A New Interpretation of Configurational Forces

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Abstract

In a recent paper [7] we interpreted configurational forces as necessary and sufficient dissipative mechanisms such that the corresponding Euler-Lagrange equations are satisfied. We now extend this argument for a dynamic elastic medium, and show that the energy flux obtained from the dynamic J integral ensures that the equations of motion hold throughout the body.

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We show that, under certain smoothness assumptions, the energy flux during an infinitesimal perturbation of the inhomogeneity position is given by the dynamic J integral if and only if linear momentum balance is preserved during the perturbation. To the best of our knowledge, such an interpretation is nowhere found in the literature, cf. [9, 5, 8].

Consider the following functional (for $\Omega \subset \mathbb{R}^3$ and $[t_1, t_2] \subset \mathbb{R}$),

$$\Pi(u_i) = \int_{t_1}^{t_2} \int_{\Omega} \{W(x_i, u_{i,j}) + T(\dot{u}_i)\} \, dV dt, \tag{1}$$

where $T(\dot{u}_i) = \frac{1}{2}\rho \dot{u}_i \dot{u}_i$ (ρ is the constant density) is the specific kinetic energy, and W is the strain energy density. The external forces are assumed to be absent. All the fields are assumed to be continuously differentiable over their respective domains. The functional (1) represents the total mechanical energy stored in an arbitrary part Ω of the body during the time interval $[t_1, t_2]$; the integrand $(W + T)$ is the total energy density. All variables are expressed in terms of their Cartesian

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components. The position and the displacement field are denoted by x_i and $u_i(x_i)$ respectively. The subscript index varies from one to three if it is a Roman letter (i, j, k, etc.), or from one to four if it is Greek (α, β, γ, etc.). The derivative of u_i with respect to x_j (at a fixed time) is written as $u_{i,j}$ and with respect to time (at a fixed x_j) as $\dot{u}_i$. We consider smooth transformations from x_i to the new independent variables $y_i = \hat{y}_i(x_i, \epsilon) = x_i + \phi_i + O(\epsilon^2)$ such that $\hat{y}_i(x_i, 0) = x_i$. Here ϵ is the transformation parameter, and ϕ_i is linear in ϵ but otherwise an arbitrary analytic function of x_i. With these transformations we restrict our attention only to the inner variations in the spatial configuration. The change in the functional Π under the assumed transformation is [10, 6]

$$\delta_x \Pi = \int_{t_1}^{t_2} \int_{\Omega} B_{\alpha,\alpha} dV dt + \int_{t_1}^{t_2} \int_{\Omega} (\sigma_{ik,k} + \rho \ddot{u}_i) u_{i,j} \phi_j dV dt, \tag{2}$$

where $\sigma_{ij} = \frac{\partial W}{\partial u_{i,j}}$ are the components of the stress tensor. The components B_α are given by

$$B_j = \{(W + T)\delta_{ij} - u_{k,i}\sigma_{kj}\} \phi_i \tag{3}$$

and

$$B_4 = -\rho u_{k,j} \dot{u}_k \phi_j \tag{4}$$

such that $B_{4,4} = \dot{B}_4$. Define (cf. Equation (80) in [3])

$$H_{ij} = (W + T)\,\delta_{ij} - u_{k,i}\sigma_{kj} \tag{5}$$

and use it to rewrite (2) as

$$\delta_x \Pi = \int_{t_1}^{t_2} \int_{\partial\Omega} H_{ij} \phi_i n_j dA dt + \int_{t_1}^{t_2} \int_{\Omega} \Big\{ \rho u_{k,j} \ddot{u}_k \phi_j - \rho \dot{u}_{k,j} \dot{u}_k \phi_j + \Psi_i u_{i,j} \phi_j \Big\} dV dt, \tag{6}$$

where $\Psi_i = \sigma_{ij,j} - \rho \ddot{u}_i$. Define

$$F = \int_{t_1}^{t_2} \int_{\partial\Omega} H_{ij} \phi_i n_j dA dt + \int_{t_1}^{t_2} \int_{\Omega} \Big\{ \rho u_{k,j} \ddot{u}_k \phi_j - \rho \dot{u}_{k,j} \dot{u}_k \phi_j \Big\} dV dt, \tag{7}$$

which, after minor manipulation, can be written as

$$F = \int_{t_1}^{t_2} \int_{\partial\Omega} E_{ij} \phi_i n_j dA dt + \int_{t_1}^{t_2} \int_{\Omega} \Big\{ T \phi_{i,i} + \rho u_{k,j} \ddot{u}_k \phi_j \Big\} dV dt, \tag{8}$$

where $E_{ij} = W\delta_{ij} - u_{k,i}\sigma_{kj}$ is Eshelby's energy momentum tensor (cf. Equation (13) in [3]). Consequently, Equation (6) is expressed as

$$\delta_x \Pi = F + \int_{t_1}^{t_2} \int_{\Omega} \Psi_i u_{i,j} \phi_j dV dt. \tag{9}$$

We now state our central proposition. Assume the displacement gradient tensor to be invertible. Then

(P): $\delta_x \Pi = F$ if and only if $\Psi_i = 0$.

