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BALANCE LAWS

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Contents

Summary

Central to any theory of continuum mechanics, are balance laws of mass, momentum, and energy. These provide us with universal relations, which should be satisfied during every process associated with the continuous body. However, to obtain general statements of these laws, several integral theorems are required. A major portion of this chapter therefore deals with divergence theorem, Stokes' theorem and transport theorems, which are then used to obtain the balance laws in the form of partial differential equations to be satisfied away from the singular surface and jump conditions at the singular surface.

In this chapter we use, unless specified otherwise, the notation introduced in the previous chapter on kinematics. In particular, $\mathcal{E}$ denotes a three dimensional Euclidean space, associated with which is its translation space $\mathcal{V}$, a three dimensional inner product vector space. We use in addition, the subscript κ or χ when referring to a reference or a spatial frame, respectively.

Let $\boldsymbol{\kappa}(\mathfrak{B}) \subset \mathcal{E}$ and $\boldsymbol{\chi}(\mathfrak{B}) \subset \mathcal{E}$ denote respectively, the fixed reference configuration and the current configuration. Let (t_1, t_2) be a fixed time interval, where $\{t_1, t_2\} \in \mathbb{R}$.

1. Integral theorems

In this subsection we state and prove the localization theorem, the divergence theorem, the Stokes' theorem, and the transport theorem for volume and surface integrals. We have employed only elementary concepts from differential geometry in proving these theorems.

Localization theorem for volume integrals Let ϕ be a continuous function defined on an open set $R \subset \mathcal{E}$. If for all closed sets $\pi \subset R$

$$\int_{\pi} \phi dV = 0, \tag{1}$$

then $\phi(\mathbf{u}) = 0$ for all $\mathbf{u} \in R$. To prove this, we start by defining

$$I_{\varepsilon} = \left| \phi(\mathbf{u}_0) - \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} \phi(\mathbf{u}) dV \right| = \left| \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} (\phi(\mathbf{u}_0) - \phi(\mathbf{u})) dV \right|, \tag{2}$$

where s_{ε} is a sphere of radius ε and volume V_{ε} centered at $\mathbf{u}_0 \in R$. A theorem in analysis (Rudin, W. *Principles of Mathematical Analysis*, 3rd Ed., McGraw-Hill (1976), page 317) yields,

$$\begin{aligned} I_{\varepsilon} &\leq \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} |\phi(\mathbf{u}_0) - \phi(\mathbf{u})| dV \\ &\leq \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} \sup_{\mathbf{u} \in s_{\varepsilon}} |\phi(\mathbf{u}_0) - \phi(\mathbf{u})| dV \\ &= \max_{\mathbf{u} \in s_{\varepsilon}} |\phi(\mathbf{u}_0) - \phi(\mathbf{u})|, \end{aligned} \tag{3}$$

where in $(3)_2$, sup can be replaced by max due to continuity and compactness of s_{ε}. Since $\phi(\mathbf{u})$ is continuous, we get $I_{\varepsilon} \to 0$ as $\varepsilon \to 0$. It then follows from Eq. (2),

$$\phi(\mathbf{u}_0) = \lim_{\varepsilon \to 0} \frac{1}{V_{\varepsilon}} \int_{s_{\varepsilon}} \phi(\mathbf{u}) dV = 0, \tag{4}$$

where the last equality is a consequence of (1). The point $\mathbf{u}_0$ can be chosen arbitrarily, and thus we can conclude that $\phi(\mathbf{u}) = 0$ for all $\mathbf{u} \in R$.

Localization theorem for surface integrals Let φ be a continuous function defined on a surface $\mathcal{F} \subset \mathcal{E}$. If for all surfaces $\varsigma \subset \mathcal{F}$

$$\int_{\varsigma} \phi dA = 0, \tag{5}$$

then $\varphi(\mathbf{u}) = 0$ for all $\mathbf{u} \in \mathcal{F}$. This can be proved using arguments similar to those used above.

Divergence theorem for smooth fields Let f, $\mathbf{p}$ and $\mathbf{P}$ be respectively, scalar, vector and tensor fields defined on $\boldsymbol{\kappa}(\mathfrak{B}) \times (t_1, t_2)$. Assume these fields to be continuously differentiable over $\boldsymbol{\kappa}(\mathfrak{B})$. Then for any part $\Omega \subset \boldsymbol{\kappa}(\mathfrak{B})$ and at any time $t \in (t_1, t_2)$

$$\int_{\Omega} (\nabla f) dV = \oint_{\partial\Omega} f \mathbf{N} dA, \tag{6}$$

$$\int_{\Omega} (\operatorname{Div} \mathbf{p}) dV = \oint_{\partial\Omega} \mathbf{p} \cdot \mathbf{N} dA, \tag{7}$$

$$\int_{\Omega} (\operatorname{Div} \mathbf{P}) dV = \oint_{\partial\Omega} \mathbf{P} \mathbf{N} dA, \tag{8}$$

where $\mathbf{N} \in \mathcal{V}_\kappa$ is the outward unit normal to the boundary $\partial\Omega$ of Ω. We outline a brief proof for (7). Let $\{\mathbf{E}_1, \mathbf{E}_2, \mathbf{E}_3\} \in \mathcal{V}_\kappa$ be an orthonormal basis for $\mathcal{V}_\kappa$. Therefore there exists $\{p_1, p_2, p_3, X_1, X_2, X_3\} \in \mathbb{R}$ such that $\mathbf{p} = p_i \mathbf{E}_i$ and $\mathbf{X} = X_i \mathbf{E}_i$, with $i \in \{1, 2, 3\}$. Consider a cuboid $\mathcal{R} = \{\mathbf{X} \in \mathcal{E}_\kappa : A < X_1 < B,\ C < X_2 < D,\ E < X_3 < F\}$, where $\{A, B, C, D, E, F\} \in \mathbb{R}$ are constants. Then the surface integral in (7), when written for the two faces of the cuboid which are orthogonal to $\mathbf{E}_1$, is

$$\int_E^F \int_C^D (p_1(B, Y, Z) - p_1(A, Y, Z)) dX_2 dX_3$$
$$= \int_E^F \int_C^D \int_A^B \frac{\partial p_1}{\partial X_1} dX_1 dX_2 dX_3, \tag{9}$$

which is obtained using the fundamental theorem of calculus (Rudin, W. *ibid.*, page 134). We can write similar relations for the surfaces of the cuboid orthogonal to $\mathbf{E}_2$ and $\mathbf{E}_3$. We get

$$\oint_{\partial\mathcal{R}} \mathbf{p} \cdot \mathbf{N} dA = \int_{\mathcal{R}} \Big(\frac{\partial p_1}{\partial X_1} + \frac{\partial p_2}{\partial X_2} + \frac{\partial p_3}{\partial X_3} \Big) dV = \int_{\mathcal{R}} (\operatorname{Div} \mathbf{p}) dV. \tag{10}$$

We have therefore proved the divergence theorem for a cuboidal region. Furthermore, we can show that it holds for regions which are obtained by smooth deformations of the cuboid and also for general regions which can be obtained by pasting together the deformed cuboids (This argument can be found in the elementary texts on calculus. For a more advanced treatment see Rudin, W. *ibid.*, page 288).
Equation (6) is obtained from (7) for a scalar $\mathbf{p}$. A proof for (8) also follows from (7). Indeed, for an arbitrary constant $\mathbf{a} \in \mathcal{V}_\kappa$,

$$\mathbf{a} \cdot \oint_{\partial\Omega} \mathbf{P}\mathbf{N} dA = \oint_{\partial\Omega} (\mathbf{P}^T \mathbf{a}) \cdot \mathbf{N} dA = \int_\Omega (\mathrm{Div}\, \mathbf{P}^T \mathbf{a}) dV = \int_\Omega (\mathrm{Div}\, \mathbf{P}) \cdot \mathbf{a} dV, \quad (11)$$

where in the last equality, the definition of the Div operator has been used. Since $\mathbf{a}$ is arbitrary, we get the desired result.

