate 1) shows the apparatus in which the films are formed, and also illustrates the construction of the spherometer. The test-plate is clamped between the two halves of the cast-iron box *A*. The lower part of this box takes the form of a shallow tray $\frac{1}{4}$ in. deep, blackened inside and supported on levelling screws, while the upper portion is simply a square frame carrying the clamping studs and enamelled white inside. A three-way cock communicates with the former and a plain tube with the latter. The film shown in the photograph represents a section of an airscrew blade. It will be noticed that a black spot has commenced to form at the top of the bubble.

The spherometer apparatus consists of a screw *B*, of 1 mm. pitch, passing through a hole in a sheet of plate glass $\frac{1}{8}$ in. thick and sufficiently large to cover the box in any possible position. It slides about on the flat upper face of the latter. The lower end of the screw carries a hard steel point *C*, tapering about 1 in 4, and its divided head moves beside a fixed vertical scale. Fixed above the screw and in its centre line is the steel recording point *D*. The record is made on a sheet of paper fixed to the board *E*, which can swing about a horizontal axis at the same height as *D*. To mark any position of the screw, it is merely necessary to prick a dot on the paper by bringing it down on the recording point.

In the auto-collimator (plate 1), light from the straight filament of the 2V. bulb *A* is reflected from the surface of the film through a V-nick *B* and a pin-hole eyepiece *C*, placed close to the lamp and shaded from direct light by a small screen. The inclinometer *D*, which measures the angle which the optical axis makes with the vertical, consists of a spirit level, of 6 ft. radius, fixed to an arm which moves over a quadrant graduated in degrees. The apparatus is mounted, by means of a stiff-jointed link, on a tripod stand weighted with lead. Fine adjustment of angle is made with a screw.

Method of using Apparatus

In using the soap-film apparatus, the test-plate and lower half of the test-box, which must both be perfectly clean, are moistened with soap solution and clamped together by means of the upper frame. The soap solution not only forms an airtight joint between the plate and box, but also serves to saturate the air within the apparatus, so that evaporation from the surface of the film is minimized. The edges of the holes are now tested with the spherometer point; if they are not parallel to the plane of motion of the glass plate they must be adjusted. A film is then drawn across the holes by means of a strip of celluloid wetted with soap solution fresh from the stock bottle and the glass cover immediately replaced. The blowing up should be done by suction from the tube in the upper frame, and not by blowing through the stopcock, as the carbon dioxide introduced by the latter method might affect the life of the film adversely. Measurements may now be made as desired. It should *be remembered that if the auto-collimator is used, the apparatus must* be levelled up beforehand.

In the case of the spherometer, the point, previously moistened with fresh solution, is set to a given height and made to touch the film at a number of positions, which are marked on the paper. This is repeated for as many contour lines as may be required. The plate need not be levelled. A contour map taken in this way is to be seen on the board (plate 1).

Usually, the use of the auto-collimator is confined to the determination of inclinations at given points on the boundary, which are marked by scratches on the plate. It is better for stress measurements than the contour-line method, since it gives the inclination directly, whereas in the other case the latter can only be found by a graphical differentiation. The use of the optical method may be extended to the finding of inclinations at points other than those on the boundary, with the help of the spherometer, in the following manner. The outline of the experimental hole is marked on the paper by means of the recording point, and the position of the point for which the stress is desired is added. The glass plate is adjusted until the recording point coincides with it. The needle is screwed down till it just touches the film, and its height is noted. It is then screwed back till the film breaks away and finally brought down again to within one- or two-thousandths of an inch of its former height. The auto-collimator is now adjusted till the image of the filament is seen in the film just below the needle-point. The reading of the inclinometer then gives the required angle.

Accuracy of Results

Strictly speaking, the soap-film surface can only be taken to represent the torsion function if its inclination γ is everywhere so small that $\sin\gamma = \tan\gamma$ to the required order of accuracy. This would mean, however, that the quantities measured would be so small as to render excessive experimental errors unavoidable. A compromise must therefore be effected. In point of fact, it has been found from experiments on sections for which the torsion function can be calculated, that the ratio of the stress at a point in any section to the stress at a point in a circular shaft, whose radius equals the value of $2A/P$ for the section, is given quite satisfactorily by the value of $\sin\gamma/\sin\mu$, where γ and μ are the respective inclinations of the corresponding films, even when γ is as much as 35°. Similarly, the volume ratio of the films has been found to be a sufficiently good approximation to the corresponding torque ratio, for a like amount of displacement.

In contour mapping, the greatest accuracy is obtained, with the apparatus at present in use, when μ is about 20°. That is to say, the displacement should be rather less than for the other two methods of experiment.

Table 1. *Showing experimental error in determining stress by means of soap films*

	Section	Radius of circle (in.)	α (deg.)	β (deg.)	α/β	$\sin\alpha/\sin\beta$	True value	Error α/β (%)	Error $\sin\alpha/\sin\beta$ (%)
1	Equilateral triangle: height, 3 in.	1·00	32·55	21·19	1·536	1·490	1·500	+2·4	−0·7
2	Square: side, 3 in.	1·5	29·11	21·34	1·364	1·337	1·350	+1·0	−1·0
3	Ellipse: semi-axes, 2 × 1 in.	1·296	30·71	24·32	1·263	1·240	1·234	+2·4	+0·5
4	Ellipse: 3 × 1 in.	1·410	31·10	24·00	1·296	1·270	1·276	+1·6	−0·5
5	Ellipse: 4 × 0·8 in.	1·196	35·35	26·58	1·331	1·293	1·286	+3·5	+0·5
6	Rectangle: 4 × 2 in.	1·333	31·70	22·36	1·418	1·380	1·395	+1·6	−1·1
7	Rectangle: 8 × 2 in.	1·60	34·83	27·23	1·279	1·247	1·245	+2·7	+0·2
*8	Infinitely long rectangle: 1 in. wide	1·00	36·42	36·19	1·006	1·005	1·000	+0·6	+0·5

* On 4 in. length.

In all soap-film measurements the experimental error is naturally greater the smaller the value of $2A/P$. Reliable results cannot be obtained, in general, if $2A/P$ is less than about half an inch, so that a shape such as a rolled I beam section could not be treated satisfactorily in an apparatus of convenient size. As a matter of fact, however, the shape of a symmetrical soap film is unaltered if it be divided by a septum or flat plate which passes through an axis of symmetry and is normal to the plane of the boundary. It is therefore only necessary to cut half the section in the test-plate and to place a normal septum of sheet metal at the line of division. This device, for the suggestion of which the authors are indebted to the late Dr C.V. Burton, may also be employed in many other cases where contour lines are so nearly normal to the septum that they are not sensibly altered by its introduction. An I beam, for instance, might be treated by dividing the web at a distance from the flange equal to two or three times the thickness of the web. It has been found advisable to carry the septum down through the hole so that it projects about $\frac{1}{8}$ in. below the underside of the plate, as, otherwise, solution collects in the corners and spoils the shape of the film.

The values set down in table 1 indicate the degree of accuracy obtainable with the auto-collimator in the determination of the maximum stresses in sections for which the torsion function is known. They also give an idea of the sizes of holes which have been found most convenient in practice. The angles given are (α) the maximum inclination at the edge of the test film, and (β) the inclination at the edge of the circular film of radius $2A/P$. They are usually the means of about five observations and are expressed in decimals of a degree.

