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KINEMATICS

Anurag Gupta

Department of Mechanical Engineering, Indian Institute of Technology, Kanpur, UP 208016, India

David J. Steigmann

Department of Mechanical Engineering, University of California, Berkeley, CA 94720, USA

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Contents

Summary

This chapter provides the basic infrastructure necessary for a rigorous study of continuum mechanics. The topics include tensor algebra and analysis, geometry and motion of continuous bodies, and singular surfaces. The concepts of tensor algebra and analysis form the language of continuum mechanics and it therefore becomes essential to have a good familiarity with them. A continuous body can demonstrate highly complicated deformations, thus requiring precise notions to characterize their geometry and motion. Singular surfaces are surfaces across which variables such as velocity and deformation suffer jump discontinuities. Understanding their kinematical behavior is a starting point in the study of many important phenomena including the propagation of shock waves, phase fronts, and grain boundaries.

1. Preliminaries

The following notation is adopted in which $\mathcal{V}$ is the translation (vector) space of a real three-dimensional Euclidean point space $\mathcal{E}$:
Lin: the linear space of linear transformations (tensors) from $\mathcal{V}$ to $\mathcal{V}$.
$InvLin$: the group of invertible tensors.

$Sym = \{\mathbf{A} \in Lin : \mathbf{A} = \mathbf{A}^T\}$, where superscript T denotes the transpose: linear space of symmetric tensors; also, the linear operation of symmetrization on Lin.
$Sym^+ = \{\mathbf{A} \in Sym : \mathbf{u} \cdot \mathbf{A}\mathbf{u} > 0\}$ for $\mathbf{u} \neq \mathbf{0}, \mathbf{u} \in \mathcal{V}$: the positive-definite tensors.
$Skw = \{\mathbf{A} \in Lin : \mathbf{A}^T = -\mathbf{A}\}$: the linear space of skew tensors; also, the linear operation of skew-symmetrization on Lin.
$Orth = \{\mathbf{A} \in InvLin : \mathbf{A}^T = \mathbf{A}^{-1}\}$, where $\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$: the group of orthogonal tensors.
$Orth^+ = \{\mathbf{A} \in Orth : J_A = 1\}$: the group of rotations.
Here and in the following chapter on balance laws, both indicial notation as well as bold notation are used to represent vector and tensor fields. The components in the indicial notation are written with respect to the Cartesian coordinate system. Indices denoted with roman alphabets vary from one to three but those denoted with Greek alphabets vary from one to two. Einstein's summation convention is assumed unless an exception is explicitly stated. Let e_{ijk} be the three dimensional permutation symbol, i.e. $e_{ijk} = 1$ or $e_{ijk} = -1$ when (i, j, k) is an even or odd permutation of $(1, 2, 3)$, respectively, and $e_{ijk} = 0$ otherwise.
The determinant and cofactor of $\mathbf{A}$ are denoted by J_A and $\mathbf{A}^*$, respectively, where $\mathbf{A}^* = J_A\mathbf{A}^{-T}$ if $\mathbf{A} \in InvLin$. It follows easily that $(\mathbf{AB})^* = \mathbf{A}^*\mathbf{B}^*$. Further, Lin is equipped with the Euclidean inner product and norm defined by $\mathbf{A} \cdot \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B}^T)$ and $|\mathbf{A}|^2 = \mathbf{A} \cdot \mathbf{A}$, respectively, where $\mathrm{tr}(\cdot)$ is the trace operator. We make frequent use of relations like $\mathbf{A} \cdot \mathbf{BC} = \mathbf{A}\mathbf{C}^T \cdot \mathbf{B} = \mathbf{C}^T \cdot \mathbf{A}^T\mathbf{B}$ and $\mathbf{AB} \cdot \mathbf{CD} = \mathbf{ABD}^T \cdot \mathbf{C}$, etc., which follow easily from $\mathrm{tr}\,\mathbf{A} = \mathrm{tr}\,\mathbf{A}^T$ and $\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$. It is well known that $Lin = Sym \oplus Skw$, the direct sum of Sym and Skw. The tensor product $\mathbf{a} \otimes \mathbf{b}$ of vectors $\{\mathbf{a}, \mathbf{b}\} \in \mathcal{V}$ is defined by $(\mathbf{a} \otimes \mathbf{b})\mathbf{v} = (\mathbf{b} \cdot \mathbf{v})\mathbf{a}$ for all $\mathbf{v}$ in $\mathcal{V}$, where $\mathbf{b} \cdot \mathbf{v}$ is the standard inner product of vectors.

2. Body, configurations, and motion

The geometrical structure of a physical body is independent of a frame of reference, and therefore the *body* (in continuum mechanics) is usually taken to be a three dimensional differential manifold. We denote such a manifold by $\mathfrak{B}$ and call its elements material points. At every material point $X \in \mathfrak{B}$ we have an associated tangent space $\mathcal{T}_X$ which is a three dimensional vector space representing a neighborhood of X. On the other hand, the body is observed and tested in a (three dimensional) Euclidean frame of reference $\mathcal{E}$, which requires us to endow the body $\mathfrak{B}$ with a class C of bijective mappings, $\boldsymbol{\chi} : \mathfrak{B} \to \mathcal{E}_\chi$ (the subscript χ is used to indicate the mapping employed). We

call these mappings the *configurations* of the body $\mathfrak{B}$. The spatial position $\boldsymbol{\chi}(X) \in \mathcal{E}_\chi$ denotes the *place* which a material point $X \in \mathfrak{B}$ occupies in $\mathcal{E}_\chi$. The translation space of $\mathcal{E}_\chi$ is a three dimensional inner product space, and is denoted by $\mathcal{V}_\chi$.
We introduce a fixed *reference configuration*, relative to which the notions of displacement and strain can be defined. Let $\boldsymbol{\kappa} \in \mathrm{C}$ be a reference configuration. The configuration $\boldsymbol{\kappa}$ need not be a configuration occupied by $\mathfrak{B}$ at any time and therefore $\boldsymbol{\kappa}$ can be arbitrary as long as it belongs to C.
The *motion* of a body $\mathfrak{B}$ is defined as a one-parameter family of configurations, $\boldsymbol{\chi}_t : \mathfrak{B} \times \mathbb{R} \to \mathcal{E}_\chi$. Such a motion assigns a place $\mathbf{x} \in \mathcal{E}_\chi$ to the material point $X \in \mathfrak{B}$ at time t. We write this as

$$\mathbf{x} = \boldsymbol{\chi}_t(X) \equiv \boldsymbol{\chi}(X, t). \tag{1}$$

The reference configuration $\boldsymbol{\kappa}$ assigns a place $\mathbf{X} \in \mathcal{E}_\kappa$ to X, so we can express $\mathbf{x}$ as a function of $\mathbf{X}$,

$$\mathbf{x} = \boldsymbol{\chi}(\boldsymbol{\kappa}^{-1}(\mathbf{X}), t) \equiv \boldsymbol{\chi}_\kappa(\mathbf{X}, t), \tag{2}$$

where $\boldsymbol{\chi}_\kappa : \mathcal{E}_\kappa \times \mathbb{R} \to \mathcal{E}_\chi$ denotes a mapping from the reference configuration to the configuration of the body at time t.
The *displacement* $\mathbf{u} : \mathfrak{B} \times \mathbb{R} \to \mathcal{V}$ ($\mathcal{V}$ can be identified with either $\mathcal{V}_\chi$ or $\mathcal{V}_\kappa$) of a material point X with respect to the reference configuration $\boldsymbol{\kappa}$ at time t is defined as

$$\mathbf{u}(X, t) = \boldsymbol{\chi}(X, t) - \boldsymbol{\kappa}(X). \tag{3}$$

The particle *velocity* $\mathbf{v} : \mathfrak{B} \times \mathbb{R} \to \mathcal{V}_\chi$ and the particle *acceleration* $\mathbf{a} : \mathfrak{B} \times \mathbb{R} \to \mathcal{V}_\chi$ are defined as

$$\mathbf{v}(X, t) = \frac{\partial}{\partial t}\boldsymbol{\chi}(X, t) \tag{4}$$

and

$$\mathbf{a}(X, t) = \frac{\partial^2}{\partial t^2}\boldsymbol{\chi}(X, t), \tag{5}$$

respectively. Displacement, particle velocity and particle acceleration can all be alternatively expressed as functions on $\boldsymbol{\kappa}(\mathfrak{B})$ by using the inverse $\boldsymbol{\kappa}^{-1} : \mathcal{E}_\kappa \to \mathfrak{B}$. Such functions exist in a one-to-one relation with the functions expressed in the equations above. We write

$$\begin{aligned} \hat{\mathbf{u}}(\mathbf{X}, t) &\equiv \mathbf{u}(\boldsymbol{\kappa}^{-1}(\mathbf{X}), t) \\ \hat{\mathbf{v}}(\mathbf{X}, t) &\equiv \mathbf{v}(\boldsymbol{\kappa}^{-1}(\mathbf{X}), t) \\ \hat{\mathbf{a}}(\mathbf{X}, t) &\equiv \mathbf{a}(\boldsymbol{\kappa}^{-1}(\mathbf{X}), t). \end{aligned} \tag{6}$$

We can similarly write these functions as

$$\begin{aligned}\tilde{\mathbf{u}}(\mathbf{x},t) &\equiv \mathbf{u}(\boldsymbol{\chi}_t^{-1}(\mathbf{x}),t)\\ \tilde{\mathbf{v}}(\mathbf{x},t) &\equiv \mathbf{v}(\boldsymbol{\chi}_t^{-1}(\mathbf{x}),t)\\ \tilde{\mathbf{a}}(\mathbf{x},t) &\equiv \mathbf{a}(\boldsymbol{\chi}_t^{-1}(\mathbf{x}),t).\end{aligned} \tag{7}$$

We define the *material time derivative* as the derivative of a function with respect to time for a fixed material point. For an arbitrary scalar function $f : \mathfrak{B} \times \mathbb{R} \to \mathbb{R}$, we denote its material time derivative as $\dot{f}$. Thus,

$$\dot{f} = \frac{\partial}{\partial t} f(X,t) \mid_X, \tag{8}$$

where the notation $|_X$ denotes the evaluation of the derivative at a fixed X. If f is instead given in terms of $\mathbf{x}$, i.e. if $f = \tilde{f}(\boldsymbol{\chi}(X,t),t)$, we write

$$\dot{f} = \frac{\partial}{\partial t} \tilde{f}(\mathbf{x},t) \mid_{\mathbf{x}} + (\operatorname{grad} \tilde{f}) \cdot \mathbf{v}, \tag{9}$$

where $\frac{\partial}{\partial t}\tilde{f}(\mathbf{x},t) \mid_{\mathbf{x}}$ is the *spatial time derivative* (at a fixed $\mathbf{x}$) and $\operatorname{grad} \tilde{f}$ is the spatial gradient (gradient is defined below). Therefore, if the particle velocity is a function of spatial position $\mathbf{x}$, then the particle acceleration is $\tilde{\mathbf{a}} = \frac{\partial}{\partial t}\tilde{\mathbf{v}}(\mathbf{x},t) \mid_{\mathbf{x}} + \mathbf{L}\mathbf{v}$, where $\mathbf{L} = \operatorname{grad} \tilde{\mathbf{v}}$ is the spatial velocity gradient.

Derivatives By *fields* we mean scalar, vector and tensor valued functions defined on position ($\mathbf{x}$ or $\mathbf{X}$) and time (t). In the following we are mainly concerned with the derivatives with respect to the position and therefore dependence of fields on time is suppressed.

