1. Surface interactions and the stress tensor. Recall the balance of linear momentum for $\Omega \subset R$:

$$\int_{\Omega} \rho_R \dot{\mathbf{v}} dV = \int_{\partial\Omega} \mathbf{t} dA + \int_{\Omega} \rho_R \mathbf{b} dV, \tag{1.1}$$

where ρ_R is the referential mass density, $\mathbf{v}$ is the material velocity, $\mathbf{t}$ is the traction vector, and $\mathbf{b}$ is the body force per unit mass. Given that a balance law of the form (1.1) exists, we now show that the surface interaction vector $\mathbf{t}$ depends on the surface only through the unit normal and moreover the dependence is linear. The first claim was introduced by Augustin-Louis Cauchy in 1823 as a hypothesis (Cauchy, A. L., *Bulletin de la Socièté Philomatique*, pp. 9-13 (1823). For an historical account see footnotes in Truesdell, C. & Toupin, R. A., *The Classical field Theories, Handbuch der Physik*, Vol III/1, Springer, Berlin (1960), Sects. 200 & 203), but was proved much later in 1957 by Walter Noll (Noll, W., The foundations of classical mechanics in the light of recent advances in continuum mechanics, pp. 266-281, *The Axiomatic Method, with Special Reference to Geometry and Physics* (Symposium at Berkeley, 1957), North-Holland Publishing Co., Amsterdam (1959)). The second claim, which is also known as the Cauchy's theorem, is based on the classical tetrahedron argument first proposed by Cauchy and is now recognized as a result of fundamental importance in continuum physics.

1.1. Cauchy's hypothesis (Noll's theorem) Let $\mathbf{N}$ be the outward unity normal to the positively oriented surface $\partial\Omega$. Then

$$\mathbf{t}(\mathbf{X}, t; \partial\Omega) = \mathbf{t}(\mathbf{X}, t; \mathbf{N}), \tag{1.2}$$

i.e. the dependence of the surface interaction vector on the surface on which it acts is only

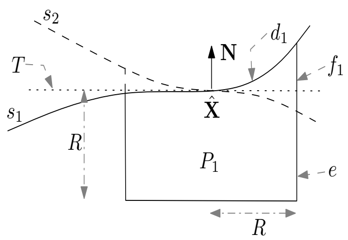

Figure 1: Two surfaces with a common tangent plane

through the normal $\mathbf{N}$. To prove this assertion let s_1 and s_2 be two surfaces in Ω such that they have a common tangent plane (denoted by T) at some $\hat{\mathbf{X}} \in s_1 \cap s_2$. Let $\mathbf{N}$ be the common unit normal to both surfaces at $\hat{\mathbf{X}}$. Let P_1 be a bounded region such that $\partial P_1 = d_1 \cup f_1 \cup e$, where d_1 is a subset of s_1, f_1 is a piece of the lateral surface of the circular cylinder with axis $\mathbf{N}$ and radius R, and e is a part of the surface of the cylinder which is common to both ∂P_1 and ∂P_2 (P_2 is the region bounded on the top by s_2). The quantities f_2 and d_2 are defined in a way similar to f_1 and d_1, respectively. Furthermore, $\partial P_2 = d_2 \cup f_2 \cup e$. If we denote the surface area of a surface s by $A(s)$ and the volume of a region P by $V(P)$, then for $a = 1, 2$,

$$\begin{aligned} A(d_a) &= \pi R^2 + o(R^2), \\ A(f_a) &= o(R^2), \\ V(P_a) &= o(R^2). \end{aligned} \tag{1.3}$$

The first of these relations is true since both d_1 and d_2 approach T as R approaches 0. Also, $A(f_a) \to 0$ as $R \to 0$.

We now apply the balance law (1.1) to regions P_1 and P_2. We obtain

$$\begin{aligned} \int_{\partial P_1} \mathbf{t}(\mathbf{X}, t; \partial P_1) dA &= \int_{P_1} \rho_R(\dot{\mathbf{v}} - \mathbf{b}) dV, \\ \int_{\partial P_2} \mathbf{t}(\mathbf{X}, t; \partial P_2) dA &= \int_{P_2} \rho_R(\dot{\mathbf{v}} - \mathbf{b}) dV. \end{aligned}$$

Subtract these two relations to get

$$\int_{d_1} \mathbf{t} dA - \int_{d_2} \mathbf{t} dA = \int_{P_1} \rho_R(\dot{\mathbf{v}} - \mathbf{b}) dV - \int_{P_2} \rho_R(\dot{\mathbf{v}} - \mathbf{b}) dV + \int_{f_2} \mathbf{t} dA - \int_{f_1} \mathbf{t} dA. \tag{1.4}$$

