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Configurational balance laws for incompatibility in stress space

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Conservation laws have been recently obtained by requiring that a positive definite functional of the stress gradient (the Euler–Lagrange equations of which are the Beltrami–Michell compatibility conditions) be invariant under certain transformations. Here these laws are extended to include body forces, thermal stresses and Kröner's incompatibility tensor as source terms in the configurational balance laws, which allows for the incompatibility in the volume to be measured from surface data. An example is presented.

Keywords: Noether's theorem; configurational balance laws; incompatibility; thermal stresses; Beltrami–Michell conditions

1. Introduction

In a recent paper, Li *et al.* (2005) obtained a class of conservation laws as a consequence of symmetries of a positive definite functional of the stress gradient in a variational principle of linear and isotropic elasticity. Such a variational principle appeared in a stress-based formulation of three-dimensional elasticity proposed by Pobedrja (1980). The equations of compatibility in terms of stresses (Beltrami–Michell) are obtained as the Euler–Lagrange equations of this variational principle. Here, we consider the case when the right-hand side of compatibility equation (1.2) is non-zero due to the incompatibility tensor (Kröner 1958, 1981) and/or the gradient of body forces, and we modify the conservation laws to include them as source terms. These balance laws are identities, in the sense that they are satisfied identically if the equations of force equilibrium and compatibility are assumed to hold pointwise inside the domain under consideration. They prove to be of considerable importance if written in an integral form since the information regarding the volume content of the incompatibility and/or the body force gradient can be obtained from the value of the field variables (and their derivatives) on the surface. Possible applications would include, for example, the incompatibility arising from a continuous distribution of dislocations or thermal stress fields, which is a topic of current interest (e.g. Bako & Groma 2005; Fujimo *et al.* 2005). In the rest of this section,

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we briefly discuss the stress formulation of the traction boundary value problem of three-dimensional linear isotropic elasticity in the absence of body forces and incompatibility.

The well-known Beltrami–Michell boundary value problem with vanishing body forces is given in terms of the symmetric Cauchy stress $\sigma_{ij}(x_k)$ by Gurtin (1972) (in Cartesian coordinates with indices ranging from 1 to 3),

$$\sigma_{ij,j} = 0, \quad \forall\, x_k \in \Omega, \tag{1.1}$$

$$\sigma_{ij,kk} + b\sigma_{kk,ij} = 0, \quad \forall\, x_k \in \Omega, \tag{1.2}$$

$$\sigma_{ij} n_j = p_i, \quad \forall\, x_k \in \partial\Omega, \tag{1.3}$$

where the Einstein's summation rule is used for repeated indices. The constant b is given in terms of the Poisson's ratio ν as $b=1/(1+\nu)$. The traction force on the boundary $\partial\Omega$ of the domain $\Omega \subset \mathbb{E}^3$ with an outward normal n_i is denoted by p_i.

Pobedrja (1980) presented an alternative formulation which renders the traction problem to be a well-posed boundary value problem of three-dimensional elasticity (see also Pobedrja & Holmatov 1982); the differential operator in the boundary value problem is self-adjoint and satisfies the Fredholm property (Kucher *et al.* 2004). For our purposes, we will only deal with the case of static, isotropic and linear elasticity and we make a special choice of the free parameters of Pobedrja so that the Poisson's ratio appears as the only independent material parameter (Li *et al.* 2005). For a compatible strain field with vanishing body forces, Pobedrja's boundary value problem can be stated as

$$\sigma_{ij,kk} + b\sigma_{kk,ij} + b\sigma_{mn,mn}\delta_{ij} + a[\sigma_{ik,kj} + \sigma_{jk,ki}] = 0, \quad \forall\, x_k \in \Omega, \tag{1.4}$$

$$\sigma_{ij} n_j = p_i, \quad \forall\, x_k \in \partial\Omega, \tag{1.5}$$

$$\sigma_{ij,j} = 0, \quad \forall\, x_k \in \partial\Omega, \tag{1.6}$$

where the constant a is given by $a=(1-\nu)/\nu(1+\nu)$. Note that equilibrium is imposed only on the boundary. The term $\sigma_{ij,j}$ can be calculated on $\partial\Omega$ by obtaining its value at points in Ω which approaches $\partial\Omega$. Pobedrja (1980) has shown that it is sufficient to satisfy the equilibrium condition over the boundary to ensure that equilibrium is satisfied in the domain. Kucher *et al.* (2004) have obtained the necessary and sufficient conditions for this formulation to be equivalent to the stress solution of the classical Navier's formulation in linear elasticity *almost everywhere* except on the boundary. Pobedrja & Radzhabov (1989) obtained a Green's function for Pobedrja's boundary value problem which has been extended to the case of anisotropy in Pobedrja (1994).

2. The Beltrami–Michell equations from a variational principle and associated conservation laws

The appendix contains a brief outline of Noether's theorem generalized to be applied to a positive definite *tensor-valued* functional. As a consequence of Noether's theorem (Noether 1918), linearly independent combinations of the

Lagrange expressions (defined as the left-hand side of Euler–Lagrange equations) become divergences (equation (A 13)). The resulting equations (A 13) are here applied to Pobedrja's functional (as defined in equation (2.3)) and thereupon conservation laws are obtained as a consequence of various symmetries of this functional by Noether's theorem.