A scalar-valued field $\phi(\mathbf{X}) : \mathcal{E}_\kappa \to \mathbb{R}$ is differentiable at $\mathbf{X}_0 \in \mathcal{U}(\mathbf{X}_0)$, where $\mathcal{U}(\mathbf{X}_0) \subset \mathcal{E}_\kappa$ is an open neighborhood of $\mathbf{X}_0$, if there exists a unique $\mathbf{c} \in \mathcal{V}_\kappa$ such that

$$\phi(\mathbf{X}) = \phi(\mathbf{X}_0) + \mathbf{c}(\mathbf{X}_0) \cdot (\mathbf{X} - \mathbf{X}_0) + o(|\mathbf{X} - \mathbf{X}_0|), \tag{10}$$

where $\frac{o(\epsilon)}{\epsilon} \to 0$ as $\epsilon \to 0$. We call $\mathbf{c}(\mathbf{X}_0) = \nabla\phi|_{\mathbf{X}_0}$ (or $\nabla\phi(\mathbf{X}_0)$) the *gradient* of ϕ at $\mathbf{X}_0$. Consider a curve $\mathbf{X}(u)$ in $\mathcal{E}_\kappa$ parameterized by $u \in \mathbb{R}$. Let $\psi(u) = \phi(\mathbf{X}(u))$ and $\mathbf{X}_1 = \mathbf{X}(u_1)$, $\mathbf{X}_0 = \mathbf{X}(u_0)$ for $\{u_1, u_0\} \in \mathbb{R}$. Then from (10),

$$\psi(u_1) - \psi(u_0) = \nabla\phi(\mathbf{X}_0) \cdot (\mathbf{X}_1 - \mathbf{X}_0) + o(|\mathbf{X}_1 - \mathbf{X}_0|). \tag{11}$$

Moreover $\mathbf{X}_1 - \mathbf{X}_0 = \mathbf{X}'(u_0)(u_1 - u_0) + o(|u_1 - u_0|)$, where $\mathbf{X}'(u_0)$ is the derivative of $\mathbf{X}$ with respect to u at $u = u_0$. Therefore, $|\mathbf{X}_1 - \mathbf{X}_0| = O(|u_1 - u_0|)$ and consequently we can rewrite (11)

$$\frac{\psi(u_1) - \psi(u_0)}{u_1 - u_0} = \nabla\phi(\mathbf{X}_0) \cdot \mathbf{X}'(u_0) + \frac{o(|u_1 - u_0|)}{u_1 - u_0}. \tag{12}$$

For $u_1 \to u_0$ we obtain *the chain rule*, $\psi'(u_0) = \nabla\phi(\mathbf{X}(u_0)) \cdot \mathbf{X}'(u_0)$, which can also be expressed as $\frac{d\phi}{du} = \nabla\phi(\mathbf{X}) \cdot \frac{d\mathbf{X}}{du}$ or

$$d\phi(\mathbf{X}) = \nabla\phi(\mathbf{X}) \cdot d\mathbf{X}. \tag{13}$$

A vector-valued field $\mathbf{v}(\mathbf{X}) : \mathcal{E}_\kappa \to \mathcal{V}$ is differentiable at $\mathbf{X}_0 \in \mathcal{U}(\mathbf{X}_0)$ if there exists a unique tensor $\mathbf{l} : \mathcal{V}_\kappa \to \mathcal{V}$ such that

$$\mathbf{v}(\mathbf{X}) = \mathbf{v}(\mathbf{X}_0) + \mathbf{l}(\mathbf{X}_0)(\mathbf{X} - \mathbf{X}_0) + \mathbf{r}, \tag{14}$$

where $|\mathbf{r}| = o(|\mathbf{X} - \mathbf{X}_0|)$. We call $\mathbf{l}(\mathbf{X}_0) = \nabla\mathbf{v}|_{\mathbf{X}_0}$ (or $\nabla\mathbf{v}(\mathbf{X}_0)$) the *gradient* of $\mathbf{v}$ at $\mathbf{X}_0$. The chain rule in this case can be obtained following the procedure preceding Eq. (13):

$$d\mathbf{v}(\mathbf{X}) = (\nabla\mathbf{v})d\mathbf{X}. \tag{15}$$

The *divergence* of a vector field is a scalar defined by

$$\mathrm{Div}\,\mathbf{v} = \mathrm{tr}(\nabla\mathbf{v}). \tag{16}$$

The *curl* of a vector field is a vector defined by

$$(\mathrm{Curl}\,\mathbf{v}) \cdot \mathbf{c} = \mathrm{Div}(\mathbf{v} \times \mathbf{c}) \tag{17}$$

for any fixed $\mathbf{c} \in \mathcal{V}$.

Differentiability of a tensor-valued function is defined in a similar manner. In particular, for a tensor $\mathbf{A}(\mathbf{X}) : \mathcal{E}_\kappa \to Lin$, we write

$$d\mathbf{A}(\mathbf{X}) = (\nabla\mathbf{A})d\mathbf{X}. \tag{18}$$

The *divergence* of $\mathbf{A}$ is the vector defined by

$$(\mathrm{Div}\,\mathbf{A}) \cdot \mathbf{c} = \mathrm{Div}(\mathbf{A}^T\mathbf{c}) \tag{19}$$

for any fixed $\mathbf{c} \in \mathcal{V}$. The *curl* of $\mathbf{A}$ is the tensor defined by

$$(\mathrm{Curl}\,\mathbf{A})\mathbf{c} = \mathrm{Curl}(\mathbf{A}^T\mathbf{c}) \tag{20}$$

for any fixed $\mathbf{c} \in \mathcal{V}$.

Finally, if the fields are expressed as functions of $\mathbf{x}$ rather than $\mathbf{X}$, the various definitions above remain valid. We instead denote the gradient, divergence and curl operators by grad, div, and curl, respectively.

3. Deformation Gradient

If the mapping $\boldsymbol{\chi}_\kappa(\mathbf{X}, t)$ is differentiable with respect to $\mathbf{X}$, then we define the deformation gradient by

$$\mathbf{F} = \nabla\boldsymbol{\chi}_\kappa. \tag{21}$$

Since $\boldsymbol{\chi}_\kappa(\mathbf{X}, t)$ is invertible for each $\mathbf{X} \in \mathcal{E}_\kappa$, the deformation gradient $\mathbf{F}$ belongs to a family of invertible linear maps from the translation space of $\mathcal{E}_\kappa$ to the translation space of $\mathcal{E}_\chi$, i.e. $\mathbf{F} \in InvLin$. This follows from the inverse function theorem (Rudin, W. *Principles of Mathematical Analysis*, 3rd Ed., McGraw-Hill (1976), page 221). For $\{\mathbf{X}, \mathbf{Y}\} \in \mathcal{E}_\kappa$ Eq. (14) becomes

$$\boldsymbol{\chi}_\kappa(\mathbf{Y}, t) = \boldsymbol{\chi}_\kappa(\mathbf{X}, t) + \mathbf{F}(\mathbf{X}, t)(\mathbf{Y} - \mathbf{X}) + \mathbf{r} \tag{22}$$

and the chain rule (15) takes the form (for fixed t)

$$d\mathbf{x} = \mathbf{F}d\mathbf{X}, \tag{23}$$

where the differentials $d\mathbf{X}$ and $d\mathbf{x}$ belong to the translation spaces $\mathcal{V}_\kappa$ at $\mathbf{X}$ and $\mathcal{V}_\chi$ at $\mathbf{x}$, respectively.

We now obtain relationships for transforming infinitesimal area and volume elements. Let $d\mathbf{X}_1 \in \mathcal{V}_\kappa$ and $d\mathbf{X}_2 \in \mathcal{V}_\kappa$ be two linearly independent infinitesimal line elements at $\mathbf{X}$. An infinitesimal area element can be constructed using these line elements, with area given by $da_\kappa = |d\mathbf{X}_1 \times d\mathbf{X}_2|$ and the associated direction given by the unit normal $\mathbf{n}_\kappa$ such that $\mathbf{n}_\kappa da_\kappa = d\mathbf{X}_1 \times d\mathbf{X}_2$. In the configuration $\boldsymbol{\chi}_t$ the line elements $d\mathbf{X}_1$ and $d\mathbf{X}_2$ are transformed into line elements $d\mathbf{x}_1 \in \mathcal{V}_\chi$ and $d\mathbf{x}_2 \in \mathcal{V}_\chi$, respectively at $\mathbf{x} = \boldsymbol{\chi}_\kappa(\mathbf{X}, t)$. We obtain, using relation (23), $d\mathbf{x}_1 = \mathbf{F}d\mathbf{X}_1$ and $d\mathbf{x}_2 = \mathbf{F}d\mathbf{X}_2$. The area element constructed using these line elements has area $da = |d\mathbf{x}_1 \times d\mathbf{x}_2|$ with unit normal $\mathbf{n}$ given by $\mathbf{n}da = d\mathbf{x}_1 \times d\mathbf{x}_2$. Therefore,

$$\begin{aligned}\mathbf{n}da &= \mathbf{F}d\mathbf{X}_1 \times \mathbf{F}d\mathbf{X}_2 \\ &= \mathbf{F}^*(d\mathbf{X}_1 \times d\mathbf{X}_2) \\ &= \mathbf{F}^*\mathbf{n}_\kappa da_\kappa.\end{aligned} \tag{24}$$

As $\mathbf{F} \in InvLin$, we have

$$\mathbf{F}^* = J_F\mathbf{F}^{-T}. \tag{25}$$

Consider a third line element $d\mathbf{X}_3 \in \mathcal{V}_\kappa$ at $\mathbf{X}$ such that the set $\{d\mathbf{X}_1, d\mathbf{X}_2, d\mathbf{X}_3\}$ is linearly independent and positively oriented. The infinitesimal volume element associated with the reference configuration is then given by $dv_\kappa =$

$d\mathbf{X}_1 \cdot d\mathbf{X}_2 \times d\mathbf{X}_3$. In configuration $\boldsymbol{\chi}_t$ the volume element at $\mathbf{x} = \boldsymbol{\chi}_\kappa(\mathbf{X}, t)$ is $dv = d\mathbf{x}_1 \cdot d\mathbf{x}_2 \times d\mathbf{x}_3$ with $d\mathbf{x}_3 = \mathbf{F}d\mathbf{X}_3$. Therefore,

$$\begin{aligned} dv &= \mathbf{F}d\mathbf{X}_1 \cdot \mathbf{F}d\mathbf{X}_2 \times \mathbf{F}d\mathbf{X}_3 \\ &= \mathbf{F}d\mathbf{X}_1 \cdot \mathbf{F}^*(d\mathbf{X}_2 \times d\mathbf{X}_3) \\ &= J_F dv_\kappa \end{aligned} \tag{26}$$

and accordingly, if $\boldsymbol{\kappa}$ is a configuration that could be attained in the course of the motion of $\mathfrak{B}$, we require $J_F > 0$ to ensure that a volume in $\boldsymbol{\kappa}$ corresponds to a volume in $\boldsymbol{\chi}$.

Material curves Consider a curve $C \subset \mathcal{E}_\kappa$ and parameterize it with a real number $s \in \mathbb{R}$ such that $C : \mathbb{R} \to \mathcal{E}_\kappa$. We call C a *material curve*. Its placement in the configuration $\boldsymbol{\chi}$ is denoted by c and we use s to parameterize it such that $c : \mathbb{R} \to \mathcal{E}_\chi$. Using the definition of the deformation gradient and assuming the mappings C and c to be differentiable, we write

$$\mathbf{x}'(s) = \mathbf{F}\mathbf{X}'(s). \tag{27}$$

If s is the arc-length on C, then the vector $\mathbf{X}'(s)$ defines a unit tangent vector (denoted $\mathbf{M}$) to the curve C at arc-length station s. Let $\mathbf{x}'(s) = \mu\mathbf{m}$ with $|\mathbf{m}| = 1$ and $\mu = |\mathbf{x}'(s)|$. Substituting these in (27), we obtain

$$\mu\mathbf{m} = \mathbf{F}\mathbf{M}. \tag{28}$$

Since $\mathbf{F} \in InvLin$, $\mathbf{F}\mathbf{M} \neq \mathbf{0}$ and therefore $\mu > 0$. We call $\mu(s, t)$ the local stretch of C. It follows from (28) that

$$\mu^2 = |\mu\mathbf{m}|^2 = \mathbf{F}\mathbf{M} \cdot \mathbf{F}\mathbf{M} = \mathbf{M} \cdot \mathbf{C}\mathbf{M}, \tag{29}$$

where $\mathbf{C} = \mathbf{F}^T\mathbf{F} : \mathcal{V}_\kappa \to \mathcal{V}_\kappa$ is the *Right Cauchy Green tensor*. The tensor $\mathbf{C}$ is symmetric and positive definite, i.e. $\mathbf{C} \in Sym^+$. Indeed, $\mathbf{C}^T = (\mathbf{F}^T\mathbf{F})^T = \mathbf{F}^T\mathbf{F} = \mathbf{C}$ and for arbitrary $\mathbf{a} \in \mathcal{V}_\kappa$, $\mathbf{a} \cdot \mathbf{C}\mathbf{a} = \mathbf{F}\mathbf{a} \cdot \mathbf{F}\mathbf{a} = |\mathbf{F}\mathbf{a}|^2 > 0$, as $J_F \neq 0$. Similarly, if we rewrite (28) as $\mu^{-1}\mathbf{M} = \mathbf{F}^{-1}\mathbf{m}$, we can arrive at the (symmetric and positive definite) *Left Cauchy Green tensor* $\mathbf{B} = \mathbf{F}\mathbf{F}^T : \mathcal{V}_\chi \to \mathcal{V}_\chi$ such that $\mu^{-2} = \mathbf{m} \cdot \mathbf{B}^{-1}\mathbf{m}$. We can use $\mathbf{C}$ to calculate the deformed length of a material curve and the deformed angle between two material curves. Given an infinitesimal element of the material curve $d\mathbf{X} = \mathbf{M}ds$, its deformed length is $|d\mathbf{x}| = \sqrt{\mathbf{F}\mathbf{M} \cdot \mathbf{F}\mathbf{M}}ds = \mu ds$ and therefore the deformed length of a material curve with reference arc-length $s_1 - s_0$ is

$$l_c(t) = \int_{s_0}^{s_1} \mu(s, t)ds. \tag{30}$$

Consider two material curves intersecting at $\mathbf{X}$ with associated unit tangent vectors $\mathbf{M}_1$ and $\mathbf{M}_2$, respectively. Let μ_1 and μ_2 be the local stretches corresponding to the two curves and let θ be the angle between the tangent vectors of the deformed curve at $\mathbf{x}$. We then write, $\mu_1\mu_2\cos\theta = \mathbf{FM}_1\cdot\mathbf{FM}_2 = \mathbf{M}_1\cdot\mathbf{CM}_2$ and, on using (29), obtain

$$\cos\theta = \frac{\mathbf{M}_1\cdot\mathbf{CM}_2}{\sqrt{(\mathbf{M}_1\cdot\mathbf{CM}_1)(\mathbf{M}_2\cdot\mathbf{CM}_2)}}. \tag{31}$$

