Let V be the three dimensional Euclidean vector space and L be the set of all linear maps from V to V. The set of real numbers is denoted by $\mathbb{R}$. Let E be the three dimensional Euclidean point space. It can be identified with V. Let R be an open subset of E.

1. Localization theorem for volume integrals. Let ϕ be a continuous function defined on an open set $R \subset E$. If for all closed sets $\pi \subset R$

$$\int_{\pi} \phi dV = 0, \tag{1.1}$$

then $\phi(\mathbf{X}) = 0$ for all $\mathbf{X} \in R$. To prove this, we start by defining

$$I_\varepsilon = \left| \phi(\mathbf{X}_0) - \frac{1}{V_\varepsilon} \int_{s_\varepsilon} \phi(\mathbf{X}) dV \right| = \left| \frac{1}{V_\varepsilon} \int_{s_\varepsilon} (\phi(\mathbf{X}_0) - \phi(\mathbf{X})) dV \right|, \tag{1.2}$$

where s_ε is a sphere of radius ε and volume V_ε centered at $\mathbf{X}_0 \in R$. A theorem in analysis (W. Rudin, *Principles of Mathematical Analysis*, 3rd Ed., McGraw-Hill (1976), p. 317) yields,

$$\begin{aligned} I_\varepsilon &\leq \frac{1}{V_\varepsilon} \int_{s_\varepsilon} |\phi(\mathbf{X}_0) - \phi(\mathbf{X})| dV \\ &\leq \frac{1}{V_\varepsilon} \int_{s_\varepsilon} \sup_{\mathbf{X} \in s_\varepsilon} |\phi(\mathbf{X}_0) - \phi(\mathbf{X})| dV \\ &= \max_{\mathbf{X} \in s_\varepsilon} |\phi(\mathbf{X}_0) - \phi(\mathbf{X})|, \end{aligned} \tag{1.3}$$

where in $(1.3)_2$, sup can be replaced by max due to continuity and compactness of s_ε. Since $\phi(\mathbf{X})$ is continuous, we get $I_\varepsilon \to 0$ as $\varepsilon \to 0$. It then follows from Eq. (1.2),

$$\phi(\mathbf{X}_0) = \lim_{\varepsilon \to 0} \frac{1}{V_\varepsilon} \int_{s_\varepsilon} \phi(\mathbf{X}) dV = 0, \tag{1.4}$$

where the last equality is a consequence of (1.1). The point $\mathbf{X}_0$ can be chosen arbitrarily, and thus we can conclude that $\phi(\mathbf{X}) = 0$ for all $\mathbf{X} \in R$.

2. Divergence theorem. Let f, $\mathbf{p}$ and $\mathbf{P}$ be respectively, scalar, vector and tensor fields defined on $R \times (t_1, t_2)$. Assume these fields to be continuously differentiable over R. Then for any part $\Omega \subset R$ and at any time $t \in (t_1, t_2)$

$$\int_{\Omega} (\nabla f) dV = \int_{\partial\Omega} f \mathbf{N} dA, \tag{2.1}$$

$$\int_{\Omega} (\mathrm{Div}\, \mathbf{p}) dV = \int_{\partial\Omega} \mathbf{p} \cdot \mathbf{N} dA, \tag{2.2}$$

$$\int_{\Omega} (\mathrm{Div}\, \mathbf{P}) dV = \int_{\partial\Omega} \mathbf{P} \mathbf{N} dA, \tag{2.3}$$

where $\mathbf{N} \in V$ is the outward unit normal to the boundary $\partial\Omega$ of Ω. We outline a brief proof for (2.2). Let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \in V$ be an orthonormal basis for V. Therefore there exists $\{p_1, p_2, p_3, X_1, X_2, X_3\} \in \mathbb{R}$ such that $\mathbf{p} = p_i \mathbf{E}_i$ and $\mathbf{X} = X_i \mathbf{E}_i$, with $i \in \{1, 2, 3\}$. Consider a cuboid $\mathcal{R} = \{\mathbf{X} \in \mathcal{E}_\kappa : A < X_1 < B,\ C < X_2 < D,\ E < X_3 < F\}$, where $\{A, B, C, D, E, F\} \in \mathbb{R}$