Divergence theorem for piecewise smooth fields Assume $\mathbf{p}$ to be piecewise continuously differentiable over $\boldsymbol{\kappa}(\mathfrak{B})$, being discontinuous across the singular surface S_t (with normal $\mathbf{N}_s$ and speed U) and smooth everywhere else. Then for a domain Ω such that $\mathcal{S} = \Omega \cap S_t \neq \emptyset$,

$$\oint_{\partial\Omega} \mathbf{p} \cdot \mathbf{N} dA = \int_\Omega (\mathrm{Div}\, \mathbf{p}) dV + \int_{\mathcal{S}} [\![\mathbf{p}]\!] \cdot \mathbf{N}_s dA. \quad (12)$$

Similar statements hold for scalar and tensor fields. We now prove (12). Let $\Omega^\pm \subset \Omega$ be such that $\Omega^+ \cup \Omega^- = \Omega$ and $\Omega^+ \cap \Omega^- = \mathcal{S}$. The normal to the surface $\mathcal{S}$ is oriented such that it points into Ω^+. Since $\mathbf{p}$ is smooth within Ω^+ and Ω^-, we can use (7) to write

$$\begin{aligned} \int_{\Omega^+} (\mathrm{Div}\, \mathbf{p}) dV &= \int_{\partial\Omega^+ \setminus \mathcal{S}} \mathbf{p} \cdot \mathbf{N} dA - \int_{\mathcal{S}} \mathbf{p}^+ \cdot \mathbf{N}_s dA, \\ \int_{\Omega^-} (\mathrm{Div}\, \mathbf{p}) dV &= \int_{\partial\Omega^- \setminus \mathcal{S}} \mathbf{p} \cdot \mathbf{N} dA + \int_{\mathcal{S}} \mathbf{p}^- \cdot \mathbf{N}_s dA, \end{aligned}$$

where $\mathbf{p}^\pm$ are the limiting values of $\mathbf{p}$ as it approaches $\mathcal{S}$ from the interior of $\Omega^\pm$. The relation (12) is obtained by adding these two equations.
If $\mathbf{q}$ is a vector field defined on $\boldsymbol{\chi}(\mathfrak{B}) \times (t_1, t_2)$ and piecewise continuously differentiable over $\boldsymbol{\chi}(\mathfrak{B})$, being discontinuous across the singular surface s_t (with normal $\mathbf{n}_s$ and speed u). Then for $\omega \subset \boldsymbol{\chi}(\mathfrak{B})$ such that $s = \omega \cap s_t \neq \emptyset$,

$$\oint_{\partial\omega} \mathbf{q} \cdot \mathbf{n} da = \int_\omega (\mathrm{div}\, \mathbf{q}) dv + \int_s [\![\mathbf{q}]\!] \cdot \mathbf{n}_s da. \quad (13)$$

The proof for (13) is similar to that of (12).

Stokes' theorem for smooth fields Let $\mathbf{p}$ and $\mathbf{P}$ be respectively, vector and tensor fields defined on $\boldsymbol{\kappa}(\mathfrak{B}) \times (t_1, t_2)$. Assume these fields to be continuously differentiable over $\boldsymbol{\kappa}(\mathfrak{B})$. Then for any surface $\mathcal{F} \subset \boldsymbol{\kappa}(\mathfrak{B})$ with normal $\mathbf{N}$ and boundary $\partial\mathcal{F}$

$$\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{p}) \cdot \mathbf{N} dA = \oint_{\partial\mathcal{F}} \mathbf{p} \cdot d\mathbf{X}, \tag{14}$$

$$\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P})^T \mathbf{N} dA = \oint_{\partial\mathcal{F}} \mathbf{P} d\mathbf{X}. \tag{15}$$

A proof for (14) can be obtained from (Rudin, W. *ibid.*, page 287). To verify (15), we use (14). Indeed, for an arbitrary constant vector $\mathbf{a} \in \mathcal{V}_\kappa$,

$$\mathbf{a} \cdot \int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P})^T \mathbf{N} dA = \int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P}^T \mathbf{a}) \cdot \mathbf{N} dA = \mathbf{a} \cdot \oint_{\partial\mathcal{F}} \mathbf{P} d\mathbf{X}, \tag{16}$$

where in the first equality, the definition of the Curl of a tensor field is used. The desired result follows upon using the arbitrariness of $\mathbf{a}$.

Stokes' theorem for piecewise smooth fields Consider $\mathbf{p}$ to be piecewise continuously differentiable over $\boldsymbol{\kappa}(\mathfrak{B})$. Assume $\mathbf{p}$ to be discontinuous across the singular surface S_t and smooth everywhere else. Let $\Gamma = \mathcal{F} \cap S_t$ be the curve of intersection. Then

$$\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{p}) \cdot \mathbf{N} dA = \oint_{\partial\mathcal{F}} \mathbf{p} \cdot d\mathbf{X} + \int_{\Gamma} [\![\mathbf{p}]\!] \cdot d\mathbf{X}. \tag{17}$$

To verify this relation start by considering two subsurfaces $\mathcal{F}^{\pm} \subset \mathcal{F}$ such that $\mathcal{F}^+ \cup \mathcal{F}^- = \mathcal{F}$ and $\mathcal{F}^+ \cap \mathcal{F}^- = \Gamma$. Since $\mathbf{p}$ is smooth in regions $\mathcal{F}^{\pm}$, we can write using (14)

$$\int_{\mathcal{F}^+} (\mathrm{Curl}\,\mathbf{p}) \cdot \mathbf{N} dA = \int_{\partial\mathcal{F}^+ \setminus \Gamma} \mathbf{p} \cdot d\mathbf{X} + \int_{\Gamma} \mathbf{p}^+ \cdot d\mathbf{X},$$

$$\int_{\mathcal{F}^-} (\mathrm{Curl}\,\mathbf{p}) \cdot \mathbf{N} dA = \int_{\partial\mathcal{F}^- \setminus \Gamma} \mathbf{p} \cdot d\mathbf{X} - \int_{\Gamma} \mathbf{p}^- \cdot d\mathbf{X}.$$

Adding these two relations we get (17). Similarly, we obtain for a piecewise continuously differentiable tensor field $\mathbf{P}$:

$$\int_{\mathcal{F}} (\mathrm{Curl}\,\mathbf{P})^T \mathbf{N} dA = \oint_{\partial\mathcal{F}} \mathbf{P} d\mathbf{X} + \int_{\Gamma} [\![\mathbf{P}]\!] d\mathbf{X}. \tag{18}$$

If $\mathbf{q}$ is a piecewise continuously differentiable vector field defined on $\boldsymbol{\chi}(\mathfrak{B}) \times (t_1, t_2)$, being discontinuous across the singular surface s_t. Consider a surface $F \subset \boldsymbol{\chi}(\mathfrak{B})$ with normal $\mathbf{n}$ and let $\gamma = F \cap s_t$. Then

$$\int_F (\operatorname{curl} \mathbf{q}) \cdot \mathbf{n} da = \oint_{\partial F} \mathbf{q} \cdot d\mathbf{x} + \int_\gamma [\![\mathbf{q}]\!] \cdot d\mathbf{x}. \tag{19}$$

The proof for (19) is similar to that of (17).