Assume all the fields to be bounded over the domain of their integration. Then,

$$\begin{aligned} \int_{P_a} \rho_R(\dot{\mathbf{v}} - \mathbf{b}) dV &\leq \max_{\mathbf{X} \in P_a} |\rho_R(\dot{\mathbf{v}} - \mathbf{b})| V(P_a), \\ \int_{f_a} \mathbf{t} dA &\leq \max_{\mathbf{X} \in f_a} |\mathbf{t}| A(f_a). \end{aligned}$$

Based on relations $(1.3)_{2,3}$, equation (1.4) can then be rewritten as

$$\int_{d_1} \mathbf{t}(\mathbf{X}, t; d_1) dA = \int_{d_2} \mathbf{t}(\mathbf{X}, t; d_2) dA + o(R^2). \tag{1.5}$$

Divide equation (1.5) throughout by πR^2 and use $(1.3)_1$. As a result obtain

$$\frac{1}{A(d_1)} \int_{d_1} \mathbf{t}(\mathbf{X}, t; d_1) dA = \frac{1}{A(d_1)} \int_{d_2} \mathbf{t}(\mathbf{X}, t; d_2) dA + \frac{o(R^2)}{\pi R^2}. \tag{1.6}$$

Since $\mathbf{t}(\mathbf{X})$ is assumed to be continuous, an application of the Mean-value theorem gives

$$\lim_{R \to 0} \frac{1}{A(d_a)} \int_{d_a} \mathbf{t}(\mathbf{X}, t; d_a) dA = \mathbf{t}(\hat{\mathbf{X}}, t; d_a), \tag{1.7}$$

where $\hat{\mathbf{X}}$ is the common point of d_1 and d_2. Therefore letting $R \to 0$ in (1.6) yields

$$\mathbf{t}(\hat{\mathbf{X}}, t; d_1) = \mathbf{t}(\hat{\mathbf{X}}, t; d_2). \tag{1.8}$$

Thus, the surface interaction vector $\mathbf{t}$ takes the same value for all surfaces with a common unit normal and therefore its dependence on the surface is only through the normal vector. The assertion (1.2) is proved.

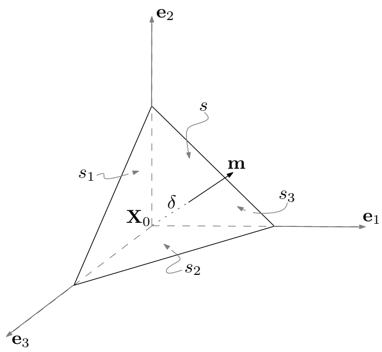

Figure 2: Tetrahedron T

1.2. Cauchy's lemma The balance law (1.1) implies that

$$\mathbf{t}(\mathbf{X}, t; -\mathbf{N}) = -\mathbf{t}(\mathbf{X}, t; \mathbf{N}). \tag{1.9}$$

This result will be used in the proof of the Cauchy's theorem. To verify this relation consider a pillbox P_ϵ of thickness ϵ, centered at $\mathbf{X}$, and with its flat surfaces parallel to $\mathbf{N}$. As we let $\epsilon \to 0$, the pillbox flattens to its middle surface S. The relation (1.1) for bounded fields then reduces to

$$\lim_{R\epsilon \to 0} \int_{\partial P_\epsilon} \mathbf{t} dA = 0 \tag{1.10}$$

or

$$\int_S (\mathbf{t}(\mathbf{N}) + \mathbf{t}(-\mathbf{N})) dA = 0. \tag{1.11}$$

Finally, shrink the disk S to the middle point $\mathbf{X}$ and use the continuity of $\mathbf{t}$ to obtain (1.9).

1.3. Cauchy's theorem The surface interaction vector $\mathbf{t}$ depends linearly on $\mathbf{N}$. Therefore, there exists a tensor $\boldsymbol{\sigma}$ such that

$$\mathbf{t}(\mathbf{X}, t; \mathbf{N}) = \boldsymbol{\sigma}(\mathbf{X}, t)\mathbf{N}. \tag{1.12}$$