We will restrict attention to the functional which has an associated Lagrangian dependent only on the stress gradient, so that for a Lagrangian of the form $L_\Omega(\sigma_{ij,k})$ with x_i and σ_{ij} being the independent and the dependent variables, respectively, equation (A 13) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\left(L_\Omega \varphi_{k\alpha} + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}(\xi_{ij\alpha} - \sigma_{ij,\ell}\varphi_{\ell\alpha})\right) = \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)(\xi_{ij\alpha} - \sigma_{ij,\ell}\varphi_{\ell\alpha}). \tag{2.1}$$

Here, the Lagrange expressions (denoted by Ψ_{ij}) involve only the second term in equation (A 10)

$$\Psi_{ij} := \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right). \tag{2.2}$$

In the above, φ_{ij} and ξ_{ijk} represent the infinitesimal generators corresponding to x_i and σ_{ij}, respectively (equations (A 5) and (A 6) in appendix A).

The variational principle for Pobedrja's stress formalism involves the following integral (Pobedrja 1980),

$$\begin{aligned} \Pi(\sigma_{ij}, \sigma_{ij,k}) = & \int_\Omega L_\Omega(\sigma_{ij,k})\mathrm{d}\Omega - \int_{\partial\Omega} \chi_{ij}\sigma_{ij}\mathrm{d}S \\ & + \int_{\partial\Omega}\left[\frac{1}{2}(\sigma_{ij,j}\sigma_{ik,k} + \sigma_{ij}n_j\sigma_{ik}n_k) - p_i\sigma_{ij}n_j\right]\mathrm{d}S, \end{aligned} \tag{2.3}$$

where $\chi_{ij} = (\partial L_\Omega / \partial \sigma_{ij,k})n_k$. The coordinate x_i denotes the position in the domain. The Lagrangian L_Ω is given as

$$L_\Omega = \left\{\frac{1}{2}\sigma_{ij,k}\sigma_{ij,k} + b\sigma_{kk,i}\sigma_{ij,j} + \frac{a}{2}(\sigma_{ik,k}\sigma_{ij,j} + \sigma_{jk,k}\sigma_{ji,i})\right\}. \tag{2.4}$$

The resulting Euler–Lagrange equations obtained as a consequence of the variation of the functional (2.3) with respect to stress (equations (A 10)–(A 12)) coincide with equations (1.4)–(1.6) and are indeed the Beltrami–Michell compatibility equations of three-dimensional elasticity. The Lagrange expressions Ψ_{ij} are the left-hand side of equation (1.4) and vanish in the absence of an incompatibility and body force gradients. If we substitute $\Psi_{ij} = 0$ in equation (2.1), we obtain a class of conservation laws from the corresponding symmetries of the problem (eqns (4.4)–(4.14) of Li *et al.* (2005)),

$$C_{k\alpha,k} = 0, \tag{2.5}$$

where

$$C_{k\alpha} \equiv \left(L_\Omega \varphi_{k\alpha} + \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}(\xi_{ij\alpha} - \sigma_{ij,\ell}\varphi_{\ell\alpha})\right). \tag{2.6}$$

In the presence of an incompatibility and/or gradient of body forces, the Beltrami–Michell equations have a non-zero right-hand side and therefore the Lagrange expressions Ψ_{ij} do not vanish. In §3, we evaluate them by using classical relations of incompatibility and equilibrium. A non-zero Ψ_{ij} will subsequently contribute as a source term in the right-hand side of equation (2.1).

3. The Euler–Lagrange equations in the presence of incompatibility and body forces

The Lagrange expressions obtained using equations (2.2) and (2.4) are given as

$$\Psi_{ij} \equiv \sigma_{ij,kk} + b\sigma_{kk,ij} + b\sigma_{mn,mn}\delta_{ij} + a(\sigma_{ik,kj} + \sigma_{jk,ki}). \tag{3.1}$$

These consist of two parts: (i) $\sigma_{ij,kk}+b\sigma_{kk,ij}$ which can be recognized from the Beltrami–Michell compatibility conditions and (ii) $\sigma_{ik,kj}$ which is the gradient of the divergence of the stress tensor. Therefore, the Lagrange expressions contain information about both compatibility and equilibrium. If the strain field is compatible (i.e. $\sigma_{ij,kk}+b\sigma_{kk,ij}=0$) and there are uniform body forces (which implies $\sigma_{ik,ki}=0$), then the Lagrange expressions vanish identically. In the following, we wish to express them in terms of Kröner's incompatibility tensor η_{ij} and the body force f_i using the relations of force equilibrium and compatibility of strain.

The force equilibrium equation in the presence of body forces is

$$\sigma_{ij,j} + f_i = 0, \tag{3.2}$$

and the strain compatibility equation in presence of the incompatibility tensor (Kröner 1958, 1981; Teodosiu 1982) is

$$-\epsilon_{ikl}\epsilon_{jmn}e_{ln,km} = \eta_{ij}, \tag{3.3}$$

where ϵ_{ikl} is the alternating tensor and e_{ij} the infinitesimal strain tensor. Recall that for the case of linear and isotropic elasticity, the constitutive relation can be written as

$$e_{kl} = \frac{1}{2\mu}\left(\sigma_{kl} - \frac{\nu}{1+\nu}\sigma_{mm}\delta_{kl}\right), \tag{3.4}$$

where μ is the shear modulus. Rewriting equation (3.3) we obtain

$$e_{kl,mm} - e_{lm,km} - e_{km,lm} + (e_{mp,mp} - e_{mm,pp})\delta_{kl} + e_{mm,kl} = \eta_{kl}, \tag{3.5}$$