The last two columns show the errors due to taking the ratio of angles and the ratio of sines respectively as giving the stress ratio.

The error is always positive for α/β, and its mean value is 1·98 %. In the case of $\sin\alpha/\sin\beta$ the average error is only 0·62 %. In only two instances does the error reach 1 %, and in both it is negative. The presence of sharp corners seems to introduce a negative error which is naturally greatest when the corners are nearest to the observation point. Otherwise, there is no evidence that the error depends to any great extent on the shape. Nos. 4, 5, 7 and 8 in the table are examples of the application of the method of normal septa.

Table 2 shows the results of volume determinations made on each of the sections 1–8 given in the previous table.

Table 2. *Showing experimental error in determining torques by means of soap films*

No.	Section	Maximum inclination (deg.)	Observed volume ratio	Calculated torque ratio	Error (%)
1	Equilateral triangle: height, 3 in.	32·06	1·953	1·985	−1·6
2	Square: side, 3 in.	30·39	1·416	1·432	−1·1
3	Ellipse: semi-axes, 2 × 1 in.	30·50	1·143	1·133	+0·9
4	Ellipse: 3 × 1 in.	31·01	2·147	2·147	0
5	Ellipse: 4 × 0·8 in.	36·12	3·041	3·020	+0·7
6	Rectangle: sides, 4 × 2 in.	31·33	1·456	1·475	−1·3
7	Rectangle: 8 × 2 in.	35·28	1·749	1·744	+0·3
*8	Infinitely long rectangle	36·00	0·858	0·848	+1·2

* On 4 in. length.

The average error is 0·89 %. In four of the eight cases considered the error is greater than 1 %, and in three of these it is negative. One may conclude that the probable error is somewhat greater than it is for the stress measurements, and that it tends to be negative. Its upper limit is probably not much in excess of 2 %. The remarks already made regarding the dependence of accuracy on the shape of the section apply equally to torque measurements.

As additional confirmation of the correctness of solutions of the torsion problem obtained by the soap-film method, some experiments on wooden beams may be cited. In the first of these, a walnut plank was shaped so that its section was exactly the same as the hole in one of the test-plates, which represented a section of an air-screw blade, of fineness ratio 10·55, having its thickest part about a third of the way from the leading edge. The value of the modulus of rigidity, N, was found by performing a torsion test on this plank, using the expression for the torque given by a soap-film experiment on the plate which was used in shaping the plank. N was found to be $0{\cdot}1355 \times 10^6$ lb. per sq.in. Five circular rods were then cut from the plank and their rigidities were measured. The mean value of N found in this way was $0{\cdot}1387 \times 10^6$, a difference of only 2·3 %.

Similar experiments were made on three lengths of spruce wing-spar, of I section. The results are set down in table 3. Column A shows the value of N, obtained by twisting the spar, using the figure for torque obtained by soap films. Columns B and C show the values of N found from round specimens cut from the thickest part of the two flanges, while column D gives the percentage difference between A and the mean of B and C.

Table 3. *Comparison of soap-film results with those of direct torsion experiments*

Spar no.	A lb. per sq.in.	B lb. per sq.in.	C lb. per sq.in.	D (%)
1	$0{\cdot}1091 \times 10^6$	$0{\cdot}1172 \times 10^6$	$0\ 1063 \times 10^6$	2·5
2	0 0873	0 0640	0·0966	8·7
3	0·1156	0 1200	0·1151	1·7

The comparatively large discrepancy in no. 2 is probably due to the extraordinarily large variation of N over this particular spar.

When contour lines have been mapped, the torque may be found from them by integration. If the graphical work is carefully done, the value found in this way is rather more accurate than the one obtained by the volumetric method. Contours may also be used to find stresses by differentiation, that is, by measuring the distance apart of the neighbouring contour lines; but here the comparison is decidedly in favour of the direct process, owing to the difficulties inseparable from graphical differentiation. The contour map is, nevertheless, a very useful means of showing the general nature of the stress distribution throughout the section in a clear and compact manner. The highly stressed parts show many lines bunched together, while few traverse the regions of low stress, and the direction of the resultant stress is shown by that of the contours at every point of the section. Furthermore, the map solves the torsion problem, not only for the boundary, but also for every section having the same shape as a contour line.

Experimental Results

The two examples which follow serve to illustrate the use of the soap-film apparatus in solving typical problems in design:

(1) It is well known that the stress at a sharp internal corner of a twisted bar is infinite or, rather, would be infinite if the elastic equations did not cease to hold when the stress becomes very high. If the internal corner is rounded off the stress is reduced; but so far no method has been devised by which the amount of reduction in strain due to a given amount of rounding can be estimated. This problem has been solved by the use of soap films.

An L-shaped hole was cut in a plate. Its arms were 5 in. long by 1 in. wide, and small pieces of sheet metal were fixed at each end, perpendicular to the shape of the hole, so as to form normal septa. The section was then practically equivalent to an angle with arms of infinite length. The radius in the internal corner was enlarged step by step, observations of the maximum inclination at the internal corner being taken on each occasion.

The inclination of the film at a point 3·5 in. from the corner was also observed, and was taken to represent the mean boundary stress in the arm, which is the same as the boundary stress at a point far from the corner. The ratio of the maximum stress at the internal corner to the mean stress in the arm was tabulated for each radius on the internal corner. The results are given in table 4.

Table 4. *Showing the effect of rounding the internal corner on the strength of a twisted L-shaped angle beam*

Radius of internal corner (in.)	Ratio: $\frac{\text{maximum stress}}{\text{stress in arm}}$	Radius of internal corner (in.)	Ratio: $\frac{\text{maximum stress}}{\text{stress in arm}}$
0·10	1·890	0 70	1·415
0·20	1·540	0·80	1·416
0·30	1·480	1·00	1·422
0·40	1·445	1·50	1·500
0 50	1·430	2·00	1·660
0·60	1·420		

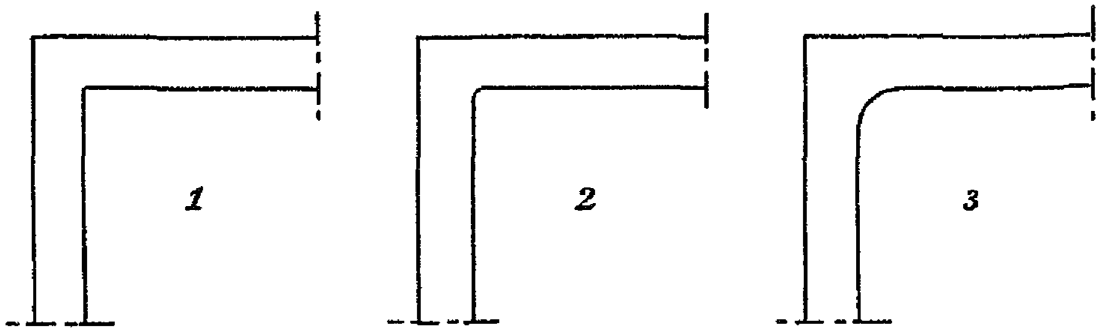


Fig. 1. Stress in internal corner.

It will be seen that the maximum stress in the internal corner does not begin to increase to any great extent till the radius of the corner becomes less than one-fifth of the thickness of the arms. A curious point which will be noticed in connection with the table is the minimum value of the ratio of the maximum stress to the stress in the arm which occurs when the radius of the corner is about 0·7 of the thickness of the arm.

