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RESEARCH NOTE

Growth and Non-Metricity in Föppl-von Kármán Shells

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Abstract The non-homogeneous Föppl-von Kármán equations for growing thin elastic shallow shells are revisited by deriving the inhomogeneity source terms directly from the non-metricity tensor associated with growth. This is in contrast with the existing literature where the source terms are obtained using the extensional and curvature growth strains after exploiting the additive decomposition of the total strain into its elastic and growth counterpart. Our framework not only establishes the additive decomposition but provides an unambiguous illustration of the geometric nature of growth in terms of a genuine material inhomogeneity measure given by the non-metricity tensor.

Keywords Föppl-von Kármán equations · Shallow shells · Growth · Non-metricity tensor

Mathematics Subject Classification 74E05 · 74K15 · 74K20 · 74K25 · 53Z05

1 Introduction

The stress and the deformation fields associated with a growing thin elastic shallow shell, such as a leaf or a flower petal [7, 8], can be determined by solving the non-homogeneous Föppl-von Kármán equations [6, 9]

$$\Delta^2 \Phi + \frac{E}{2}[\mathrm{w}, \mathrm{w}] = -E\left(\lambda^g - \frac{1}{2}\left[\mathrm{w}^0, \mathrm{w}^0\right]\right) \quad \text{and} \tag{1a}$$

$$D\Delta^2 \mathrm{w} - [\mathrm{w}, \Phi] = -D\left(\Omega^g - \Delta^2 \mathrm{w}^0\right), \tag{1b}$$

where $\Phi(\theta^\alpha)$ is the Airy stress function which determines the equilibrated stress field through $\sigma^{\alpha\beta} = e^{\alpha\mu} e^{\beta\nu} \Phi_{,\mu\nu}$, with $e^{11} = e^{22} = 0$, $e^{12} = -e^{21} = 1$, $\{\theta^1, \theta^2\} \in \mathbb{R}^2$ is the surface parametrization, and the subscript comma denotes ordinary spatial derivative with respect to θ^α; Δ^2 is the biharmonic operator on $\mathbb{R}^2$, defined as $\Delta^2 f = \Delta(f_{,11} + f_{,22}) =$

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$f_{,1111} + 2f_{,1122} + f_{,2222}$, and $[\cdot,\cdot]$ is the Monge-Ampère bracket, defined by $[f, g] = e^{\alpha\beta}e^{\mu\nu} f_{,\alpha\mu} g_{,\beta\nu} = f_{,11}g_{,22} + f_{,22}g_{,11} - 2f_{,12}g_{,12}$, for sufficiently differentiable real-valued functions f and g; $\mathrm{w}(\theta^\alpha)$ and $\mathrm{w}^0(\theta^\alpha)$ are the height functions associated with the mid-surface of the deformed and the reference shape, respectively (in fact, with $\mathrm{w}^0(\theta^\alpha) = 0$ we recover the corresponding plate equations); $E = E^\star h$ is the two-dimensional (2D) Young's modulus and $D = E^\star h^3/(12(1-\nu^2))$ is the bending modulus, where $E^\star$ and ν are the Young's modulus and Poisson's ratio, respectively, of the homogeneous, isotropic, linear elastic material of the underlying three-dimensional (3D) shell, of constant thickness h, which is being approximated by the 2D shell theory; further details on the notation are given in the subsequent section. Here, and elsewhere, the fields are assumed to be sufficiently differentiable as required. The source terms λ^g and Ω^g contain information regarding the growth of the elastic shell; they are related to growth strain fields, given in terms of an extensional growth strain $E^g_{\alpha\beta}$ and a bending growth strain $\Lambda^g_{\alpha\beta}$, as [6, 9, 13]

$$\lambda^g = e^{\alpha\beta}e^{\mu\lambda}E^g_{\alpha\mu,\beta\lambda} = E^g_{11,22} + E^g_{22,11} - 2E^g_{12,12} \quad \text{and} \tag{2a}$$

$$\Omega^g = \left(\nu\delta^{\mu\nu}\delta^{\alpha\beta} + (1-\nu)\delta^{\mu\alpha}\delta^{\nu\beta}\right)\Lambda^g_{\mu\nu,\alpha\beta}. \tag{2b}$$

Equations (1a)–(1b), with source terms given by (2a)–(2b), were first derived by Mansfield [9] (in the context of thermal strains) from the classical (compatible) Föppl-von Kármán shallow shell theory [4] under appropriately scaled additive decompositions of the (compatible) total extensional and bending strains into (individually incompatible) elastic and growth counterparts. The same equations were later obtained by Lewicka et al. [6] as the Euler-Lagrange equations of an appropriate Γ-limit of the elastic energy functional of a "thin" three-dimensional (3D) incompatible elastic body as its thickness goes to zero, under appropriate scalings of the incompatible extensional and curvature growth strains that constitute the incompatible strain field of the underlying 3D body.