and using equations (3.2) and (3.4) we get

$$\sigma_{kl,mm} + b(\sigma_{mm,kl} - \delta_{kl}\sigma_{mm,nn}) - f_{m,m}\delta_{kl} + (f_{k,l} + f_{l,k}) = 2\mu\eta_{kl}, \tag{3.6}$$

where upon contracting the free indices we obtain

$$-b\delta_{kl}\sigma_{mm,nn} = \frac{1}{1-\nu}(f_{m,m} + 2\mu\eta_{mm})\delta_{kl}, \tag{3.7}$$

and therefore we rewrite equation (3.6) as

$$\sigma_{kl,mm} + b\sigma_{mm,kl} = -\frac{\nu}{1-\nu} f_{m,m}\delta_{kl} - f_{k,l} - f_{l,k} - \frac{2\mu}{1-\nu}\eta_{mm}\delta_{kl} + 2\mu\eta_{kl}, \tag{3.8}$$

which is the modified Beltrami–Michell relation in the presence of an incompatible strain field and the gradient of body forces. We now use this relation with the gradient of the equilibrium equation (3.2) to write the modified Euler–Lagrange equation as

$$\sigma_{ij,kk} + b\sigma_{kk,ij} + b\sigma_{mn,mn}\delta_{ij} + a[\sigma_{ik,kj} + \sigma_{jk,ki}]$$
$$= -\frac{2\mu}{1-\nu}\eta_{mm}\delta_{ij} + 2\mu\eta_{ij} - \frac{1+\nu^2}{1-\nu^2} f_{m,m}\delta_{ij} - \frac{1+\nu^2}{\nu+\nu^2}(f_{i,j} + f_{j,i}), \tag{3.9}$$

where the Lagrange expressions Ψ_{ij}, instead of vanishing, are now evaluated by the right-hand side of equation (3.9). The above relation extends equation (1.4) to include body forces and the incompatibility tensor. The body forces appear only in the form of gradients, and therefore, a uniform body force distribution would not contribute to the right-hand side of the above equation.

4. Balance laws for incompatibility

It is evident from equation (2.1) that in the case of non-vanishing Lagrange expressions, we do not have conservation laws in a strict sense of vanishing divergences. Therefore, we call them *balance laws* instead of conservation laws. The Lagrange expressions then act as source terms for the flux, which is represented by the divergence term. We obtain various balance laws corresponding to the symmetries of the variational principle. The symmetry transformations involving the independent variable x_i give rise to the configurational balance equations, since the transformations induce a change in the configuration of the material space.

The various balance laws obtained from Pobedrja's formulation are different from the existing ones (Maugin 1993; Gurtin 1999; Kienzler & Herrmann 2000). The earlier configurational laws were obtained as a result of the change in the internal energy functional with respect to a transformation of the material configuration. The corresponding conserved quantities were therefore interpreted as *configurational forces*. On the other hand, the integral involved in the Pobedrja's variational principle has a physical meaning not of energy but of a positive definite expression involving quadratic terms in the stress gradient (Pobedrja & Holmatov 1982). The order of the differential operator in the obtained laws is higher than the energy-based laws, and it can be checked easily that they cannot be obtained merely by differentiating previously known laws. If we consider Eshelby's (1975) energy momentum tensor, the expression for the Eshelby tensor involves quadratic terms in the stress. In contrast, the corresponding quantity here is $\tilde{X}_{k\alpha}$ (see equation (4.4)) which is obtained by using the translational invariance of Pobedrja's variational principle and involves terms quadratic in the stress gradient. Therefore, it is not possible to simply differentiate the quadratic terms in the stress to obtain the quadratic terms in its gradient without introducing other higher order derivatives of the stress.

In the following, we obtain various balance laws from the symmetries expressing translation, rotational, scaling and pre-stress invariance of the problem. The general form of these balance laws follows from equation (2.1):

$$C_{k\alpha,k} = \Psi_{ij}(\xi_{ij\alpha} - \sigma_{ij,\ell}\varphi_{\ell\alpha}), \tag{4.1}$$

where the quantities $C_{k\alpha}$ are as given in equation (2.6) and the Lagrange expressions Ψ_{ij} are evaluated from the right-hand side of relation (3.9). We denote the source term $\Psi_{ij}(\xi_{ij\alpha} - \sigma_{ij,\ell}\varphi_{\ell\alpha})$ above by F_α. The source F_α can be further decomposed into two parts, with contributions involving the incompatibility tensor and the body force gradients, respectively. Such a decomposition is possible as a result of equation (3.9) and we further write

$$F_\alpha = F_\alpha^i + F_\alpha^b, \tag{4.2}$$

where F_α^i is the source term containing the incompatibility and F_α^b, the gradient of the body forces (superscripts on F do not represent an index).