Finally, we introduce two definitions of extensional strain: The first, denoted $\mathbf{e}_C$ and defined by $\mathbf{e}_C = \frac{1}{2}(\mathbf{C}-\mathbf{1})$, yields $\frac{1}{2}(\mu^2-1) = \mathbf{M}\cdot\mathbf{e}_C\mathbf{M}$, where $\mathbf{1}\in Lin$ is the identity transformation. Therefore, $\mathbf{e}_C : \mathcal{V}_\kappa \to \mathcal{V}_\kappa$ characterizes the relative local stretch with respect to the reference configuration. It is known as the *relative Lagrange strain* or the *Green-St. Venant strain*. Alternatively, to characterize local stretch relative to the current configuration, we define $\mathbf{e}_B = \frac{1}{2}(\mathbf{1}-\mathbf{B}^{-1})$, and obtain $\frac{1}{2}(1-\mu^{-2}) = \frac{1}{2}\mathbf{m}\cdot(\mathbf{1}-\mathbf{B}^{-1})\mathbf{m}$. The tensor $\mathbf{e}_B : \mathcal{V}_\chi \to \mathcal{V}_\chi$ is called the *relative Eulerian strain* or the *Almansi-Hamel strain tensor*. The two strain tensors are related by $\mathbf{e}_C = \mathbf{F}^T\mathbf{e}_B\mathbf{F}$.
Using Eqs. (3) and (21), we can obtain the deformation gradient from the displacement field, $\mathbf{F} = \mathbf{1} + \nabla\mathbf{u}$. For small deformations $|\nabla\mathbf{u}| \ll 1$ and consequently $\mu \approx 1$ and $\mathbf{e}_C \approx \frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T)$ ($\approx$ denotes the small deformation approximation). The two strain measures are asymptotically coincident in this approximation.

Principal stretches We would now like to identify the material curves along which the local stretch assumes extreme values and obtain these extremals from $\mathbf{C}$. Define $f(\mathbf{M}) = \mathbf{M}\cdot\mathbf{CM}$ at fixed $\mathbf{C}$. We therefore have $f(\mathbf{M}) > 0$ (from (29)), for $\mathbf{M}\in\mathcal{S} = \{\mathbf{v}\in\mathcal{V}_\kappa : |\mathbf{v}| = 1\}$. Since $f(\mathbf{M})$ is a continuous function, defined on a compact set, a theorem in analysis (Rudin, W. *ibid.*, page 89) yields the existence of $\mathbf{M}_1\in\mathcal{S}$ and $\mathbf{M}_2\in\mathcal{S}$ such that $f(\mathbf{M}_1) = \min_{\mathbf{M}\in\mathcal{S}} f(\mathbf{M}) \equiv \lambda_1^2$ and $f(\mathbf{M}_2) = \max_{\mathbf{M}\in\mathcal{S}} f(\mathbf{M}) \equiv \lambda_2^2$, respectively. Our aim is to compute λ_1^2 and λ_2^2 for a given $\mathbf{C}$. These are extremal values of $f(\mathbf{M})$ and thus render $f(\mathbf{M})$ stationary, i.e. $df(\mathbf{M}) = 0$, or $\mathbf{CM}\cdot d\mathbf{M} = 0$ for $\mathbf{M}\in\{\mathbf{M}_1,\mathbf{M}_2\}$. Furthermore, the identity $\mathbf{M}\cdot\mathbf{M} = 1$ implies $\mathbf{M}\cdot d\mathbf{M} = 0$ and therefore $d\mathbf{M}\perp\mathbf{M}$ at each $\mathbf{M}\in\mathcal{S}$. Since $\mathcal{S}$ is a two dimensional manifold with $d\mathbf{M}$ belonging to its tangent space, the vector $\mathbf{M}$ represents the unit vector normal to $\mathcal{S}$ at $\mathbf{M}\in\mathcal{S}$. As a result of these arguments, for some $\mu_1,\mu_2\in\mathbb{R}$ we can write, $\mathbf{CM}_1 = \mu_1\mathbf{M}_1$ and $\mathbf{CM}_2 = \mu_2\mathbf{M}_2$. Evidently, μ_1 and μ_2 are equal to λ_1^2 and λ_2^2, respectively ($\mu_1 = \mathbf{M}_1\cdot\mathbf{CM}_1 = f(\mathbf{M}_1) = \lambda_1^2$, etc.), the largest and smallest eigenvalues of $\mathbf{C}$, respectively, and thus

$$\mathbf{CM}_1 = \lambda_1^2\mathbf{M}_1, \text{ and } \mathbf{CM}_2 = \lambda_2^2\mathbf{M}_2. \tag{32}$$

In general, for $\lambda \in \mathbb{R}$ and $\mathbf{M} \in \mathcal{S}$, we can solve the eigenvalue problem $\mathbf{CM} = \lambda^2\mathbf{M}$ to obtain three real values for λ^2. If $\{\mathbf{E}_A\}$ is an orthonormal basis for $\mathcal{V}_\kappa$ and if we set $C_{AB} = \mathbf{E}_A \cdot \mathbf{CE}_B$, then we can conclude that the eigenvalues bound the diagonal entries of the matrix $\{C_{AB}\}$; i.e.

$$\lambda_1^2 \leq \min\{C_{11}, C_{22}, C_{33}\} \leq \max\{C_{11}, C_{22}, C_{33}\} \leq \lambda_2^2. \tag{33}$$

Two theorems for symmetric tensors According to the *spectral theorem*, for every $\mathbf{A} \in Sym$, there exists an orthonormal basis $\{\mathbf{u}_i\} \in \mathcal{V}(i = 1, 2, 3)$ and numbers $\lambda_i \in \mathbb{R}$ such that

$$\mathbf{A} = \sum_{i=1}^{3} \lambda_i \mathbf{u}_i \otimes \mathbf{u}_i. \tag{34}$$

The numbers λ_i are the principal values associated with the tensor $\mathbf{A}$ and can be obtained as the roots of the characteristic equation $\det(\mathbf{A} - \lambda\mathbf{1}) = 0$ with $\lambda \in \mathbb{R}$. We now prove this assertion. Let λ and $\mathbf{u}$ be a principal value (eigenvalue) and the corresponding principal vector (eigenvector) associated with $\mathbf{A}$. Allow them to be complex, i.e. $\lambda = a + ib$ and $\mathbf{u} = \mathbf{a} + i\mathbf{b}$ for some $\{a, b\} \in \mathbb{R}$ and $\{\mathbf{a}, \mathbf{b}\} \in \mathbb{R}^3$ with $i = \sqrt{-1}$. Therefore $\mathbf{Au} = \lambda\mathbf{u}$. We also have $\mathbf{A}\bar{\mathbf{u}} = \bar{\lambda}\bar{\mathbf{u}}$, where an over-bar represents the complex conjugate. Since $\mathbf{A}$ is symmetric, we can write $\mathbf{u} \cdot \mathbf{A}\bar{\mathbf{u}} = \bar{\mathbf{u}} \cdot \mathbf{Au}$ or $0 = (\lambda - \bar{\lambda})\mathbf{u} \cdot \bar{\mathbf{u}}$. This implies $\lambda = \bar{\lambda}$, as $\mathbf{u} \cdot \bar{\mathbf{u}} > 0$. We now have to prove the existence of orthonormal $\{\mathbf{u}_i\}$ such that (34) holds. For eigenvalues $\lambda_1,\ \lambda_2$ and their corresponding eigenvectors $\mathbf{u}_1,\ \mathbf{u}_2$, we have $\mathbf{Au}_1 = \lambda_1\mathbf{u}_1$ and $\mathbf{Au}_2 = \lambda_2\mathbf{u}_2$. As $\mathbf{A}$ is symmetric, $\mathbf{u}_1 \cdot \mathbf{Au}_2 = \mathbf{u}_2 \cdot \mathbf{Au}_1$ and thus $0 = (\lambda_1 - \lambda_2)\mathbf{u}_1 \cdot \mathbf{u}_2$. If $\lambda_1 \neq \lambda_2$, then $\mathbf{u}_1$ and $\mathbf{u}_2$ are mutually orthogonal. Therefore if $\{\lambda_i\}$ are distinct, $\{\mathbf{u}_i\}$ necessarily forms an orthonormal set. If $\lambda_1 \neq \lambda_2 = \lambda_3$, then $\mathbf{u}_1 \cdot \mathbf{u}_2 = 0$. Define $\mathbf{u}_3 = \mathbf{u}_1 \times \mathbf{u}_2$, so that $\{\mathbf{u}_i\}$ is orthonormal. The vector $\mathbf{u}_3$ is the third principal vector of $\mathbf{A}$. Indeed $\mathbf{Au}_3 = \sum_{i=1}^{3}(\mathbf{u}_i \cdot \mathbf{Au}_3)\mathbf{u}_i = \sum_{i=1}^{3}(\mathbf{u}_3 \cdot \mathbf{Au}_i)\mathbf{u}_i = (\mathbf{u}_3 \cdot \mathbf{Au}_3)\mathbf{u}_3$ where in the first equality, the vector $\mathbf{Au}_3$ is expressed in terms of the basis vectors $\{\mathbf{u}_i\}$. In the second equality, the symmetry of $\mathbf{A}$ is used and in the third equality, the relations $\mathbf{Au}_\alpha = \lambda_\alpha\mathbf{u}_\alpha$ $(\alpha = 1, 2)$ and the orthonormality of $\{\mathbf{u}_i\}$ are employed. Finally, if $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$, we can pick any orthonormal basis in $\mathcal{V}$ and in this case $\mathbf{A} = \lambda\mathbf{1}$.
According to the *square root theorem*, for every $\mathbf{A} \in Sym^+$, there exists a unique tensor $\mathbf{G} \in Sym^+$ such that $\mathbf{A} = \mathbf{G}^2$. By the spectral theorem we have a representation (34) for $\mathbf{A}$ with $\lambda_i > 0$ (due to the positive definiteness of $\mathbf{A}$). Define $\mathbf{G} = \sum_{i=1}^{3}\sqrt{\lambda_i}\mathbf{u}_i \otimes \mathbf{u}_i$. Then, $\mathbf{G}^2 = \mathbf{GG} = \sum_{i=1}^{3}\sqrt{\lambda_i}(\mathbf{Gu}_i) \otimes \mathbf{u}_i =$

$\sum_{i=1}^{3}\lambda_i \mathbf{u}_i \otimes \mathbf{u}_i = \mathbf{A}$ and it is obvious that $\mathbf{G}$ is symmetric and positive definite. To prove uniqueness we assume that there exists a symmetric and positive definite tensor $\hat{\mathbf{G}}$ such that $\mathbf{G}^2 = \mathbf{A} = \hat{\mathbf{G}}^2$ and show that $\mathbf{G} = \hat{\mathbf{G}}$. Let $\mathbf{u}$ be an eigenvector of $\mathbf{A}$ with eigenvalue $\lambda > 0$. Then $(\mathbf{G}^2 - \lambda\mathbf{1})\mathbf{u} = \mathbf{0}$ or $(\mathbf{G} + \sqrt{\lambda}\mathbf{1})\mathbf{v} = \mathbf{0}$, where $\mathbf{v} = (\mathbf{G} - \sqrt{\lambda}\mathbf{1})\mathbf{u}$. This requires $\mathbf{v} = \mathbf{0}$ as otherwise $-\sqrt{\lambda}$ becomes an eigenvalue of $\mathbf{G}$, contradicting the positive definiteness of $\mathbf{G}$. Therefore $\mathbf{G}\mathbf{u} = \sqrt{\lambda}\mathbf{u}$ and similarly $\hat{\mathbf{G}}\mathbf{u} = \sqrt{\lambda}\mathbf{u}$. Thus $\mathbf{G}\mathbf{u}_i = \hat{\mathbf{G}}\mathbf{u}_i$ and since an arbitrary vector $\mathbf{f}$ can be expressed as a linear combination of $\{\mathbf{u}_i\}$, we obtain $\mathbf{G}\mathbf{f} = \hat{\mathbf{G}}\mathbf{f}$. This implies $\mathbf{G} = \hat{\mathbf{G}}$.

