Week 2 Discussion Handout

$\mathcal{V}$ is the three dimensional Euclidean space, which is a three dimensional vector space equipped with the Euclidean inner product $\mathbf{x} \cdot \mathbf{y} = x_i\, y_i$, where $\mathbf{x}$, $\mathbf{y} \in \mathcal{V}$ and $\mathbf{z} = z_i\, \mathbf{e}_i$ is the representation of any $\mathbf{z} \in \mathcal{V}$ in the standard orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathcal{V}$. This inner product induces the Euclidean norm on $\mathcal{V}$: $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$. With this norm, $\mathcal{V}$ becomes a normed vector space. Elements of the set Lin of all linear transformations from $\mathcal{V}$ into itself are called second order tensors. We denote vectors by lowercase boldfaced letters and second order tensors by uppercase boldfaced letters. Sym, Skw and Orth$^+$ are subsets of Lin containing symmetric, skew and proper orthogonal (or rotation) tensors respectively.

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1. $\mathbf{Ax} \cdot \mathbf{y} = \mathbf{Bx} \cdot \mathbf{y}\, \forall\, \mathbf{x}, \mathbf{y} \in \mathcal{V} \Rightarrow \mathbf{A} = \mathbf{B}$. [Hint: Use the definition of norm of a vector and the notion of equality of two second order tensors.]

   Tensor product

   Given $\mathbf{a}$, $\mathbf{b} \in \mathcal{V}$, their tensor product $\mathbf{a} \otimes \mathbf{b}$ is a second order tensor (linear transformation from $\mathcal{V}$ into itself) defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{x} = (\mathbf{x} \cdot \mathbf{b})\mathbf{a}$. It has the following properties.

2. $(\mathbf{a} \otimes \mathbf{b})^{\mathrm{T}} = \mathbf{b} \otimes \mathbf{a}$. [Hint: Use the definition of tensor product.]

3. For any $\mathbf{S} \in$ Lin, $\mathbf{S}(\mathbf{a} \otimes \mathbf{b}) = \mathbf{Sa} \otimes \mathbf{b}$, $(\mathbf{a} \otimes \mathbf{b})\mathbf{S} = \mathbf{a} \otimes \mathbf{S}^{\mathrm{T}}\mathbf{b}$. [Hint: Use the definition of composition of two tensors: $\mathbf{ST}(\mathbf{x}) = \mathbf{S}(\mathbf{Tx})\, \forall \mathbf{x} \in \mathcal{V}$.]

4. $(\mathbf{a} \otimes \mathbf{b})(\mathbf{c} \otimes \mathbf{d}) = (\mathbf{b} \cdot \mathbf{c})\, \mathbf{a} \otimes \mathbf{d}$. [Hint: Use above results.]

5. $I_1(\mathbf{a} \otimes \mathbf{b}) = \mathbf{a} \cdot \mathbf{b}$, $I_2(\mathbf{a} \otimes \mathbf{b}) = 0$ and $I_3(\mathbf{a} \otimes \mathbf{b}) = 0$. [Hint: Use the definition of principal invariants.]

6. For non-zero $\mathbf{a}$, $\mathbf{b} \in \mathcal{V}$, range of $\mathbf{a} \otimes \mathbf{b}$ = span of $\mathbf{a}$ and null space of $\mathbf{a} \otimes \mathbf{b}$ = orthogonal complement of $\mathbf{b}$. Hence, Rank$(\mathbf{a} \otimes \mathbf{b}) = 1$. Use this fact and 12 to justify the choice of $\{\mathbf{e}_i \otimes \mathbf{e}_j\}$, for i, $j = 1, 2, 3$, as a standard 'orthonormal' basis of Lin.. [Hint: Use the definition of tensor product.]