We now obtain balance laws for the aforementioned invariances. First, we obtain the balance law due to translational invariance. The infinitesimal generators are $\varphi_{i\alpha}=\delta_{i\alpha}$ and $\xi_{ij\alpha}=0$. We then write the corresponding balance law using equations (2.6), (3.9) and (4.1) as

$$\tilde{X}_{k\alpha,k} = F_\alpha^{ti} + F_\alpha^{tb}, \tag{4.3}$$

where (relation (4.28) from Li *et al.* (2005)),

$$\tilde{X}_{k\alpha} = \frac{1}{2}\sigma_{\ell m,n}\sigma_{\ell m,n}\delta_{k\alpha} - \sigma_{ij,\alpha}\sigma_{ij,k} - b\sigma_{ik,\alpha}\sigma_{qq,i}, \tag{4.4}$$

and

$$F_\alpha^{ti} = \left(\frac{2\mu}{1-\nu}\eta_{mm}\delta_{kl} - 2\mu\eta_{kl}\right)\sigma_{kl,\alpha},$$

$$F_\alpha^{tb} = \left(\frac{1+\nu^2}{1-\nu^2}f_{m,m}\delta_{kl} + \frac{1+\nu^2}{\nu+\nu^2}(f_{k,l}+f_{l,k})\right)\sigma_{kl,\alpha}. \tag{4.5}$$

Balance law (4.3) is a consequence of the translational symmetry of the integral given in equation (2.3).

For rotational symmetry, the infinitesimal generators are $\varphi_{i\alpha}=\epsilon_{ji\alpha}x_j$ and $\epsilon_{ij\alpha}=\epsilon_{ki\alpha}\sigma_{kj}+\epsilon_{lj\alpha}\sigma_{il}$ (eqns (4.36) and (4.37) in Li *et al.* (2005)). Using equations (2.6), (3.9) and (4.1), we write the balance law as

$$\tilde{R}_{k\alpha,k} = F_\alpha^{ri} + F_\alpha^{rb}, \tag{4.6}$$

where (relation (4.5) in Li *et al.* (2005))

$$\tilde{R}_{k\alpha} = \frac{1}{2}\sigma_{\ell m,n}\sigma_{\ell m,n}\epsilon_{pk\alpha}x_p + (\epsilon_{\ell i\alpha}\sigma_{\ell j} + \epsilon_{\ell j\alpha}\sigma_{i\ell} - \sigma_{ij,\ell}\epsilon_{m\ell\alpha}x_m)\left(\sigma_{ij,k} + \frac{b}{2}(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})\right),$$

and

$$F_\alpha^{ri} = \left(-\frac{2\mu}{1-\nu}\eta_{mm}\delta_{ij} + 2\mu\eta_{ij}\right)(\epsilon_{ki\alpha}\sigma_{kj} + \epsilon_{lj\alpha}\sigma_{il} - \sigma_{ij,l}\epsilon_{kl\alpha}x_k),$$

$$F_\alpha^{rb} = \left(-\frac{1+\nu^2}{1-\nu^2}f_{m,m}\delta_{kl} - \frac{1+\nu^2}{\nu+\nu^2}(f_{k,l} + f_{l,k})\right)(\epsilon_{ki\alpha}\sigma_{kj} + \epsilon_{lj\alpha}\sigma_{il} - \sigma_{ij,l}\epsilon_{kl\alpha}x_k).$$

Another symmetry of the problem is obtained by an appropriate scaling. The infinitesimal generators for the scaling symmetry are $\varphi_{i\alpha}=x_i$ and $\xi_{ij\alpha}=-(1/2)\sigma_{ij}$ with $\alpha=1$ (refer eqns (4.44) and (4.45) in Li *et al.* (2005)). Therefore, the balance law is given by (using equations (2.6), (3.9) and (4.1))

$$\tilde{S}_{k,k} = F^{si} + F^{sb}, \tag{4.7}$$

where (relation (4.6) in Li *et al.* (2005))

$$\tilde{S}_k = -\sigma_{\ell m,n}\sigma_{\ell m,n}x_k + (2\sigma_{ij,\ell}x_\ell + \sigma_{ij})\left(\sigma_{ij,k} + \frac{b}{2}(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})\right),$$

and

$$F^{si} = \left(-\frac{2\mu}{1-\nu}\eta_{mm}\delta_{kl} + 2\mu\eta_{kl}\right)\left(\frac{1}{2}\sigma_{kl} + \sigma_{kl,m}x_m\right),$$

$$F^{sb} = \left(-\frac{1+\nu^2}{1-\nu^2}f_{m,m}\delta_{kl} - \frac{1+\nu^2}{\nu+\nu^2}(f_{k,l} + f_{l,k})\right)\left(\frac{1}{2}\sigma_{kl} + \sigma_{kl,m}x_m\right).$$

Finally, we consider a transformation by incrementing the stress by a constant tensor which leaves the stress gradient invariant. The corresponding infinitesimal generators are $\varphi_{i\alpha}=0$ and $\xi_{ij\alpha}=c_{ij}$ where $\alpha=1$ and c_{ij} is an arbitrary constant symmetric matrix. We obtain the balance law as (using equations (2.6), (3.9) and (4.1))

$$\tilde{P}_{k,k} = F^{pi} + F^{pb}, \tag{4.8}$$

where (relation (4.7) in Li *et al.* (2005))

$$\tilde{P}_k = \left(\sigma_{ij,k} + \frac{b}{2}(\delta_{jk}\sigma_{qq,i} + \delta_{ik}\sigma_{qq,j})\right)c_{ij},$$

and

$$F^{pi} = \left(-\frac{2\mu}{1-\nu}\eta_{mm}\delta_{kl} + 2\mu\eta_{kl}\right)c_{kl},$$

$$F^{pb} = \left(-\frac{1+\nu^2}{1-\nu^2}f_{m,m}\delta_{kl} - \frac{1+\nu^2}{\nu+\nu^2}(f_{k,l}+f_{l,k})\right)c_{kl}.$$

This law has no correspondence in classical conservation laws since it is obtained as a result of the tensorial extension of Noether's theorem. If c_{ij} is not a constant tensor, then equation (4.8) is modified to

$$\tilde{P}_{k,k} - \frac{\partial L_\Omega}{\partial \sigma_{ij,k}}c_{ij,k} = F^{pi} + F^{pb}, \tag{4.9}$$

the validity of which can be verified by a direct substitution. Equation (4.9) can also be expressed in an integral form in the region of analyticity.