Polar decomposition theorem Every $\mathbf{F} \in InvLin$ can be uniquely decomposed in terms of tensors $\{\mathbf{U}, \mathbf{V}\} \in Sym^+$ and $\mathbf{R} \in Orth$ such that

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}. \tag{35}$$

The first of these equalities can be proved by using the right Cauchy Green tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$. By the square root theorem there exists a unique symmetric positive definite tensor $\mathbf{U}$ such that $\mathbf{U}^2 = \mathbf{C}$. Define $\mathbf{R} = \mathbf{F}\mathbf{U}^{-1}$. It follows, that $\mathbf{R}^T\mathbf{R} = \mathbf{1}$. If $\det \mathbf{F} > 0$ then $\det \mathbf{R} = 1$ (since $\det \mathbf{F} = \det \mathbf{U} = \sqrt{\det \mathbf{C}}$), and therefore $\mathbf{R}$ is a proper orthogonal tensor. The relation $\mathbf{F} = \mathbf{V}\mathbf{R}$ can be proved similarly via the left Cauchy Green tensor $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. Relation (35) allows us to decompose a deformation into a stretch and a pure rotation. Let unit vectors $\mathbf{M} \in \mathcal{V}_\kappa$ and $\mathbf{m} \in \mathcal{V}_\chi$ and the scalar μ be such that $\mu\mathbf{m} = \mathbf{F}\mathbf{M}$ (from Eq. (28)). Using the decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ and defining $\bar{\mathbf{M}} = \mathbf{U}\mathbf{M}$ we find that $\mu = |\bar{\mathbf{M}}|$ and $\mathbf{m} = \frac{\mathbf{R}\bar{\mathbf{M}}}{|\bar{\mathbf{M}}|}$. Thus, $\mathbf{F}$ stretches (and rotates) $\mathbf{M}$ to $\bar{\mathbf{M}}$, and then rotates $\bar{\mathbf{M}}$ to the direction of $\mathbf{m}$. If μ is a principal value of $\mathbf{U}$ with $\mathbf{M}$ as the corresponding principal vector, then $\mathbf{U}\mathbf{M} = \mu\mathbf{M}$. As a result, $\bar{\mathbf{M}} = \mu\mathbf{M}$, and therefore in such a case, $\mathbf{F}$ stretches $\mathbf{M}$ to $\mu\mathbf{M}$, and then rotates it to $\mu\mathbf{m}$, with $\mathbf{m} = \mathbf{R}\mathbf{M}$. Consider three material curves intersecting at $\mathbf{X} \in \mathcal{E}_\kappa$ with mutually orthogonal tangent vectors $\{\mathbf{M}_i\}$. If $\{\mathbf{M}_i\}$ coincide with the principal vectors of $\mathbf{U}$ at $\mathbf{X}$, then the curves will undergo (locally) a pure stretch, followed by a rigid rotation, with tangent vectors to the deformed curves remaining mutually orthogonal at $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X})$. If on the other hand $\{\mathbf{M}_i\}$ are not the principal vectors of $\mathbf{U}$, then $\mathbf{U}\mathbf{M}_i$ is no longer parallel to $\mathbf{M}_i$ and consequently $\mathbf{U}$ changes the angle between $\{\mathbf{M}_i\}$. The tangent vectors to the deformed curves, therefore, are not orthogonal at $\mathbf{x}$. The decomposition can be understood in the opposite order (rotation followed by a stretch) if we consider the $\mathbf{V}\mathbf{R}$ decomposition instead of the $\mathbf{R}\mathbf{U}$ decomposition of $\mathbf{F}$.

Principal invariants The characteristic equation for $\mathbf{A} \in Lin$ is

$$0 = \det(\mathbf{A} - \lambda\mathbf{1}) = -\lambda^3 + \lambda^2 I_1(\mathbf{A}) - \lambda I_2(\mathbf{A}) + I_3(\mathbf{A}), \tag{36}$$

where

$$\begin{aligned} I_1(\mathbf{A}) &= \operatorname{tr}\mathbf{A} \\ I_2(\mathbf{A}) &= \operatorname{tr}\mathbf{A}^* = \frac{1}{2}[(\operatorname{tr}\mathbf{A})^2 - \operatorname{tr}\mathbf{A}^2] \\ I_3(\mathbf{A}) &= \det\mathbf{A} \end{aligned} \tag{37}$$

are the *principal invariants* of $\mathbf{A}$. A physically meaningful interpretation of these invariants follows by identifying $\mathbf{A}$ with $\mathbf{U} \in Sym^+$ which appears in the polar decomposition (35) of the deformation gradient. In terms of the eigenvalues of $\mathbf{U}$ (denoted by $\lambda_i > 0$), we obtain from (37), $I_1(\mathbf{U}) = \lambda_1 + \lambda_2 + \lambda_3$, $I_2(\mathbf{U}) = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3$ and $I_3(\mathbf{U}) = \lambda_1\lambda_2\lambda_3$. Therefore, if the edges of a unit cube are aligned with the eigenvectors of $\mathbf{U}$, then $I_1(\mathbf{U})$ is the sum of the lengths of three mutually orthogonal edges after deformation, $I_2(\mathbf{U})$ is the sum of the areas of three mutually orthogonal sides after deformation, and $I_3(\mathbf{U})$ is the deformed volume.
According to the *Cayley-Hamilton theorem*, $\mathbf{A}$ satisfies its own characteristic equation, i.e.

$$-\mathbf{A}^3 + I_1(\mathbf{A})\mathbf{A}^2 - I_2(\mathbf{A})\mathbf{A} + I_3(\mathbf{A})\mathbf{1} = \mathbf{0}. \tag{38}$$

We now prove this theorem. Let $\mathbf{D} = ((\mathbf{A} - \lambda\mathbf{1})^*)^T$, where $\lambda \in \mathbb{R}$ is such that $\det(\mathbf{A} - \lambda\mathbf{1}) \neq 0$ but otherwise arbitrary. Since $\mathbf{A} - \lambda\mathbf{1}$ is invertible, we have $\mathbf{D} = \det(\mathbf{A} - \lambda\mathbf{1})(\mathbf{A} - \lambda\mathbf{1})^{-1}$ or $\mathbf{D}(\mathbf{A} - \lambda\mathbf{1}) = \det(\mathbf{A} - \lambda\mathbf{1})\mathbf{1}$. The right hand side of this relation is cubic in λ and the term $\mathbf{A} - \lambda\mathbf{1}$ is linear in λ. Therefore $\mathbf{D}$ has to be quadratic in λ (by a theorem on factorization of polynomials). Let $\mathbf{D} = \mathbf{D}_0 + \mathbf{D}_1\lambda + \mathbf{D}_2\lambda^2$ for some $\mathbf{D}_0$, $\mathbf{D}_1$ and $\mathbf{D}_2$. Then $(\mathbf{D}_0 + \mathbf{D}_1\lambda + \mathbf{D}_2\lambda^2)(\mathbf{A} - \lambda\mathbf{1}) = \det(\mathbf{A} - \lambda\mathbf{1})\mathbf{1} = (-\lambda^3 + \lambda^2 I_1 - \lambda I_2 + I_3)\mathbf{1}$. Matching coefficients of various powers of λ between the first and the last term and eliminating $\mathbf{D}_0$, $\mathbf{D}_1$ and $\mathbf{D}_2$ from these, we get the required relation (38). The coefficients of all the powers of λ have to vanish since otherwise we would obtain a polynomial (of order 3) in λ, which could then be solved to obtain roots for λ, contradicting the premise that $\lambda \in \mathbb{R}$ is arbitrary.

Velocity gradient We can use the chain rule for differentiation to write the gradient of the velocity field with respect to $\mathbf{X}$ as

$$\nabla\hat{\mathbf{v}}(\mathbf{X}, t) = \mathbf{L}\mathbf{F}, \tag{39}$$

where $\mathbf{L} = \operatorname{grad}\tilde{\mathbf{v}} : \mathcal{V}_\chi \to \mathcal{V}_\chi$ is the *spatial velocity gradient.* Under sufficient continuity of the motion we have $\nabla\hat{\mathbf{v}} = \dot{\mathbf{F}}$ and therefore $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$. We can decompose $\mathbf{L}$ into $\mathbf{D} \in Sym$ (*rate of deformation tensor*) and $\mathbf{W} \in Skw$ (*vorticity tensor*). The material time derivative of the right and the left Cauchy-Green tensor can be obtained as

$$\dot{\mathbf{C}} = 2\mathbf{F}^T\mathbf{D}\mathbf{F}, \ \dot{\mathbf{B}} = \mathbf{L}\mathbf{B} + \mathbf{B}\mathbf{L}^T. \tag{40}$$

Indeed, $\dot{\mathbf{C}} = \dot{\mathbf{F}}^T\mathbf{F} + \mathbf{F}^T\dot{\mathbf{F}} = \mathbf{F}^T\mathbf{L}^T\mathbf{F} + \mathbf{F}^T\mathbf{L}\mathbf{F}$ and $\dot{\mathbf{B}} = \dot{\mathbf{F}}\mathbf{F}^T + \mathbf{F}\dot{\mathbf{F}}^T = \mathbf{L}\mathbf{F}\mathbf{F}^T + \mathbf{F}\mathbf{F}^T\mathbf{L}^T$.
For a fixed material curve with unit tangent vector $\mathbf{M}$ recall relation (28), i.e. $\mu\mathbf{m} = \mathbf{F}\mathbf{M}$. As a result

$$\frac{\dot{\mu}}{\mu} = \mathbf{m} \cdot \mathbf{D}\mathbf{m}, \tag{41}$$

where we have used $\dot{\mathbf{M}} = 0$, $\mathbf{m} \cdot \dot{\mathbf{m}} = 0$ (which follows from $\mathbf{m} \cdot \mathbf{m} = 1$) and $\mathbf{m} \cdot \mathbf{W}\mathbf{m} = 0$ (since $\mathbf{m} \cdot \mathbf{W}\mathbf{m} = \mathbf{W}^T\mathbf{m} \cdot \mathbf{m} = -\mathbf{W}\mathbf{m} \cdot \mathbf{m}$). We also obtain $\mu\dot{\mathbf{m}} = \mu\mathbf{L}\mathbf{m} - \dot{\mu}\mathbf{m}$, which on using (41) and the decomposition of $\mathbf{L}$ into symmetric and skew parts, reduces to

$$\dot{\mathbf{m}} = \mathbf{D}\mathbf{m} - (\mathbf{m} \cdot \mathbf{D}\mathbf{m})\mathbf{m} + \mathbf{W}\mathbf{m}. \tag{42}$$

If $\mathbf{m}$ should coincide with a principal vector of $\mathbf{D}$ with principal value γ, then $\mathbf{D}\mathbf{m} = \gamma\mathbf{m}$. The relations (41) and (42) in this case give $\gamma = \frac{\dot{\mu}}{\mu} = (\ln\mu)^{\cdot}$ and $\dot{\mathbf{m}} = \mathbf{W}\mathbf{m}$, respectively. Therefore, when the unit tangent $\mathbf{m}$ to the deformed material curve instantaneously aligns with a principal vector of $\mathbf{D}$, the corresponding principal value of $\mathbf{D}$ is the rate of the natural logarithm of the stretch associated with the material curve. Moreover, the vorticity tensor $\mathbf{W}$ then characterizes the spin of the material element instantaneously aligned with a principal vector.
Associated with $\mathbf{W} \in Skw$ there exists a vector $\mathbf{w} \in \mathcal{V}_\chi$ (the axial vector of $\mathbf{W}$) such that, $\mathbf{W}\mathbf{a} = \mathbf{w} \times \mathbf{a}$ for all $\mathbf{a} \in \mathcal{V}_\chi$. This fact can be proved by first obtaining the canonical form for a skew tensor. The characteristic equation for $\mathbf{W}$ has three roots and therefore at least one of them is real (complex roots occur in a pair). Let this real eigenvalue be λ and let $\mathbf{f} \in \mathcal{V}_\chi$ be the corresponding eigenvector. Then $\mathbf{W}\mathbf{f} = \lambda\mathbf{f}$. But this implies $\lambda = \mathbf{W}\mathbf{f} \cdot \mathbf{f} = 0$ and so $\mathbf{W}\mathbf{f} = 0$. Choose $\{\mathbf{g}, \mathbf{h}\} \in \mathcal{V}_\chi$ such that $\{\mathbf{f}, \mathbf{g}, \mathbf{h}\}$ forms a right handed orthonormal basis for $\mathcal{V}_\chi$. The canonical form for $\mathbf{W}$ is then given by

$$\mathbf{W} = \omega(\mathbf{h} \otimes \mathbf{g} - \mathbf{g} \otimes \mathbf{h}), \tag{43}$$

where $\omega = \mathbf{h} \cdot \mathbf{W}\mathbf{g}$. The canonical form (43) can been proved by remembering that $\mathbf{W}^T = -\mathbf{W}$, $\mathbf{W}\mathbf{f} = \mathbf{0}$ and $\mathbf{a} \cdot \mathbf{W}\mathbf{a} = 0$ for all $\mathbf{a} \in \mathcal{V}_\chi$. Then $\mathbf{W} = \mathbf{W}\mathbf{1} =$