Remark (Surface divergence theorem) Consider a vector field p continuously differentiable over the surface $S \subset \boldsymbol{\kappa}(\mathfrak{B})$ (with unit normal $\mathbf{N}$ and mean curvature H) for a fixed time interval (t_1, t_2). Then

$$\oint_{\partial S} \mathrm{p} \cdot \boldsymbol{\nu} dL = \int_S (\operatorname{Div}^S \mathrm{p} + 2H \mathrm{p} \cdot \mathbf{N}) dA, \tag{20}$$

where $\boldsymbol{\nu}$ is the outer unit normal to ∂S such that $(\mathbf{N}, \boldsymbol{\nu}, \mathbf{t})$ form a positively-oriented orthogonal triad at ∂S with $\mathbf{t}$ being the tangent vector along ∂S. Moreover, if p is tangential, i.e. $\mathbb{P}\mathrm{p} = \mathrm{p}$, then $\mathrm{p} \cdot \mathbf{N} = 0$ and (20) reduces to

$$\oint_{\partial S} \mathrm{p} \cdot \boldsymbol{\nu} dL = \int_S \operatorname{Div}^S \mathrm{p} dA. \tag{21}$$

We now prove (20). By definition $\boldsymbol{\nu} = \mathbf{t} \times \mathbf{N}$ and therefore we can use Stokes' theorem to rewrite the term on the left hand side of Eq. (20) as

$$\begin{aligned} \oint_{\partial S} \mathrm{p} \cdot \boldsymbol{\nu} dL &= \oint_{\partial S} \mathrm{p} \cdot (\mathbf{t} \times \mathbf{N}) dL \\ &= \oint_{\partial S} (\mathbf{N} \times \mathrm{p}) \cdot \mathbf{t} dL \\ &= \int_S \operatorname{Curl}(\mathbf{N} \times \mathrm{p}) \cdot \mathbf{N} dA. \end{aligned} \tag{22}$$

Use the identity $\operatorname{Curl}(\mathbf{N} \times \mathrm{p}) = \operatorname{Div}(\mathbf{N} \otimes \mathrm{p} - \mathrm{p} \otimes \mathbf{N})$ to get

$$\operatorname{Curl}(\mathbf{N} \times \mathrm{p}) \cdot \mathbf{N} = (\nabla \mathbf{N})^T \mathbf{N} \cdot \mathrm{p} - (\mathrm{p} \cdot \mathbf{N}) \operatorname{Div} \mathbf{N} + \nabla \mathrm{p} \cdot \mathbb{P}. \tag{23}$$

But $(\nabla \mathbf{N})^T \mathbf{N} = \mathbf{0}$ (follows from $\mathbf{N} \cdot \mathbf{N} = 1$) and $\nabla \mathrm{p} \cdot \mathbb{P} = \operatorname{tr}(\nabla \mathrm{p} \mathbb{P}) = \operatorname{Div}^S \mathrm{p}$. Furthermore, $\operatorname{Div} \mathbf{N} = -2H$ (using a result from the chapter on kinematics). Therefore we can rewrite (23) to get

$$\operatorname{Curl}(\mathbf{N} \times \mathrm{p}) \cdot \mathbf{N} = 2H(\mathrm{p} \cdot \mathbf{N}) + \operatorname{Div}^S \mathrm{p}. \tag{24}$$

Substituting this into (22) yields (20).

Transport theorem for volume integrals with smooth fields Let P and Q denote a scalar, vector or tensor field continuously differentiable on $\boldsymbol{\kappa}(\mathfrak{B}) \times (t_1, t_2)$ and $\boldsymbol{\chi}(\mathfrak{B}) \times (t_1, t_2)$, respectively. Then for arbitrary parts $\Omega \subset \boldsymbol{\kappa}(\mathfrak{B})$, $\omega \subset \boldsymbol{\chi}(\mathfrak{B})$ and at any time $t \in (t_1, t_2)$

$$\frac{d}{dt}\int_\Omega P dV = \int_\Omega \dot{P} dV, \tag{25}$$

$$\frac{d}{dt}\int_\omega Q dv = \int_\omega \frac{\partial Q}{\partial t} dv + \int_{\partial\omega} Q(\mathbf{v}\cdot\mathbf{n}) da. \tag{26}$$

Since Ω is fixed with respect to time and P is smooth over Ω, the time derivative and the volume integral in the left hand side of (25) can commute to give the right hand side of the equation. Equation (26) can be proved by first transforming the volume ω to a fixed reference volume, say Ω. We get

$$\begin{aligned} \frac{d}{dt}\int_\omega Q dv &= \frac{d}{dt}\int_\Omega Q J dV \\ &= \int_\omega \dot{Q} dv + \int_\omega Q(\operatorname{div}\mathbf{v}) dv, \end{aligned} \tag{27}$$

where J is the Jacobian associated with the mapping which transforms Ω to ω and $\dot{J} = J(\operatorname{div}\mathbf{v})$. Equation (26) follows from (27) upon recalling the definition of the material time derivative and using the divergence theorem.

Transport theorem for volume integrals with piecewise smooth fields Let Ω be such that $\mathcal{S} = \Omega \cap S_t \neq \emptyset$. Then for a P which is discontinuous across S_t but smooth everywhere else,

$$\frac{d}{dt}\int_\Omega P dV = \int_\Omega \dot{P} dV - \int_{\mathcal{S}} U[\![P]\!] dA. \tag{28}$$

We now prove this relation. Recall surface parametrization introduced at the end of the chapter on kinematics. In a small neighborhood, say $\Omega_{\mathcal{S}}$, of the singular surface $\mathcal{S}$ we parameterize the domain by coordinates $\{\xi_1, \xi_2, \zeta\}$ such that for $\mathbf{X} \in \Omega_{\mathcal{S}}$ we can write $\mathbf{X} = \hat{\mathbf{X}}(\xi_1, \xi_2, t) + \zeta(t)\mathbf{N}(\xi_1, \xi_2, t)$, where $\hat{\mathbf{X}} \in \mathcal{S}$ and $\{\xi_1, \xi_2\}$ are convected. Let $-\varsigma < \zeta(t) < \varsigma$, where $\varsigma \in \mathbb{R}^+$ is constant. The position of the singular surface is indicated by $\zeta = 0$ and it is assumed that the surface $\mathcal{S}$ remains inside $\Omega_{\mathcal{S}}$ during the instantaneous motion. Obtain

$$
\begin{aligned}
\frac{d}{dt}\int_{\Omega} P dV &= \frac{d}{dt}\int_{\Omega\backslash\Omega_{\mathcal{S}}} P dV + \frac{d}{dt}\int_{\Omega_{\mathcal{S}}} P dV \\
&= \int_{\Omega\backslash\Omega_{\mathcal{S}}} \dot{P} dV + \int_{(\xi_1,\xi_2)} \frac{d}{dt}\left(\int_{-\varsigma}^{\varsigma} P j_A d\zeta\right) dA_{\xi}, \\
&= \int_{\Omega\backslash\Omega_{\mathcal{S}}} \dot{P} dV \\
&\quad + \int_{(\xi_1,\xi_2)} \left\{\frac{d}{dt}\left(\int_{-\varsigma}^{\zeta_1(t)} P j_A d\zeta + \int_{\zeta_2(t)}^{\varsigma} P j_A d\zeta\right)\right\}_{\zeta_1,\zeta_2=0} dA_{\xi},
\end{aligned}
$$

where j_A is the Jacobian related to the change of coordinates. On the singular surface, $\zeta_1 = \zeta_2 = 0$, $\dot{\zeta}_1 = \dot{\zeta}_2 = U$, $j_A = \xi$ and $dA = \xi dA_{\xi}$, where ξ is the surface Jacobian. Taking the limit $|\varsigma| \to 0$ we obtain the desired result. The infinitesimal area of the surface in terms of the new coordinates can be obtained by using Nanson's formula, $\mathbf{N}dA = j_A \mathbf{A}^{-T}\hat{\mathbf{N}} dA_{\xi}$, where $\hat{\mathbf{N}} = \mathbf{N}$ and $\mathbf{A}$ is the gradient of the map from the new coordinates to $\mathbf{X}$. For the considered transformation this formula reduces to $dA = j_A dA_{\xi}$.
Let ω be such that $s = \omega \cap s_t \neq \emptyset$. Then for a Q which is discontinuous across s_t but smooth everywhere else,

$$
\frac{d}{dt}\int_{\omega} Q dv = \int_{\omega}\left(\frac{\partial Q}{\partial t} + \operatorname{div}(Q\mathbf{v})\right) dv - \int_{s}(u[\![Q]\!] - [\![Q\mathbf{v}]\!]\cdot\mathbf{n}) da, \tag{29}
$$

where $u = U|(\mathbf{F}^{\pm})^T\mathbf{n}| + \mathbf{n}\cdot\mathbf{v}^{\pm}$ is the spatial speed of the singular surface s_t. This relation can be proved by first transforming ω to Ω and then using (28). We get

$$
\frac{d}{dt}\int_{\omega} Q dv = \int_{\Omega} (JQ)^{\cdot} dV - \int_{S} U[\![JQ]\!] dA. \tag{30}
$$

