1. Notes on compatibility equations and stress functions. We will use Stokes' theorem to develop strain compatibility equations in linear elasticity as well as to introduce the concept of Airy stress functions.

E denotes the three-dimensional Euclidean point space; $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ form an orthonormal basis in a Cartesian coordinate system.

1.1 A corollary from Stokes' theorem

Let $\mathbf{v} = v_i \mathbf{e}_i$ be a smooth vector field on a simply-connected region $\Omega \subset E$. A region is simply-connected if any closed curve contained in it can be continuously shrunk to one of its point without leaving the region; therefore a sphere with a central cavity is simply-connected but a hollow cylinder with open ends is not. In two dimensions, the region should be essentially free of holes. Let Γ be an arbitrary closed curve in Ω such that it bounds a smooth surface S in Ω. Then by Stokes' theorem

$$\int_S \operatorname{curl} \mathbf{v} \cdot \mathbf{N} dA = \oint_\Gamma \mathbf{v} \cdot d\mathbf{X}, \tag{1.1}$$

where $\mathbf{N}$ is the normal associated with S and $(\operatorname{curl} \mathbf{v})_i = e_{ijk} v_{k,j}$.

If $\operatorname{curl} \mathbf{v} = \mathbf{0}$, then $e_{ijk} v_{k,j} = 0$ implying that $v_{k,j} = v_{j,k}$ or $(\nabla \mathbf{v}) = (\nabla \mathbf{v})^T$, i.e. the gradient of a vector, whose curl is zero, is a symmetric tensor. Moreover (1.1) implies that

$$\oint_\Gamma \mathbf{v} \cdot d\mathbf{X} = 0, \tag{1.2}$$

for all closed curves in Ω. Therefore

$$\int_{\mathbf{X}_0}^{\mathbf{Y}} \mathbf{v} \cdot d\mathbf{X}, \tag{1.3}$$

where $\mathbf{X}_0$ is some fixed point in Ω, is independent of the path from $\mathbf{X}_0$ to $\mathbf{Y} \in \Omega$. Indeed, consider two paths along the curves Γ_1 and Γ_2 such that their initial point is $\mathbf{X}_0$ and end point is $\mathbf{Y}$ and that $\Gamma_1 \cup \Gamma_2 = \Gamma$ is a closed curve in Ω. The path independence of (1.3) then follows from (1.2) (write the integral in (1.2) as a sum of two line integrals). Therefore, (1.3) defines a function of $\mathbf{Y}$ alone and we can posit the existence of a scalar function ϕ which satisfies

$$\phi(\mathbf{Y}) = \phi(\mathbf{X}_0) + \int_{\mathbf{X}_0}^{\mathbf{Y}} \mathbf{v} \cdot d\mathbf{X}, \tag{1.4}$$

or equivalently $\mathbf{v}(\mathbf{X}) = \nabla \phi(\mathbf{X})$. Conversely, if $\mathbf{v} = \nabla \phi$ then $\operatorname{curl} \mathbf{v} = \mathbf{0}$. We have proved that $\operatorname{curl} \mathbf{v} = \mathbf{0}$ is necessary and sufficient for the existence of a scalar field ϕ such that $\mathbf{v} = \nabla \phi$.

We will also require the two dimensional version of this result. Consider the vector $\mathbf{v}$ such that $v_3 = 0$. Moreover, assume that v_1 and v_2 depend only on X_1 and X_2. In such a case $\operatorname{curl} \mathbf{v} = \mathbf{0}$ is reduced to a single equation given by

$$\frac{\partial v_2}{\partial X_1} - \frac{\partial v_1}{\partial X_2} = 0. \tag{1.5}$$

According to our result above, this is equivalent to the existence of a scalar field (say F) such that

$$v_1 = \frac{\partial F}{\partial X_1} \text{ and } v_2 = \frac{\partial F}{\partial X_2}. \tag{1.6}$$

It should be noted that the above corollary is valid only for simply-connected domains.

