Week 6 Discussion Handout

Strain compatibility conditions for a simply connected body

At every point in the current configuration $\mathcal{C}$, which is a simply connected domain,

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\varepsilon} = \mathbf{0}. \tag{1}$$

In some Cartesian basis, the above vector equation looks like the following 81 scalar equations

$$e_{ijk}\, e_{lmn}\, \varepsilon_{jm,kn} = 0, \tag{2}$$

out of which only the following 6 are independent.

$$\begin{aligned}
\varepsilon_{11,22} + \varepsilon_{22,11} - 2\,\varepsilon_{12,12} &= 0,\\
\varepsilon_{22,22} + \varepsilon_{33,22} - 2\,\varepsilon_{23,32} &= 0,\\
\varepsilon_{33,11} + \varepsilon_{11,33} - 2\,\varepsilon_{31,13} &= 0,\\
(\varepsilon_{12,3} - \varepsilon_{23,1} + \varepsilon_{31,2})_{,1} - \varepsilon_{11,23} &= 0,\\
(\varepsilon_{23,1} - \varepsilon_{31,2} + \varepsilon_{12,3})_{,2} - \varepsilon_{22,31} &= 0,\\
(\varepsilon_{31,2} - \varepsilon_{12,3} + \varepsilon_{23,1})_{,3} - \varepsilon_{33,12} &= 0.
\end{aligned}$$

For a compatible strain field $\boldsymbol{\varepsilon}$, the displacement field $\boldsymbol{u}$ for a simply connected body can be written as

$$\boldsymbol{u}(\mathbf{x}) = \int_{\mathbf{x_0}}^{\mathbf{x}} \Big(\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y} - (\mathbf{x} - \mathbf{y}) \times \operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y}\Big), \tag{3}$$

modulo a rigid deformation (which is of the form $\boldsymbol{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x} - \mathbf{x}_0)$, for constant $\boldsymbol{u}_0$ and $\boldsymbol{\omega}_0$). $\mathbf{x}_0$ is some fixed point in $\mathcal{C}$. curl inside the integral sign is with respect to the variable $\mathbf{y}$. The integral in the above expression for displacement is independent of the path that joins $\mathbf{x}_0$ and $\mathbf{x}$, which can be embodied into the fact that the following cyclic integral along any closed curve C in $\mathcal{C}$ is zero:

$$\oint_C \Big(\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y} - (\mathbf{x} - \mathbf{y}) \times \operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y}\Big) = \mathbf{0}. \tag{4}$$

The above expression gives the global compatibility condition for a simply connected body.

Strain compatibility conditions for a multiply connected body

An $(n+1)$-tuply connected body has n holes and, consequently, there always exist n closed curves C_i, $i = 1, \ldots, n$, which cannot be continuously shrunk to a point without leaving the body, nor they can be deformed continuously into one another without leaving the body. Such closed curves C_i are called irreducible.

At every point in the current configuration $\mathcal{C}$, which is now an $(n + 1)$-tuply connected domain,

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\varepsilon} = \mathbf{0} \tag{5}$$

and, additionally,

$$\oint_{C_i} \Big(\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y} - (\mathbf{x} - \mathbf{y}) \times \operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y}\Big) = \mathbf{0} \tag{6}$$

for all irreducible curves C_i, $i = 1, \ldots, n$.

Under the above hypothesis, the displacement field $\boldsymbol{u}$ for a multiply connected body can be written as

$$\boldsymbol{u}(\mathbf{x}) = \int_{\mathbf{x_0}}^{\mathbf{x}} \Big(\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y} - (\mathbf{x} - \mathbf{y}) \times \operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\, d\mathbf{y}\Big), \tag{7}$$

modulo a rigid deformation (which is of the form $\boldsymbol{u}_0 + \boldsymbol{\omega}_0 \times (\mathbf{x} - \mathbf{x}_0)$, for constant $\boldsymbol{u}_0$ and $\boldsymbol{\omega}_0$). $\mathbf{x}_0$ is, again, some fixed point in $\mathcal{C}$.

Example 1

For a simply connected body, we wish to find out the temperature distribution $T(\mathbf{x})$ that gives a compatible thermal strain field.

If the thermal conductivity is homogeneous and isotropic, we can write down the thermal strain as $\varepsilon_{ij} = \beta\, T\, \delta_{ij}$, where β is the constant coefficient of thermal expansion. Putting this strain field in (1) implies

$$T_{,ij} + T_{,kk}\, \delta_{ij} = 0.$$

Taking trace of this expression implies $T_{,kk} = 0$. Hence, $T_{,ij} = 0$. The general solution is $T(\mathbf{x}) = a + b_i\, x_i$, where a and b_i, $i = 1, 2, 3$, are integration constants. Thus, for a simply connected body, the temperature distribution must be necessarily linear in order to have a compatible thermal strain field.