The term $U[\![JQ]\!]$ can be expanded as

$$
\begin{aligned}
U[\![JQ]\!] &= (Q^+u^+J^+|(\mathbf{F}^+)^{-T}\mathbf{N}|) - (Q^-u^-J^-|(\mathbf{F}^-)^{-T}\mathbf{N}|) \\
&= (u[\![Q]\!] - [\![Q\mathbf{v}]\!]\cdot\mathbf{n})|(\mathbf{F}^-)^*\mathbf{N}|, \qquad (31)
\end{aligned}
$$

where $u^{\pm} = u - \mathbf{n}\cdot\mathbf{v}^{\pm}$. Relations $U = u^{\pm}|(\mathbf{F}^{\pm})^{-T}\mathbf{N}|$ and $|(\mathbf{F}^+)^*\mathbf{N}| = |(\mathbf{F}^-)^*\mathbf{N}|$ have also been used. Equation (29) follows immediately after substituting (31) into (30).

Transport theorem for surface integrals with smooth fields Let p be a scalar, vector or tensor field continuously differentiable on $S_t \times (t_1, t_2)$. Then, for an arbitrary surface $\mathcal{S} \subset S_t$

$$\frac{d}{dt}\int_{\mathcal{S}} \mathrm{p} dA = \int_{\mathcal{S}} (\mathring{\mathrm{p}} - 2\mathrm{p}UH) dA, \tag{32}$$

where $\mathbf{N}$, U, and H are the unit normal, normal velocity, and the mean curvature associated with S_t, respectively. We prove this relation using the surface parametrization outlined in the chapter on kinematics. We assume that p can be extended to the small neighborhood $\Omega_{\mathcal{S}}$, and use the same symbol to denote its extension. Obtain

$$\begin{aligned} \frac{d}{dt}\int_{\mathcal{S}} \mathrm{p} dA &= \{\frac{d}{dt}\int_{(\xi_1,\xi_2)} \mathrm{p}(\mathbf{X}(\xi_\alpha, \zeta(t)), t) j_A dA_\xi\}_{\zeta=0} \\ &= \int_{(\xi_1,\xi_2)} \{(\dot{\mathrm{p}} j_A + j_A \dot{\zeta}(\nabla \mathrm{p} \cdot \mathbf{N}) + \mathrm{p}\dot{j_A}) dA_\xi\}_{\zeta=0} \\ &= \int_{\mathcal{S}} \{\dot{\mathrm{p}} + \dot{\zeta}\nabla \mathrm{p} \cdot \mathbf{N} + \mathrm{p}\dot{j_A} j_A^{-1}\}_{\zeta=0} dA. \end{aligned} \tag{33}$$

At the surface, $\zeta = 0$, we have $\dot{\zeta} = U$, $j_A = \xi$ and $\dot{j_A} = -2UH\xi$. Substituting these into (33) and recalling the definition of the normal time derivative, we obtain (32).

2. Surface interactions

Let $\mathbf{p}$, $\mathbf{r}$ and $\mathbf{s}$ be continuously differentiable vector fields on $\boldsymbol{\kappa}(\mathfrak{B}) \times (t_1, t_2)$. A global (or integral) *balance law* is a relation of the following form: For an arbitrary $\Omega \subset \boldsymbol{\kappa}(\mathfrak{B})$,

$$\frac{d}{dt}\int_{\Omega} \mathbf{p} dV = \int_{\Omega} \mathbf{r} dV + \int_{\partial\Omega} \mathbf{s} dA. \tag{34}$$

This relation expresses the integral form of the balance of the change in the quantity $\mathbf{p}$ with a volume supply/sink density $\mathbf{r}$ and a surface interaction $\mathbf{s}$. Given that a balance law of the form (34) exists, we now show that the surface interaction vector $\mathbf{s}$ depends on the surface only through the unit normal and moreover the dependence is linear. The first claim was introduced by Cauchy in 1823 (Cauchy, A. L., *Bulletin de la Sociètè Philomatique*, pp. 9-13 (1823).

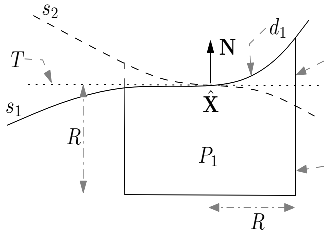

Figure 1: Two surfaces with a common tangent plane

For an historical account see footnotes in Truesdell, C. & Toupin, R. A., *The Classical field Theories, Handbuch der Physik*, Vol III/1, Springer, Berlin (1960), Sects. 200 & 203) as a hypothesis and was proved much later in 1959 by Noll (Noll, W., The foundations of classical mechanics in the light of recent advances in continuum mechanics, pp. 266-281, *The Axiomatic Method, with Special Reference to Geometry and Physics* (Symposium at Berkeley, 1957), North-Holland Publishing Co., Amsterdam (1959)). The second claim, which is also known as the Cauchy's theorem, is based on the classical tetrahedron argument first proposed by Cauchy and is now recognized as a result of fundamental importance in continuum physics. The proofs below can be easily reproduced for cases when the fields are scalar or tensorial in nature.

Cauchy's hypothesis (Noll's theorem) Let $\mathbf{N}$ be the outward unity normal to the positively oriented surface $\partial\Omega$. Then

$$\mathbf{s}(\mathbf{X}, t; \partial\Omega) = \mathbf{s}(\mathbf{X}, t; \mathbf{N}), \tag{35}$$

i.e. the dependence of the surface interaction vector on the surface on which it acts is only through the normal $\mathbf{N}$. To prove this assertion let s_1 and s_2 be two surfaces in $\boldsymbol{\kappa}(\mathfrak{B})$ such that they have a common tangent plane (denoted by T) at some $\hat{\mathbf{X}} \in s_1 \cap s_2$. Let $\mathbf{N}$ be the common unit normal to both surfaces at $\hat{\mathbf{X}}$. Let P_1 be a bounded region such that $\partial P_1 = d_1 \cup f_1 \cup e$, where d_1 is a subset of s_1, f_1 is a piece of the lateral surface of the circular cylinder with axis $\mathbf{N}$ and radius R, and e is a part of the surface of the cylinder which is common to both ∂P_1 and ∂P_2 (P_2 is the region bounded on the top by s_2). The quantities f_2 and d_2 are defined in a way similar to f_1 and d_1, respectively. Furthermore, $\partial P_2 = d_2 \cup f_2 \cup e$. If we denote the surface area of a surface s by $A(s)$ and the volume of a region P by $V(P)$, then for $a = 1, 2$,

$$\begin{aligned} A(d_a) &= \pi R^2 + o(R^2), \\ A(f_a) &= o(R^2), \\ V(P_a) &= o(R^2). \end{aligned} \tag{36}$$

The first of these relations is true since both d_1 and d_2 approach T as R approaches 0. Also, $A(f_a) \to 0$ as $R \to 0$.