1.2 Strain compatibility

Given a single-valued continuously differentiable displacement field, the strain field (for small deformations) is defined as the symmetric part of the displacement gradient tensor. If however we are given a (smooth) symmetric tensor, then what are the necessary and sufficient conditions that there exist a single-valued continuously differentiable vector field whose symmetric gradient is equal to the given tensor field? The answer is provided by the strain compatibility equations for simply-connected domains. For multiply-connected domains, additional conditions need to be imposed for the single-valuedness of the displacement.

For a displacement field $\mathbf{u} = u_i \mathbf{e}_i$, the strain field $\boldsymbol{\epsilon} = \epsilon_{ij} \mathbf{e}_i \otimes \mathbf{e}_j$ is given by

$$u_{i,j} + u_{j,i} = 2\epsilon_{ij}. \tag{1.7}$$

Take another derivative of the above equation (w.r.t X_p) and then multiply the whole equation by e_{rip}. Use $e_{rip} u_{j,ip} = 0$ to obtain

$$e_{rip} u_{i,jp} = 2 e_{rip} \epsilon_{ij,p}. \tag{1.8}$$

Differentiate this again w.r.t X_q and multiply both sides with e_{sjq}. The resulting equation is

$$e_{rip} e_{sjq} \epsilon_{ij,pq} = 0 \tag{1.9}$$

or equivalently $\text{curl}(\text{curl}\, \boldsymbol{\epsilon}) = \mathbf{0}$. The relations (1.9) are called strain compatibility conditions. We have shown them to be necessary for (1.7) to hold true. Since e_{rip} is skew w.r.t. i and p index, (1.9) implies that the tensor $e_{sjq}\epsilon_{ij,pq}$ will be symmetric w.r.t. i and p, i.e. $e_{sjq}(\epsilon_{ij,pq} - \epsilon_{pj,iq}) = 0$. This in turn implies that $\epsilon_{ij,pq} - \epsilon_{pj,iq}$ must be symmetric w.r.t. j and q, i.e.

$$\epsilon_{ij,pq} - \epsilon_{pj,iq} = \epsilon_{iq,pj} - \epsilon_{pq,ij}. \tag{1.10}$$

This is an equivalent form of strain compatibility conditions. These are 81 equations, out of which only 6 are independent. These are

$$\begin{aligned}
\epsilon_{11,22} + \epsilon_{22,11} - 2\epsilon_{12,12} &= 0, \\
\epsilon_{22,33} + \epsilon_{33,22} - 2\epsilon_{23,23} &= 0, \\
\epsilon_{11,33} + \epsilon_{33,11} - 2\epsilon_{13,13} &= 0, \\
(\epsilon_{12,3} - \epsilon_{23,1} + \epsilon_{31,2})_{,1} - \epsilon_{11,23} &= 0, \\
(\epsilon_{23,1} - \epsilon_{31,2} + \epsilon_{12,3})_{,2} - \epsilon_{22,31} &= 0, \text{ and} \\
(\epsilon_{31,2} - \epsilon_{12,3} + \epsilon_{23,1})_{,3} - \epsilon_{33,12} &= 0.
\end{aligned}$$

We will now show that (1.10) (where ϵ_{ij} is symmetric) is also a sufficient condition for the existence of u_i for simply connected domains. Define a third order tensor with components I_{ijk} as

$$I_{ijk} = \epsilon_{ik,j} - \epsilon_{jk,i}. \tag{1.11}$$

It follows immediately that $I_{ijk} = -I_{jik}$. Moreover, note that $I_{ijk,l} - I_{ijl,k} = 0$ is equivalent to (1.10). Therefore if we assume (1.10) then, using results from previous section, we can define a second order tensor field $\mathbf{w} = w_{ij}\mathbf{e}_i \otimes \mathbf{e}_j$ such that