Indeed, if $\Psi_i = 0$ then, (9) implies $\delta_x \Pi = F$. On the other hand if $\delta_x \Pi = F$ then, owing to the arbitrariness of domains $[t_t, t_2]$ and Ω, and of the vector ϕ_i, we have $u_{i,j}\Psi_i = 0$. Hence $\Psi_i = 0$ (since $u_{i,j}$ is invertible). Note that the displacement gradient tensor becomes null when $u_i = 0$ (i.e., in the un-deformed configuration) and therefore not invertible.

We exploit this proposition to provide a novel interpretation of the dynamic J integral in the following remark.

Remark 1. (Interpretation for the dynamic J integral) Consider the transformation with ϕ_i independent of x_i, i.e., let $\phi_i = \epsilon a_i$, where a_i are constant. This represents a translation with respect to x_i. Define J_i^d such that $F = \epsilon J_i^d a_i$, where F was introduced in (7). We obtain

$$J_i^d = \int_{t_1}^{t_2} \int_{\partial\Omega} H_{ij} n_j dA dt + \int_{t_1}^{t_2} \int_{\Omega} \Big\{ \rho u_{k,i} \ddot{u}_k - \rho \dot{u}_{k,i} \dot{u}_k \Big\} dV dt. \tag{10}$$

This expression is identified as the dynamic J integral [3, 9]. We now interpret proposition (P) for the case at hand. The energy released (or the change in total energy), with respect to an infinitesimal translation ϵa_i, is given by the dynamic J integral J_i^d if and only if $\Psi_i = 0$ (i.e., if the equations of motion, $\sigma_{ij,j} - \rho \ddot{u}_i = 0$, are satisfied). Only those material points which represent an inhomogeneity contribute towards the energy released. We thus associate the (non-trivial) change in the energy with the motion of inhomogeneities [7]. *Therefore, the energy flux, given by J_i^d, ensures that the equations of motion are satisfied even after a small perturbation of the inhomogeneity position.*

Remark 2. (Comparison with the Lagrangian formulation) The relations $\Psi_i = 0$ are obtained as the Euler-Lagrange equations of the functional

$$\Pi^L(u_i) = \int_{t_1}^{t_2} \int_{\Omega} \{W(x_i, u_{i,j}) - T(\dot{u}_i)\} \, dV dt. \tag{11}$$

There is no difference between this functional and (1) in the case of vanishing kinetic energy. Otherwise the functional Π should be employed to calculate the energy release rate as the variation $\delta_x \Pi$ has a physical meaning of the energy released due to the variation of spatial coordinates. There is no such interpretation for $\delta_x \Pi^L$. In any case, one can obtain

$$\delta_x \Pi^L = F^L + \int_{t_1}^{t_2} \int_{\Omega} \Psi_i u_{i,j} \phi_j dV dt, \tag{12}$$

where

$$F^L = \int_{t_1}^{t_2}\int_{\partial\Omega} E_{ij}\phi_j n_i dA dt + \int_{t_1}^{t_2}\int_{\Omega}\Big\{-T\phi_{i,i} + \rho u_{k,j}\ddot{u}_k\phi_j\Big\} dV dt. \tag{13}$$

A proposition similar to (P) follows. Note that $F \neq F^L$ in general, and therefore using a Lagrangian framework to calculate energy release rates would lead to erroneous results. However, if ϕ_i is independent of x_i then $F = F^L$, as can be checked easily. In certain texts (e.g., §5.6 in [5]), the dynamic energy release rates are motivated from the conservation laws obtained via the application of Noether's theorem to a functional of the type (11) [3, 4, 8]. This leads to the correct expression for the dynamic J integral, but only because ϕ_i is independent of x_i. For a related discussion see §7.8 in [9].

Remark 3. The equivalence established in the proposition (P) holds only when the displacement field is smooth over the whole domain. That this is not true otherwise can be demonstrated by following the arguments from Ball [1, 2]. For example, if the displacement field is singular on a set of measure zero, then the weak form of $\Psi_i = 0$ does not hold true, but the weak form of $\delta_x\Pi = F$ holds (cf. Example 3 in [2]).

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