In this research note we interpret λ^g and Ω^g directly in terms of the material non-metricity tensor field $\mathcal{Q}$ of the materially inhomogeneous shell without invoking the notion of growth strains $\boldsymbol{E}^g$ and $\boldsymbol{\Lambda}^g$. We derive

$$\lambda^g = \frac{1}{2}e^{\alpha\mu}e^{\beta\nu}\mathcal{Q}_{\alpha\mu\beta,\nu} \quad \text{and} \tag{3a}$$

$$\Omega^g = -\frac{1}{2}\left(\nu\mathcal{Q}_{3\alpha\alpha,\beta\beta} + (1-\nu)\mathcal{Q}_{3\alpha\beta,\alpha\beta}\right), \tag{3b}$$

where $\mathcal{Q}_{\mu\alpha\beta}$ and $\mathcal{Q}_{3\alpha\beta}$ are the only non-metricity components which make their appearance due to the underlying Kirchhoff-Love kinematics. Recall that, for a non-Riemannian (geometric) material space, a non-zero non-metricity tensor leads to non-preservation of the inner product between the tangent vectors during parallel transport along material curves [12, 13]. Such geometric spaces have been the basis for a unified continuum theory of distributed metric anomalies such as biological growth, thermal deformation, distributed point defects [1, 5]. There have also been alternative proposals for constructing a geometric theory of growth in shells [14]. As we shall argue, expressing growth in terms of non-metricity tensor, which is a genuine invariant measure of material inhomogeneity (in addition to torsion and curvature), yields an unambiguous geometrical characterization of the underlying growth. Secondly, the notion of non-metricity is more general than that of growth strains, since the form of the latter depends on the assumed shell kinematics; for instance, we will need additional strain measures to deal with higher-order shell theories, but growth from the standpoint of non-metricity will remain unchanged.