The invariance of integral (2.3) for the above considered symmetries has been proved in Li *et al.* (2005). The surface term in integral (2.3) remains invariant under the above transformations. This can be verified using the following condition which should be satisfied identically by the infinitesimal generators for checking invariance of the surface integral (eqn (3.44) from Li *et al.* (2005))

$$\left(\frac{\partial L_\Gamma}{\partial x_i}\varphi_{i\alpha} + \chi_{ij}\xi_{ij\alpha}\right) + L_\Gamma\left(\frac{\mathrm{d}\varphi_{i\alpha}}{\mathrm{d}x_i} - \frac{\partial\varphi_{i\alpha}}{\partial x_j}n_i n_j\right) = 0, \tag{4.10}$$

where

$$L_\Gamma = -\chi_{ij}\sigma_{ij} - \left[p_i\sigma_{ij}n_j - \frac{1}{2}(\sigma_{ij,j}\sigma_{ik,k} + \sigma_{ij}n_j\sigma_{ik}n_k)\right]. \tag{4.11}$$

The balance laws presented in this section can also be expressed in an integral form. Take any arbitrary simply connected domain $\omega \subset \Omega$ with a smooth boundary $\partial\omega$. The domain ω is assumed to contain no singularities and the integrands are assumed to be analytic in the domain. We then integrate equation (4.1) over ω and use the divergence theorem to obtain

$$\int_{\partial\omega} C_{k\alpha}m_k\mathrm{d}S = \int_\omega (F^i_\alpha + F^b_\alpha)\mathrm{d}\omega, \tag{4.12}$$

where m_k is the outward unit normal to $\partial\omega$. We specialize this identity to specific symmetries and obtain corresponding integral laws. It is clear from the integral equation (4.12) that, in the case of vanishing source terms, we obtain a surface-independent integral. Such a surface-independent integral represents the change in the integral Π which is a positive definite functional of the stress gradient given by equation (2.3), with respect to the corresponding transformation. For example, if we consider the change in Π due to the transformation in the dependent variable σ_{ij} by a constant increment, then we have

$$\frac{\delta\Pi}{\delta\sigma_{ij}}c_{ij} = \int_{\partial\omega}\tilde{P}_k m_k\mathrm{d}S, \tag{4.13}$$

which may assist our understanding of the physical nature of the integral Π depending on the interpretation of the surface term.

In the presence of source terms, equation (4.12) gives us a means to evaluate the volume integrals that quantify the strength of the incompatibility or the body force gradients in the domain from the values of the field variables on the surface. In §5, we illustrate this point by demonstrating some applications in the theory of continuous distribution of dislocations and theory of thermal stresses. We also present an example of heat flow in a domain containing a spherical cavity, which does not naturally contain singularities at the origin.

5. Applications

(a) Continuous distribution of dislocations

Assume the domain to contain a continuous distribution of dislocations (Kröner 1958, 1981; Teodosiu 1982). Let α_{ij} denote the dislocation density; the Kröner incompatibility η_{ij} is related to the dislocation density as (Kröner 1958, 1981)

$$\eta_{ij} = \frac{1}{2}(\epsilon_{ikl}\alpha_{lj,k} + \epsilon_{jkl}\alpha_{li,k}). \tag{5.1}$$

Therefore, $\eta_{ii}=\epsilon_{ikl}\alpha_{li,k}$. We now discuss some results obtained using the integral form of balance law (4.8). The domain of integration is assumed to be free of singularities so that the functions involved will be analytic. In the absence of body forces, we write

$$\int_{\partial\omega} \tilde{P}_k \mathrm{d}S_k - \int_{\omega} F^{pi} \mathrm{d}\omega = 0, \tag{5.2}$$

where we have used the divergence theorem and the notation $\mathrm{d}S_k = m_k \mathrm{d}S$. Substituting the expressions for $\tilde{P}_k$ and F^{pi} we obtain

$$\int_{\partial\omega} (\sigma_{ij,k} + b\delta_{jk}\sigma_{qq,i}) c_{ij} \mathrm{d}S_k + \int_{\omega} \left(\frac{2\mu}{1-\nu}\eta_{mm}\delta_{ij} - 2\mu\eta_{ij} \right) c_{ij} \mathrm{d}\Omega = 0. \tag{5.3}$$

A measure of the incompatibility can be obtained from this relation. Let

$$\hat{\eta}_{ij} = -\frac{2\mu}{1-\nu}\eta_{mm}\delta_{ij} + 2\mu\eta_{ij}, \tag{5.4}$$

which is a symmetric tensor and, therefore, can be diagonalized. By choosing an appropriate c_{ij}, one can identify all the diagonal components separately. For example, let e_{ak}, $a=1, 2, 3$ be the three normalized eigenvectors of $\hat{\eta}_{ij}$. Then $c_{ij}=e_{1i}e_{1j}$ will separate out the first diagonal component of $\hat{\eta}_{ij}$. Therefore, we determine the volume integral of $\hat{\eta}_{ij}$ by calculating the stress gradient on the surface.

