Let V be the three dimensional Euclidean vector space and L be the set of all linear maps from V to V. The set of real numbers is denoted by $\mathbb{R}$. The identity element in L is denoted by $\mathbf{I}$.

Two theorems for symmetric tensors According to the *spectral theorem*, for every symmetric tensor $\mathbf{A} \in L$, there exists an orthonormal basis $\{\mathbf{u}_i\} \in V (i = 1, 2, 3)$ and numbers $\lambda_i \in \mathbb{R}$ such that

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i. \tag{0.1}$$

The numbers λ_i are the principal values associated with the tensor $\mathbf{A}$ and can be obtained as the roots of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ with $\lambda \in \mathbb{R}$. We now prove this assertion. Let λ and $\mathbf{u}$ be a principal value (eigenvalue) and the corresponding principal vector (eigenvector) associated with $\mathbf{A}$. Allow them to be complex, i.e. $\lambda = a + ib$ and $\mathbf{u} = \mathbf{a} + i\mathbf{b}$ for some $\{a, b\} \in \mathbb{R}$ and $\{\mathbf{a}, \mathbf{b}\} \in V$ with $i = \sqrt{-1}$. Therefore $\mathbf{A}\mathbf{u} = \lambda\mathbf{u}$. We also have $\mathbf{A}\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$, where an over-bar represents the complex conjugate. Since $\mathbf{A}$ is symmetric, we can write $\mathbf{u} \cdot \mathbf{A}\bar{\mathbf{u}} = \bar{\mathbf{u}} \cdot \mathbf{A}\mathbf{u}$ or $0 = (\lambda - \bar{\lambda})\mathbf{u} \cdot \bar{\mathbf{u}}$. This implies $\lambda = \bar{\lambda}$, as $\mathbf{u} \cdot \bar{\mathbf{u}} > 0$. We now have to prove the existence of orthonormal $\{\mathbf{u}_i\}$ such that (0.1) holds. For eigenvalues $\lambda_1,\ \lambda_2$ and their corresponding eigenvectors $\mathbf{u}_1,\ \mathbf{u}_2$, we have $\mathbf{A}\mathbf{u}_1 = \lambda_1\mathbf{u}_1$ and $\mathbf{A}\mathbf{u}_2 = \lambda_2\mathbf{u}_2$. As $\mathbf{A}$ is symmetric, $\mathbf{u}_1 \cdot \mathbf{A}\mathbf{u}_2 = \mathbf{u}_2 \cdot \mathbf{A}\mathbf{u}_1$ and thus $0 = (\lambda_1 - \lambda_2)\mathbf{u}_1 \cdot \mathbf{u}_2$. If $\lambda_1 \neq \lambda_2$, then $\mathbf{u}_1$ and $\mathbf{u}_2$ are mutually orthogonal. Therefore if $\{\lambda_i\}$ are distinct, $\{\mathbf{u}_i\}$ necessarily forms an orthonormal set. If $\lambda_1 \neq \lambda_2 = \lambda_3$, then $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$. Define $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, so that $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a right handed orthonormal set. The vector $\mathbf{u}_3$ is the third principal vector of $\mathbf{A}$. Indeed $\mathbf{A}\mathbf{u}_3 = \sum_{i=1}^{3}(\mathbf{u}_i \cdot \mathbf{A}\mathbf{u}_3)\mathbf{u}_i = \sum_{i=1}^{3}(\mathbf{u}_3 \cdot \mathbf{A}\mathbf{u}_i)\mathbf{u}_i = (\mathbf{u}_3 \cdot \mathbf{A}\mathbf{u}_3)\mathbf{u}_3$ where in the first equality, the vector $\mathbf{A}\mathbf{u}_3$ is expressed in terms of the basis vectors $\{\mathbf{u}_i\}$. In the second equality, the symmetry of $\mathbf{A}$ is used and in the third equality, the relations $\mathbf{A}\mathbf{u}_\alpha = \lambda_\alpha\mathbf{u}_\alpha$ $(\alpha = 1, 2)$ and the orthonormality of $\{\mathbf{u}_i\}$ are employed. Finally, if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, we can pick any orthonormal basis in V and in this case $\mathbf{A} = \lambda\mathbf{I}$.