$\mathbf{W}(\mathbf{f} \otimes \mathbf{f} + \mathbf{g} \otimes \mathbf{g} + \mathbf{h} \otimes \mathbf{h}) = \mathbf{W}\mathbf{g} \otimes \mathbf{g} + \mathbf{W}\mathbf{h} \otimes \mathbf{h}$. Note that $\mathbf{W}\mathbf{g} = \omega\mathbf{h}$, since $\mathbf{W}\mathbf{g} \cdot \mathbf{f} = -\mathbf{g} \cdot \mathbf{W}\mathbf{f} = 0$ and $\mathbf{W}\mathbf{g} \cdot \mathbf{g} = 0$. Similarly $\mathbf{W}\mathbf{h} = -\omega\mathbf{g}$. This completes the proof.
Let $\mathbf{w} = \omega\mathbf{f}$. Then on using (43) for arbitrary $\mathbf{a}$, we obtain $\mathbf{W}\mathbf{a} = \omega((\mathbf{g}\cdot\mathbf{a})\mathbf{h} - (\mathbf{h}\cdot\mathbf{a})\mathbf{g}) = \omega((\mathbf{g}\cdot\mathbf{a})(\mathbf{f}\times\mathbf{g}) - (\mathbf{h}\cdot\mathbf{a})(\mathbf{h}\times\mathbf{f})) = \omega\mathbf{f}\times((\mathbf{f}\cdot\mathbf{a})\mathbf{f} + (\mathbf{g}\cdot\mathbf{a})\mathbf{g} + (\mathbf{h}\cdot\mathbf{a})\mathbf{h}) = \mathbf{w} \times \mathbf{a}$.
If $\mathbf{W}$ is the skew part of the spatial velocity gradient, then the axial vector $\mathbf{w}$ is given in terms of the velocity field $\mathbf{v}$ as

$$\mathbf{w} = \frac{1}{2} \operatorname{curl} \tilde{\mathbf{v}}. \tag{44}$$

The vector $\mathbf{w}$ is also called the vorticity vector. This relation can be proved by considering two constant but otherwise arbitrary vectors $\mathbf{g}$ and $\mathbf{h}$. Then $2\mathbf{W}\mathbf{g} \cdot \mathbf{h} = ((\operatorname{grad} \tilde{\mathbf{v}}) - (\operatorname{grad} \tilde{\mathbf{v}})^T)\mathbf{g} \cdot \mathbf{h} = \operatorname{div}((\tilde{\mathbf{v}} \cdot \mathbf{h})\mathbf{g} - (\tilde{\mathbf{v}} \cdot \mathbf{g})\mathbf{h}) = \operatorname{div}(\tilde{\mathbf{v}} \times (\mathbf{g} \times \mathbf{h})) = \operatorname{curl} \tilde{\mathbf{v}} \cdot \mathbf{g} \times \mathbf{h} = (\operatorname{curl} \tilde{\mathbf{v}} \times \mathbf{g}) \cdot \mathbf{h}$. Using the arbitrariness of $\mathbf{h}$ and the relation $\mathbf{W}\mathbf{g} = \mathbf{w} \times \mathbf{g}$ we obtain Eq. (44).
Finally, we interpret the off-diagonal terms of $\mathbf{D}$ on an orthogonal basis. Consider two intersecting material curves with tangent vectors $\mathbf{M}_1$ and $\mathbf{M}_2$ at the point of intersection. In the current configuration, they map to $\mathbf{m}_1$ and $\mathbf{m}_2$ with local stretches μ_1 and μ_2, respectively. Let $\cos\theta = \mathbf{m}_1 \cdot \mathbf{m}_2$. Then, $(\sin\theta)\dot{\theta} = (\mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_1 + \mathbf{m}_2 \cdot \mathbf{D}\mathbf{m}_2)(\mathbf{m}_1 \cdot \mathbf{m}_2) - 2\mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_2$, where relation (42) has been used. If $\sin\theta = 1$ (i.e. $\mathbf{m}_1 \cdot \mathbf{m}_2 = 0$), we have $\dot{\theta} = -2\mathbf{m}_1 \cdot \mathbf{D}\mathbf{m}_2$. Therefore the off-diagonal terms of $\mathbf{D}$ on an orthogonal basis are proportional to the rate of change of the angle between tangents to the deformed material curves instantaneously aligned with the orthogonal elements of the basis.

Homogeneous deformation We call a deformation homogeneous if the associated deformation gradient $\mathbf{F}(\mathbf{X}, t)$ is independent of $\mathbf{X}$. The motion then takes the simple form

$$\mathbf{x} = \boldsymbol{\chi}_\kappa(\mathbf{X}, t) = \mathbf{F}(t)\mathbf{X} + \mathbf{c}(t) \tag{45}$$

with $\mathbf{c} \in \mathcal{V}_\chi$. Thus, for $\mathbf{Y} \in \mathcal{E}_\kappa$,

$$\mathbf{y} = \boldsymbol{\chi}_\kappa(\mathbf{Y}, t) = \mathbf{F}(t)\mathbf{Y} + \mathbf{c}(t) \tag{46}$$

and therefore,

$$\boldsymbol{\chi}_\kappa(\mathbf{Y}, t) = \boldsymbol{\chi}_\kappa(\mathbf{X}, t) + \mathbf{F}(t)(\mathbf{Y} - \mathbf{X}). \tag{47}$$

Comparing this to Eq. (22) we note that every deformation is approximated by a homogeneous deformation in any sufficiently small neighborhood of a material point. Homogeneous deformation is characterized by several properties:

(i) Material planes deform into planes and parallel planes map to parallel planes. A material plane is represented by $\mathbf{X} \cdot \mathbf{N} = D$, where $\mathbf{X} \in \mathcal{V}_\kappa$ represents a vector extending from the origin to points on the plane, $\mathbf{N}$ is the constant normal of the plane, and D is the (constant) perpendicular distance from the origin to the plane. As a result we obtain, using (45), $D = \mathbf{F}^{-1}(\mathbf{x} - \mathbf{c}) \cdot \mathbf{N}$ or $\mathbf{x} \cdot \mathbf{F}^{-T}\mathbf{N} = d$, where $d = D + \mathbf{c} \cdot \mathbf{F}^{-T}\mathbf{N}$ is the perpendicular distance from the origin (in $\mathcal{E}_\chi$) to the deformed plane. The vector $\mathbf{F}^{-T}\mathbf{N}$ is parallel to the deformed normal (see Eq. (24)).
(ii) Straight material lines deform into straight lines and parallel lines map to parallel lines. The equation of a straight material line is given by $\mathbf{X} = S\mathbf{M} + \mathbf{X}_0$ for some $S \in \mathbb{R}^+$, $\mathbf{X}_0 \in \mathcal{V}_\kappa$ (arc-length), where $\mathbf{M}$ is the constant (unit) tangent to the line. Using (45) we can obtain $\mathbf{x} = \mathbf{F}\mathbf{X} + \mathbf{c} = s\mathbf{m} + \mathbf{x}_0$, where $\mathbf{m} = \frac{\mathbf{FM}}{|\mathbf{FM}|}$, $s = S|\mathbf{FM}|$ and $\mathbf{x}_0 = \mathbf{F}\mathbf{X}_0 + \mathbf{c}$. This too, is the equation of a straight line.
(iii) A spherical material surface is mapped to an ellipsoidal surface. Let $\mathbf{p}_0 = \mathbf{Y} - \mathbf{X}$ and $\mathbf{p} = \mathbf{y} - \mathbf{x}$. Thus for a homogeneous deformation, we have, according to (47), $\mathbf{p} = \mathbf{F}\mathbf{p}_0$. A spherical material surface can be represented by $\mathbf{p}_0 \cdot \mathbf{p}_0 = 1$ (with $\mathbf{X}$ as the center and $\mathbf{Y} - \mathbf{X}$ as the radius vector). This can be rewritten as, $\mathbf{F}^{-1}\mathbf{p} \cdot \mathbf{F}^{-1}\mathbf{p} = 1$ or $\mathbf{p} \cdot \mathbf{B}^{-1}\mathbf{p} = 1$. In spectral form, let $\mathbf{B} = \sum_{i=1}^{3} \lambda_i^2 \mathbf{v}_i \otimes \mathbf{v}_i$. Then the relation $\mathbf{p} \cdot \mathbf{B}^{-1}\mathbf{p} = 1$ represents an ellipsoid with axes $\mathbf{v}_i \in \mathcal{V}_\chi$ and semi-axis lengths $\lambda_i \in \mathbb{R}$.

4. Rotation tensors and rigid body motion

Rotations A tensor $\mathbf{Q} : \mathcal{V} \to \mathcal{V}$ is orthogonal if for arbitrary $\{\mathbf{a}, \mathbf{b}\} \in \mathcal{V}$ we have $\mathbf{Qa} \cdot \mathbf{Qb} = \mathbf{a} \cdot \mathbf{b}$. As a result, $\mathbf{Q}^T\mathbf{Q} = \mathbf{1} = \mathbf{Q}\mathbf{Q}^T$ and $|\det \mathbf{Q}| = 1$. If $\det \mathbf{Q} = 1$ then $\mathbf{Q}$ is called *proper orthogonal* or a *rotation*. Note that $\mathbf{Q}^T(\mathbf{Q} - \mathbf{1}) = -(\mathbf{Q} - \mathbf{1})^T$ and therefore if $\mathbf{Q}$ is a rotation, $\det(\mathbf{Q} - \mathbf{1}) = 0$. Thus, there exists a nonzero $\mathbf{f} \in \mathcal{V}$ such that $\mathbf{Qf} = \mathbf{f}$. The vector $\mathbf{f}$ is called the axis of $\mathbf{Q}$ and is unaffected by the action of $\mathbf{Q}$. We can take $\mathbf{f}$ to be a unit vector without any loss of generality. Let $\{\mathbf{f}, \mathbf{g}, \mathbf{h}\} \in \mathcal{V}$ be a right handed orthonormal basis for $\mathcal{V}$. Then $\mathbf{Qg} \cdot \mathbf{f} = \mathbf{Qg} \cdot \mathbf{Qf} = \mathbf{g} \cdot \mathbf{Q}^T\mathbf{Qf} = \mathbf{g} \cdot \mathbf{f} = 0$ and similarly $\mathbf{Qh} \cdot \mathbf{f} = 0$. In addition, $\mathbf{Qg} \cdot \mathbf{Qh} = \mathbf{g} \cdot \mathbf{h} = 0$. Furthermore, $\mathbf{f} \cdot \mathbf{Qg} \times \mathbf{Qh} = \mathbf{Qf} \cdot \mathbf{Qg} \times \mathbf{Qh} = \mathbf{f} \cdot \mathbf{g} \times \mathbf{h}$ (since $\frac{\mathbf{Qf} \cdot \mathbf{Qg} \times \mathbf{Qh}}{\mathbf{f} \cdot \mathbf{g} \times \mathbf{h}} \equiv \det \mathbf{Q} = 1$). Therefore $\{\mathbf{f}, \mathbf{Qg}, \mathbf{Qh}\}$ forms a right handed orthonormal basis in $\mathcal{V}$. As shown above, $\mathbf{Qg}$ is orthogonal to $\mathbf{f}$, and consequently we can write $\mathbf{Qg} = a\mathbf{g} + b\mathbf{h}$ for some $\{a, b\} \in \mathbb{R}$. But $|\mathbf{Qg}| = \sqrt{\mathbf{Qg} \cdot \mathbf{Qg}} = \sqrt{\mathbf{g} \cdot \mathbf{g}} = 1$. Therefore, there exists $\theta \in \mathbb{R}$ such that $a = \cos\theta$ and $b = \sin\theta$. We then obtain $\mathbf{Qh} = \mathbf{f} \times \mathbf{Qg} = -\sin\theta\mathbf{g} + \cos\theta\mathbf{h}$. We write $\mathbf{Q} = \mathbf{Q1} = \mathbf{Q}(\mathbf{f} \otimes \mathbf{f} + \mathbf{g} \otimes \mathbf{g} + \mathbf{h} \otimes \mathbf{h})$, or

$$
\begin{aligned}
\mathbf{Q} &= \mathbf{f}\otimes\mathbf{f} + \mathbf{Q}\mathbf{g}\otimes\mathbf{g} + \mathbf{Q}\mathbf{h}\otimes\mathbf{h} \\
&= \mathbf{f}\otimes\mathbf{f} + \cos\theta(\mathbf{g}\otimes\mathbf{g} + \mathbf{h}\otimes\mathbf{h}) + \sin\theta(\mathbf{h}\otimes\mathbf{g} - \mathbf{g}\otimes\mathbf{h}). \qquad (48)
\end{aligned}
$$