In fig. 1 is shown a diagram representing the appearance of these sections of angle-irons.

No. 1 is the angle-iron for which the radius of the corner is one-tenth of the thickness of the arm. This angle is distinctly weak at the corner.

In no. 2 the radius is one-fifth of the thickness. This angle-iron is nearly as strong as it can be. Very little increase in strength is effected by rounding off the corner more than this. No. 3 is the angle with minimum ratio of stress in corner to stress in arm.

A further experiment was made to determine the extent of the region of high stress in angle-iron no. 1. For this purpose contour lines were mapped, and from

Table 5. *Showing the rate of falling off of the stress in the internal corner of the angle-iron*

Distance from boundary (in.)	Ratio: $\frac{\text{stress at point}}{\text{boundary stress in arm}}$	Distance from boundary (in.)	Ratio: $\frac{\text{stress at point}}{\text{boundary stress in arm}}$
0·00	1·89	0·30	0·49
0·05	1·36	0·40	0·24
0·10	1·12	0·50	0·00
0·20	0·77		

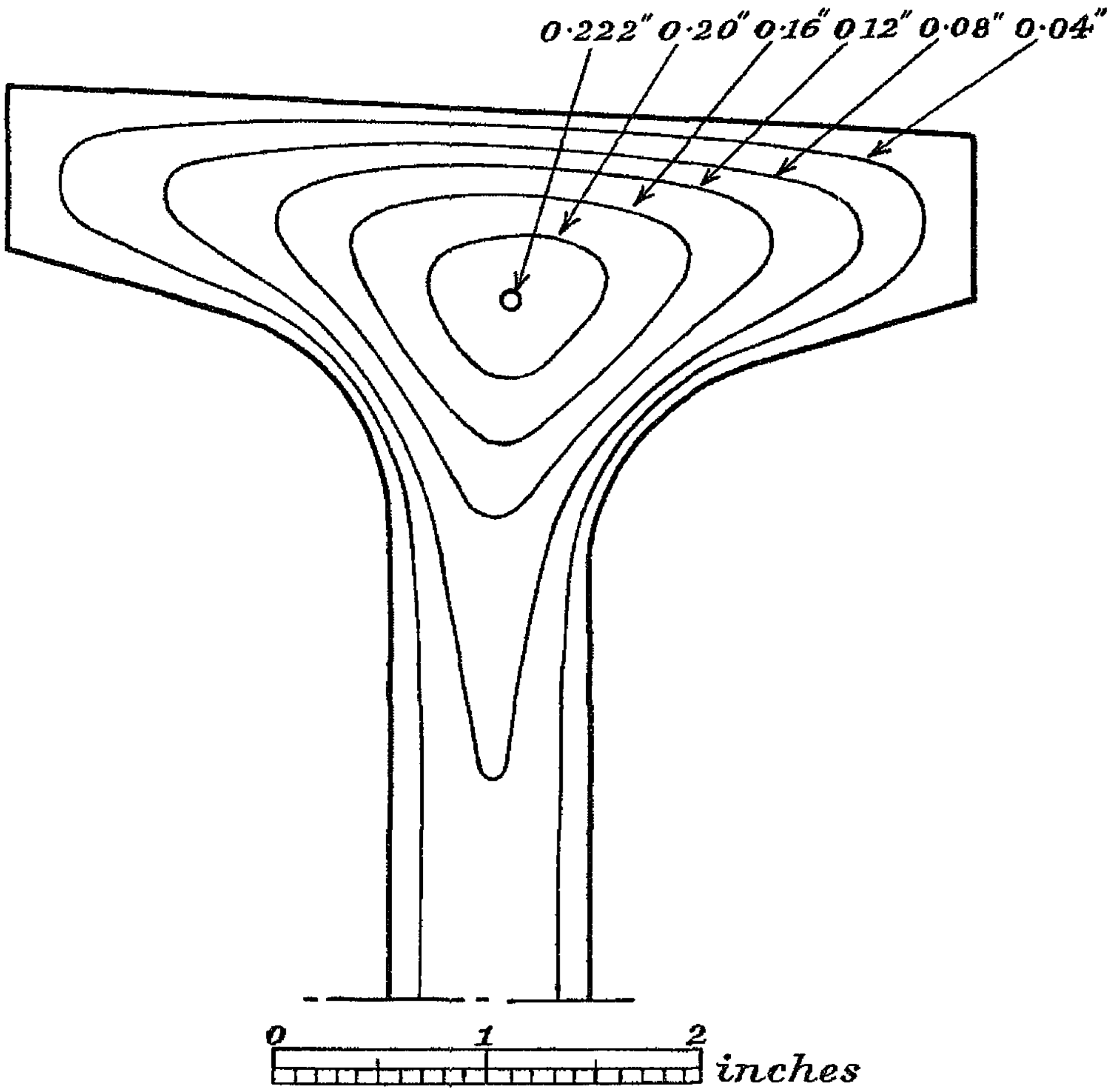


Fig. 2. Lines of shearing stress in the torsion of a wooden spar to scale. The figures give the heights of the contour-lines of the corresponding soap film. Stress at any point = 2·70 $N\tau \sin \nu$ lb. per sq.in., where ν is the inclination of the film. Torque on half section = 4·09 $N\tau$ in.lb.

these the slope of the bubble was found at a number of points on the line of symmetry of the angle-iron. Hence the stresses at these points were deduced. The results are given in table 5.

It will be seen that the stress falls off so rapidly that its maximum value is to all intents and purposes a matter of no importance, if the material is capable of yielding. If the material is brittle and not ductile a crack would, of course, start at the point of maximum stress and penetrate the section.

(2) The diagram shown in fig. 2, which represents the half-section of a wooden wing spar, is a good example of the contour-line method. The close grouping of the lines near the internal radii, denoting high stress, is immediately evident. The projecting parts of the flange are lightly stressed and contribute little to the torsional stiffness. The stress at the middle of the upper face is, however, considerable, being in fact next in order of magnitude below that in the radii. The stress near the middle of the web is practically constant and equal to that in a very long rectangular section of the same thickness under the same twist.

A further point of interest is that the 'unstressed fibre' is very near the centre of the largest circle which can be inscribed in the section. It will also be observed that the three points of greatest stress are almost coincident with the points of contact of the circle. The maximum stress is about 1·89 times the mean boundary stress.

The figures given below the diagram for the values of the stress and torque on the section fully confirm the generally accepted notions regarding the extreme weakness of I beams in torsion.

General Deductions from the Soap-film Analogy

One of the greatest advantages of considering the torsion problem from the soap-film standpoint arises from the circumstance that it is very much easier to form a mental picture of a soap bubble than it is to visualize the complicated system of shear-strains in a twisted bar. It cannot be too strongly urged that the surest way of forming a clear idea of the nature of the torsion problem is to blow a few soap films on boundaries of various shapes. This can be done with the simplest of apparatus; the holes may be cut in plates of thin sheet metal, which can be luted on to the top of a biscuit tin with vaseline or soft soap. To blow the films up it is only necessary to bore a hole in the bottom of the tin and stand it in a vessel containing water. Two sections may readily be compared by cutting them in the same plate. A simple way of estimating inclinations is to view the image of the eye in the film and adjust the arm of a clinograph so that it lies along the line of sight. Black spots, as previously mentioned, may be observed if arrangements are made to cover the films with a sheet of glass, in order to exclude dust and air currents.