Note. Example of a non-linear temperature distribution over a hollow cylindrical body (doubly connected) which produces locally compatible thermal strain field would be $T(r,\, \theta,\, z) = Ar^2 \cos 2\theta$, where A is some constant.

Example 2

Consider simple shear of a simply connected cube in x_1-x_2 plane. We have seen that the displacement field for such a deformation looks like $\boldsymbol{u} = \lambda\, x_2\, \mathbf{e}_1$, with λ constant. If $|\lambda| << 1$ (a measure of small deformation), the small strain field in the Cartesian basis looks like

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} 0 & \frac{\lambda}{2} & 0 \\ \frac{\lambda}{2} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

which is a constant field and, hence, satisfies the compatibility conditions (2).

Taking $\mathbf{x}_0 = \mathbf{0}$, expression (3) gives

$$\boldsymbol{u}(\mathbf{x}) = \frac{\lambda}{2} x_2\, \mathbf{e}_1 + \frac{\lambda}{2} x_1\, \mathbf{e}_2.$$

Here is an example of a strain field that corresponds to two different displacement fields. As expected, these displacement fields must be related through some rigid body deformation, which in this case is an infinitesimal rigid rotation of amount $-\frac{\lambda}{2}$ about x_3 axis:

$$\lambda\, x_2\, \mathbf{e}_1 = \Big(-\frac{\lambda}{2} \mathbf{e}_3 \times \mathbf{x} \Big) + \frac{\lambda}{2} x_2\, \mathbf{e}_1 + \frac{\lambda}{2} x_1\, \mathbf{e}_2.$$

Example 3

Consider the following non-linear small strain field over a simply connected body, given in some Cartesian basis,

$$[\boldsymbol{\varepsilon}] = \begin{bmatrix} 0 & \frac{\lambda}{2} x_1 x_2 & 0 \\ \frac{\lambda}{2} x_1 x_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad \lambda \neq 0,\ |\lambda| << 1.$$

This strain field does not satisfy the condition (2) ($\varepsilon_{11,22} + \varepsilon_{22,11} - 2\, \varepsilon_{12,12} = -\lambda \neq 0$). Hence, the above strain field is an example of an incompatible strain field over a simply connected body.

Example 4

Let us consider now a long thin elastic tube (a model of an artery) which is a doubly connected domain. Let the inner and outer radii be r_1 and r_2 respectively, with $|r_1 - r_2| << 1$ (thin-ness). Take the standard cylindrical coordinate system $\{r_1 < r < r_2,\, \theta \in [0,\, 2\pi],\, z \in \mathbb{R}\}$ and consider the following small strain field

$$\varepsilon_{rr} = 0,\ \varepsilon_{\theta\theta} = k,\ \varepsilon_{33} = 0, \qquad k = \text{cosntant},\ k \neq 0,\ |k| << 1.$$

The above is an example of an axisymmetric plane strain field (ε_{rr}, $\varepsilon_{\theta\theta}$ are functions of r only and all other strain components are zero). For such fields,

$$\operatorname{curl} \boldsymbol{\varepsilon} = \left(\frac{\partial \varepsilon_{\theta\theta}}{\partial r} - \frac{\varepsilon_{rr} - \varepsilon_{\theta\theta}}{r} \right) \mathbf{e}_3 \otimes \mathbf{e}_\theta.$$

For our case, $\operatorname{curl} \boldsymbol{\varepsilon} = \frac{k}{r} \mathbf{e}_3 \otimes \mathbf{e}_\theta$. To verify the local compatibility condition (5) for multiply connected bodies, we need to take another curl. Now, since $\operatorname{curl} \boldsymbol{\varepsilon}$ is not an axisymmetric field, we cannot use the above formula for curl. From the general formula of curl for planar fields (planar fields are fields which are functions of r and θ only) in cylindrical coordinates, we can calculate that

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\varepsilon} = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\,(\operatorname{curl}\boldsymbol{\varepsilon})_{3\theta}\Big)\mathbf{e}_3\otimes\mathbf{e}_3 = \frac{1}{r}\frac{\partial}{\partial r}(k)\mathbf{e}_3\otimes\mathbf{e}_3 = \mathbf{0}.$$

Hence, the strain field is locally compatible. It still remains to verify the global condition (6) to conclude about strain compatibility.

To calculate the cyclic integral in (6), we can choose the circle $x_1^2 + x_2^2 = a^2$, of radius $a \in (r_1, r_2)$, in $z = 0$ plane, which, for this particular example, is an irreducible curve. Let $\mathbf{y} = a\,\mathbf{e}_r(\tilde{\theta})$. Then $d\mathbf{y} = a\,d\mathbf{e}_r(\tilde{\theta}) = a\,\mathbf{e}_\theta(\tilde{\theta})\,d\tilde{\theta}$. Further, let $\mathbf{x} = a\,\mathbf{e}_r(\theta)$.