$$w_{ij}(\mathbf{X}) = w_{ij}^0 + \int_{\mathbf{X}_0}^{\mathbf{X}} I_{ijk} d\hat{X}_k, \text{ or } w_{ij,k} = I_{ijk}, \tag{1.12}$$

where $\mathbf{w}^0 = \mathbf{w}(\mathbf{X}_0)$. The tensor $\mathbf{w}$ is skew (due to skew symmetry of I_{ijk} with respect to i and j). Let $v_{ik} = \epsilon_{ik} + w_{ik}$. Then

$$v_{ik,l} - v_{il,k} = 0, \tag{1.13}$$

which follows upon using $(1.12)_2$ and (1.11). Thus, according to the corollary from Stokes' theorem, there exist a vector field $\mathbf{u} = u_i\mathbf{e}_i$ such that

$$u_i(\mathbf{X}) = u_i(\mathbf{X}_0) + \int_{\mathbf{X}_0}^{\mathbf{X}} (\epsilon_{ik} + w_{ik}) d\hat{X}_k, \text{ or } u_{i,k} = (\epsilon_{ik} + w_{ik}). \tag{1.14}$$

As a result (1.7) holds and u_i qualifies as a displacement field. We can rewrite $(1.14)_1$ such that the integrand is given purely in terms of strain field. For a fixed point $\mathbf{X}$, the chain rule gives $w_{ik}d\hat{X}_k = w_{ik}d(\hat{X}_k - X_k) = d(w_{ik}(\hat{X}_k - X_k)) - dw_{ik}(\hat{X}_k - X_k)$, where $dw_{ik} = I_{ikl}d\hat{X}_l$; the later relation follows from $(1.12)_2$. As a result $(1.14)_1$ becomes

$$u_i(\mathbf{X}) = u_i(\mathbf{X}_0) + w_{ij}(\mathbf{X}_0)(X_j - X_{0j}) + \int_{\mathbf{X}_0}^{\mathbf{X}} U_{ij}(\hat{\mathbf{X}}) d\hat{X}_j, \tag{1.15}$$

where $U_{ij}(\hat{\mathbf{X}}) = \epsilon_{ij}(\hat{\mathbf{X}}) + (X_l - \hat{X}_l)I_{ilj}(\hat{\mathbf{X}})$ (note: $U_{ij}(\hat{\mathbf{X}})d\hat{X}_j$ are Cartesian components of a vector given by $\big(\boldsymbol{\epsilon}(\hat{\mathbf{X}}) - (\mathbf{X} - \hat{\mathbf{X}}) \times (\operatorname{curl} \boldsymbol{\epsilon}) d\hat{\mathbf{X}}\big)$. The above relation (called Cesàro integral) provides us with an explicit relation to solve for displacement field from a given strain field. If $\epsilon_{ij}(\mathbf{X}) = 0$ then $u_i(\mathbf{X}) = a_i + w_{ij}^0 X_j$, where $a_i = u_i(\mathbf{X}_0) - w_{ij}^0 X_{0j}$. Therefore displacement field reduces down to a rigid body displacement for vanishing strain field. Moreover if two displacement fields have a common strain field, then they differ by a rigid body displacement. The single-valuedness of the displacement field is guaranteed by the part independent integral in (1.15), whose path-independence is equivalent to compatibility equations (1.9).

The situation is more involved for a multiply-connected domain. The conditions (1.9) are still necessary for the existence of a single-valued continuously differentiable displacement field. They are however not sufficient. To this end, we consider a three-dimensional multiply-connected body as shown in Figure 1 (it can be thought of as a potato with three holes drilled across). We note that any closed curve within the body is either reducible (to a point without leaving the body), for e.g. curve A, or irreducible, for e.g. curves C_1 and C_2. Consider an arbitrary point