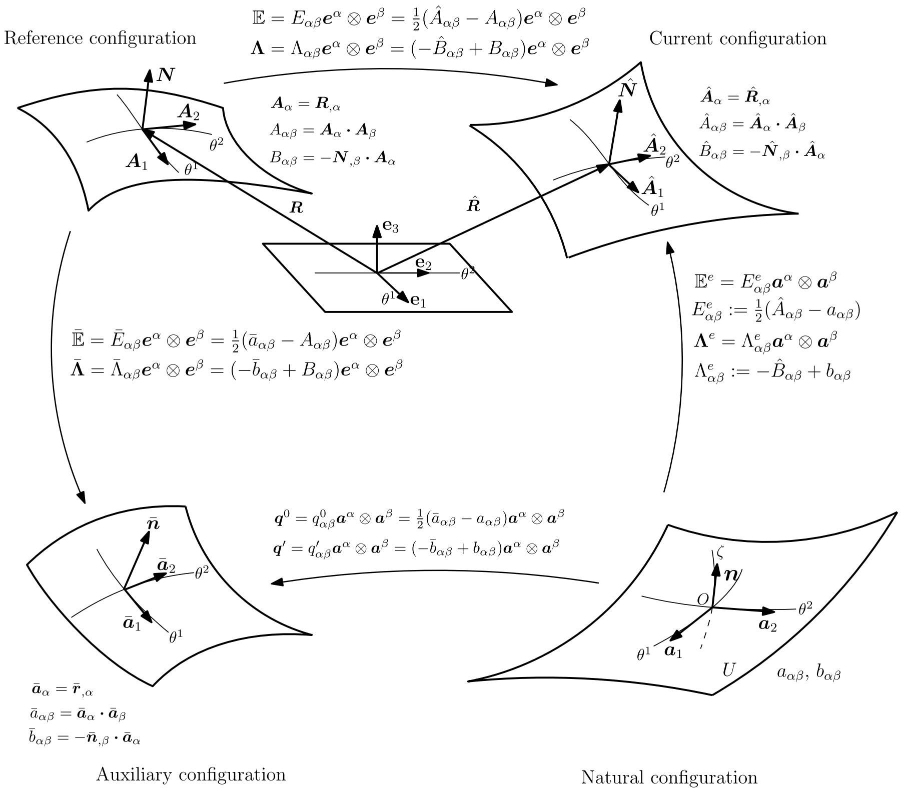


Fig. 1 Various configurations of the mid-surface of the shell

2 Non-Riemannian Geometry of the Natural Configuration

Let ω, representing the mid-surface of the shell, be a 2D simply connected bounded manifold, with piecewise smooth boundary $\partial\omega$, homeomorphic to a closed disc in $\mathbb{R}^2$. Let $\mathscr{B} = \omega \times [-h/2, h/2] \subset \mathbb{R}^3$, for some real constant $h > 0$, be the cylindrical closed neighborhood of ω, representing the sufficiently thin 3D shell. The sufficient thinness of the shell is understood in terms of the relation $\epsilon = (h/L) \ll 1$, where L is a characteristic linear dimension of the mid-surface, e.g., the minimum wavelength of the deformation pattern on ω for all deformations under consideration [4]. Let θ^α be the natural coordinate system on ω, and let ζ be the transverse coordinate along the thickness direction. We use small case Greek indices $\alpha, \beta, \mu \ldots$, etc., to take values from the set $\{1, 2\}$ and small case Roman indices $i, j, k \ldots$ etc., from the set $\{1, 2, 3\}$.

In Fig. 1, various configurations and strain fields of the shallow elastic shell are illustrated describing the kinematics of the mid-surface. The pairs $(A_{\alpha\beta}, B_{\alpha\beta})$ and $(\hat{A}_{\alpha\beta}, \hat{B}_{\alpha\beta})$, belonging to $\mathsf{S}_2^+ \times \mathsf{S}_2$, where S_2 and S_2^+ are the sets of symmetric and symmetric positive-definite 2×2 matrices on $\mathbb{R}^2$, denote the first and the second fundamental forms of the reference configuration $\boldsymbol{R}(\omega) \subset \mathbb{R}^3$ and the current configuration $\hat{\boldsymbol{R}}(\omega) \subset \mathbb{R}^3$ of the mid-surface, respectively. They satisfy the Gauss and Codazzi-Mainardi equations [13]

$$K_{1212} + \left[B_{11}B_{22} - B_{12}^2\right] = 0, \quad -\partial_2 B_{11} + \partial_1 B_{12} = 0, \quad -\partial_2 B_{21} + \partial_1 B_{22} = 0; \tag{4a}$$

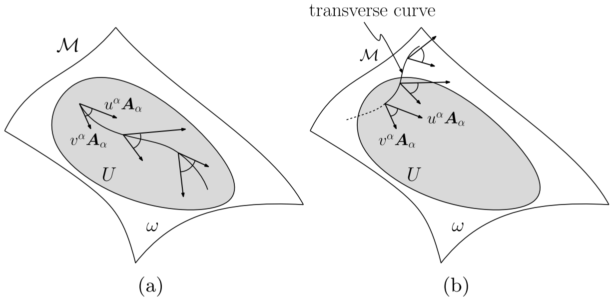


Fig. 2 (**a**) The change in angle between two tangent vectors **u** and **v**, during parallel transportation along a surface curve, due to non-zero $Q_{\mu\alpha\beta}$. (**b**) The change in angle between two tangent vectors **u** and **v**, during parallel transportation along a normal curve, due to non-zero Q_{3ij}. (**c**) The change in length of a transverse vector, during parallel transportation along a surface curve, due to non-zero $Q_{\mu i3}$

$$\hat{K}_{1212} + \left[\hat{B}_{11}\hat{B}_{22} - \hat{B}_{12}^2\right] = 0, \quad -\hat{\partial}_2\hat{B}_{11} + \hat{\partial}_1\hat{B}_{12} = 0, \quad -\hat{\partial}_2\hat{B}_{21} + \hat{\partial}_1\hat{B}_{22} = 0, \tag{4b}$$

where K_{1212} and $\hat{K}_{1212}$ are the Riemann-Christoffel curvatures associated with the metrics $A_{\alpha\beta}$ and $\hat{A}_{\alpha\beta}$, respectively; ∂ and $\hat{\partial}$ denote the covariant derivatives with respect to the induced Levi-Civita connections $\Gamma^\mu_{\alpha\beta}$ and $\hat{\Gamma}^\mu_{\alpha\beta}$, respectively.