   Inner product in Lin

   $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{AB}^{\mathrm{T}})$, where $\mathrm{tr}(\mathbf{A})$ is the trace of $\mathbf{A} \in$ Lin and has been defined in class. This inner product induces a norm on Lin: $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$. Trace is a linear function and it has the following two properties:

7. $\mathrm{tr}(\mathbf{S}^{\mathrm{T}})$ =$\mathrm{tr}(\mathbf{S})$. [Hint: Use the definition trace.]

8. $\mathrm{tr}(\mathbf{ST})$ =$\mathrm{tr}(\mathbf{TS})$. [Hint: Use the definition trace.]

   Linearity of trace and 8 confirm that the above definition of inner product is indeed an inner product. [Check!]

   These further implies

9. $\mathrm{tr}(\mathbf{A}) = \mathbf{I} \cdot \mathbf{A}$. [Hint: Use the definition of inner product, 7 and 8.]

10. $\mathbf{R} \cdot (\mathbf{ST}) = (\mathbf{S}^{\mathrm{T}}\mathbf{R}) \cdot \mathbf{T} = \mathbf{RT}^{\mathrm{T}} \cdot \mathbf{S}$. [Hint: Use the definition of inner product, 7 and 8.]

11. $\mathbf{u} \cdot \mathbf{Sv} = \mathbf{S} \cdot (\mathbf{u} \otimes \mathbf{v})$. [Hint: Use the definition of inner product, 3, 7 and 8.]

12. $(\mathbf{a} \otimes \mathbf{b}) \cdot (\mathbf{c} \otimes \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})$. [Hint: Use the definition of inner product, 4, 7 and 8.]

Few more important facts

13. If $\mathbf{S} \in \mathrm{Sym}$ and $\mathbf{T} \in \mathrm{Lin}$, then $\mathbf{S} \cdot \mathbf{T} = \mathbf{S} \cdot \mathbf{T}^{\mathrm{T}} = \mathbf{S} \cdot \{\frac{1}{2}(\mathbf{T} + \mathbf{T}^{\mathrm{T}})\}$. [Hint: Use 7, 8, 9 and 10.]

14. If $\mathbf{W} \in \mathrm{Skw}$ and $\mathbf{T} \in \mathrm{Lin}$, then $\mathbf{W} \cdot \mathbf{T} = -\mathbf{W} \cdot \mathbf{T}^{\mathrm{T}} = \mathbf{W} \cdot \{\frac{1}{2}(\mathbf{T} - \mathbf{T}^{\mathrm{T}})\}$. [Hint: Use 7, 8, 9 and 10.]

15. For $\mathbf{S} \in \mathrm{Sym}$ and $\mathbf{W} \in \mathrm{Skw}$, $\mathbf{S} \cdot \mathbf{W} = 0$. [Hint: Use 13 and 14.]

16. If $\mathbf{A} \cdot \mathbf{B} = 0 \; \forall \, \mathbf{B} \in \mathrm{Lin}$, then $\mathbf{A} = \mathbf{0}$. [Hint: Choose $\mathbf{B}$ as $\mathbf{A}$.]

17. If $\mathbf{A} \cdot \mathbf{S} = 0 \; \forall \, \mathbf{S} \in \mathrm{Sym}$, then $\mathbf{A} \in \mathrm{Skw}$. [Hint: Use 15 and 16.]

18. If $\mathbf{A} \cdot \mathbf{W} = 0 \; \forall \, \mathbf{W} \in \mathrm{Skw}$, then $\mathbf{A} \in \mathrm{Sym}$. [Hint: Use 15 and 16.]

ϵ-δ relations

19.
$$\epsilon_{ijk}\epsilon_{lmn} = \det \begin{bmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{bmatrix}.$$

20. $\epsilon_{ijp}\epsilon_{lmp} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$. [Hint: Use 19.]

21. $\epsilon_{ipq}\epsilon_{lpq} = 2\delta_{il}$. [Hint: Use 20.]

22. $\epsilon_{ijk}\epsilon_{ijk} = 6$. [Hint: Use 21.]