With the aid of simple apparatus of this kind the truth of theorems, such as those contained in the following list, may be readily demonstrated:

(*a*) The stress distribution (and therefore the torque) for any section is independent of the axis of twist. This is easily seen, since the shape of the soap film is completely determined by the boundary and the value of $4S/p$. Hence the torque on a number of bars clamped together at their ends may be found by adding the separate torques which would be necessary to twist each through the same angle. This, as in other cases, applies to torsion only. It will be realized that in practice there will be bending stresses which must be taken account of in the usual way.

(*b*) Any addition of material to a section must increase the torque, and vice versa, so long as the distribution of material in the original section is unaltered.

(*c*) Any cut made in a section, whether it decreases the area or not, must decrease the torque.

(*d*) The stress at any point of the boundary of a section is never less than the boundary stress in a circular bar under the same twist, whose radius is equal to that of the circle inscribed in the section, which touches the boundary at the point in question.

More generally, if one section lie entirely inside another, so as to touch it at two or more points, the stresses in the inner figure are less than those in the outer one at the points of contact; if the two figures are approximately congruent in the neighbourhood of the points of contact, the difference between the stresses is small. The maximum stress in a section is not greater than $2aN\tau$, where a is the radius of the largest inscribed circle, unless the boundary is concave, that is, re-entrant.

(*e*) If a concave part of the boundary approximates to a sharp corner, the stress at this point may be very high, and if the curvature is infinite then the stress is also theoretically infinite, whatever be the situation of the corner with respect to the rest of the section. Actually, of course, if the material is ductile, we can only deduce that the stresses at such a corner are above the elastic limit.

(*f*) It is a consequence of (*e*) that it does not necessarily follow that the making of a cut in a section will reduce its strength, whether material is removed or not. As an example of this, one may quote the case of an angle-iron in which the internal corner is quite sharp. It is well known in practice that this will often fracture. It may be strengthened, however, by reducing the section, planing out a semicircular groove at the root of the angle-iron.

(*g*) There can be no discontinuous changes of stress anywhere in a section, excepting only those parts of the boundary where the curvature is infinite (concave or convex sharp corners).

(*h*) The maximum stress occurs at or near one of the points of contact of the largest inscribed circle, and not, in general, at the point of the boundary nearest the centroid, as has been hitherto assumed. An exception may occur if, at some other part of the boundary, the curvature is (algebraically) considerably less (that is, the boundary is more concave) than it is at this point.

(*i*) If a section which is long compared with its greatest thickness be bent so that its area and the length of its median line are unchanged, its torque will not be greatly changed thereby. For instance, the torsional stiffness of a metal plate is practically unaltered by folding or rolling it up into the form of an L or a split tube. Soap-film experiments show, in fact, that there is a diminution of less than 5 % when the inner radius of the boundary is not less than the thickness at the bend.

(*j*) The 'unstressed fibre', which is situated at the point corresponding with the maximum ordinate of the soap film, is near the centre of the largest circle which can be inscribed in the section.

In general, the inscribed circle has a maximum value wherever it touches the boundary at more than two points, and there is usually an unstressed fibre near the centre of each of these circles. Between each pair of maximum ordinates on the soap film, however, there is a 'minimax' point, which is near the centre of the corresponding minimum inscribed circle. This fibre in the bar is also unstressed.

(k) The 'lines of shearing stress' round the unstressed fibres of the first sort are initially ellipses, and round those of the second sort hyperbolae, from which shapes they gradually approximate to that of the boundary. Notions of this sort are useful in practice, because it is possible, with their help, to sketch in the general nature of the lines of shearing stress for any section.

Approximate Formulae for Torques and Stresses

The torque on any section is given by

$$T = N\tau C,$$

where C is a quantity of the fourth degree in the unit of length, which may be called the torsional stiffness of the section.

In the case of a circular shaft, in which there is no distortion of cross-sections, C is equal to the polar moment of inertia, so that we have

$$C = \tfrac{1}{2}Ar^2,$$

where r is the radius of the circle.

In the general case we may put

$$C = \tfrac{1}{2}Ak^2.$$

k is a length, which, by analogy with the circle, may be called the 'equivalent torsional radius' of the section.

It is seen (see Appendix, p. 20) that the mean stress round the boundary of any section is equal to the boundary stress in a circular shaft whose radius equals the quantity $2A/P$, which we have called h. This result suggested that some fairly simple approximate relation might be found between h and k.

When this idea was tested by application to known results, it became immediately evident that the fraction k/h was not very different from unity for a large number of sections. It was observed, however, that the presence of sharp outwardly projecting corners tended to make k greater than h, while the opposite effect was noticed in the case of sections whose length was great compared with their greatest thickness. For instance, h for the square is equal to a, the radius of the inscribed circle, whereas k is about 6 % greater. In the equilateral triangle h is still equal to a, while k is 9 % greater. For long rectangles and ellipses, however, k is considerably less than h.

At first sight, since, in many sections, these two effects are operating simultaneously, it might be thought that their separation, with a view to formulating a method of finding k empirically, would be a matter of some difficulty. It has been accomplished, however, by a process of successive trial, with the result that the empirical treatment about to be described has been evolved. The curves giving the values of the constants were found by plotting the values they should have for all the sections, for which a solution has been obtained, in order to get the correct result, and then drawing the best curves through these points.

If the figure contains sharp, outwardly projecting corners, construct a new figure by rounding off each corner with a radius r, which is a certain fraction of a, the radius of the largest inscribed circle. The value of this fraction depends on the angle θ, turned through by the tangent to the boundary in passing round the corner in question. In fig. 3 (p. 15), r/a is shown graphically as a function of θ/π, and, in addition, a table of values is subjoined:

θ/π	r/a	θ/π	r/a
0·0	1·00	0·6	0·375
0·1	0·93	0·7	0·270
0·2	0·85	0·8	0·210
0·3	0·75	0·9	0·170
0·4	0·625	1·0	0·155
0·5	0·500		

If the area of this new figure be called A_1, and its perimeter P_1, the value of $2A_1/P_1$ is a close approximation to the k of the original boundary, subject to the second modification, which must be made for long sections.

It is not difficult to see that a certain amount of common sense may be required in applying the above rule. For instance, if the figure has a projection which is slightly rounded instead of being quite sharp, the value of r is that which would be used if the projection did run out to a sharp point. In most cases of this sort, however, it is found that the correction makes little difference.

The criterion, which has been adopted for fixing the value of the correction factor for long sections, is the fraction a/h. Where this is appreciably less than unity, the stiffness calculated by the process already described should be multiplied by the correction factor K, which is given in table 6, and which is also shown graphically in fig. 4.

The expression for C now takes the form

$$C=\tfrac{1}{2}KA\left(\frac{2A_1}{P_1}\right)^2.$$

This formula is quite satisfactory for figures such as triangles, squares, ellipses, etc., in which a has one maximum value only, which may be called 'simple sections', but if a has more than one maximum the solution in its present form is ambiguous, and it is necessary to split the section up into two or more parts, which will be referred to as the 'components' of the original figure. The stiffness of each component must be found separately and the total stiffness obtained by addition.