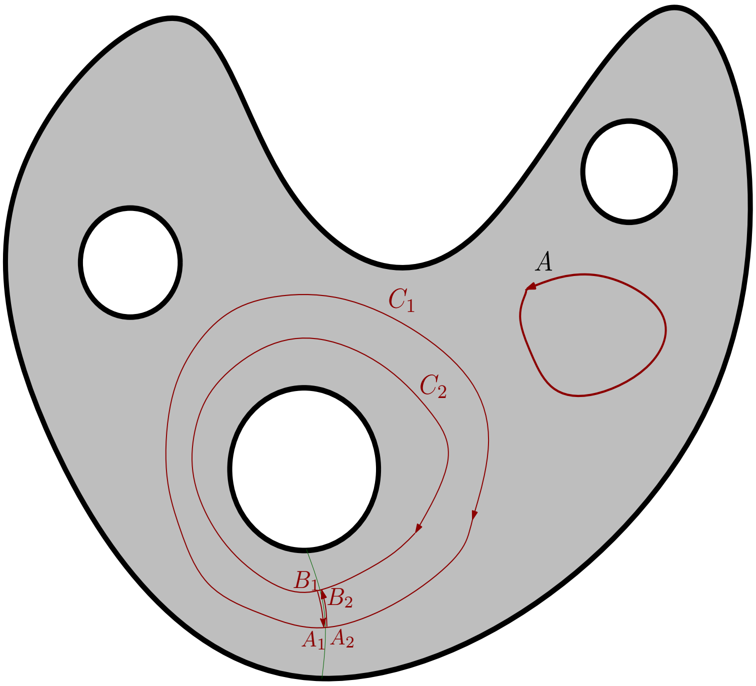

Figure 1: A multiply-connected body.

in the body; there will always exist infinitely many reducible curves containing it. Therefore, starting with strain compatibility equation (1.9), we can always construct a displacement field given by (1.15). The integral therein has to be path-independent for the single-valuedness of displacement. This is indeed so for simply-connected domains, for which the path-independence is equivalent to compatibility equations. However, for single-valuedness in a multiply-connected domain we would need to impose the path-independence, i.e.

$$\oint_C U_{ij}(\hat{\mathbf{X}}) d\hat{X}_j = 0, \tag{1.16}$$

for any irreducible curve C in the body. This needs to be enforced for any one irreducible curve around every hole; therefore these are three additional relations which need to be enforced for a body with three holes as shown in the figure, corresponding to each hole. The integral in (1.16) is invariant for any two irreducible curves around the same hole, i.e. (see figure)

$$\oint_{C_1} U_{ij}(\hat{\mathbf{X}}) d\hat{X}_j = \oint_{C_2} U_{ij}(\hat{\mathbf{X}}) d\hat{X}_j. \tag{1.17}$$

Thus enforcing (1.16) around C_1 implies that it is satisfied for any irreducible curve around the particular hole. That (1.17) is true can be shown by considering a simply-connected domain enclosed by the curve constructed as following (see figure): start at A_1, go around C_1 to reach A_2, go to B_2, go around C_2 (in the opposite direction as shown in the figure) to reach B_1, go to A_1. The points A_1 and B_1 are infinitesimally close to A_2 and B_2, respectively. It is as if we have made a cut in the hollow cylinder to make it simply-connected. The integral (1.16) vanishes in this domain because of strain compatibility; moreover the continuity of strains and its gradients will ensure the contribution from the integral over A_2B_2 and B_1A_1 to cancel. We are finally left with (1.17). Note that the validity of this equation rests on the validity of strain compatibility equation (1.9).

Summarizing our result, the strain compatibility relations (1.9) are necessary and sufficient for the existence of a single-valued and continuously differentiable displacement field (given a strain field) for a simply-connected domains. For multiply-connected domains, with say n holes, the strain field should additionally satisfy n equations given by (1.16) (one equation for each hole, the choice of irreducible curve around a hole is arbitrary).

[References: i) A. E. H. Love, The mathematical theory of elasticity, fourth edition, Dover, 1944, §156A. ii) B. A. Boley and J. H. Weiner, Theory of thermal stresses, Dover, 1997, pp. 84-100.]