We also express these relations in terms of the dislocation density. Equation (5.3) can be rewritten using (5.1) in the form

$$\int_{\partial\omega} (\sigma_{ij,k} + b\delta_{jk}\sigma_{qq,i}) c_{ij} \mathrm{d}S_k + \int_{\omega} \left(\frac{2\mu}{1-\nu}\epsilon_{mkl}\alpha_{lm,k}\delta_{ij} - 2\mu\epsilon_{ikl}\alpha_{lj,k} \right) c_{ij} \mathrm{d}\Omega = 0, \tag{5.5}$$

where we have used the symmetry of c_{ij}. The above relation, therefore, provides a way to obtain a volumetric measure of the gradient of the dislocation density, from the values of the stress gradients on the surface enclosing the domain of interest. By close measurements of the stress at points near the surface, the gradient may be experimentally measured with some degree of accuracy. The above illustration can be repeated for other quantities obtained in §5. A proper application of these results lies, of course, in the hands of the experimentalist.

(b) A thermoelastic example

We consider a linear thermoelastic problem for a simply connected domain with zero body forces and a specified temperature distribution T. The stress formulation for such a problem is given by Boley & Weiner (1997),

$$\sigma_{ij,j} = 0, \quad \forall\, x_k \in \Omega, \tag{5.6}$$

$$\sigma_{ij,kk} + b\sigma_{kk,ij} = -2\mu\alpha\left(\frac{1+\nu}{1-\nu}\, T_{,kk}\delta_{ij} - T_{,ij}\right), \quad \forall\, x_k \in \Omega, \tag{5.7}$$

$$\sigma_{ij} n_j = p_i, \quad \forall\, x_k \in \partial\Omega, \tag{5.8}$$

where α is the coefficient of linear thermal expansion. We would illustrate an application by interpreting the temperature field as a source for the incompatibility. We begin by denoting

$$\hat{\eta}_{ij} = -2\mu\alpha\left(\frac{1+\nu}{1-\nu}\, T_{,kk}\delta_{ij} + T_{,ij}\right). \tag{5.9}$$

Therefore, integral equation (5.3) becomes

$$\int_{\partial\omega} (\sigma_{ij,k} + b\delta_{jk}\sigma_{qq,i}) c_{ij} \mathrm{d}S_k = \int_{\omega} \hat{\eta}_{ij} c_{ij} \mathrm{d}\Omega, \tag{5.10}$$

where $\hat{\eta}_{ij}$ is given by equation (5.9) above. If c_{ij} is not a constant tensor then the above integral equation is modified using equation (4.9) to

$$\int_{\partial\omega} (\sigma_{ij,k} + b\delta_{jk}\sigma_{qq,i}) c_{ij} \mathrm{d}S_k - \int_{\omega} (\sigma_{ij,k} + b\delta_{ij}\sigma_{mm,k}) c_{ij,k} \mathrm{d}\Omega = \int_{\omega} \hat{\eta}_{ij} c_{ij} \mathrm{d}\Omega. \tag{5.11}$$

The above equation gives a measure by which to detect the second-order gradient of the temperature field inside the domain by measuring the first-order stress gradient on the boundary enclosing the domain. We would next present an example to illustrate this point. Consider the classical problem of a spherical cavity in an uniform heat flow (Florence & Goodier 1959). We employ spherical coordinates (r, θ, ϕ) $(\theta \in (0, \pi)$ and $(\phi \in (0, 2\pi))$ for our discussion. The harmonic function representing the temperature field with a uniform temperature gradient τ is given by

$$T(r, \theta) = \tau\left(r + \frac{R^3}{2r^2}\right)\cos\theta, \tag{5.12}$$

where R is the radius of the cavity. The corresponding stress field with a traction-free cavity is (non-zero components only)

$$\sigma_r = -2s\left(\frac{R^2}{r^2} - \frac{R^4}{r^4}\right)\cos\theta, \tag{5.13}$$

$$\sigma_\theta = \sigma_\phi = -s\left(\frac{R^2}{r^2} + \frac{R^4}{r^4}\right)\cos\theta, \tag{5.14}$$

$$\sigma_{r\theta} = -s\left(\frac{R^2}{r^2} - \frac{R^4}{r^4}\right)\sin\theta, \tag{5.15}$$

where

$$s = \frac{\mu}{2}\left(\frac{1+\nu}{1-\nu}\right)\alpha\tau R. \tag{5.16}$$

We choose $c_{ij} = e_{ri}e_{\theta j} + e_{\theta i}e_{rj}$, where $(e_{ri}, e_{\theta i}, e_{\phi i})$ form the triad of the orthogonal unit vectors in the spherical coordinate system. We will need to verify the validity of relation (5.11) as an identity for the above stress field. The temperature field as given by equation (5.12) implies $T_{,kk} = 0$ and

$$T_{,ij} = \frac{3}{2}\tau\frac{R^3}{r^4}\cos\theta(2e_{ri}e_{rj} + \tan\theta(e_{ri}e_{\theta j} + e_{\theta i}e_{rj}) - e_{\theta i}e_{\theta j} - e_{\phi i}e_{\phi j}). \tag{5.17}$$