We now apply the balance law (34) to regions P_1 and P_2. We obtain

$$\begin{aligned} \int_{\partial P_1} \mathbf{s}(\mathbf{X}, t; \partial P_1) dA &= \int_{P_1} (\dot{\mathbf{p}} - \mathbf{r}) dV, \\ \int_{\partial P_2} \mathbf{s}(\mathbf{X}, t; \partial P_2) dA &= \int_{P_2} (\dot{\mathbf{p}} - \mathbf{r}) dV. \end{aligned}$$

Subtract these two relations to get

$$\int_{d_1} \mathbf{s} dA - \int_{d_2} \mathbf{s} dA = \int_{P_1} (\dot{\mathbf{p}} - \mathbf{r}) dV - \int_{P_2} (\dot{\mathbf{p}} - \mathbf{r}) dV + \int_{f_2} \mathbf{s} dA - \int_{f_1} \mathbf{s} dA. \tag{37}$$

Assume all the fields to be bounded over the domain of their integration. Then,

$$\begin{aligned} \int_{P_a} (\dot{\mathbf{p}} - \mathbf{r}) dV &\leq \max_{\mathbf{X} \in P_a} |\dot{\mathbf{p}} - \mathbf{r}| V(P_a), \\ \int_{f_a} \mathbf{s} dA &\leq \max_{\mathbf{X} \in f_a} |\mathbf{s}| A(f_a). \end{aligned}$$

Based on relations $(36)_{2,3}$, Eq. (37) can then be rewritten as

$$\int_{d_1} \mathbf{s}(\mathbf{X}, t; d_1) dA = \int_{d_2} \mathbf{s}(\mathbf{X}, t; d_2) dA + o(R^2). \tag{38}$$

Divide Eq. (38) throughout by πR^2 and use $(36)_1$. As a result obtain

$$\frac{1}{A(d_1)} \int_{d_1} \mathbf{s}(\mathbf{X}, t; d_1) dA = \frac{1}{A(d_1)} \int_{d_2} \mathbf{s}(\mathbf{X}, t; d_2) dA + \frac{o(R^2)}{\pi R^2}. \tag{39}$$

Since $\mathbf{s}(\mathbf{X})$ is assumed to be continuous, an application of the Mean-value theorem gives

$$\lim_{R \to 0} \frac{1}{A(d_a)} \int_{d_a} \mathbf{s}(\mathbf{X}, t; d_a) dA = \mathbf{s}(\hat{\mathbf{X}}, t; d_a), \tag{40}$$

where $\hat{\mathbf{X}}$ is the common point of d_1 and d_2. Therefore letting $R \to 0$ in (39) yields

$$\mathbf{s}(\hat{\mathbf{X}}, t; d_1) = \mathbf{s}(\hat{\mathbf{X}}, t; d_2). \tag{41}$$

Thus, the surface interaction vector $\mathbf{s}$ takes the same value for all surfaces with a common unit normal and therefore its dependence on the surface is only through the normal vector. The assertion (35) is proved.

Cauchy's lemma The balance law (34) implies that

$$\mathbf{s}(\mathbf{X}, t; -\mathbf{N}) = -\mathbf{s}(\mathbf{X}, t; \mathbf{N}). \tag{42}$$

This result will be used in the proof of the Cauchy's theorem. To verify this relation consider a pillbox P_ϵ of thickness ϵ, centered at $\mathbf{X}$, and with its flat surfaces parallel to $\mathbf{N}$. As we let $\epsilon \to 0$, the pillbox flattens to its middle surface S. The relation (34) for bounded fields then reduces to

$$\lim_{R\epsilon \to 0} \int_{\partial P_\epsilon} \mathbf{s} dA = 0 \tag{43}$$

or

$$\int_S (\mathbf{s}(\mathbf{N}) + \mathbf{s}(-\mathbf{N})) dA = 0. \tag{44}$$

Finally, shrink the disk S to the middle point $\mathbf{X}$ and use the continuity of $\mathbf{s}$ to obtain (42).

Cauchy's theorem The surface interaction vector $\mathbf{s}$ depends linearly on $\mathbf{N}$. Therefore, there exists a tensor $\mathbf{S}$ such that

$$\mathbf{s}(\mathbf{X}, t; \mathbf{N}) = \mathbf{S}(\mathbf{X}, t)\mathbf{N}. \tag{45}$$

We now prove this theorem. Consider a tetrahedron $T \subset \boldsymbol{\kappa}(\mathfrak{B})$ with vertex $\mathbf{X}_0 \in \boldsymbol{\kappa}(\mathfrak{B})$. The surface of the tetrahedron normal to the axis $\mathbf{e}_i$ is denoted by s_i. Let δ be the distance along the unit normal $\mathbf{m}$ from the vertex to the fourth surface s (see figure 2). Then, the volume of the tetrahedron $V(T)$ and the surface area $A(s)$ of the face s can be calculated as respectively, $c_1\delta^3$ and $c_2\delta^2$, where $\{c_1, c_2\} \in \mathbb{R}^+$ are constants. The area of the remaining faces (given by $A(s_i)$) can be obtained from $A(s)$:

$$A(s_i) = (\mathbf{m} \cdot \mathbf{e}_i) A(s). \tag{46}$$

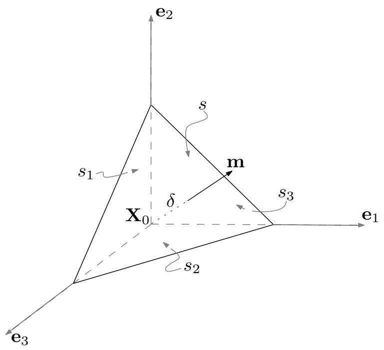

Figure 2: Tetrahedron T

This relation can be verified by first noting, using the divergence theorem, that $\int_{\partial T} \mathbf{N} dA = 0$, where ∂T is piecewise smooth. Since $\mathbf{N}$ is constant on each face of T, (46) follows.
We will now use the balance law (34) and the assumption of the continuity of the fields to arrive at the relation (45). The balance law when restricted to the tetrahedron T implies

$$\Big| \int_{\partial T} \mathbf{s} dA \Big| = \Big| \int_T (\dot{\mathbf{p}} - \mathbf{r}) dV \Big| \leq \int_T |\dot{\mathbf{p}} - \mathbf{r}| dV \leq kV(T), \tag{47}$$

where $k = \max_{\mathbf{X} \in T} |\dot{\mathbf{p}} - \mathbf{r}|$ is finite. Therefore,

$$\begin{aligned} O(\delta) = \frac{1}{A(s)} \int_{\partial T} \mathbf{s} dA &= \frac{1}{A(s)} \Big(\int_s \mathbf{s}(\mathbf{X}; \mathbf{m}) dA + \sum_{i=1}^{3} \int_{s_i} \mathbf{s}(\mathbf{X}; -\mathbf{e}_i) dA \Big) \\ &= \frac{1}{A(s)} \Big(\int_s \mathbf{s}(\mathbf{X}; \mathbf{m}) dA - \sum_{i=1}^{3} \int_{s_i} \mathbf{s}(\mathbf{X}; \mathbf{e}_i) dA \Big), \end{aligned} \tag{48}$$

where the last equality is a consequence of Cauchy's lemma. By the Mean-value theorem, for continuous $\mathbf{s}$, we obtain

$$\begin{aligned} \int_s \mathbf{s}(\mathbf{X}; \mathbf{m}) dA &= A(s) \mathbf{s}(\tilde{\mathbf{X}}; \mathbf{m}), \\ \int_{s_i} \mathbf{s}(\mathbf{X}; \mathbf{e}_i) dA &= A(s_i) \mathbf{s}(\tilde{\mathbf{X}}_i; \mathbf{e}_i) \end{aligned} \tag{49}$$

for some $\tilde{\mathbf{X}} \in s$ and $\tilde{\mathbf{X}}_i \in s_i$, respectively. Let $\delta \to 0$. Then $\tilde{\mathbf{X}} \to \mathbf{X}_0$ and