1.3 Examples

(Courtesy Ayan)

Example 1 For a simply connected body, we wish to find out the temperature distribution $T(\mathbf{x})$ that gives a compatible thermal strain field.

If the thermal conductivity is homogeneous and isotropic, we can write down the thermal strain as $\varepsilon_{ij} = \beta\, T\, \delta_{ij}$, where β is the constant coefficient of thermal expansion. Putting this strain field in (1.9) implies

$$T_{,ij} + T_{,kk}\, \delta_{ij} = 0.$$

Taking trace of this expression implies $T_{,kk} = 0$. Hence, $T_{,ij} = 0$. The general solution is $T(\mathbf{x}) = a + b_i\, x_i$, where a and b_i, $i = 1, 2, 3$, are integration constants. Thus, for a simply connected body, the temperature distribution must be necessarily linear in order to have a compatible thermal strain field.

Note. Example of a non-linear temperature distribution over a hollow cylindrical body (doubly connected) which produces locally compatible thermal strain field would be $T(r,\, \theta,\, z) = Ar^2 \cos 2\theta$, where A is some constant.

Example 2 Consider simple shear of a simply connected cube in x_1-x_2 plane. We have seen that the displacement field for such a deformation looks like $\boldsymbol{u} = \lambda\, x_2\, \mathbf{e}_1$, with λ constant. If $|\lambda| << 1$ (a measure of small deformation), the small strain field in the Cartesian basis looks like

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} 0 & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is a constant field and, hence, satisfies the compatibility conditions (1.9).

Example 3 Consider the following non-linear small strain field over a simply connected body, given in some Cartesian basis,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} 0 & \frac{\lambda}{2}x_1x_2 & 0 \\ \frac{\lambda}{2}x_1x_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda \neq 0,\, |\lambda| << 1.$$

This strain field does not satisfy the condition (1.9) ($\varepsilon_{11,22} + \varepsilon_{22,11} - 2\,\varepsilon_{12,12} = -\lambda \neq 0$). Hence, the above strain field is an example of an incompatible strain field over a simply connected body.

Example 4 Let us consider now a long thin elastic tube (a model of an artery) which is a doubly connected domain. Let the inner and outer radii be r_1 and r_2 respectively, with $|r_1 - r_2| << 1$ (thin-ness). Take the standard cylindrical coordinate system $\{r_1 < r < r_2,\, \theta \in [0,\, 2\pi],\, z \in \mathbb{R}\}$ and consider the following small strain field

$$\varepsilon_{rr} = 0,\ \varepsilon_{\theta\theta} = k,\ \varepsilon_{33} = 0, \qquad k = \text{cosntant},\ k \neq 0,\, |k| << 1.$$

The above is an example of an axisymmetric plane strain field (ε_{rr}, $\varepsilon_{\theta\theta}$ are functions of r only and all other strain components are zero). For such fields,

$$\operatorname{curl} \boldsymbol{\varepsilon} = \left(\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\varepsilon_{rr} - \varepsilon_{\theta\theta}}{r} \right) \mathbf{e}_3 \otimes \mathbf{e}_\theta.$$

For our case, $\operatorname{curl} \boldsymbol{\varepsilon} = \dfrac{k}{r} \mathbf{e}_3 \otimes \mathbf{e}_\theta$. To verify the strain compatibility conditions we need to take another curl. Now, since $\operatorname{curl} \boldsymbol{\varepsilon}$ is not an axi-symmetric field, we cannot use the above formula for curl. From the general formula of curl for planar fields (planar fields are fields which are functions of r and θ only) in cylindrical coordinates, we can calculate that

$$\operatorname{curl} \operatorname{curl} \boldsymbol{\varepsilon} = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\,(\operatorname{curl} \boldsymbol{\varepsilon})_{3\theta} \Big) \mathbf{e}_3 \otimes \mathbf{e}_3 = \frac{1}{r}\frac{\partial}{\partial r}(k)\mathbf{e}_3 \otimes \mathbf{e}_3 = \mathbf{0}.$$

Hence, the strain field is locally compatible. It still remains to verify the condition (1.16) to conclude single-valuedness of the displacement.