In order to evolve a method of division, it is to be noted that the process already described is based on the equality of the resultant air pressure and surface-tension forces acting on the analogous soap film. If the film is divided by a series of 'normal septa', which are so arranged that they are everywhere at right angles to the contour lines which they cut, the equilibrium equations are in no way altered, and the

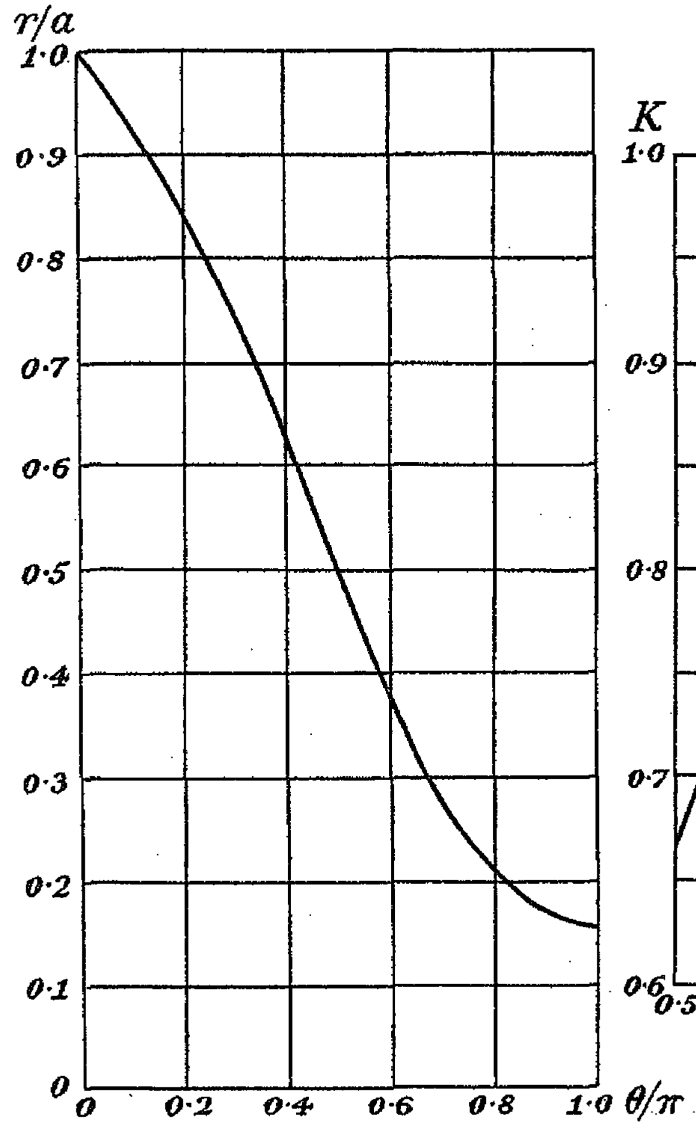


Fig. 3. Values of r/a in terms of θ/π.

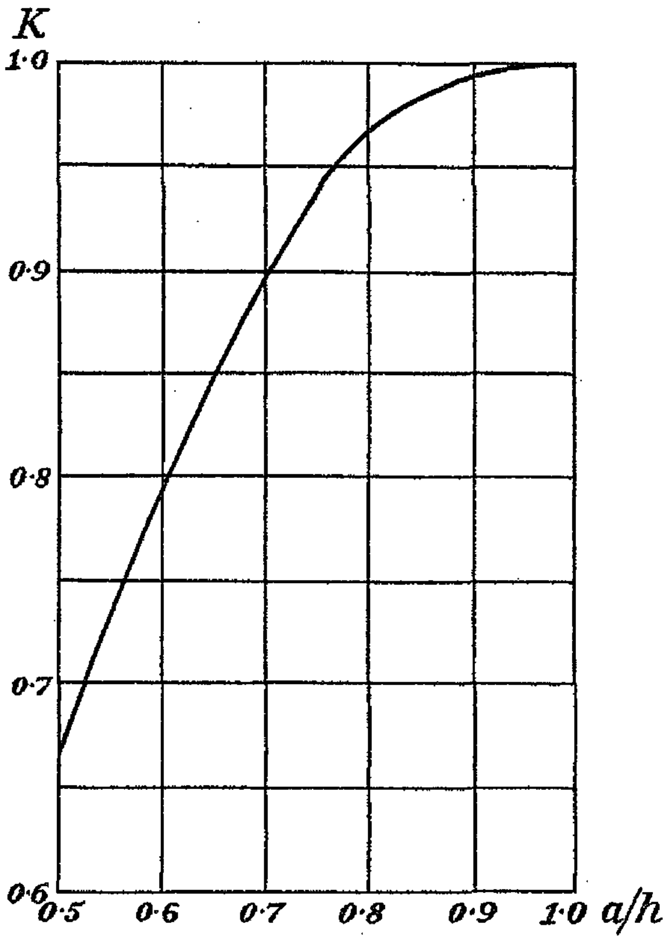


Fig. 4. Values of torque factor K in

$$T = KN\tau\frac{\overline{2A_1^2}}{P_1^2}A.$$

Table 6

a/h	K	a/h	K
1·00	1·000	0·70	0·897
0·95	0·998	0·65	0·848
0·90	0·994	0·60	0·793
0·85	0·984	0·55	0·732
0·80	0·966	0·50	0·667
0·75	0·938		

theorem is still true of each separate part of the film. Hence, if the section is divided in this way, the empirical treatment explained above should be applicable to each component. It is to be noted, however, that the term 'perimeter' must be taken to mean that part of the boundary of the component which formed part of the perimeter of the original figure. The remainder is not, strictly speaking, part of the boundary at all. It remains to formulate rules for the division of these 'compound' sections.

Imagine a circle to be drawn in the section so as to *touch* the boundary at two points. Now let the centre of this circle move through the figure, the radius being varied simultaneously so that there is always contact at two points. At some places