This expression is known as the *Rodrigues' representation formula*. Several interesting facts regarding rotations are now stated:
(i) Every rotation $\mathbf{Q}$ ($\mathbf{Q} \neq \mathbf{1}$) has a unique axis. Let $\mathbf{v} \in \mathcal{V}$ be such that $\mathbf{Q}\mathbf{v} = \mathbf{v}$. Using (48) we obtain $(\mathbf{v}\cdot\mathbf{g}\cos\theta - \mathbf{v}\cdot\mathbf{h}\sin\theta)\mathbf{g} + (\mathbf{v}\cdot\mathbf{g}\sin\theta + \mathbf{v}\cdot\mathbf{h}\cos\theta)\mathbf{h} = (\mathbf{v}\cdot\mathbf{g})\mathbf{g} + (\mathbf{v}\cdot\mathbf{h})\mathbf{h}$, which in turn results into a system of two simultaneous equations,

$$
\begin{pmatrix} \cos\theta - 1 & -\sin\theta \\ \sin\theta & \cos\theta - 1 \end{pmatrix} \begin{pmatrix} \mathbf{v}\cdot\mathbf{g} \\ \mathbf{v}\cdot\mathbf{h} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \qquad (49)
$$

Assume $\{\mathbf{v}\cdot\mathbf{g}, \mathbf{v}\cdot\mathbf{h}\} \neq \{0, 0\}$. This requires $0 = (\cos\theta - 1)^2 + \sin^2\theta = 2(1 - \cos\theta)$ or $\cos\theta = 1$. In such a case (48) reduces to $\mathbf{Q} = \mathbf{f}\otimes\mathbf{f} + \mathbf{g}\otimes\mathbf{g} + \mathbf{h}\otimes\mathbf{h} = \mathbf{1}$. Thus for $\mathbf{Q} \neq \mathbf{1}$ we require $\{\mathbf{v}\cdot\mathbf{g}, \mathbf{v}\cdot\mathbf{h}\} = \{0, 0\}$ and thus $\mathbf{v} = (\mathbf{v}\cdot\mathbf{f})\mathbf{f}$, i.e. $\mathbf{v}$ is parallel to $\mathbf{f}$.
(ii) There exists $\mathbf{W} \in Skw$ such that its axial vector coincides with the axis of $\mathbf{Q}$ and moreover

$$
\mathbf{Q} = \mathbf{1} + \sin\theta\mathbf{W} + (1 - \cos\theta)\mathbf{W}^2. \qquad (50)
$$

To see this, let $\mathbf{W} = \mathbf{h}\otimes\mathbf{g} - \mathbf{g}\otimes\mathbf{h}$. Then according to (43), $\mathbf{W}$ is skew with $\omega = \mathbf{W}\mathbf{g}\cdot\mathbf{h} = 1$ and axial vector $\mathbf{w} = \omega\mathbf{f} = \mathbf{f}$, which is also the axis of $\mathbf{Q}$. Consider $\mathbf{W}^2 = \mathbf{W}\mathbf{h}\otimes\mathbf{g} - \mathbf{W}\mathbf{g}\otimes\mathbf{h} = -(\mathbf{g}\otimes\mathbf{g} + \mathbf{h}\otimes\mathbf{h})$ and thus $\mathbf{f}\otimes\mathbf{f} = \mathbf{1} + \mathbf{W}^2$. On substituting these relations in (48) we get the required formula (50).
(iii) Every rotation $\mathbf{Q}$ is expressible as $\mathbf{Q} = \exp(\theta\mathbf{W})$, with $\mathbf{W} \in Skw$ as defined in (ii) above and $\theta \in \mathbb{R}$. The exponential of a tensor is defined by the power series $\exp\mathbf{W} = \sum\limits_{n=0}^{\infty} \frac{1}{n!}\mathbf{W}^n$. Using this definition we can expand $\exp(\theta\mathbf{W})$,

$$
\exp(\theta\mathbf{W}) = \sum_{n=0}^{\infty} \frac{\theta^n}{n!}\mathbf{W}^n = \mathbf{1} + \sum_{m=1}^{\infty} \frac{\theta^{2m}}{2m!}\mathbf{W}^{2m} + \sum_{m=0}^{\infty} \frac{\theta^{2m+1}}{(2m+1)!}\mathbf{W}^{2m+1}. \qquad (51)
$$

Note that $\mathbf{W}^{2m} = (-1)^{m+1}\mathbf{W}^2$ and $\mathbf{W}^{2m+1} = (-1)^m\mathbf{W}$ ($m = 1, 2, 3, ..$). Both of these claims can be proved using induction. The relation (51) then takes the form

$$
\exp(\theta\mathbf{W}) = \mathbf{1} + \Big(\sum_{m=0}^{\infty} \frac{(-1)^m\theta^{2m+1}}{(2m+1)!}\Big)\mathbf{W} + \Big(\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\theta^{2m}}{2m!}\Big)\mathbf{W}^2. \qquad (52)
$$

Recalling that $\sin\theta = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!}\theta^{2m+1}$ and $(1-\cos\theta) = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{2m!}\theta^{2m}$, and substituting these in (52) we obtain an expression for $\exp(\theta\mathbf{W})$ which coincides with the right hand side of (50).
(*iv*) For a fixed $\mathbf{W} \in Skw$, the rotation $\mathbf{Q}(\theta) = \exp(\theta\mathbf{W})$ uniquely solves the following initial value problem:

$$\mathbf{Q}'(\theta) = \mathbf{W}\mathbf{Q}(\theta), \quad \mathbf{Q}(0) = \mathbf{1}. \tag{53}$$

Using definition (51) for $\exp(\theta\mathbf{W})$, we can obtain $\mathbf{Q}'(\theta) = \sum_{n=1}^{\infty} \frac{\theta^{n-1}}{(n-1)!}\mathbf{W}^n = \mathbf{W}\sum_{n=0}^{\infty} \frac{\theta^n}{(n)!}\mathbf{W}^n$ and therefore $\mathbf{Q}'(\theta) = \mathbf{W}\mathbf{Q}$. It is easy to see that $\mathbf{Q}(0) = \mathbf{1}$. Thus, the rotation $\mathbf{Q}$ satisfies (53). The uniqueness of the solution for (53) results from the theory of ordinary differential equations (Coddington, E. A. & Levinson, N. *Theory of Ordinary Differential Equations*, Krieger (1984)).

Rigid body motion The motion of a body $\mathfrak{B}$ is *rigid* if the distance between any pair of material points remains invariant. For arbitrary $\{\mathbf{X}, \mathbf{Y}\} \in \mathcal{E}_\kappa$ we then have

$$|\boldsymbol{\chi}_\kappa(\mathbf{Y}, t) - \boldsymbol{\chi}_\kappa(\mathbf{X}, t)| = |\mathbf{Y} - \mathbf{X}|. \tag{54}$$

We now show that the rigid body motion is homogeneous and the associated deformation gradient is a proper orthogonal tensor. Fix $\mathbf{X}$ and then differentiate both sides of (54) with respect to $\mathbf{Y}$. We obtain $\mathbf{F}^T(\mathbf{Y}, t)(\boldsymbol{\chi}_\kappa(\mathbf{Y}, t) - \boldsymbol{\chi}_\kappa(\mathbf{X}, t)) = \mathbf{Y} - \mathbf{X}$. Now fix $\mathbf{Y}$ and differentiate this relation with respect to $\mathbf{X}$. We obtain $\mathbf{F}^T(\mathbf{Y}, t)\mathbf{F}(\mathbf{X}, t) = \mathbf{1}$ for all $\{\mathbf{X}, \mathbf{Y}\} \in \mathcal{E}_\kappa$. Set $\mathbf{X} = \mathbf{Y}$ to obtain $\mathbf{F}^T(\mathbf{X}, t)\mathbf{F}(\mathbf{X}, t) = \mathbf{1}$. Since $\det\mathbf{F} > 0$, it follows that $\det\mathbf{F} = 1$. We have thus proved that $\mathbf{F}$ is proper orthogonal. Furthermore, $\mathbf{F}(\mathbf{X}, t) = \mathbf{F}^{-T}(\mathbf{Y}, t) = \mathbf{F}(\mathbf{Y}, t)$, where the second equality is a consequence of the orthogonality of $\mathbf{F}$. Since $\mathbf{X}$ and $\mathbf{Y}$ are arbitrary, we conclude that $\mathbf{F}$ is homogeneous. Denote $\mathbf{F}(\mathbf{X}, t) = \mathbf{Q}(t)$, where $\mathbf{Q} \in Orth^+$. Equation (45) then takes the form

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{X} + \mathbf{c}(t), \tag{55}$$

where $\mathbf{x} = \boldsymbol{\chi}_\kappa(\mathbf{X}, t)$. The spatial velocity gradient in this case is given by $\mathbf{L}(t) = \dot{\mathbf{Q}}\mathbf{Q}^T$, which is skew. Therefore the rate of deformation tensor, which is the symmetric part of $\mathbf{L}$, vanishes i.e. $\mathbf{D} = \mathbf{0}$ and the vorticity tensor is $\mathbf{W}(t) = \dot{\mathbf{Q}}\mathbf{Q}^T$. The relation $\text{grad}\,\tilde{\mathbf{v}} = \mathbf{W}(t)$ can then be integrated to get

$$\tilde{\mathbf{v}}(\mathbf{x}, t) = \mathbf{W}(t)\mathbf{x} + \mathbf{d}(t) \tag{56}$$

for some $\mathbf{d} \in \mathcal{V}_\chi$. This follows directly from (55).

We have shown that the vanishing of the rate of deformation tensor, i.e. $\mathbf{D} = \mathbf{0}$, is a necessary condition for rigid motion. The proof for sufficiency is now given. An equivalent condition for rigid body motion is obtained by taking the material time derivative of (54), yielding

$$(\hat{\mathbf{v}}(\mathbf{Y}, t) - \hat{\mathbf{v}}(\mathbf{X}, t)) \cdot (\boldsymbol{\chi}_\kappa(\mathbf{Y}, t) - \boldsymbol{\chi}_\kappa(\mathbf{X}, t)) = 0. \tag{57}$$

We will obtain (57) by assuming $\mathbf{L}$ to be skew and thus prove our assertion. Let $\mathbf{y} = \boldsymbol{\chi}_\kappa(\mathbf{Y}, t)$ and $\mathbf{x} = \boldsymbol{\chi}_\kappa(\mathbf{X}, t)$ be two points in some open neighborhood of $\mathcal{E}_\chi$. The equation of a straight line $L \subset \mathcal{E}_\chi$ connecting $\mathbf{x}$ and $\mathbf{y}$ is given by $\mathbf{z}(u) = \mathbf{x} + u(\mathbf{y} - \mathbf{x})$, where $0 \leq u \leq 1$. Then,

$$\hat{\mathbf{v}}(\mathbf{Y}, t) - \hat{\mathbf{v}}(\mathbf{X}, t) = \int_{\boldsymbol{\kappa} \circ \boldsymbol{\chi}^{-1}(L)} (\nabla \hat{\mathbf{v}}) d\mathbf{Z} = \int_{\boldsymbol{\kappa} \circ \boldsymbol{\chi}^{-1}(L)} \mathbf{L}\mathbf{F} d\mathbf{Z} = \int_L \mathbf{L}(\mathbf{z}) d\mathbf{z} \tag{58}$$

or

$$\hat{\mathbf{v}}(\mathbf{Y}, t) - \hat{\mathbf{v}}(\mathbf{X}, t) = \int_0^1 \mathbf{L}(\mathbf{z}(u)) \mathbf{z}'(u) du = \int_0^1 \mathbf{L}(\mathbf{z}(u))(\mathbf{y} - \mathbf{x}) du \tag{59}$$

and therefore

$$(\hat{\mathbf{v}}(\mathbf{Y}, t) - \hat{\mathbf{v}}(\mathbf{X}, t)) \cdot (\boldsymbol{\chi}_\kappa(\mathbf{Y}, t) - \boldsymbol{\chi}_\kappa(\mathbf{X}, t)) = \int_0^1 (\mathbf{y} - \mathbf{x}) \cdot \mathbf{L}(\mathbf{y} - \mathbf{x}) d\mathbf{u}, \tag{60}$$

which vanishes because $\mathbf{L}$ is skew. This completes the proof.