To calculate the cyclic integral, we can choose the circle $x_1^2 + x_2^2 = a^2$, of radius $a \in (r_1,\, r_2)$, in $z = 0$ plane, which, for this particular example, is an irreducible curve. Let $\mathbf{y} = a\,\mathbf{e}_r(\tilde{\theta})$. Then $d\mathbf{y} = a\, d\mathbf{e}_r(\tilde{\theta}) = a\,\mathbf{e}_\theta(\tilde{\theta})\, d\tilde{\theta}$. Further, let $\mathbf{x} = a\,\mathbf{e}_r(\theta)$.

Hence,

$$\begin{aligned} & \boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y} - (\mathbf{x} - \mathbf{y}) \times \operatorname{curl} \boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y} = \\ & \Big\{ k\, \mathbf{e}_\theta(\tilde{\theta}) \otimes \mathbf{e}_\theta(\tilde{\theta}) \Big\}\, a\,\mathbf{e}_\theta(\tilde{\theta})\, d\tilde{\theta} - \Big(a\,\mathbf{e}_r(\theta) - a\,\mathbf{e}_r(\tilde{\theta}) \Big) \times \left(\Big\{ \frac{k}{a}\mathbf{e}_3 \otimes \mathbf{e}_\theta(\tilde{\theta}) \Big\}\ a\,\mathbf{e}_\theta(\tilde{\theta})\, d\tilde{\theta} \right) \\ = \ & ka\,\mathbf{e}_\theta(\tilde{\theta})\, d\tilde{\theta} - \Big(a\,\mathbf{e}_r(\theta) - a\,\mathbf{e}_r(\tilde{\theta}) \Big) \times \Big(k\,\mathbf{e}_3\, d\tilde{\theta} \Big) \\ = \ & \Big[ka\,\mathbf{e}_\theta(\tilde{\theta}) - \big(-ka\,\mathbf{e}_\theta(\theta) + ka\,\mathbf{e}_\theta(\tilde{\theta}) \big) \Big]\, d\tilde{\theta} \\ = \ & ka\,\mathbf{e}_\theta(\theta)\, d\tilde{\theta}. \end{aligned}$$

Choose $\mathbf{x}_0 = a\,\mathbf{e}_r(0)$. Observe that $\mathbf{x}_0 = a\,\mathbf{e}_r(\tilde{\theta} = 0)$ and $\mathbf{x} = a\,\mathbf{e}_r(\tilde{\theta} = \theta)$. Thus, we have

$$
\begin{aligned}
\boldsymbol{u}(\mathbf{x}) &= \int_{\mathbf{x}_0}^{\mathbf{x}} \Big(\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} - (\mathbf{x}-\mathbf{y}) \times \operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y}\Big) \\
&= \int_{\tilde{\theta}=0}^{\tilde{\theta}=\theta} ka\,\mathbf{e}_\theta(\theta)\,d\tilde{\theta} \\
&= ka\theta\;\mathbf{e}_\theta(\theta),
\end{aligned}
$$

and

$$
\begin{aligned}
\oint_{x_1^2+x_2^2=a^2} \Big(\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} - (\mathbf{x}-\mathbf{y}) \times \operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y}\Big) &= \int_{\tilde{\theta}=0}^{\tilde{\theta}=2\pi} ka\,\mathbf{e}_\theta(\theta)\,d\tilde{\theta} \\
&= 2\pi ka\,\mathbf{e}_\theta(\theta) \neq \mathbf{0}.
\end{aligned}
$$

Hence, the given strain field is not compatible, though it is compatible locally. It can be seen that, in this case, the displacement field is multi-valued, e.g. the coordinates $(r,\,\theta,\,z) = (a,\,0,\,0)$ and $(r,\,\theta,\,z) = (a,\,2\pi,\,0)$ represent the same material point but its displacement has both the values $\mathbf{0}$ and $2\pi ka\,\mathbf{e}_\theta(2\pi) = 2\pi ka\,\mathbf{e}_\theta(0)$. In fact, the displacement field is discontinuous at every point in the doubly connected domain we are considering. At any point $(r,\,\theta,\,z)$ in the domain, $\boldsymbol{u}$ has a discontinuity of amount $2\pi kr\,\mathbf{e}_\theta(\theta)$.