$\tilde{\mathbf{X}}_i \to \mathbf{X}_0$. As a result, Eqs. (46), (48) and (49) yield

$$\mathbf{s}(\mathbf{X}_0; \mathbf{m}) = (\mathbf{m} \cdot \mathbf{e}_i)\mathbf{s}(\mathbf{X}_0; \mathbf{e}_i), \tag{50}$$

where summation over i is implicit. As the choice of the vertex $\mathbf{X}_0$ and the unit normal $\mathbf{m}$ is arbitrary, the relation (50) holds for all $\mathbf{X} \in \boldsymbol{\kappa}(\mathfrak{B})$ and all unit vectors. Equation (50) shows that $\mathbf{s}$ is linear in $\mathbf{m}$. Therefore there exists a tensor $\mathbf{S}$ such that

$$\mathbf{s}(\mathbf{X}, t; \mathbf{m}) = \mathbf{S}(\mathbf{X}, t)\mathbf{m} \tag{51}$$

for all $\mathbf{X} \in \boldsymbol{\kappa}(\mathfrak{B})$ and any unit vector $\mathbf{m}$. The proof is complete.
Note that we have restricted our attention to only continuously differentiable fields defined on domains with piecewise smooth boundaries. Much research has been done in the past fifty years to investigate these results under less stringent smoothness requirements. Such considerations are indeed necessary for many practical problems in mechanics such as those involving shocks, fracture, dislocations and corner singularities (For a recent contribution, where the past work is carefully reviewed, see Schuricht, F., A new mathematical foundation for contact interactions in continuum physics, *Archive of Rational Mechanics and Analysis*, 184(3), pp. 495-551 (2007)).

3. Balance laws and jump conditions

We now obtain local statements of the fundamental balance laws in continuum mechanics. The fields are allowed to be piecewise continuously differentiable so that they may suffer jump discontinuities across a surface in the domain over which they are defined. Consider an arbitrary part of the body, $\mathfrak{S} \subset \mathfrak{B}$, whose placement in the reference and current configurations is denoted by $\Omega = \boldsymbol{\kappa}(\mathfrak{S})$ and $\omega = \boldsymbol{\chi}(\mathfrak{S})$, respectively. Let $\mathcal{S} = \Omega \cap S_t$, where S_t is the singular surface in $\boldsymbol{\kappa}(\mathfrak{B})$ with normal $\mathbf{N}_s$ and speed U. Correspondingly let $s = \omega \cap s_t$, where s_t is the singular surface in $\boldsymbol{\chi}(\mathfrak{B})$ with normal $\mathbf{n}_s$ and speed u.

Balance of mass Define a *mass* function $m \in \mathbb{R}$ such that:
(*i*) $m(\mathfrak{S}) \geq 0,\ \forall \mathfrak{S} \subset \mathfrak{B}$,
(*ii*) $m(\emptyset) = 0$ and
(*iii*) Let $\{\mathfrak{S}_i\}_{i=1}^{\infty}$ be a disjoint family of subsets of the body $\mathfrak{B}$, i.e. $\mathfrak{S}_i \cap \mathfrak{S}_j = \emptyset$, $i \neq j$. Then $m(\bigcup_{i=1}^{\infty} \mathfrak{S}_i) = \sum_{i=1}^{\infty} m(\mathfrak{S}_i)$.

Therefore the function m is a measure on $\mathfrak{B}$. Denote by V and v respectively, the volume of $\mathfrak{S}$ in the reference configuration and the current configuration. Define the density of mass in the reference and the current configuration by

$$0 < \rho_\kappa(\mathbf{X}, t) = \lim_{V \to 0} \frac{m(\mathfrak{S}, t)}{V} \tag{52}$$

and

$$0 < \rho(\mathbf{x}, t) = \lim_{v \to 0} \frac{m(\mathfrak{S}, t)}{v}, \tag{53}$$

respectively, where $\mathbf{X} \in \Omega$ and $\mathbf{x} \in \omega$. The existence of limits is assumed in the above definitions. The mass of the part $\mathfrak{S} \subset \mathfrak{B}$ is then given by

$$m(\mathfrak{S}, t) = \int_\Omega \rho_\kappa(\mathbf{X}, t) dV = \int_\omega \rho(\mathbf{x}, t) dv. \tag{54}$$

The reference mass density can be related to the current density of mass by using the Jacobian $J_F = \det \mathbf{F} > 0$, such that $J_F dV = dv$ (and $d\mathbf{x} = \mathbf{F} d\mathbf{X}$), and the localization theorem in (54). We get

$$\rho_\kappa = J_F \rho. \tag{55}$$

Assuming an absence of diffusion and any external source of mass, we express the law of balance of mass as

$$\dot{m}(\mathfrak{S}, t) = 0 \tag{56}$$

or from (54)

$$\frac{d}{dt} \int_\Omega \rho_\kappa(\mathbf{x}, t) dv = 0, \tag{57}$$

which, on using the transport theorem (28), reduces to

$$\int_\Omega \dot{\rho}_\kappa dV - \int_{\mathcal{S}} U [\![\rho_\kappa]\!] dA = 0. \tag{58}$$

We can choose Ω such that $\mathcal{S} = \emptyset$. Thereupon using the localization theorem we obtain

$$\dot{\rho}_\kappa = 0 \tag{59}$$

outside the singular surface. The referential mass density is therefore independent of time. For $\mathcal{S} \neq \emptyset$, substitution of (59) in (58) reduces it to a surface integral. Using the arbitrariness of $\mathcal{S}$, the localization theorem for surface integrals then yields the following jump condition at the singular surface

$$U[\![\rho_\kappa]\!] = 0, \tag{60}$$

i.e. either the normal speed vanishes or the referential mass density is continuous across S_t.
The spatial form of the balance law reads

$$\frac{d}{dt}\int_\omega \rho(\mathbf{x}, t) dv = 0, \tag{61}$$

which, on using the transport theorem (29) and the localization theorem, yields

$$\frac{\partial \rho}{\partial t} + \text{div}(\rho \mathbf{v}) = 0 \tag{62}$$

outside the singular surface and

$$(u[\![\rho]\!] - [\![\rho \mathbf{v}]\!] \cdot \mathbf{n}_s) = 0 \tag{63}$$

on the singular surface s_t.

Balance of linear and angular momentum We assume that the forces acting on $\mathfrak{S}$ are either contact forces or body forces. A *contact force* arises from the contact of two parts of $\mathfrak{B}$, say $\mathfrak{S}_1$ and $\mathfrak{S}_2$. The force exerted by $\mathfrak{S}_2$ on $\mathfrak{S}_1$ is given by

$$\mathbf{F}_c(\mathfrak{S}_1, \mathfrak{S}_2, t) = \int_I \mathbf{p} dA = \int_i \mathbf{t} da, \tag{64}$$

where $I = \boldsymbol{\kappa}(\mathfrak{S}_1) \cap \boldsymbol{\kappa}(\mathfrak{S}_2)$ and $i = \boldsymbol{\chi}(\mathfrak{S}_1) \cap \boldsymbol{\chi}(\mathfrak{S}_2)$. The vector $\mathbf{p}$ is the contact force per unit area of $\partial\Omega$ (Piola traction force) and $\mathbf{t}$ is the contact force per unit area of $\partial\omega$ (Cauchy traction force). A *body force* arises from the interaction of $\mathfrak{S}$ with sources external to $\mathfrak{S}$ (e.g. gravitational force). It can be of two kinds: one due to effects exterior to $\mathfrak{B}$ and the other due to effects due to the matter in $\mathfrak{B} \setminus \mathfrak{S}$. It acts on the particles comprising the body and has a form

$$\mathbf{F}_b(\mathfrak{S}, t) = \int_\Omega \rho_\kappa \mathbf{b} dV = \int_\omega \rho \mathbf{b} dv, \tag{65}$$

where $\mathbf{b} = \hat{\mathbf{b}}(\mathbf{X}, t) = \tilde{\mathbf{b}}(\mathbf{x}, t)$ is the body force per unit mass. The total force on $\mathfrak{S}$ can then be written as

$$\mathbf{F}(\mathfrak{S}, t) = \mathbf{F}_c(\mathfrak{S}, \mathfrak{B} \setminus \mathfrak{S}, t) + \mathbf{F}_b(\mathfrak{S}, t). \tag{66}$$