5. Singular surfaces

By a singular surface, we refer to a surface in the body across which jump discontinuities are allowed for various fields (and their derivatives) which otherwise are continuous in the body. The jump of a field (say Ψ) across a singular surface is denoted by

$$[\![\Psi]\!] = \Psi^+ - \Psi^-, \tag{61}$$

where Ψ^+ and Ψ^- are the limit values of Ψ as it approaches the singular surface from either side. The '+' side is taken to be the one into which the normal to the surface points. Let Φ be another piecewise continuous field. The following relation, which can be verified by direct substitution using (61), will find much use in our later developments

$$[\![\Phi\Psi]\!] = [\![\Phi]\!]\langle\Psi\rangle + \langle\Phi\rangle[\![\Psi]\!], \tag{62}$$

where

$$\langle\Psi\rangle = \frac{\Psi^+ + \Psi^-}{2}. \tag{63}$$

A two dimensional surface which evolves in time is given by

$$S_t = \{\mathbf{X} \in \boldsymbol{\kappa}(\mathfrak{B}) : \phi(\mathbf{X}, t) = 0\}, \tag{64}$$

where $\phi : \boldsymbol{\kappa}(\mathfrak{B}) \times \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function. The referential normal to the surface and the referential normal velocity are defined by

$$\begin{aligned} \mathbf{N}(\mathbf{X}, t) &= \frac{\nabla \phi}{|\nabla \phi|} \quad \text{and} \\ U(\mathbf{X}, t) &= -\frac{\dot{\phi}}{|\nabla \phi|} \ , \end{aligned} \tag{65}$$

respectively. The second of these definitions is motivated towards the end of this section. An immediate consequence of these definitions is

$$\dot{\mathbf{N}} = -(\mathbf{1} - \mathbf{N} \otimes \mathbf{N})\nabla U - U(\nabla \mathbf{N})\mathbf{N}. \tag{66}$$

Indeed, we have from $(65)_1$

$$\dot{\mathbf{N}} = \frac{\nabla \dot{\phi}}{|\nabla \phi|} - \frac{\nabla \phi}{|\nabla \phi|^2}\left(\frac{\nabla \phi}{|\nabla \phi|} \cdot \nabla \dot{\phi}\right) = \frac{\nabla \dot{\phi}}{|\nabla \phi|}(\mathbf{1} - \mathbf{N} \otimes \mathbf{N}) \quad \text{and} \tag{67}$$

$$\nabla \mathbf{N} = \frac{\nabla^2 \phi}{|\nabla \phi|} - \frac{\nabla \phi}{|\nabla \phi|^2} \otimes \left(\nabla^2 \phi \frac{\nabla \phi}{|\nabla \phi|}\right) = \frac{\nabla^2 \phi}{|\nabla \phi|} - \mathbf{N} \otimes \frac{(\nabla^2 \phi)\mathbf{N}}{|\nabla \phi|}. \tag{68}$$

On the other hand, $(65)_2$ yields

$$\nabla U = -\frac{\nabla \dot{\phi}}{|\nabla \phi|} + \frac{\dot{\phi}}{|\nabla \phi|^2}\left(\nabla^2 \phi \frac{\nabla \phi}{|\nabla \phi|}\right) = -\frac{\nabla \dot{\phi}}{|\nabla \phi|} - U\frac{(\nabla^2 \phi)\mathbf{N}}{|\nabla \phi|}, \tag{69}$$

where $\nabla^2 \phi = \nabla(\nabla \phi) \in Sym$. Combining these relations we obtain (66). The tensor $\mathbf{1} - \mathbf{N} \otimes \mathbf{N}$ is the orthogonal projection onto $\mathcal{V}_\kappa$ and is denoted by $\mathbb{P}$. It is easy to check that $\mathbb{P}^T = \mathbb{P}$ and $\mathbb{P}\mathbb{P} = \mathbb{P}$.

Derivatives We now define surface derivatives for scalar, vector and tensor valued functions which are defined on the surface S_t. Let $\mathfrak{f}$ denote a scalar, vector or tensor valued function on S_t. The function $\mathfrak{f}$ is differentiable at $\mathbf{X} \in S_t$ if $\mathfrak{f}$ has an extension f to a neighborhood N of $\mathbf{X}$, which is differentiable at $\mathbf{X}$ in the classical sense and is equal to $\mathfrak{f}$ for $\mathbf{X} \in S_t$. The *surface gradient* of $\mathfrak{f}$ at $\mathbf{X} \in S_t$ is then defined by

$$\nabla^S \mathfrak{f}(\mathbf{X}) = \nabla f(\mathbf{X})\mathbb{P}(\mathbf{X}). \tag{70}$$

Let $\mathrm{v} : S_t \to \mathcal{V}$ and $\mathrm{A} : S_t \to Lin$ be respectively, vector and tensor valued functions on the surface S_t. We define the *surface divergence* as a scalar field $\mathrm{Div}^S \mathrm{v}$ and a vector field $\mathrm{Div}^S \mathrm{A}$ by

$$\begin{aligned} \mathrm{Div}^S \mathrm{v} &= \mathrm{tr}(\nabla^S \mathrm{v}) \\ \mathbf{c} \cdot \mathrm{Div}^S \mathrm{A} &= \mathrm{Div}^S(\mathrm{A}^T \mathbf{c}) \end{aligned} \tag{71}$$

for any fixed $\mathbf{c} \in \mathcal{V}$. Moreover, we call v *tangential* if $\mathbb{P}\mathrm{v} = \mathrm{v}$ and A *superficial* if $\mathrm{A}\mathbb{P} = \mathrm{A}$. We define the *curvature tensor* L by (the normal $\mathbf{N}$ and its extension to a neighborhood of S_t are both denoted by the same symbol)

$$\mathrm{L} = -\nabla^S \mathbf{N}, \tag{72}$$

or $\mathrm{L} = -\nabla\mathbf{N}(\mathbf{1} - \mathbf{N} \otimes \mathbf{N})$. Therefore

$$\mathrm{tr}\,\mathrm{L} = -\,\mathrm{Div}\,\mathbf{N} + (\nabla\mathbf{N})\mathbf{N} \cdot \mathbf{N} = -\,\mathrm{Div}\,\mathbf{N}, \tag{73}$$

where we have used $(\nabla\mathbf{N})^T\mathbf{N} = \mathbf{0}$, which follows from $\mathbf{N} \cdot \mathbf{N} = 1$. Since $\nabla^S\mathbf{N}\mathbb{P} = \nabla\mathbf{N}\mathbb{P}\mathbb{P} = \nabla\mathbf{N}\mathbb{P} = \nabla^S\mathbf{N}$, the curvature tensor is superficial. Furthermore, using (68) we have

$$\begin{aligned} \mathrm{L} &= -\nabla\mathbf{N}(\mathbf{1} - \mathbf{N} \otimes \mathbf{N}) \\ &= -\left(\frac{\nabla^2\phi}{|\nabla\phi|} - \mathbf{N} \otimes \frac{(\nabla^2\phi)\mathbf{N}}{|\nabla\phi|}\right)(\mathbf{1} - \mathbf{N} \otimes \mathbf{N}) \\ &= \frac{-1}{|\nabla\phi|}\{\nabla^2\phi - \mathbf{N} \otimes (\nabla^2\phi)\mathbf{N} - (\nabla^2\phi)\mathbf{N} \otimes \mathbf{N} + \big((\nabla^2\phi)\mathbf{N} \cdot \mathbf{N}\big)\,\mathbf{N} \otimes \mathbf{N}\} \end{aligned}$$

and consequently we infer that $\mathrm{L} = \mathrm{L}^T$ and $\mathrm{L}\mathbf{N} = \mathbf{0}$. Therefore, $\mathbf{N}$ is a principal direction of L with the corresponding principal value being zero. Since L is symmetric, the spectral theorem implies that it has three real eigenvalues with mutually orthogonal eigenvectors. We have already obtained zero as an eigenvalue (with $\mathbf{N}$ as the eigenvector). Let the other eigenvalues be κ_1 and κ_2, whose corresponding eigenvectors lie in the plane normal to $\mathbf{N}$. The mean and the Gaussian curvature associated with the surface are then defined as

$$H = \frac{1}{2}(\kappa_1 + \kappa_2), \quad \text{and } K = \kappa_1\kappa_2, \tag{74}$$

respectively.

A function $\boldsymbol{\varphi} : (t-\varepsilon, t+\varepsilon) \to \mathcal{E}_\kappa$, $\varepsilon > 0$, is said to be a *normal curve* through $\mathbf{X} \in S_t$ at time t if for each $\tau \in (t-\varepsilon, t+\varepsilon)$,

$$\boldsymbol{\varphi}'(\tau) = U(\boldsymbol{\varphi}(\tau), \tau)\mathbf{N}(\boldsymbol{\varphi}(\tau), \tau). \tag{75}$$

The function $\boldsymbol{\varphi}(\tau)$ is therefore the position parameterized by τ. We define the *normal time derivative* of a function on S_t by

$$\mathring{\mathrm{v}}(\mathbf{X}, t) = \frac{d\mathrm{v}(\boldsymbol{\varphi}(\tau), \tau)}{d\tau}\Big|_{\tau=t}. \tag{76}$$

The relation (66) can therefore be written as $\mathring{\mathbf{N}} = -\nabla^S U$.

Remark: We will assume that an extension of a surface field to a neighborhood of the surface exists, and will abuse the notation to use the same symbol for the field and its extension.

Compatibility conditions Central to the discussion on the kinematics of singular surfaces are the compatibility conditions which relate the deformation gradient and the velocity field across the singular surface. Consider a closed material curve $C \subset \mathcal{E}_\kappa$ such that it intersects the singular surface S_t at two points, say $\mathbf{p}_1$ and $\mathbf{p}_2$. Let A_C be the area bounded by C and let $\Gamma = A_C \cap S_t$ be the line of intersection of this area with the singular surface. We parameterize Γ by arc-length u such that the curve Γ extends from $\mathbf{p}_2$ to $\mathbf{p}_1$.
In general we can write

$$\mathbf{b} = \oint_C \mathbf{F} d\mathbf{X}, \tag{77}$$

where a non-zero $\mathbf{b} \in \mathcal{E}_\chi$ arises when $\mathbf{F}$ is incompatible (we assume for now that $\mathbf{F}$ is not expressible as a gradient). In dislocation theory $\mathbf{b}$ is referred to as the Burgers vector associated with the closed curve C. The integration in the above relation is well defined since we assume $\mathbf{F}$ to be singular only over a set of zero Lebesgue measure (a finite number of points on a continuous line constitute such a set). According to Stokes' theorem with a singular surface,

$$\mathbf{b} = \oint_C \mathbf{F} d\mathbf{X} = \int_{A_C} (\mathrm{Curl}\, \mathbf{F})^T \mathbf{N}_A dA - \int_\Gamma [\![\mathbf{F}]\!] d\mathbf{X}, \tag{78}$$

where $\mathbf{N}_A$ is the unit normal associated with the area A_C. A proof of this theorem is given in the following chapter on balance laws. We discuss two consequences of the above relation:
(*i*) Let $\mathbf{b} = \mathbf{0}$. Then, $\mathrm{Curl}\, \mathbf{F} = \mathbf{0}$ in $\boldsymbol{\kappa}(\mathfrak{B}) \setminus S_t$. We can show this by choosing a C such that $\Gamma = \emptyset$. The arbitrariness of A_C (and thus of $\mathbf{N}_A$) and the localization theorem for surface integrals (see the following chapter) then imply $\mathrm{Curl}\, \mathbf{F} = \mathbf{0}$ for all $\mathbf{X} \in \boldsymbol{\kappa}(\mathfrak{B}) \setminus S_t$. Equation (78) now reduces to

$$\mathbf{0} = \int_\Gamma [\![\mathbf{F}]\!] d\mathbf{X}. \tag{79}$$

Use the parametrization of Γ to write $d\mathbf{X} = \mathbf{s}du$, where $\mathbf{s} \in T_{S_t(\mathbf{X})}$ is a unit vector in the tangent plane $T_{S_t(\mathbf{X})}$ to S_t at $\mathbf{X}$. The curve C can be arbitrarily chosen and therefore Γ is arbitrary. Use the arbitrariness of Γ to localize (79), and obtain

$$[\![\mathbf{F}]\!]\mathbf{s} = 0 \tag{80}$$

for all $\mathbf{s} \in T_{S_t(\mathbf{X})}$. Thus, there exists a vector $\mathbf{k} \in \mathcal{V}_\chi$ such that

$$[\![\mathbf{F}]\!] = \mathbf{k} \otimes \mathbf{N} \tag{81}$$

on S_t, which is *Hadamard's compatibility condition* for the deformation gradient.