are constants. Then the surface integral in (2.2), when written for the two faces of the cuboid which are orthogonal to $\mathbf{E}_1$, is

$$\int_E^F \int_C^D (p_1(B,Y,Z) - p_1(A,Y,Z))dX_2 dX_3$$
$$= \int_E^F \int_C^D \int_A^B \frac{\partial p_1}{\partial X_1} dX_1 dX_2 dX_3, \tag{2.4}$$

which is obtained using the fundamental theorem of calculus (W. Rudin, *ibid.*, p. 134). We can write similar relations for the surfaces of the cuboid orthogonal to $\mathbf{E}_2$ and $\mathbf{E}_3$. We get

$$\int_{\partial\mathcal{R}} \mathbf{p}\cdot\mathbf{N}dA = \int_{\mathcal{R}} \Big(\frac{\partial p_1}{\partial X_1} + \frac{\partial p_2}{\partial X_2} + \frac{\partial p_3}{\partial X_3}\Big)dV = \int_{\mathcal{R}} (\mathrm{Div}\,\mathbf{p})dV. \tag{2.5}$$

We have therefore proved the divergence theorem for a cuboidal region. Furthermore, we can show that it holds for regions which are obtained by smooth deformations of the cuboid and also for general regions which can be obtained by pasting together the deformed cuboids (This argument can be found in the elementary texts on calculus. For a more advanced treatment see W. Rudin, *ibid.*, p. 288).

Equation (2.1) is obtained from (2.2) for a scalar $\mathbf{p}$. A proof for (2.3) also follows from (2.2). Indeed, for an arbitrary constant $\mathbf{a} \in V$,

$$\mathbf{a}\cdot\int_{\partial\Omega} \mathbf{P}\mathbf{N}dA = \int_{\partial\Omega} (\mathbf{P}^T\mathbf{a})\cdot\mathbf{N}dA = \int_{\Omega} (\mathrm{Div}\,\mathbf{P}^T\mathbf{a})dV = \int_{\Omega} (\mathrm{Div}\,\mathbf{P})\cdot\mathbf{a}dV, \tag{2.6}$$

where in the last equality, the definition of the Div operator has been used. Since $\mathbf{a}$ is arbitrary, we get the desired result.

3. Stokes' theorem. Let $\mathbf{p}$ and $\mathbf{P}$ be respectively, vector and tensor fields defined on $R \times (t_1, t_2)$. Assume these fields to be continuously differentiable over R. Then for any surface $\mathcal{F} \subset R$ with normal $\mathbf{N}$ and boundary $\partial\mathcal{F}$

$$\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{p})\cdot\mathbf{N}dA = \oint_{\partial\mathcal{F}} \mathbf{p}\cdot d\mathbf{X}, \tag{3.1}$$
$$\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P})^T\mathbf{N}dA = \oint_{\partial\mathcal{F}} \mathbf{P}d\mathbf{X}. \tag{3.2}$$

A proof for (3.1) can be obtained from (Rudin, W. *ibid.*, page 287). To verify (3.2), we use (3.1). Indeed, for an arbitrary constant vector $\mathbf{a} \in V$,

$$\mathbf{a}\cdot\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P})^T\mathbf{N}dA = \int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P}^T\mathbf{a})\cdot\mathbf{N}dA = \mathbf{a}\cdot\oint_{\partial\mathcal{F}} \mathbf{P}d\mathbf{X}, \tag{3.3}$$

where in the first equality, the definition of the Curl of a tensor field is used. The desired result follows upon using the arbitrariness of $\mathbf{a}$.