Associated with these forces are *moments*. The moments of the contact force and the body force with respect to an arbitrary point $\mathbf{x}_0 \in \mathcal{E}$ are respectively,

$$\begin{aligned} \mathbf{M}_c(\mathfrak{S}_1, \mathfrak{S}_2, t; \mathbf{x}_0) &= \int_I (\mathbf{x} - \mathbf{x}_0) \times \mathbf{p} dA = \int_i (\mathbf{x} - \mathbf{x}_0) \times \mathbf{t} da, \text{ and} \\ \mathbf{M}_b(\mathfrak{S}, t; \mathbf{x}_0) &= \int_\Omega \rho_\kappa (\mathbf{x} - \mathbf{x}_0) \times \mathbf{b} dV = \int_\omega \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{b} dv. \end{aligned} \tag{67}$$

The total moment acting upon $\mathfrak{S}$ is

$$\mathbf{M}(\mathfrak{S}, t; \mathbf{x}_0) = \mathbf{M}_c(\mathfrak{S}, \mathfrak{B} \setminus \mathfrak{S}, t; \mathbf{x}_0) + \mathbf{M}_b(\mathfrak{S}, t; \mathbf{x}_0). \tag{68}$$

The linear momentum of $\mathfrak{S} \subset \mathfrak{B}$ is given by

$$\mathbf{G}(\mathfrak{S}, t) = \int_\Omega \rho_\kappa \mathbf{v} dV = \int_\omega \rho \mathbf{v} dv. \tag{69}$$

The balance of linear momentum can be stated in the form of Euler's first postulate of motion: The rate of change of linear momentum of $\mathfrak{S}$ is equal to the total force acting on $\mathfrak{S}$, i.e.

$$\dot{\mathbf{G}}(\mathfrak{S}, t) = \mathbf{F}(\mathfrak{S}, t). \tag{70}$$

The referential (or material) form of the balance of linear momentum obtained by substituting definitions (66) and (69) into (70) is

$$\frac{d}{dt} \int_\Omega \rho_\kappa \mathbf{v} dV = \int_{\partial\Omega} \mathbf{p} dA + \int_\Omega \rho_\kappa \mathbf{b} dV. \tag{71}$$

By Noll's and Cauchy's theorems there exists a tensor field $\mathbf{P}$ such that $\mathbf{p} = \mathbf{PN}$. The tensor $\mathbf{P}$ is called the Piola-Kirchhoff stress tensor. Use the transport theorem (28) and the divergence theorem (12) to get

$$\int_\Omega \rho_\kappa \dot{\mathbf{v}} dV - \int_\mathcal{S} U \rho_\kappa [\![\mathbf{v}]\!] dA = \int_\Omega \operatorname{Div} \mathbf{P} dV + \int_\mathcal{S} [\![\mathbf{P}]\!] \mathbf{N}_s dA + \int_\Omega \rho_\kappa \mathbf{b} dV, \tag{72}$$

where we have also used (59) and (60). Since Ω is arbitrary, we can choose it such that $\mathcal{S} = \emptyset$. The localization theorem then yields the local form for the balance of linear momentum

$$\rho_\kappa \dot{\mathbf{v}} = \operatorname{Div} \mathbf{P} + \rho_\kappa \mathbf{b}, \tag{73}$$

which holds outside the singular surface. Now consider $\mathcal{S} \neq \emptyset$. Substitute (73) in (72) and use the arbitrariness of $\mathcal{S}$ to use the localization theorem to obtain the jump condition across S_t

$$U \rho_\kappa [\![\mathbf{v}]\!] + [\![\mathbf{P}]\!] \mathbf{N}_s = \mathbf{0}. \tag{74}$$

The spatial form of these equations can be obtained in a similar manner. We write the spatial form of the balance of linear momentum as

$$\frac{d}{dt} \int_\omega \rho \mathbf{v} dv = \int_{\partial\omega} \mathbf{t} da + \int_\omega \rho \mathbf{b} dv. \tag{75}$$

By Noll's and Cauchy's theorems there exists a tensor field $\mathbf{T}$ such that $\mathbf{t} = \mathbf{T}\mathbf{n}$. The tensor $\mathbf{T}$ is called the Cauchy stress tensor. The local form of the balance law can be now obtained using the transport theorem (29), the divergence theorem (13) and the localization theorem. We obtain outside the singular surface and on the singular surface respectively,

$$\rho \dot{\mathbf{v}} = \operatorname{div} \mathbf{T} + \rho \mathbf{b} \tag{76}$$

and

$$j_s [\![\mathbf{v}]\!] + [\![\mathbf{T}]\!] \mathbf{n}_s = \mathbf{0}, \tag{77}$$

where

$$j_s = \frac{\rho_\kappa U}{|(\mathbf{F}^\pm)^* \mathbf{N}_s|}$$

is the flux of mass through the singular surface. Rewrite (77) as

$$\rho^\pm (u - \mathbf{n}_s \cdot \mathbf{v}^\pm) [\![\mathbf{v}]\!] + [\![\mathbf{T}]\!] \mathbf{n}_s = \mathbf{0}. \tag{78}$$

The moment of momentum of $\mathfrak{S} \subset \mathfrak{B}$ relative to an arbitrary $\mathbf{x}_0 \in \mathcal{E}$ is given by

$$\mathbf{H}(\mathfrak{S}, t; \mathbf{x}_0) = \int_\Omega \rho_\kappa (\mathbf{x} - \mathbf{x}_0) \times \mathbf{v} dV = \int_\omega \rho (\mathbf{x} - \mathbf{x}_0) \times \mathbf{v} dv. \tag{79}$$

The balance of angular momentum in the form of Euler's second postulate of motion is the following: The rate of change of moment of momentum of $\mathfrak{S}$ is equal to the total moment acting on $\mathfrak{S}$, i.e.

$$\dot{\mathbf{H}}(\mathfrak{S}, t; \mathbf{x}_0) = \mathbf{M}(\mathfrak{S}, t; \mathbf{x}_0). \tag{80}$$

Substituting Eqs. (68) and (79) into (80) we get the referential form of the balance of angular momentum

$$\frac{d}{dt}\int_{\Omega} \rho_\kappa (\mathbf{x} - \mathbf{x}_0) \times \mathbf{v} dV = \int_{\partial\Omega} (\mathbf{x} - \mathbf{x}_0) \times \mathbf{p} dA + \int_{\Omega} \rho_\kappa (\mathbf{x} - \mathbf{x}_0) \times \mathbf{b} dV. \tag{81}$$

The local form outside the singular surface is

$$\rho_\kappa ((\mathbf{x} - \mathbf{x}_0) \times \mathbf{v})^{\cdot} = \mathrm{Div}((\mathbf{x} - \mathbf{x}_0) \times \mathbf{P}) + \rho_\kappa (\mathbf{x} - \mathbf{x}_0) \times \mathbf{b}, \tag{82}$$

where for any $\mathbf{y} \in \mathcal{V}$, $(\mathbf{y} \times \mathbf{P})_{il} = e_{ijk} y_j P_{kl}$. On using (73), (82) leads to

$$\mathbf{P}\mathbf{F}^T = \mathbf{F}\mathbf{P}^T. \tag{83}$$

The jump condition is

$$U \rho_\kappa [\![(\mathbf{x} - \mathbf{x}_0) \times \mathbf{v}]\!] + [\![(\mathbf{x} - \mathbf{x}_0) \times \mathbf{P}\mathbf{N}_s]\!] = \mathbf{0}, \tag{84}$$

which can be rewritten as

$$\langle \mathbf{x} - \mathbf{x}_0 \rangle \times (U \rho_\kappa [\![\mathbf{v}]\!] + [\![\mathbf{P}]\!] \mathbf{N}_s) + [\![\mathbf{x}]\!] \times (U \rho_\kappa \langle \mathbf{v} \rangle + \langle \mathbf{P} \rangle \mathbf{N}_s) = \mathbf{0}. \tag{85}$$

