Week 4 Discussion Handout

Definitions

Let f, $\mathbf{v}$ and $\mathbf{A}$ be a scalar, a vector and a second order tensor fields, respectively, defined on an open domain U of $\mathcal{V}$ and continuously differentiable on U. Let $\mathbf{c} \in \mathcal{V}$ be fixed. Then we have the following definitions. Last definition assumes twice continuously differentiable f and $\mathbf{u}$.

- $\text{div } \mathbf{v} = \text{tr } \nabla\mathbf{v}$.
- $(\text{curl } \mathbf{v}) \cdot \mathbf{c} = \text{div}(\mathbf{v} \times \mathbf{c})$.
- $(\text{div } \mathbf{A}) \cdot \mathbf{c} = \text{div}(\mathbf{A}^{\text{T}}\mathbf{c})$.
- $(\text{curl } \mathbf{A})\, \mathbf{c} = \text{curl}(\mathbf{A}^{\text{T}}\mathbf{c})$.
- $\Delta\, f = \text{div}(\nabla\, f)$, $\Delta\, \mathbf{v} = \text{div}(\nabla\, \mathbf{v})$ and $(\Delta\, \mathbf{A})\, \mathbf{c} = \Delta(\mathbf{Sc})$ (Δ denotes the Laplacian operator).

Components in some standard basis ($\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$)

- $(\nabla f)_i = f_{,i}$.
- $(\nabla\mathbf{v})_{ij} = v_{i,j}$.
- $\text{div } \mathbf{v} = v_{i,i}$.
- $(\text{curl } \mathbf{v})_i = e_{ijk}\, v_{k,j}$.
- $(\text{div } \mathbf{A})_i = A_{ij,j}$.
- $(\text{curl } \mathbf{A})_{ij} = e_{imn}\, A_{jn,m}$.
- $\Delta\, f = f_{,ii}$.
- $(\Delta\, \mathbf{v})_i = v_{i,jj}$.
- $(\Delta\, \mathbf{A})_{ij} = A_{ij,kk}$.

Some useful identities

Let ϕ, $\mathbf{v}$, $\mathbf{w}$ and $\mathbf{S}$ be continuously differentiable fields with ϕ scalar valued, $\mathbf{v}$ and $\mathbf{w}$ vector valued, and $\mathbf{S}$ tensor valued. Then we have the following identities. First six of these are applications of product rule (or the Leibniz's rule). 7 to 14 assume the corresponding fields to be twice continuously differentiable.

1. $\nabla(\phi\mathbf{v}) = \phi\, \nabla\mathbf{v} + \mathbf{v} \otimes \nabla\phi$
2. $\text{div}(\phi\mathbf{v}) = \phi \text{ div } \mathbf{v} + \mathbf{v} \cdot \nabla\phi$
3. $\nabla(\mathbf{v} \cdot \mathbf{w}) = (\nabla\mathbf{w})^{\text{T}}\, \mathbf{v} + (\nabla\mathbf{v})^{\text{T}}\, \mathbf{w}$
4. $\text{div}(\mathbf{v} \otimes \mathbf{w}) = \mathbf{v} \text{ div } \mathbf{w} + (\nabla\mathbf{v})\, \mathbf{w}$
5. $\text{div}(\mathbf{S}^{\text{T}}\mathbf{v}) = \mathbf{S} \cdot \nabla\mathbf{v} + \mathbf{v} \cdot \text{div } \mathbf{S}$
6. $\text{div}(\phi\mathbf{S}) = \phi \text{ div } \mathbf{S} + \mathbf{S}\, \nabla\phi$
7. $\text{curl } \nabla\phi = \mathbf{0}$.
8. $\text{div curl } \mathbf{v} = 0$.
9. $\text{curl curl } \mathbf{v} = \nabla \,\text{div}\, \mathbf{v} - \Delta\, \mathbf{u}$.
10. $\text{curl } \nabla\mathbf{u} = \mathbf{0}$.
11. $\text{curl}(\nabla\mathbf{u}^{\text{T}}) = \nabla \,\text{curl}\, \mathbf{u}$.

12. div curl $\mathbf{S}$=curl div $\mathbf{S}^{\mathrm{T}}$.

13. div (curl $\mathbf{S})^{\mathrm{T}} = \mathbf{0}$.

14. (curl curl $\mathbf{S})^{\mathrm{T}}$ = curl curl $\mathbf{S}^{\mathrm{T}}$.

Exercises

1. Prove using divergence theorem: $\displaystyle\int_{\partial\Omega} \mathbf{v} \otimes \mathbf{n}\, dA = \int_{\Omega} \nabla \mathbf{v}\, dV$.

2. Show that $\mathrm{vol}(\Omega) = \dfrac{1}{3} \displaystyle\int_{\partial\Omega} \mathbf{x} \cdot \mathbf{n}\, dA$, where $\mathbf{x}$ denotes the position vector of a point in Ω.

3. Let $\mathbf{A}$ be a second order tensor field that satisfies div $\mathbf{A} = \mathbf{0}$ over some open region $\mathcal{R}$ of $\mathcal{V}$. Show that

$$\int_{\partial\Omega} \mathbf{x} \times \mathbf{A}\mathbf{n}\, dA = \mathbf{0} \qquad \text{for all regular parts } \Omega \text{ inside } \mathcal{R}$$

implies that $\mathbf{A} \in \mathrm{Sym}$.