Relation (5.9) can be used to obtain

$$\hat{\eta}_{ij}(e_{ri}e_{\theta j} + e_{\theta i}e_{rj}) = -6\mu\alpha\tau\frac{R^3}{r^4}\sin\theta, \tag{5.18}$$

which can then be used to evaluate

$$\int_\omega \hat{\eta}_{ij}c_{ij}\mathrm{d}\Omega = \int_0^{2\pi}\int_0^{\pi}\int_R^r \hat{\eta}_{ij}c_{ij}r^2\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\phi = -6\pi^2\mu\tau\alpha R^3\left(\frac{1}{R} - \frac{1}{r}\right), \tag{5.19}$$

which gives the value for the right-hand side integral of equation (5.11). The integrand for the left-hand side integral can be obtained by using the stress components given above and noting that the normal to the surface is $\pm e_{ri}$. The gradient of the stress tensor is obtained using the following relation:

$$\sigma_{ij,k} = \left(\frac{\partial\sigma^{ab}}{\partial z^c} + \sigma^{db}\Gamma^a_{dc} + \sigma^{ad}\Gamma^b_{dc}\right)g_{ai}g_{bj}g^c_k, \tag{5.20}$$

where for the spherical coordinate system, $g_{1i} = e_{ri}$, $g_{2i} = re_{\theta i}$, $g_{3i} = r\sin\theta e_{\phi i}$, $g^1_i = e_{ri}$, $g^2_i = (1/r)e_{\theta i}$, $g^3_i = (1/r\sin\theta)e_{\phi i}$. In the present case, the only non-zero components of the Christoffel tensor are $\Gamma^1_{22} = -r$, $\Gamma^1_{33} = -r\sin^2\theta$, $\Gamma^2_{12} = \Gamma^2_{21} = (1/r)$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{13} = \Gamma^3_{31} = (1/r)$, $\Gamma^3_{32} = \Gamma^3_{23} = \cot\theta$. The stress components in the above equation are given by $\sigma^{11} = \sigma_r$, $\sigma^{12} = \sigma^{21} = (1/r)\sigma_{r\theta}$, $\sigma^{22} = (1/r^2)\sigma_\theta$, $\sigma^{13} = \sigma^{31} = (1/r\sin\theta)$, $\sigma^{23} = \sigma^{32} = (1/r^2\sin\theta)\sigma_{\theta\phi}$, $\sigma^{33} = (1/r^2\sin^2\theta)\sigma_\phi$. Using these we obtain

$$(\sigma_{ij,k}e_{rk})(e_{ri}e_{\theta j} + e_{\theta i}e_{rj}) = 2r\sigma^{12}_{,r} + 2\sigma^{12} = 4s\left(\frac{R^2}{r^3} - 2\frac{R^4}{r^5}\right)\sin\theta. \tag{5.21}$$

Another calculation yields

$$\sigma_{mm,k}\delta_{ij}e_{rk}(e_{ri}e_{\theta j}+e_{\theta i}e_{rj}) = 4s\frac{R^2}{r^3}\sin\theta. \tag{5.22}$$

Therefore, we evaluate the integral

$$\int_{\partial\omega}(\sigma_{ij,k}+b\delta_{jk}\sigma_{qq,i})c_{ij}\mathrm{d}S_k = -4\pi^2 sR^2\left(\frac{1}{R}-\frac{1}{r}\right)(1+b) + 8\pi^2 sR^4\left(\frac{1}{R^3}-\frac{1}{r^3}\right), \tag{5.23}$$

where $\mathrm{d}S_k = -e_{rk}r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ for the internal surface (at radius R) and $\mathrm{d}S_k = -e_{rk}r^2\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$ at the outer surface (at radius r). To complete the calculations, we would need to evaluate the gradient of the tensor c_{ij}. We use equation (5.20) and note that the only non-zero components are $c^{12}=c^{21}=1/r$. We then obtain the following:

$$c_{ij,k} = \frac{2}{r}e_{\theta i}e_{\theta j}e_{\theta k}+\frac{1}{r}(e_{\phi i}e_{\theta j}+e_{\theta i}e_{\phi j})e_{\phi k}-\frac{2}{r}e_{ri}e_{rj}e_{\theta k}+\frac{\cot\theta}{r}(e_{\phi i}e_{rj}+e_{ri}e_{\phi j})e_{\phi k}, \tag{5.24}$$

and subsequently

$$\sigma_{ij,k}c_{ij,k} = -12s\frac{R^2}{r^4}\sin\theta - 4s\frac{R^2}{r^4}\frac{\cos^2\theta}{\sin\theta}+16s\frac{R^4}{r^6}\sin\theta+8s\frac{R^4}{r^6}\frac{\cos^2\theta}{\sin\theta}, \tag{5.25}$$

and

$$\sigma_{mm,k}\delta_{ij}c_{ij,k} = 12s\frac{R^2}{r^4}\sin\theta+8s\frac{R^2}{r^4}\frac{\cos^2\theta}{\sin\theta}. \tag{5.26}$$

Using these results we evaluate the integral

$$\int_{\omega}(\sigma_{ij,k}+b\delta_{ij}\sigma_{mm,k})c_{ij,k}\mathrm{d}\Omega = (20b-16)\pi^2 sR^2\left(\frac{1}{R}-\frac{1}{r}\right) + 8\pi^2 sR^4\left(\frac{1}{R^3}-\frac{1}{r^3}\right), \tag{5.27}$$

and therefore

$$\int_{\partial\omega}(\sigma_{ij,k}+b\delta_{jk}\sigma_{qq,i})c_{ij}\mathrm{d}S_k - \int_{\omega}(\sigma_{ij,k}+b\delta_{ij}\sigma_{mm,k})c_{ij,k}\mathrm{d}\Omega = -6\pi^2\mu\tau\alpha R^3\left(\frac{1}{R}-\frac{1}{r}\right). \tag{5.28}$$

The fact that from (5.28) we retrieve (5.19) attests to the validity of relation (5.11) as an identity. Thus, we have verified that a certain measure of incompatibility in the volume can be obtained by a surface integral.