Jump conditions (60) and (74) imply that the term $(U \rho_\kappa \mathbf{v} + \mathbf{P}\mathbf{N}_s)$ is continuous across $\mathcal{S}$, thereby reducing (85) to

$$[\![\mathbf{x}]\!] \times (U \rho_\kappa \mathbf{v}^{\pm} + \mathbf{P}^{\pm} \mathbf{N}_s) = \mathbf{0}, \tag{86}$$

where the superscript $\pm$ indicates that either of the limits can be used. For a motion which continuous across the singular surface,i.e. $[\![\mathbf{x}]\!] = \mathbf{0}$, this results into a trivial relation, and therefore is of no consequence. But for $[\![\mathbf{x}]\!] \neq \mathbf{0}$, (86) provides us with an additional jump condition to be satisfied across the singular surface.
The corresponding spatial form of the Eq. (83) is

$$\mathbf{T} = \mathbf{T}^T. \tag{87}$$

Remark: The balance of angular momentum implies the balance of linear momentum. Let $\mathbf{c}$ be an arbitrary vector. Rewrite Eq. (81) after replacing $\mathbf{x}_0$ by $(\mathbf{x}_0 + \mathbf{c})$. Subtract (81) from this equation to get

$$\frac{d}{dt}\int_{\Omega} \rho_\kappa \mathbf{c} \times \mathbf{v} dV = \int_{\partial\Omega} \mathbf{c} \times \mathbf{p} dA + \int_{\Omega} \rho_\kappa \mathbf{c} \times \mathbf{b} dV. \tag{88}$$

Since $\mathbf{c}$ is arbitrary, we get the desired result.

Balance of energy We restrict our attention to systems where the energy is supplied to the body either through mechanical work (done by contact and body forces) or via a supply of heat. We assume that the supply of heat to $\mathfrak{S}$ has two sources. The *contact heating* supplied to $\mathfrak{S}_1$ by $\mathfrak{S}_2$ through their surface of contact is

$$H_c(\mathfrak{S}_1, \mathfrak{S}_2, t) = \int_I q dA = \int_i h da, \tag{89}$$

where $q \in \mathbb{R}$ and $h \in \mathbb{R}$ are heat flux per unit area of $I = \boldsymbol{\kappa}(\mathfrak{S}_1) \cap \boldsymbol{\kappa}(\mathfrak{S}_2)$ and $i = \boldsymbol{\chi}(\mathfrak{S}_1) \cap \boldsymbol{\chi}(\mathfrak{S}_2)$, respectively. The *external supply of heat* to $\mathfrak{S}$ is received from sources external to the body and is given by

$$H_e(\mathfrak{S}, t) = \int_{\Omega} \rho_\kappa r dV = \int_{\omega} \rho r dv, \tag{90}$$

where $r = \hat{r}(\mathbf{X}, t) = \tilde{r}(\mathbf{x}, t)$ is the rate of heat supply to $\mathfrak{S}$ per unit mass of $\mathfrak{S}$. Therefore, the total heat supply to $\mathfrak{S}$ is

$$H(\mathfrak{S}, t) = H_c(\mathfrak{S}, \mathfrak{B} \setminus \mathfrak{S}, t) + H_e(\mathfrak{S}, t). \tag{91}$$

The total energy U of the part $\mathfrak{S}$ of the body at any time consists of the kinetic energy and the internal energy of $\mathfrak{S}$

$$U(\mathfrak{S}, t) = \int_{\Omega} \frac{1}{2}\rho_\kappa \mathbf{v} \cdot \mathbf{v} dV + \int_{\Omega} \rho_\kappa e dV, \tag{92}$$

where e is the internal energy per unit mass of $\mathfrak{S}$. The balance of total energy is the *first law of thermodynamics* which postulates that a time-change in total energy of $\mathfrak{S}$ is balanced by the supply of the mechanical power and the heat to (or from) $\mathfrak{S}$:

$$\dot{U}(\mathfrak{S}, t) = P(\mathfrak{S}, t) + H(\mathfrak{S}, t), \tag{93}$$

where the mechanical power P is of the form

$$P(\mathfrak{S},t) = \int_{\partial\Omega} \mathbf{p}\cdot\mathbf{v}dA + \int_{\Omega} \rho_\kappa \mathbf{b}\cdot\mathbf{v}dV. \tag{94}$$

Upon substituting definitions (92), (94), and (91) into (93) we obtain the referential form of the balance of energy

$$\frac{d}{dt}\int_{\Omega} \rho_\kappa (e + \frac{1}{2}|\mathbf{v}|^2)dV = \int_{\partial\Omega} (\mathbf{v}\cdot\mathbf{PN} + q)dA + \int_{\Omega} \rho_\kappa (\mathbf{v}\cdot\mathbf{b} + r)dV. \tag{95}$$

By Noll's and Cauchy's theorems, the balance law (95) implies the existence of a vector $\mathbf{q}$ such that $q = -\mathbf{q}\cdot\mathbf{N}$ (the $-$ sign is conventional). The vector $\mathbf{q}$ is the referential heat flux vector (we can similarly argue for the existence of a vector $\mathbf{h}$, the spatial heat flux vector, such that $h = -\mathbf{h}\cdot\mathbf{n}$). The local form of the balance of energy can be obtained upon using the transport theorem (28), the divergence theorem (12), and the localization theorem. We get

$$\rho_\kappa (e + \frac{1}{2}|\mathbf{v}|^2)^{\cdot} = \mathrm{Div}(\mathbf{P}^T\mathbf{v} - \mathbf{q}) + \rho_\kappa(\mathbf{v}\cdot\mathbf{b} + r) \tag{96}$$

outside the singular surface and

$$-U\rho_\kappa [\![e + \frac{1}{2}|\mathbf{v}|^2]\!] = [\![\mathbf{P}^T\mathbf{v} - \mathbf{q}]\!] \cdot \mathbf{N}_s \tag{97}$$

on the singular surface S_t (the conditions for mass balance, (59) and (60), have also been used). These two equations, on using the local forms of the balance of linear momentum, reduce to

$$\rho_\kappa \dot{e} = \mathbf{P}\cdot\dot{\mathbf{F}} - \mathrm{Div}\,\mathbf{q} + \rho_\kappa r \tag{98}$$

and

$$U\rho_\kappa [\![e]\!] = -[\![\mathbf{v}]\!] \cdot \langle \mathbf{P} \rangle \mathbf{N}_s + [\![\mathbf{q}]\!] \cdot \mathbf{N}_s, \tag{99}$$

respectively. For a coherent interface, rewrite (99) as

$$U\rho_\kappa [\![e]\!] = U\langle \mathbf{P} \rangle \cdot [\![\mathbf{F}]\!] + [\![\mathbf{q}]\!] \cdot \mathbf{N}_s. \tag{100}$$

The spatial form of these balance equations can be derived in a similar manner. We obtain

$$\rho\dot{e} = \mathbf{T}\cdot\mathbf{L} - \mathrm{div}\,\mathbf{h} + \rho r \tag{101}$$

outside the singular surface s_t in $\boldsymbol{\chi}(\mathfrak{B})$ and

$$-j_s[\![e]\!] = \langle \mathbf{T} \rangle [\![\mathbf{v}]\!] \cdot \mathbf{n}_s - [\![\mathbf{h}]\!] \cdot \mathbf{n}_s \tag{102}$$

on the singular surface s_t.

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