We now prove this theorem. Consider a tetrahedron $T \subset \Omega$ with vertex $\mathbf{X}_0 \in \Omega$. The surface of the tetrahedron normal to the axis $\mathbf{e}_i$ is denoted by s_i. Let δ be the distance along the unit normal $\mathbf{m}$ from the vertex to the fourth surface s (see figure 2). Then, the volume of the tetrahedron $V(T)$ and the surface area $A(s)$ of the face s can be calculated as respectively, $c_1\delta^3$ and $c_2\delta^2$, where $\{c_1, c_2\} \in \mathbb{R}^+$ are constants. The area of the remaining faces (given by $A(s_i)$) can be obtained from $A(s)$:

$$A(s_i) = (\mathbf{m} \cdot \mathbf{e}_i) A(s). \tag{1.13}$$

This relation can be verified by first noting, using the divergence theorem, that $\int_{\partial T} \mathbf{N} dA = 0$, where ∂T is piecewise smooth. Since $\mathbf{N}$ is constant on each face of T, (1.13) follows.

We will now use the balance law (1.1) and the assumption of the continuity of the fields to arrive at the relation (1.12). The balance law when restricted to the tetrahedron T implies

$$\Big|\int_{\partial T}\mathbf{t}dA\Big| = \Big|\int_T \rho_R(\dot{\mathbf{v}}-\mathbf{b})dV\Big| \leq \int_T |\rho_R(\dot{\mathbf{v}}-\mathbf{b})|dV \leq kV(T), \tag{1.14}$$

where $k = \max_{\mathbf{X}\in T}|\rho_R(\dot{\mathbf{v}}-\mathbf{b})|$ is finite. Therefore,

$$\begin{aligned} O(\delta) = \frac{1}{A(s)}\int_{\partial T}\mathbf{t}dA &= \frac{1}{A(s)}\Big(\int_s \mathbf{t}(\mathbf{X};\mathbf{m})dA + \sum_{i=1}^{3}\int_{s_i}\mathbf{t}(\mathbf{X};-\mathbf{e}_i)dA\Big) \\ &= \frac{1}{A(s)}\Big(\int_s \mathbf{t}(\mathbf{X};\mathbf{m})dA - \sum_{i=1}^{3}\int_{s_i}\mathbf{t}(\mathbf{X};\mathbf{e}_i)dA\Big), \end{aligned} \tag{1.15}$$

where the last equality is a consequence of the Cauchy's lemma. By the Mean-value theorem, for continuous $\mathbf{t}$, we obtain

$$\begin{aligned} \int_s \mathbf{t}(\mathbf{X};\mathbf{m})dA &= A(s)\mathbf{t}(\tilde{\mathbf{X}};\mathbf{m}), \\ \int_{s_i}\mathbf{t}(\mathbf{X};\mathbf{e}_i)dA &= A(s_i)\mathbf{t}(\tilde{\mathbf{X}}_i;\mathbf{e}_i) \end{aligned} \tag{1.16}$$

for some $\tilde{\mathbf{X}} \in s$ and $\tilde{\mathbf{X}}_i \in s_i$, respectively. Let $\delta \to 0$. Then $\tilde{\mathbf{X}} \to \mathbf{X}_0$ and $\tilde{\mathbf{X}}_i \to \mathbf{X}_0$. As a result, equations (1.13), (1.15) and (1.16) yield

$$\mathbf{t}(\mathbf{X}_0;\mathbf{m}) = (\mathbf{m}\cdot\mathbf{e}_i)\mathbf{t}(\mathbf{X}_0;\mathbf{e}_i), \tag{1.17}$$

where summation over i is implicit. As the choice of the vertex $\mathbf{X}_0$ and the unit normal $\mathbf{m}$ is arbitrary, the relation (1.17) holds for all $\mathbf{X} \in R$ and all unit vectors. Equation (1.17) shows that $\mathbf{t}$ is linear in $\mathbf{m}$. Therefore there exists a tensor $\boldsymbol{\sigma}$, the stress tensor, such that

$$\mathbf{t}(\mathbf{X},t;\mathbf{m}) = \boldsymbol{\sigma}(\mathbf{X},t)\mathbf{m} \tag{1.18}$$

for all $\mathbf{X} \in R$ and any unit vector $\mathbf{m}$. The proof is complete.

Note that we have restricted our attention to only continuously differentiable fields defined on domains with piecewise smooth boundaries. Much research has been done in the past fifty years to investigate these results under less stringent smoothness requirements. Such considerations are indeed necessary for many practical problems in mechanics such as those involving shocks, fracture, dislocations and corner singularities (For a recent contribution, where the past work is carefully reviewed, see Schuricht, F., A new mathematical foundation for contact interactions in continuum physics, *Archive of Rational Mechanics and Analysis*, 184(3), pp. 495-551 (2007)).