According to the *square root theorem*, for every positive definite symmetric tensor $\mathbf{A} \in L$, there exists a unique positive definite symmetric tensor $\mathbf{G} \in L$ such that $\mathbf{A} = \mathbf{G}^2$. By the spectral theorem we have a representation (0.1) for $\mathbf{A}$ with $\lambda_i > 0$ (due to the positive definiteness of $\mathbf{A}$). Define $\mathbf{G} = \sum_{i=1}^{3}\sqrt{\lambda_i}\mathbf{u}_i \otimes \mathbf{u}_i$. Then, $\mathbf{G}^2 = \mathbf{G}\mathbf{G} = \sum_{i=1}^{3}\sqrt{\lambda_i}(\mathbf{G}\mathbf{u}_i) \otimes \mathbf{u}_i = \sum_{i=1}^{3}\lambda_i\mathbf{u}_i \otimes \mathbf{u}_i = \mathbf{A}$ and it is obvious that $\mathbf{G}$ is symmetric and positive definite. To prove uniqueness we assume that there exists a symmetric and positive definite tensor $\hat{\mathbf{G}}$ such that $\mathbf{G}^2 = \mathbf{A} = \hat{\mathbf{G}}^2$ and show that $\mathbf{G} = \hat{\mathbf{G}}$. Let $\mathbf{u}$ be an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda > 0$. Then $(\mathbf{G}^2 - \lambda\mathbf{I})\mathbf{u} = \mathbf{0}$ or $(\mathbf{G} + \sqrt{\lambda}\mathbf{I})\mathbf{v} = \mathbf{0}$, where $\mathbf{v} = (\mathbf{G} - \sqrt{\lambda}\mathbf{I})\mathbf{u}$. This requires $\mathbf{v} = \mathbf{0}$ as otherwise $-\sqrt{\lambda}$ becomes an eigenvalue of $\mathbf{G}$, contradicting the positive definiteness of $\mathbf{G}$. Therefore $\mathbf{G}\mathbf{u} = \sqrt{\lambda}\mathbf{u}$ and similarly $\hat{\mathbf{G}}\mathbf{u} = \sqrt{\lambda}\mathbf{u}$. Thus $\mathbf{G}\mathbf{u}_i = \hat{\mathbf{G}}\mathbf{u}_i$ and since an arbitrary vector $\mathbf{f}$ can be expressed as a linear combination of $\{\mathbf{u}_i\}$, we obtain $\mathbf{G}\mathbf{f} = \hat{\mathbf{G}}\mathbf{f}$. This implies $\mathbf{G} = \hat{\mathbf{G}}$.

Polar decomposition theorem Every invertible tensor $\mathbf{F} \in L$ can be uniquely decomposed in terms of symmetric positive definite tensors $\{\mathbf{U}, \mathbf{V}\} \in L$ and a orthogonal tensor $\mathbf{R} \in L$ such that

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}. \tag{0.2}$$

The first of these equalities can be proved by using the right Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. By the square root theorem there exists a unique symmetric positive definite tensor $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{C}$. Define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. It follows, that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$. If $\det \mathbf{F} > 0$ then $\det \mathbf{R} = 1$ (since $\det \mathbf{F} = \det \mathbf{U} = \sqrt{\det \mathbf{C}}$), and therefore $\mathbf{R}$ is a proper orthogonal tensor. The relation $\mathbf{F} = \mathbf{VR}$ can be proved similarly via the left Cauchy Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$.

Principal invariants The characteristic equation for $\mathbf{A} \in L$ is

$$0 = \det(\mathbf{A} - \lambda\mathbf{I}) = -\lambda^3 + \lambda^2 I_1(\mathbf{A}) - \lambda I_2(\mathbf{A}) + I_3(\mathbf{A}), \tag{0.3}$$

where

$$\begin{aligned} I_1(\mathbf{A}) &= \operatorname{tr} \mathbf{A} \\ I_2(\mathbf{A}) &= \operatorname{tr} \mathbf{A}^* = \frac{1}{2}[(\operatorname{tr} \mathbf{A})^2 - \operatorname{tr} \mathbf{A}^2] \\ I_3(\mathbf{A}) &= \det \mathbf{A} \end{aligned} \tag{0.4}$$

are the *principal invariants* of $\mathbf{A}$. According to the *Cayley-Hamilton theorem*, $\mathbf{A}$ satisfies its own characteristic equation, i.e.

$$-\mathbf{A}^3 + I_1(\mathbf{A})\mathbf{A}^2 - I_2(\mathbf{A})\mathbf{A} + I_3(\mathbf{A})\mathbf{I} = \mathbf{0}. \tag{0.5}$$

We now prove this theorem. Let $\mathbf{D} = ((\mathbf{A} - \lambda\mathbf{I})^*)^T$, where $\lambda \in \mathbb{R}$ is such that $\det(\mathbf{A} - \lambda\mathbf{I}) \neq 0$ but otherwise arbitrary. Since $\mathbf{A} - \lambda\mathbf{I}$ is invertible, we have $\mathbf{D} = \det(\mathbf{A} - \lambda\mathbf{I})(\mathbf{A} - \lambda\mathbf{I})^{-1}$ or $\mathbf{D}(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})\mathbf{I}$. The right hand side of this relation is cubic in λ and the term $\mathbf{A} - \lambda\mathbf{I}$ is linear in λ. Therefore $\mathbf{D}$ has to be quadratic in λ (by a theorem on factorization of polynomials). Let $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1\lambda + \mathbf{D}_2\lambda^2$ for some $\mathbf{D}_0$, $\mathbf{D}_1$ and $\mathbf{D}_2$. Then $(\mathbf{D}_0 + \mathbf{D}_1\lambda + \mathbf{D}_2\lambda^2)(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} - \lambda\mathbf{I})\mathbf{I} = (-\lambda^3 + \lambda^2 I_1 - \lambda I_2 + I_3)\mathbf{I}$. Matching coefficients of various powers of λ between the first and the last term and eliminating $\mathbf{D}_0$, $\mathbf{D}_1$ and $\mathbf{D}_2$ from these, we get the required relation (0.5). The coefficients of all the powers of λ have to vanish since otherwise we would obtain a polynomial (of order 3) in λ, which could then be solved to obtain roots for λ, contradicting the premise that $\lambda \in \mathbb{R}$ is arbitrary.