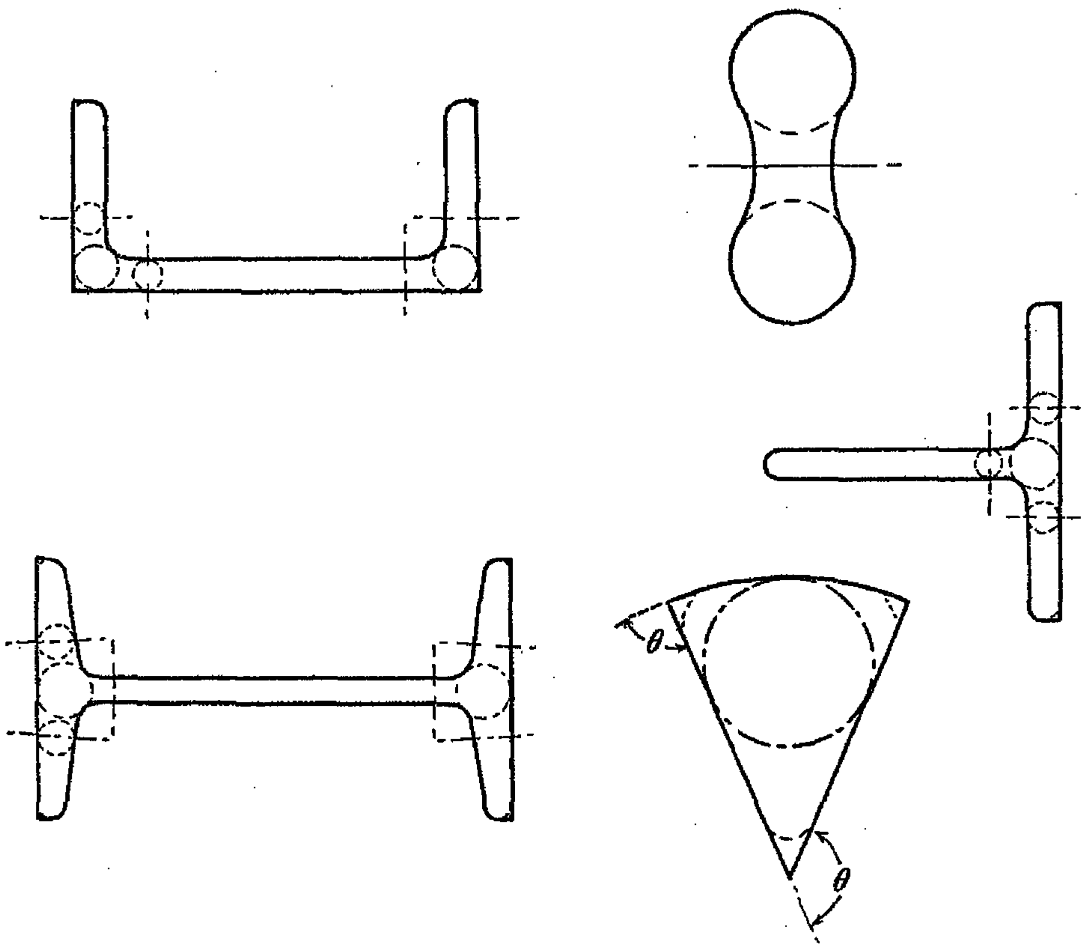
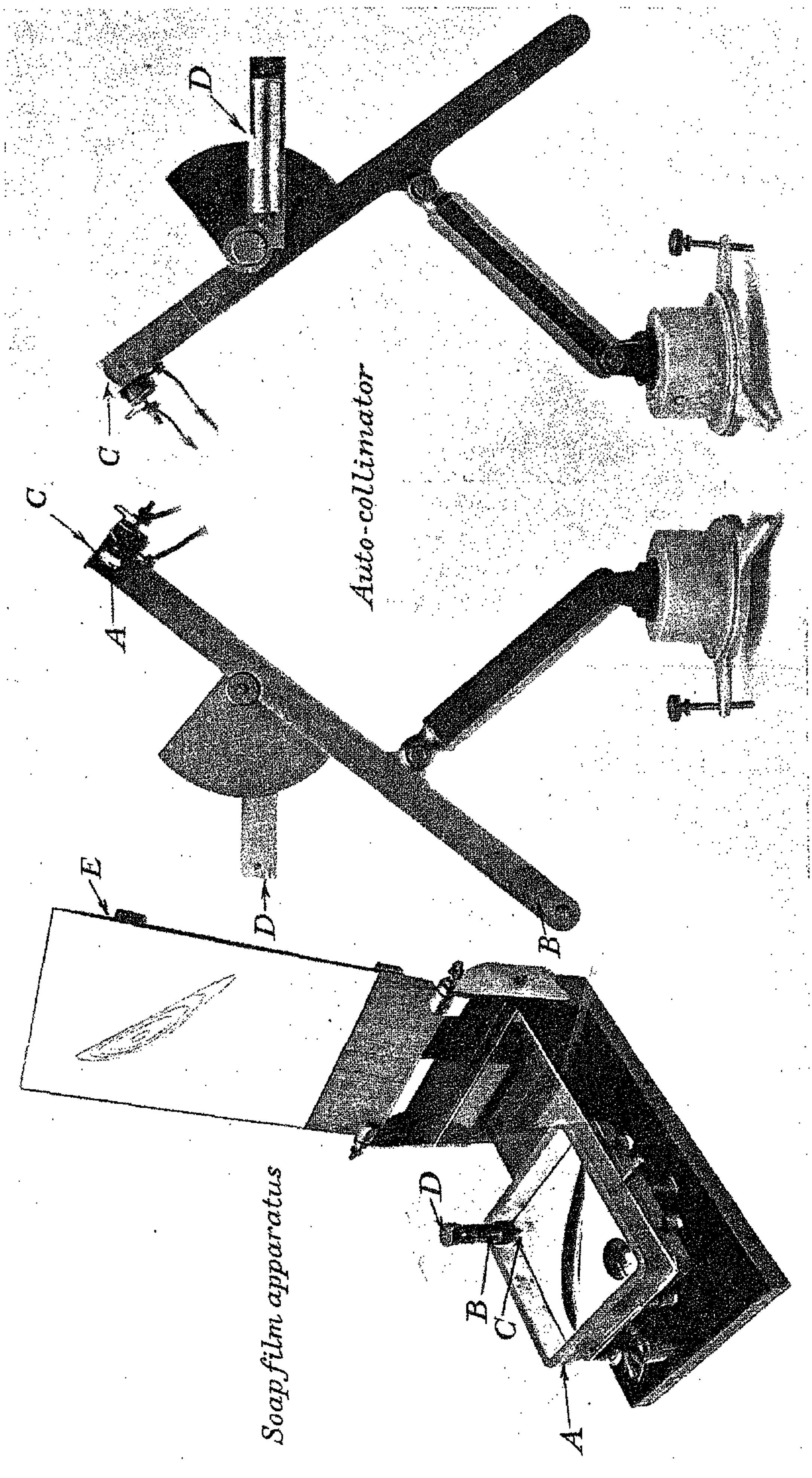


Fig. 5. Subdivision of compound sections.

the circle will touch at three (or more) points. It is then an 'inscribed circle of maximum radius', and between every such pair of maxima there must be a position where the radius is a minimum. The section should be divided by straight lines passing through the points of contact of these minimum circles.

In some cases, such as the web of an I beam, there is a long thin parallel portion, and the position of the minimum circle is indeterminate. Here the line of division should be at a distance from the commencement of the parallel part equal to half its thickness. The portion of the web cut out may be treated separately as part of an infinitely long thin rectangle, the torsional properties of which are well known. If the piece cut off is 'closed' at the other end (e.g. the arm of an angle), it may be treated as a separate component. It is advisable to cut off all long, thin, projecting parts in the same way, even though the sides are not quite parallel. The tapering flanges of I beams may be cited in illustration.

Fig. 5 shows some typical examples of the subdivision of compound sections, and also illustrates the rounding off of sharp corners. The I beam, for instance, has seven



(*Facing p.* 16)

components, the channel five, and the tee four. In the 45° sector there is only one component. The angle turned through at the apex is 135°, so that $\theta/\pi = 0{\cdot}75$. Hence, from table 5 (p. 10), $r = 0{\cdot}24a$ (a is the radius of the chain-dotted circle). At the other two corners $\theta/\pi = 0{\cdot}5$, and hence $r = 0{\cdot}5a$.