This simple example illustrates the important role that the topological properties (e.g. connectedness) of a material body play in theory of elasticity. The strain field which is compatible over a simply connected body may be incompatible over a multiply connected body.

Example 5 Consider a simply connected body and a non-homogeneous isotropic expansion of it given by the strain field

$$\varepsilon_{ij}(\mathbf{x}) = \alpha\,|\mathbf{x}|\delta_{ij}, \qquad \alpha = \text{constant},\ \alpha > 0,\ |\alpha| << 1,$$

expressed in some Cartesian basis. Such strain fields are used to model tumor growth. This strain field does not satisfy the compatibility conditions and, hence, is not compatible. In fact, incompatibility in this case can be also inferred from example 1, where it has been shown that the only non-homogeneous and isotropic strain field which is compatible over a simply connected domain must be linear in space variables. But $|\mathbf{x}|$ is a non-linear function of $\mathbf{x}$.

1.4 Airy stress function

Our aim is to show that equilibrium equations for stress in two dimensions (with body forces derived from a potential) is equivalent to existence of a potential, the Airy stress function, which is related to stresses by (1.24).

The equilibrium equations in two dimensions (i.e. for plane stress and plane strain problems) are given by

$$\sigma_{11,1} + \sigma_{12,2} + f_1 = 0,\ \text{and}\ \sigma_{21,1} + \sigma_{22,2} + f_2 = 0, \tag{1.18}$$

where f_α denotes the body force. Assume that there exists a potential $\Omega(X_1, X_2)$ such that $f_\alpha = -\frac{\partial \Omega}{\partial X_\alpha}$. The equilibrium equations then take the form

$$(\sigma_{11} - \Omega)_{,1} + \sigma_{12,2} = 0, \text{ and } \sigma_{21,1} + (\sigma_{22} - \Omega)_{,2} = 0. \tag{1.19}$$

According to the discussion at the end of first section, there exists potentials φ and χ such that (cf. (1.5) and (1.6))

$$\sigma_{11} - \Omega = \frac{\partial \varphi}{\partial X_2}, \text{ and } \sigma_{12} = -\frac{\partial \varphi}{\partial X_1}, \tag{1.20}$$

$$\sigma_{21} = -\frac{\partial \chi}{\partial X_2}, \text{ and } \sigma_{22} - \Omega = \frac{\partial \chi}{\partial X_1}. \tag{1.21}$$

Use $\sigma_{21} = \sigma_{12}$ to write

$$\frac{\partial \varphi}{\partial X_1} - \frac{\partial \chi}{\partial X_2} = 0. \tag{1.22}$$

Therefore, there exists a potential $\Phi(X_1, X_2)$ such that (cf. (1.5) and (1.6))

$$\varphi = \frac{\partial \Phi}{\partial X_2}, \text{ and } \chi = \frac{\partial \Phi}{\partial X_1}. \tag{1.23}$$

The potential Φ is called the Airy stress function. Substitute (1.23) into (1.20) and (1.21) to write

$$\sigma_{11} - \Omega = \Phi_{,22}, \text{ and } \sigma_{12} = -\Phi_{12}, \text{ and } \sigma_{22} - \Omega = \Phi_{,11}. \tag{1.24}$$

Conversely, if relations (1.24) are satisfied then equilibrium equations (1.18) are satisfied. This can be checked by direct substitution.