The local neighborhoods of the current configuration are relaxed through the elastic extensional strain field $E^e_{\alpha\beta}$ and elastic curvature strain field $\Lambda^e_{\alpha\beta}$, the energetic duals of the stress $\boldsymbol{\sigma}$ and the bending moment $\boldsymbol{M}$, respectively, so that the functions $a_{\alpha\beta} = \hat{A}_{\alpha\beta} - 2E^e_{\alpha\beta}$ and $b_{\alpha\beta} = \hat{B}_{\alpha\beta} + \Lambda^e_{\alpha\beta}$ constitute the *natural first fundamental form* and the *natural second fundamental form*, respectively, of the relaxed or natural (stress-free, moment-free) configuration of the material surface. Unlike the reference and the current configuration, the natural configuration, in general, cannot be realized as a connected isometric embedding of the mid-surface ω in $\mathbb{R}^3$ as a whole, and, as a result, the natural fundamental forms will not satisfy the conventional Gauss and Codazzi-Mainardi equations. However, the natural configuration can be realized as an appropriate 2D projection of an isometric embedding $\boldsymbol{\chi} : \mathscr{B} \to \mathbb{M}^3$ in a hypothetical 3D non-Riemannian space $\mathbb{M}^3$ equipped with an inner product $\langle \cdot, \cdot \rangle$ and a material connection $\mathfrak{L}$ [5]. Let $\boldsymbol{g}$ be the induced non-Riemannian metric on $\mathscr{B}$ coming from this embedding, with components g_{ij} with respect to the natural coordinates (θ^α, ζ). We assume that the material connection $\mathfrak{L}$ has vanishing torsion and Riemann-Christoffel curvature tensors (hence, no dislocations and disclinations). The essential non-Riemannian nature of the material surface is encoded in the third-order non-metricity tensor $\tilde{\mathcal{Q}}$ with covariant components $\tilde{Q}_{kij} = -g_{ij;k}$, where the subscript semicolon represents the covariant derivative with respect to the material connection $\mathfrak{L}$ [12]. The non-metricity tensor field $\mathcal{Q} = \tilde{\mathcal{Q}}|_{\boldsymbol{\chi}(\omega)}$ on the natural configuration $\boldsymbol{\chi}(\omega)$, with covariant components $Q_{kij} = \tilde{Q}_{kij}(\theta^\alpha, \zeta = 0)$, measures the non-metricity on the mid-surface ω.

The geometrical meaning of Q_{ijk}, as a source for the non-preservation of the inner product of tangent vectors (with respect to $\boldsymbol{g}$) under parallel transport (with respect to $\mathfrak{L}$), is illustrated in Fig. 2. In particular, considering $u^1\boldsymbol{A}_1$ and $v^2\boldsymbol{A}_2$ as two fixed orthogonal vectors at a point $\theta^\alpha \in \omega$, Q_{111} would change the length of $u^1\boldsymbol{A}_1$ under its parallel transport along θ^1-direction; Q_{222} would change the length of $v^2\boldsymbol{A}_2$ under its parallel transport along θ^2-direction; Q_{122} would change the length of $v^2\boldsymbol{A}_2$ under its parallel transport along θ^1-direction (Q_{211} vice-versa); $Q_{121}(= Q_{112})$ would change the angle between $u^1\boldsymbol{A}_1$ and $v^2\boldsymbol{A}_2$ under their parallel transport along θ^1-direction; $Q_{212}(= Q_{221})$ would similarly change the angle under their parallel transport along θ^2-direction; Q_{311} would change the length of $u^1\boldsymbol{A}_1$ under its parallel transport along ζ-direction; Q_{322} would change the length of $v^2\boldsymbol{A}_2$ under its parallel transport along ζ-direction; and finally $Q_{312}(=Q_{321})$ would change the angle between $u^1\boldsymbol{A}_1$ and $v^2\boldsymbol{A}_2$ under their parallel transport along ζ-direction.