In the case of certain sections, another form of expression may be arrived at by a more direct method. Consider a soap film on a long narrow slit of varying width. If the rate of change of width with length is nowhere large, we may neglect the longitudinal curvature $\delta^2 z/\delta x^2$ of the film, and put the transverse curvature $\delta^2 z/\delta y^2$

Table 7

Section	C (formula)	C (calculated)	Error (%)
Square: side $2s$	$2{\cdot}249S^4$	$2{\cdot}249s^4$	0
Rectangle: sides $2b$, $3b$	$4{\cdot}710b^4$	$4{\cdot}698b^4$	$+0{\cdot}26$
sides $2b$, $4b$	$7{\cdot}320b^4$	$7{\cdot}318b^4$	$-0{\cdot}03$
sides $2b$, $10b$	$23{\cdot}15b^4$	$23{\cdot}31b^4$	$-0{\cdot}68$
sides $2b$, $2l$ ($l \to \infty$)	$\frac{16}{3}lb^3$	$\frac{16}{3}lb^3$	0
Ellipse: axes $2b$, $3b$	$3{\cdot}250b^4$	$3{\cdot}260b^4$	$-0{\cdot}31$
axes $2b$, $4b$	$5{\cdot}035b^4$	$5{\cdot}025b^4$	$+0{\cdot}20$
axes $2b$, $10b$	$15{\cdot}14b^4$	$15{\cdot}10b^4$	$+0{\cdot}26$
axes $2b$, $2l$ ($l \to \infty$)	$3{\cdot}235lb^3$	πlb^3	$+3{\cdot}00$
Equilateral triangle: side $2s$	$0{\cdot}3476S^4$	$0{\cdot}3464S^4$	$+0{\cdot}27$
45° sector: radius R	$0{\cdot}01810R^4$	$0{\cdot}01815R^4$	$-0{\cdot}27$
90° sector: radius R	$0{\cdot}0830R^4$	$0{\cdot}0824R^4$	$+0{\cdot}73$
Curtate sector: 180°, $R_1 = 2R_0$	$1{\cdot}355R_0^4$	$1{\cdot}369R_0^4$	$-1{\cdot}02$

equal to a constant R^{-1}, say. If the width at a distance x from one end be y, and the total length l, we readily obtain the volume, V, of the film in the form

$$V = \frac{1}{12R}\int_0^l y^3\,dx.$$

This result must be exact for an indefinitely long rectangle, whence we have, by comparison with the known stiffness of the latter,

$$C = \frac{1}{3}\int_0^l y^3\,dx = I, \quad \text{say,}$$

for the torsional stiffness of any long thin section.

A consideration of the case of ellipses suggests the modification

$$C = \frac{I}{1 + 4\dfrac{I}{Al^2}}$$

to allow for the longitudinal curvature of the figure.

The expression is now exact for all ellipses, whatever their fineness ratio, and, as will be seen, its error is within the limits of accuracy of soap-film measurements, for sections such as those of airscrew blades, down to a fineness ratio of two, at least.

The formula may also be applied, though with somewhat less accuracy, to thin sections having a curved median line, provided that x is measured along the latter and y at right angles thereto.

Tables 7, 8 and 9 have been prepared to indicate the degrees of accuracy which may be expected in the application of the preceding formulae.

Table 8

Section		C (formula)	C (soap film)
Wing spar:	$2\frac{1}{2} \times 1\frac{1}{4}$ in. (I section)	0·0678 in.^{4}	0 0680 in.^{4}
	$3 \times 1\frac{1}{2}$ in. (I section)	0·1042 in.^{4}	0·1051 in.^{4}
Angle:	3×3 in.	0·478 in.^{4}	0·487 in.^{4}
Airscrew section:	A	11·70 in.^{4}	11·72 in.^{4}
	B	7·44 in.^{4}	7·50 in.^{4}
	C	2·42 in.^{4}	2·38 in.^{4}
	D	0·846 in.^{4}	0 835 in.^{4}

Table 9

Section		C (formula)	C (experiment)
Angle:	$1·175 \times 1·175$ in.	0·01234 in.^{4}	0 01284 in.^{4}
	$1·00 \times 1·00$ in.	0·00440 in.^{4}	0·00455 in.^{4}
Tee:	$1·58 \times 1·58$ in.	0 01451 in.^{4}	0·01481 in.^{4}
I beam:	$5·01 \times 8·02$ in.	1·160 in.^{4}	1·140 in.^{4}
	$3·01 \times 3·00$ in.	0·1179 in.^{4}	0·1082 in.^{4}
	$1·75 \times 4·78$ in.	0 0702 in.^{4}	0·0635 in.^{4}
Channel:	$0·97 \times 2·00$ in.	0·0175 in.^{4}	0·0139 in.^{4}

In table 7, comparison is made with the results of Saint-Venant's exact analysis; in table 8 the second column of values has been obtained from soap-film measurements; while in table 9 the results of the method are compared with those of some direct torsion experiments on rolled beams, carried out by Mr E. G. Ritchie.*

It will be seen that all the figures in table 9 show good agreement with the exception of those referring to the last three beams. In view of the remarks made by the author cited in regard to the want of homogeneity of rolled beams, and more particularly the comparative weakness of the metal in the internal radii, the discrepancy in these cases cannot be considered unsatisfactory.

The method of calculating C should be chosen according to the nature of the section.

If there is only one maximum inscribed circle, and the section is not a long thin one, proceed by the method of rounding off sharp corners and finding $2A_1/P_1$, etc.

If the section is compound, divide it into its components and then proceed as before. Alternatively, if some of the components are thin compared with their length, they may be dealt with by finding $\int y^3 dx$.

If the median line of the section is long in comparison with the greatest thickness, straighten out the median line where necessary and use the $\int y^3 dx$ method.

* *A Study of the Circular Arc Bow Girder*, by Gibson and Ritchie (Constable and Co. 1914).

Estimation of Stresses

The empirical calculation of the stress at any given point of a section is naturally a matter of greater difficulty than the determination of torques. If the section contains no re-entrant angles, the stresses at the three points of contact of the inscribed circle of maximum radius a are usually given sufficiently well by the expression

$$\frac{2a}{1+m^2}\left[1+0{\cdot}15\left(m^2-\frac{a}{\rho}\right)\right],$$

where m is the quantity $\pi a^2/A$ and ρ is the radius of curvature of the boundary.

In the case of a 'compound' section, the formula may be applied to each component separately.

Table 10

Section	Stress/$N\tau$ (formula)	Stress/$N\tau$ (true)
Ellipse: axes $2a$, $2b$	$\frac{2ab^2}{(a^2+b^2)}$	$\frac{2ab^2}{(a^2+b^2)}$
Square: side $2s$	$1{\cdot}35s$	$1{\cdot}35s$
Rectangle: sides $2s$, $3s$	$1{\cdot}64s$	$1{\cdot}69s$
sides $2s$, $4s$	$1{\cdot}77s$	$1{\cdot}86s$
sides $2s$, $8s$	$1{\cdot}94s$	$1{\cdot}99s$
sides $2s$	$2{\cdot}00s$	$2{\cdot}00s$
Equilateral triangle: sides $2\sqrt{3}s$	$1{\cdot}53s$	$1{\cdot}50s$
Wing spar (I) $a=1{\cdot}05$ in.	2·14	2·13*
Wing spar (I) $a=1{\cdot}27$ in.	2·60	2·58*

* Determined by soap films.

The mean value of the stress round the boundary of any component is accurately equal to $2N\tau A/P$. By combining this value with those obtained for the maximum stresses, and bearing in mind the general properties of soap films, it is possible to sketch in a boundary stress diagram for the component, with sufficient accuracy for most purposes.

Obviously the formula cannot be expected to apply to points where the boundary is concave—that is, re-entrant angles, since it fails to differentiate between an acute re-entrant angle and an obtuse one. It is possible to devise a formula which will take account of this angle and which will fit any assigned number of observed results within, say, 4 or 5 %, but such a formula naturally becomes more complicated as its range of application is increased, and hence the practical utility of such generalization is doubtful. Probably the most satisfactory way of dealing with re-entrant sections is to make soap-film measurements and to deduce, from these, formulae or curves which apply to one particular class of figure only.