Skew tensors

23. $\mathbf{W} \in \mathrm{Skw} \Leftrightarrow \mathbf{W}\mathbf{x} \cdot \mathbf{x} = 0$ for all $\mathbf{x} \in \mathcal{V}$. [Hint: Use the definition of a skew tensor and symmetricity of the inner product.]

24. $\mathbf{W} \in \mathrm{Skw} \Rightarrow \exists$ unique $\mathbf{w} \in \mathcal{W}$ such that $\mathbf{W}\mathbf{x} = \mathbf{w} \times \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$. Conversely, for any $\mathbf{0} \neq \mathbf{w} \in \mathcal{V}$, $\exists$ unique $\mathbf{W} \in \mathrm{Skw}$ such that $\mathbf{W}\mathbf{x} = \mathbf{w} \times \mathbf{x}$ for all $\mathbf{x} \in \mathcal{V}$. $\mathbf{w}$ is called the axial vector of $\mathbf{W}$.

25. Let $\mathbf{W}$ be a skew tensor and $\mathbf{w}$ be its axial vector. Then $I_1(\mathbf{W}) = 0$, $I_2(\mathbf{W}) = |\mathbf{w}|^2$ and $I_3(\mathbf{W}) = 0$. Deduce that $\mathbf{W}$ has only one real eigenvalue which is zero. [Hint: Use the definitions and properties of principal invariants.]

Proper orthogonal tensors

26. For $\mathbf{Q} \in \mathrm{Orth}^+$, $\exists$ a unit vector $\mathbf{p}$, called the axis of $\mathbf{Q}$, such that $\mathbf{Q}\mathbf{p} = \mathbf{p} = \mathbf{Q}^{\mathrm{T}}\mathbf{p}$. Further, $\exists$ right handed orthonormal basis $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ ($\mathbf{p}$ as before) and an angle $\theta \in (0, 2\pi]$ such that $\mathbf{Q}$ has the representation

$$\mathbf{Q} = \mathbf{p} \otimes \mathbf{p} + \cos\theta \, (\mathbf{q} \otimes \mathbf{q} + \mathbf{r} \otimes \mathbf{r}) + \sin\theta \, (\mathbf{r} \otimes \mathbf{q} - \mathbf{q} \otimes \mathbf{r}).$$

Deduce that $I_1(\mathbf{Q}) = I_2(\mathbf{Q}) = 1 + 2 \cos\theta$ and if $\theta \neq 0, 2\pi$, $\mathbf{Q}$ has only one real eigenvalue which is 1.

Differentiability of scalar, vector and tensor valued fields

Let us identify first the Euclidean point space $\mathcal{E}$ (set of all points in our physical space) with the three dimensional Euclidean vector space $\mathcal{V}$ by fixing a suitable origin $o \in \mathcal{E}$. Then each point $x \in \mathcal{E}$ can be identified with a three dimensional Euclidean vector $\mathbf{x} \in \mathcal{V}$ as $\mathbf{x} = x - o$. *Fields* are, by definition, functions on $\mathcal{E}$ or on subsets of $\mathcal{E}$. But with this identification, we can consider *fields* as functions on $\mathcal{V}$ or on subsets of $\mathcal{V}$.

Definition 1. Let U be an open subset of $\mathcal{V}$. A function $f : U \to \mathbb{R}$ (i.e. a scalar valued field) is differentiable at $\mathbf{x} \in U$ if there exists a linear map $A : \mathcal{V} \to \mathbb{R}$ such that

$$\lim_{|\mathbf{h} \to 0|} \frac{f(\mathbf{x}+\mathbf{h}) - f(\mathbf{x}) - A(\mathbf{h})}{|\mathbf{h}|} = 0.$$

According to the Riesz representation theorem of linear algebra, the linear map $A : \mathcal{V} \to \mathbb{R}$ has the unique representation $A(\mathbf{h}) = \nabla f\big|_{\mathbf{x}} \cdot \mathbf{h}$, where $\nabla f\big|_{\mathbf{x}}$ is a unique fixed vector. The vector $\nabla f\big|_{\mathbf{x}}$ is called the gradient (or the Fréchet derivative) of f at $\mathbf{x}$.