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Appendix A

In the following, we briefly introduce Noether's theorem for a functional involving tensorial arguments. For details please refer to Li *et al.* (2005). Consider the functional

$$\Pi(\sigma_{ij}, \sigma_{ij,k}) = \int_{\Omega} L_{\Omega}(x_i, \sigma_{ij}, \sigma_{ij,k})\mathrm{d}\Omega + \int_{\partial\Omega} L_{\Gamma}(x_i, \sigma_{ij}, \sigma_{ij,k})\mathrm{d}S. \tag{A 1}$$

Assume that we are given an r-parameter family of invertible transformations on coordinate variable x_i and Cartesian tensor field σ_{ij},

$$\bar{x}_i = \bar{x}_i(x_i, \sigma_{ij}, \epsilon_\alpha), \tag{A 2}$$

$$\bar{\sigma}_{ij} = \bar{\sigma}_{ij}(x_i, \sigma_{ij}, \epsilon_\alpha), \tag{A 3}$$

where $\epsilon_\alpha = (\epsilon_1, \epsilon_2, \ldots, \epsilon_r)$ such that

$$\bar{x}_i|_{\epsilon_\alpha=0} = \bar{x}_i(x_i, \sigma_{ij}, 0) = x_i \qquad \bar{\sigma}_{ij}|_{\epsilon_\alpha=0} = \bar{\sigma}_{ij}(x_i, \sigma_{ij}, 0) = \sigma_{ij}. \tag{A 4}$$

We expand these transformations about ϵ_α to get

$$\bar{x}_i = x_i + \varphi_{i\alpha}(x_i, \sigma_{ij})\epsilon_\alpha + o(\epsilon), \tag{A 5}$$

$$\bar{\sigma}_{ij} = \sigma_{ij} + \xi_{ij\alpha}(x_i, \sigma_{ij})\epsilon_\alpha + o(\epsilon), \tag{A 6}$$

where

$$\varphi_{i\alpha}(x_i, \sigma_{ij}) := \left.\frac{\partial \bar{x}_i}{\partial \epsilon_\alpha}\right|_{\epsilon_\alpha=0}, \tag{A 7}$$

$$\xi_{ij\alpha}(x_i, \sigma_{ij}) := \left.\frac{\partial \bar{\sigma}_{ij}}{\partial \epsilon_\alpha}\right|_{\epsilon_\alpha=0}, \tag{A 8}$$

represent the infinitesimal generators corresponding to x_i and σ_{ij}, respectively (Olver 2000, p. 27).

Taking variations of the functional (A 1) with respect to stress, we obtain

$$\delta\Pi = \int_{\Omega} \left(\frac{\partial L_\Omega}{\partial \sigma_{ij}} - \frac{\mathrm{d}}{\mathrm{d}x_k}\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right)\delta\sigma_{ij}\mathrm{d}\Omega + \int_{\partial\Omega}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}} n_k - \frac{\partial L_\Gamma}{\partial \sigma_{ij}}\right)\delta\sigma_{ij}\mathrm{d}S \tag{A 9}$$

$$-\int_{\partial\Omega}\frac{\partial L_\Gamma}{\partial \sigma_{ij,k}}\delta\sigma_{ij,k}\mathrm{d}S,$$

the stationarity of which gives us the following Euler–Lagrangian equations

$$\frac{\partial L_\Omega}{\partial \sigma_{ij}} - \frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial \sigma_{ij,k}}\right) = 0, \quad \forall\, x_i \in \Omega, \tag{A 10}$$

$$\frac{\partial L_\Omega}{\partial \sigma_{ij,k}} n_k + \frac{\partial L_\Gamma}{\partial \sigma_{ij}} = 0, \quad \forall\, x_i \in \partial\Omega, \tag{A 11}$$

$$\frac{\partial L_\Gamma}{\partial \sigma_{ij,k}} = 0, \quad \forall\, x_i \in \partial\Omega. \tag{A 12}$$

By Lagrange expressions, we refer to the left-hand side of the above Euler–Lagrange equations.

As a result of Noether's theorem generalized for tensors, the invariance of the integral (A 1) implies

$$\frac{\mathrm{d}}{\mathrm{d}x_k}\left(L_\Omega\varphi_{k\alpha}+\frac{\partial L_\Omega}{\partial\sigma_{ij,k}}\xi_{ij\alpha}-\frac{\partial L_\Omega}{\partial\sigma_{ij,k}}\sigma_{ij,\ell}\varphi_{\ell\alpha}\right)+\left(\frac{\partial L_\Omega}{\partial\sigma_{ij}}-\frac{\mathrm{d}}{\mathrm{d}x_k}\left(\frac{\partial L_\Omega}{\partial\sigma_{ij,k}}\right)\right)\left(\xi_{ij\alpha}-\sigma_{ij,\ell}\varphi_{\ell\alpha}\right)=0. \tag{A 13}$$

Therefore, (linear independent) combinations of Lagrange expressions become divergences. In case the second term in equation (A 13) vanishes as a result of relation (A 10), we obtain r divergence-free quantities.

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