It should be mentioned, however, that the formula given has been found to agree with soap-film measurements on a number of re-entrant sections, in which the angle is approximately a right angle, when ρ is not very small. I beams, channels and tees are examples of such sections, to which the formula may be applied. It should be borne in mind, however, that ρ is now negative.

The stress at any point of a rolled standard section may be taken to be $2aN\tau$, where a is the radius of the inscribed circle which touches at that point, except at places near the end of a flange, where the stress is smaller. The same thing holds for figures such as airscrew sections, when the fineness ratio is greater than about eight.

MATHEMATICAL APPENDIX

The solution of the problem of torsion can be made to depend (see the book referred to in the introduction) on the finding of a function, ψ, of x and y, the co-ordinates of points on the cross-section, which satisfies the partial differential equation

$$\frac{\delta^2\psi}{\delta x^2}+\frac{\delta^2\psi}{\delta y^2}+2=0 \tag{1}$$

at all points of the cross-section, and is zero at all points on the bounding curve.

Consider the equations which represent the surface of a soap film stretched over a hole of the same size and shape as the cross-section of the twisted bar, cut in a flat plate, the film being slightly displaced from the plane of the plate by a small pressure p.

If S be the surface tension of the soap solution, the equation of the surface of the film is

$$\frac{\delta^2 z}{\delta x^2}+\frac{\delta^2 z}{\delta y^2}+\frac{p}{2S}=0, \tag{2}$$

where z is the displacement of the film and x and y are the same as before. Round the boundary, of course, $z=0$.

It will be seen that if z is measured to such a scale that $\psi=4Sz/p$, then the two equations are identical. It appears, therefore, that the value of ψ corresponding with any values of x and y can be found by measuring the quantities p/S and z on the soap film.

To put the matter in another light, the soap film is a graphical representation of the function ψ for the given cross-section. Actual values of ψ can be obtained from it by multiplying the ordinates by $4S/p$.

If N is the modulus of rigidity of the material and τ the twist per unit length of the bar, the shear stress at any point of the cross-section can be found by multiplying the slope of the ψ surface at the point by $N\tau$, so that, if γ is the inclination of the bubble to the plane of the plate, the stress is

$$f_s=\frac{4S}{p}N\tau\gamma. \tag{3}$$

The torque T on the bar is given by

$$T=2N\tau\iint\psi\,dx\,dy$$

or

$$T=\frac{8S}{p}N\tau V, \tag{4}$$

where V is the volume enclosed between the film surface and the plane of the plate.

The contour lines of the soap film in planes parallel to the plate correspond to the 'lines of shearing stress' in the twisted bar, that is, they run parallel to the direction of the resultant shear stress at every point of the section.

It is evident that the torque on and stresses in a twisted bar of any section whatever may be obtained by measuring soap films in these respects.

In order to obtain quantitative results, it is necessary to find the value of $4S/p$ in each experiment. This might be done by measuring S and p directly, but a much simpler plan is to determine the curvature of a film, made with the same soap solution, stretched over a circular hole and subjected to the same pressure difference, p, between its two surfaces, as the test film.

The curvature of the circular film may be measured by observing the maximum inclination of the film to the plane of its boundary.

If this angle be called μ, then

$$\frac{4S}{p}=\frac{h}{\sin\mu}, \tag{5}$$

where h is the radius of the circular boundary.

The most convenient way of ensuring that the two films shall be under the same pressure is to make the circular hole in the same plate as the experimental hole.

It is evident that, since the two films have the same constant $4S/p$, we may, by comparing inclinations at any desired points, find the ratio of the stresses at the corresponding points of the cross-section of the bar under investigation to the stresses in a circular shaft of radius h under the same twist. Equally, we can find the ratio of the torques on the two bars by comparing the displaced volumes of the soap films. This is, in fact, the form which the investigations usually take.

As a matter of fact, the value of $4S/p$ can be found from the test film itself by integrating γ, its inclination, round the boundary. If A be the area of the cross-section, then the equilibrium of the film requires that

$$\int 2S\sin\gamma\, ds=pA. \tag{6}$$

This equation may be written in the form

$$\frac{4S}{P}=2\times\frac{\text{area of cross-section}}{(\text{perimeter of cross-section})\times(\text{mean value of }\sin\gamma)}. \tag{7}$$

By measuring γ all round the boundary the mean value of $\sin\gamma$ can be found, and hence $4S/p$ may be determined. This is, however, more laborious in practice than the use of the circular standard.

It is evident that if the radius of the circular hole be made equal to the value of $2A/P$, where A is the area and P the perimeter of the test hole, then $\sin\mu$ = mean value of $\sin\gamma$. It is convenient to choose the radius of the circular hole so that it satisfies this condition, in order that the quantities measured on the two films may be of the same order of magnitude.

The corresponding theorem in the torsion problem states that the mean stress round the boundary of a twisted bar is equal to the stress at the boundary of a circular shaft of radius $2A/P$. It is shown in the text that this property can be made the basis of a method of approximating to the torsional stiffness of any bar by calculation.

Symbols and Formulae used in the Paper

N = modulus of rigidity of material.

τ = twist of bar in radians per unit of length.

A = area of cross-section of bar.

P = length of perimeter of cross-section.

$h = 2A/P$.

f_s = shear stress in bar.

f_c = shear stress in circular bar of radius h under twist τ.

T = torque applied to bar.

T_1 = torque applied to circular bar to give twist τ.

γ = inclination of soap film blown on a hole of the same shape as the twisted bar.

μ = inclination of film blown on a circular hole of radius h.

V = displaced volume of the test-film.

V_1 = displaced volume of the circular film.

S = surface tension of soap solution.

p = pressure difference causing displacement.

$C = T/N\tau$.

k = 'equivalent torsional radius.'

a = radius of inscribed circle.

r = radius for rounding projecting corners.

θ = angle turned through at a corner by the tangent to the boundary.

A_1 = area of modified section, when the corners have been rounded off.

P_1 = perimeter of modified section.

K = torque correction factor.

$I = \frac{1}{3}\int_0^l y^3\,dx$, the integration being taken along the median line of the section (l = length of median line).

$m = \frac{\pi a^2}{A}$.

ρ = radius of curvature of boundary of section.

(1) $f_s = \frac{4S}{p} N\tau\gamma$.

(2) $T = \frac{8S}{p} N\tau V$.

(3) $\frac{4S}{p} = \frac{h}{\sin\mu}$.

(4) $\frac{f_s}{f_c}=\frac{\sin\gamma}{\sin\mu}$, for any pair of points on the sections.

(5) $\frac{T}{T_1}=\frac{V}{V_1}$.

(6) $C=\frac{1}{2}KA\left(\frac{2A_1}{P_1}\right)^2$ for a simple section or for any component of a compound section.

(7) $C=\dfrac{I}{1+\dfrac{4I}{Al^2}}$ for a long thin section.

(8) Stresses at points of contact of inscribed circles of maximum radius a

$$f_s=\frac{2aN\tau}{1+m^2}\left[1+0\cdot15\left(m^2-\frac{a}{\rho}\right)\right].$$

(9) Mean stress round the boundary of any section

$$f_s=\frac{2A}{P}N\tau.$$

(10) Stress at any point of the boundary of a rolled standard section

$$f_s=2aN\tau$$

(a is the radius of the inscribed circle which touches at the point in question).