(*ii*) Let $\mathrm{Curl}\,\mathbf{F} = \mathbf{0}$ in $\boldsymbol{\kappa}(\mathfrak{B}) \setminus S_t$. Therefore there exists a vector field $\boldsymbol{\chi}_\kappa$ such that $\mathbf{F} = \nabla\boldsymbol{\chi}_\kappa$ away from S_t. Note that $\boldsymbol{\chi}_\kappa$ might still suffer a jump across S_t. Let $C^+ \cup C^- = C$, where C^+ and C^- are two disjoint parts of C which lie on the '+' and '−' side of S_t, respectively. The '+' side is the one into which the normal $\mathbf{N}$ points. Therefore,

$$\begin{aligned}\oint_C \mathbf{F}d\mathbf{X} &= \int_{C^+} \mathbf{F}d\mathbf{X} + \int_{C^-} \mathbf{F}d\mathbf{X}\\ &= \boldsymbol{\chi}_2^+ - \boldsymbol{\chi}_1^+ + \boldsymbol{\chi}_1^- - \boldsymbol{\chi}_2^-\\ &= [\![\boldsymbol{\chi}_\kappa]\!]_2 - [\![\boldsymbol{\chi}_\kappa]\!]_1 = -\int_\Gamma [\![\boldsymbol{\chi}_\kappa]\!]'(u)du,\end{aligned} \tag{82}$$

where $\boldsymbol{\chi}_2^+ = \boldsymbol{\chi}_\kappa^+(\mathbf{p}_2)$ etc. The negative sign in the last term above arises due to the orientation of Γ, which extends from $\mathbf{p}_2$ to $\mathbf{p}_1$. On the other hand we have in this case, from (78),

$$\oint_C \mathbf{F}d\mathbf{X} = -\int_\Gamma [\![\mathbf{F}]\!]d\mathbf{X}. \tag{83}$$

Since $[\![\boldsymbol{\chi}_\kappa]\!]'(u) = \nabla[\![\boldsymbol{\chi}_\kappa]\!]\mathbf{s} = \nabla^S[\![\boldsymbol{\chi}_\kappa]\!]\mathbf{s}$ (as $\mathbb{P}\mathbf{s} = \mathbf{s}$), we obtain, on comparing Eqs. (82) and (83) and using the arbitrariness of Γ

$$[\![\mathbf{F}]\!]\mathbf{s} = \nabla^S[\![\boldsymbol{\chi}_\kappa]\!]\mathbf{s} \tag{84}$$

for all $\mathbf{s} \in T_{S_t(\mathbf{X})}$. Thus, there exists a vector $\mathbf{k} \in \mathcal{E}_\chi$ such that

$$[\![\mathbf{F}]\!] = \mathbf{k} \otimes \mathbf{N} + \nabla^S[\![\boldsymbol{\chi}_\kappa]\!] \tag{85}$$

on S_t, which is the modified compatibility condition for the deformation gradient in the case when $\boldsymbol{\chi}_\kappa$ suffers a jump on the singular surface. If $[\![\boldsymbol{\chi}_\kappa]\!] =$ constant then Eq. (85) reduces to Hadamard's compatibility condition (81).

To obtain the compatibility condition for the velocity field, we apply the definition of the normal time derivative (cf. (75), (76)) on fields $\boldsymbol{\chi}_\kappa^+$ and $\boldsymbol{\chi}_\kappa^-$. We obtain

$$(\boldsymbol{\chi}_\kappa^+)^\circ = \frac{d\boldsymbol{\chi}_\kappa(\boldsymbol{\varphi}(\tau),\tau)}{d\tau}\Big|_{\tau=t^+} = U\mathbf{F}^+\mathbf{N} + \mathbf{v}^+ \tag{86}$$

and

$$(\boldsymbol{\chi}_\kappa^-)^\circ = \frac{d\boldsymbol{\chi}_\kappa(\boldsymbol{\varphi}(\tau),\tau)}{d\tau}\Big|_{\tau=t^-} = U\mathbf{F}^-\mathbf{N} + \mathbf{v}^-, \tag{87}$$

where $\tau \in (t, t+\varepsilon)$ in (86) and $\tau \in (t-\varepsilon, t)$ in (87). Subtracting these relations we get the compatibility condition for the velocity field,

$$[\![\mathbf{v}]\!] + U[\![\mathbf{F}]\!]\mathbf{N} = [\![\boldsymbol{\chi}_\kappa]\!]^\circ. \tag{88}$$

For $[\![\boldsymbol{\chi}_\kappa]\!] = const.$ (including the case when $\boldsymbol{\chi}_\kappa$ is continuous, i.e. $[\![\boldsymbol{\chi}_\kappa]\!] = \mathbf{0}$) this condition reduces to

$$[\![\mathbf{v}]\!] + U[\![\mathbf{F}]\!]\mathbf{N} = \mathbf{0} \tag{89}$$

or equivalently

$$U[\![\mathbf{F}]\!] = -[\![\mathbf{v}]\!] \otimes \mathbf{N}. \tag{90}$$

Surface deformation gradient For a continuous motion across the surface S_t, we have $[\![\boldsymbol{\chi}_\kappa(\mathbf{X},t)]\!] = \mathbf{0}$ for $\mathbf{X} \in S_t$, and in this case we can define the *surface deformation gradient* F and the *surface normal velocity* v by

$$\mathrm{F} = \nabla^S \boldsymbol{\chi}_\kappa, \;\; \mathrm{v} = \mathring{\boldsymbol{\chi}}_\kappa. \tag{91}$$

It is then easy to check that

$$\mathrm{F} = \mathbf{F}^\pm \mathbb{P}, \;\; \mathrm{v} = \hat{\mathbf{v}}^\pm + U\mathbf{F}^\pm\mathbf{N}, \tag{92}$$

where on the right hand side above, $\pm$ indicates that either $+$ or $-$ can be used, a fact which can be verified using the compatibility conditions (81) and (89).

The tensor F as defined above satisfies $\det \mathrm{F} = 0$ and $\mathrm{F}^* \neq \mathbf{0}$. That $\det \mathrm{F} = 0$ can be verified using $(92)_1$ and $\det \mathbb{P} = 0$. The cofactor F^* of F is defined by $\mathrm{F}^*(\mathbf{a}\times\mathbf{b}) = \mathrm{F}\mathbf{a} \times \mathrm{F}\mathbf{b}$ for arbitrary vectors $\{\mathbf{a},\mathbf{b}\} \in \mathcal{V}_\kappa$. Let $\{\mathbf{t}_1,\mathbf{t}_2\} \in T_{S_t(\mathbf{X})}$ be two unit vectors in the tangent plane to S_t at $\mathbf{X} \in S_t$, such that $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{N}\}$ form an orthonormal basis at $\mathbf{X}$. Then,

$$\begin{aligned} \mathrm{F}^*\mathbf{N} = \mathrm{F}^*(\mathbf{t}_1 \times \mathbf{t}_2) &= \mathrm{F}\mathbf{t}_1 \times \mathrm{F}\mathbf{t}_2 \\ &= \mathbf{F}^\pm\mathbf{t}_1 \times \mathbf{F}^\pm\mathbf{t}_2 \\ &= (\mathbf{F}^\pm)^*\mathbf{N}, \end{aligned} \tag{93}$$

where in the second equality, relation $(92)_1$ has been used. Furthermore, it is straightforward to check that $\mathrm{F}^*\mathbf{t}_\alpha = \mathbf{0}$ $(\alpha = 1, 2)$, since $\mathrm{F}\mathbf{N} = \mathbf{0}$. Therefore F^* remains non-zero as long as $(\mathbf{F}^\pm)^*\mathbf{N}$ does not vanish. Note that $|\mathrm{F}^*\mathbf{N}|$ is equal to the ratio of the infinitesimal areas (on the singular surface) in the current and the reference configuration. This follows immediately from Eqs. (24) and (93).

Surface parametrization Consider $\mathbf{X} \in \mathcal{E}_\kappa$ in a neighborhood of S_t. We can then find a point $\hat{\mathbf{X}} \in S_t$ such that

$$\mathbf{X} = \hat{\mathbf{X}} + \zeta\mathbf{N}, \tag{94}$$

where $\zeta(t) \in \mathbb{R}$ is a scalar. We parameterize the surface S_t by using a local coordinate system (ξ_1, ξ_2), where $\{\xi_1, \xi_2\} \in \mathbb{R}$. In terms of the new variables, $\mathbf{X} = \mathbf{X}(\xi_1, \xi_2, \zeta)$, $\hat{\mathbf{X}} = \hat{\mathbf{X}}(\xi_1, \xi_2, t)$ and $\mathbf{N} = \hat{\mathbf{N}}(\xi_1, \xi_2, t)$. Let $\hat{\mathbf{X}}_{,\alpha} = \boldsymbol{\xi}_\alpha$ for $\alpha = 1, 2$. We assume that the parametrization is such that the triad $\{\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \mathbf{N}\}$ forms an orthonormal basis at $\hat{\mathbf{X}}$. In a sufficiently small neighborhood of $\mathbf{X} = \mathbf{X}(\xi_1, \xi_2, \zeta)$ it is possible to invert this to obtain $\boldsymbol{\xi} \equiv (\xi_1, \xi_2, \zeta) = \boldsymbol{\xi}(\mathbf{X})$. Use (94) to obtain the differential of $\mathbf{X}$,

$$d\mathbf{X} = (\boldsymbol{\xi}_\alpha + \zeta\mathbf{N}_{,\alpha})d\xi_\alpha + \mathbf{N}d\zeta. \tag{95}$$

If we identify with $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, the principal directions of L (recall that $\mathbf{N}$ is the third principal direction, cf. (72) and the paragraph preceding (74)), we have $\mathbf{N}_{,\alpha} = -\kappa_\alpha\boldsymbol{\xi}_\alpha$ (no summation implied over α), where κ_α are the principal curvatures associated with the surface. Therefore, if $\mathbf{A}$ is the gradient of the map taking $\boldsymbol{\xi}$ to $\mathbf{X}$ then $d\mathbf{X} = \mathbf{A}d\boldsymbol{\xi}$ and it follows from (95) that

$$\mathbf{A} = \xi_{11}(1 - \kappa_1\zeta)\boldsymbol{\xi}_1 \otimes \boldsymbol{\xi}_1 + \xi_{22}(1 - \kappa_2\zeta)\boldsymbol{\xi}_2 \otimes \boldsymbol{\xi}_2 + \mathbf{N} \otimes \mathbf{N}, \tag{96}$$

where $\xi_{\alpha\alpha} = \boldsymbol{\xi}_\alpha \cdot \boldsymbol{\xi}_\alpha$ (no summation). Let $\xi = \xi_{11}\xi_{22}$. Thus

$$j_A \equiv \det \mathbf{A} = \xi(1 - 2\zeta H + \zeta^2 K), \tag{97}$$

where H and K are defined in (74).
Taking the differential of the function ϕ, $d\phi = \nabla\phi \cdot d\mathbf{X} + \dot{\phi}dt$, and substituting in it the expression for $d\mathbf{X}$ from (95) for a point near the surface, we obtain

$$d\phi = \nabla\phi \cdot ((\boldsymbol{\xi}_\alpha + \zeta\mathbf{N}_{,\alpha})d\xi_\alpha + \mathbf{N}d\zeta) + \dot{\phi}dt. \tag{98}$$

On the surface, we have $\zeta = 0$ and $d\phi = 0$. Consequently we obtain

$$\begin{aligned} 0 &= \nabla\phi \cdot \boldsymbol{\xi}_\alpha d\xi_\alpha + \nabla\phi \cdot \mathbf{N}d\zeta + \dot{\phi}dt \\ &= \nabla\phi \cdot \boldsymbol{\xi}_\alpha d\xi_\alpha + \nabla\phi \cdot \mathbf{N}\dot{\zeta}dt + \dot{\phi}dt \end{aligned} \tag{99}$$

on S_t, and noting the independence of $d\xi_\alpha$ and dt, we recover relations (65) along with the identification of $\dot{\zeta}$ with U.

Singular surface in the current configuration The image of the singular surface S_t in the current configuration is give by

$$\begin{aligned} s_t = \boldsymbol{\chi}_\kappa(S_t, t) &= \{\mathbf{x} \in \boldsymbol{\chi}(\mathfrak{B}) : \psi(\mathbf{x}, t) = 0\}, \quad \text{where} \\ & \quad \psi(\boldsymbol{\chi}_\kappa(\mathbf{X}, t), t) = \phi(\mathbf{X}, t). \end{aligned} \tag{100}$$

The scalar function $\psi : \boldsymbol{\chi}(\mathfrak{B}) \times \mathbb{R} \to \mathbb{R}$ is assumed to be continuously differentiable with respect to its arguments. The normal to the surface s_t and the spatial normal velocity are defined by (cf. (65))

$$\begin{aligned} \mathbf{n} &= \frac{\operatorname{grad}\psi}{|\operatorname{grad}\psi|}, \quad \text{and} \\ u &= -\frac{1}{|\operatorname{grad}\psi|}\frac{\partial\psi}{\partial t}, \end{aligned} \tag{101}$$

respectively, where $\frac{\partial\psi}{\partial t}$ is the spatial time derivative of ψ at a fixed $\mathbf{x}$. Differentiate $(100)_2$ to obtain

$$\begin{aligned} \nabla\phi &= (\mathbf{F}^{\pm})^T \operatorname{grad}\psi, \quad \text{and} \\ \dot{\phi} &= \operatorname{grad}\psi \cdot \mathbf{v}^{\pm} + \frac{\partial\psi}{\partial t}. \end{aligned} \tag{102}$$

The following relations can then be obtained on combining (65), (101), and (102):

$$\begin{aligned} \mathbf{n} &= \frac{(\mathbf{F}^{\pm})^{-T}\mathbf{N}}{|(\mathbf{F}^{\pm})^{-T}\mathbf{N}|}, \quad \text{and} \\ u &= \mathbf{n} \cdot \mathbf{v}^{\pm} + \frac{U}{|(\mathbf{F}^{\pm})^{-T}\mathbf{N}|}. \end{aligned} \tag{103}$$

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