Definition 2. Let U be an open subset of $\mathcal{V}$. A function $\mathbf{f} : U \to \mathcal{V}$ (i.e. a vector valued field) is differentiable at $\mathbf{x} \in U$ if there exists a linear map $\mathbf{A} : \mathcal{V} \to \mathcal{V}$ such that

$$\lim_{|\mathbf{h} \to 0|} \frac{|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{A}(\mathbf{h})|}{|\mathbf{h}|} = 0.$$

According to the Riesz representation theorem of linear algebra, the linear map $\mathbf{A} : \mathcal{V} \to \mathcal{V}$ has the unique representation $\mathbf{A}(\mathbf{h}) = \nabla \mathbf{f}\big|_{\mathbf{x}} \mathbf{h}$, where $\nabla \mathbf{f}\big|_{\mathbf{x}}$ is a unique fixed second order tensor. The second order tensor $\nabla \mathbf{f}\big|_{\mathbf{x}}$ is called the gradient (or the Fréchet derivative) of $\mathbf{f}$ at $\mathbf{x}$.

Definition 3. Let U be an open subset of $\mathcal{V}$. A function $\mathbf{F} : U \to \mathrm{Lin}$ (i.e. a tensor valued field) is differentiable at $\mathbf{x} \in U$ if there exists a linear map $\mathcal{A} : \mathcal{V} \to \mathrm{Lin}$ such that

$$\lim_{|\mathbf{h} \to 0|} \frac{|\mathbf{F}(\mathbf{x}+\mathbf{h}) - \mathbf{F}(\mathbf{x}) - \mathcal{A}(\mathbf{h})|}{|\mathbf{h}|} = 0.$$

According to the Riesz representation theorem of linear algebra, the linear map $\mathcal{A} : \mathcal{V} \to \mathrm{Lin}$ has the unique representation $\mathcal{A}(\mathbf{h}) = \nabla \mathbf{F}\big|_{\mathbf{x}} \mathbf{h}$, where $\nabla \mathbf{F}\big|_{\mathbf{x}}$ is a unique fixed third order tensor (a third order tensor, roughly speaking, is a linear map from $\mathcal{V}$ to Lin). The third order tensor $\nabla \mathbf{F}\big|_{\mathbf{x}}$ is called the gradient (or the Fréchet derivative) of $\mathbf{F}$ at $\mathbf{x}$.

Alternate and more useful definitions

We will denote by Φ a scalar or vector or tensor valued field. The nature of $\nabla\Phi$ (i.e. whether it is a vector or a second order tensor or a third order tensor) will be clear from the context.

Alternate Definition 1. For fixed $\mathbf{x} \in U \subseteq \mathcal{V}$, if the limit

$$\lim_{t \to 0} \frac{|\Phi(\mathbf{x} + t\,\mathbf{h}) - \Phi(\mathbf{x})|}{t}$$

exists for all $\mathbf{h} \in \mathcal{V}$ and depends continuously on $\mathbf{h}$, then

$$\nabla\Phi\big|_{\mathbf{x}}\mathbf{h} = \frac{d}{dt}\bigg|_{t=0} \Phi(\mathbf{x} + t\,\mathbf{h}).$$

Alternate Definition 2. If $d\Phi$ and $d\mathbf{x}$ denote the respective total differential increment in Φ and $\mathbf{x}$, then the linear map $\nabla\Phi\big|_{\mathbf{x}}$, defined by $d\Phi = \nabla\Phi\big|_{\mathbf{x}}[d\mathbf{x}]$, is called the gradient of Φ at $\mathbf{x} \in U \subseteq \mathcal{V}$.