Hence,

$$\begin{aligned}
\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} - (\mathbf{x}-\mathbf{y})\times\operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} &= \Big\{k\,\mathbf{e}_\theta(\tilde{\theta})\otimes\mathbf{e}_\theta(\tilde{\theta})\Big\}\,a\,\mathbf{e}_\theta(\tilde{\theta})\,d\tilde{\theta} - \Big(a\,\mathbf{e}_r(\theta) - a\,\mathbf{e}_r(\tilde{\theta})\Big)\times\Big(\Big\{\frac{k}{a}\mathbf{e}_3\otimes\mathbf{e}_\theta(\tilde{\theta})\Big\}\,a\,\mathbf{e}_\theta(\tilde{\theta})\,d\tilde{\theta}\Big)\\
&= ka\,\mathbf{e}_\theta(\tilde{\theta})\,d\tilde{\theta} - \Big(a\,\mathbf{e}_r(\theta) - a\,\mathbf{e}_r(\tilde{\theta})\Big)\times\Big(k\,\mathbf{e}_3\,d\tilde{\theta}\Big)\\
&= \Big[ka\,\mathbf{e}_\theta(\tilde{\theta}) - \big(-ka\,\mathbf{e}_\theta(\theta) + ka\,\mathbf{e}_\theta(\tilde{\theta})\big)\Big]\,d\tilde{\theta}\\
&= ka\,\mathbf{e}_\theta(\theta)\,d\tilde{\theta}.
\end{aligned}$$

Choose $\mathbf{x}_0 = a\,\mathbf{e}_r(0)$. Observe that $\mathbf{x}_0 = a\,\mathbf{e}_r(\tilde{\theta}=0)$ and $\mathbf{x} = a\,\mathbf{e}_r(\tilde{\theta}=\theta)$. Thus, we have

$$\begin{aligned}
\boldsymbol{u}(\mathbf{x}) &= \int_{\mathbf{x}_0}^{\mathbf{x}}\Big(\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} - (\mathbf{x}-\mathbf{y})\times\operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y}\Big)\\
&= \int_{\tilde{\theta}=0}^{\tilde{\theta}=\theta} ka\,\mathbf{e}_\theta(\theta)\,d\tilde{\theta}\\
&= ka\theta\;\mathbf{e}_\theta(\theta),
\end{aligned}$$

and

$$\begin{aligned}
\oint_{x_1^2+x_2^2=a^2}\Big(\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y} - (\mathbf{x}-\mathbf{y})\times\operatorname{curl}\boldsymbol{\varepsilon}(\mathbf{y})\,d\mathbf{y}\Big) &= \int_{\tilde{\theta}=0}^{\tilde{\theta}=2\pi} ka\,\mathbf{e}_\theta(\theta)\,d\tilde{\theta}\\
&= 2\pi ka\,\mathbf{e}_\theta(\theta) \neq \mathbf{0}.
\end{aligned}$$

Hence, the given strain field is not compatible, though it is compatible locally. It can be seen that, in this case, the displacement field is multi-valued, e.g. the coordinates $(r,\,\theta,\,z) = (a,\,0,\,0)$ and $(r,\,\theta,\,z) = (a,\,2\pi,\,0)$ represent the same material point but its displacement has both the values $\mathbf{0}$ and $2\pi ka\,\mathbf{e}_\theta(2\pi) = 2\pi ka\,\mathbf{e}_\theta(0)$. In fact, the displacement field is discontinuous at every point in the doubly connected domain we are considering. At any point $(r,\,\theta,\,z)$ in the domain, $\boldsymbol{u}$ has a discontinuity of amount $2\pi kr\,\mathbf{e}_\theta(\theta)$.

This simple example illustrates the important role that the topological properties (e.g. connectedness) of a material body play in theory of elasticity. The strain field which is compatible over a simply connected body may be incompatible over a multiply connected body.

Example 5

Consider a simply connected body and a non-homogeneous isotropic expansion of it given by the strain field

$$\varepsilon_{ij}(\mathbf{x}) = \alpha\,|\mathbf{x}|\delta_{ij}, \qquad \alpha = \text{constant},\ \alpha > 0,\ |\alpha| << 1,$$

expressed in some Cartesian basis. Such strain fields are used to model tumor growth. This strain field does not satisfy the compatibility condition (1) and, hence, is not compatible. In fact, incompatibility in this case can be also inferred from example 1, where it has been shown that the only non-homogeneous and isotropic strain field which is compatible over a simply connected domain must be linear in space variables. But $|\mathbf{x}|$ is a non-linear function of $\mathbf{x}$.