The natural fundamental form pair $(a_{\alpha\beta}, b_{\alpha\beta}) \in \mathsf{S}_2^+ \times \mathsf{S}_2$ satisfy the so-called incompatible Gauss and Codazzi-Mainardi equations [13]

$$k_{1212} + \left[b_{11}b_{22} - b_{12}^2\right] = I_1, \quad -\nabla_2 b_{11} + \nabla_1 b_{12} = I_2, \quad -\nabla_2 b_{21} + \nabla_1 b_{22} = I_3, \tag{5}$$

where

$$I_1 = -2\nabla_{[1} M_{2]12} + 2b_{1[1}M_{2]32} + 2b_{2[1}M_{2]13} - 2M_{[1|\alpha 2|}M^{\alpha}_{2]1} - 2M_{[1|32|}M^{3}_{2]1}, \tag{6a}$$

$$I_2 = -2\nabla_{[1} M_{2]13} + 2b_{1[1}M_{2]33} - 2b^{\rho}_{[1}M_{2]1\rho} - 2M_{[1|\alpha 3|}M^{\alpha}_{2]1} - 2M_{[1|33|}M^{3}_{2]1}, \tag{6b}$$

$$I_3 = -2\nabla_{[1} M_{2]23} + 2b_{2[1}M_{2]33} - 2b^{\rho}_{[1}M_{2]2\rho} - 2M_{[1|\alpha 3|}M^{\alpha}_{2]2} - 2M_{[1|33|}M^{3}_{2]2}, \tag{6c}$$

are the incompatibility measures;[1] k_{1212} is the Gaussian curvature of the material metric $a_{\alpha\beta}$, and ∇ denotes covariant derivative with respect to the Levi-Civita connection $s^{\mu}_{\alpha\beta}$ induced by $a_{\alpha\beta}$. The functions $M_{ijk}(\theta^\alpha)$ are obtained by restricting the metric anomaly tensor $\tilde{M}_{ijk}(\theta^i) = (1/2)(\tilde{Q}_{ikj} - \tilde{Q}_{kji} + \tilde{Q}_{jik})$ on ω, i.e., $M_{kij}(\theta^\alpha) = \tilde{M}_{kij}(\theta^\alpha, \zeta = 0)$. Also, $\tilde{M}^k_{ij} = g^{kp}\tilde{M}_{ijp}$, with $[g^{ij}] = [g_{ij}]^{-1}$, where the components of the material metric g_{ij} on the embedded image $\boldsymbol{\chi}(\mathscr{B}) \subset \mathbb{M}^3$ are defined as

$$g_{\alpha\beta} = \langle \boldsymbol{\chi}_{,\alpha}, \boldsymbol{\chi}_{,\beta} \rangle = a_{\alpha\beta} - 2\zeta b_{\alpha\beta} + \zeta^2 c_{\alpha\beta}, \tag{7a}$$

$$g_{\alpha 3} = g_{3\alpha} = \langle \boldsymbol{\chi}_{,\alpha}, \boldsymbol{\chi}_{,3} \rangle = 0, \quad \text{and} \tag{7b}$$

$$g_{33} = \langle \hat{\boldsymbol{\chi}}_{,3}, \hat{\boldsymbol{\chi}}_{,3} \rangle = 1 \tag{7c}$$

in terms of the natural fundamental forms in accordance with the Kirchhoff-Love kinematical assumptions. Here, $c_{\alpha\beta} = a^{\mu\nu}b_{\alpha\mu}b_{\beta\nu}$ is the third fundamental form of the natural configuration. The components $\tilde{M}^i_{jk}$ measure the difference between the material connection and the Levi-Civita connection induced by the material metric, $\tilde{M}^i_{jk} = L^i_{jk} - s^i_{jk}$ [12].

3 The Auxiliary Material Space and the Specification of Growth

The non-metricity tensor $\tilde{Q}_{ijk}$, in the absence of dislocations and disclinations, necessarily satisfies the Bianchi-Padova relation [12]

$$\tilde{Q}_{[j|kl|;i]} = \left(\tilde{Q}_{jkl,i} + L^p_{jk}\tilde{Q}_{ipl} + L^p_{jl}\tilde{Q}_{ipk}\right)_{[ji]} = 0. \tag{8}$$

In the above expression, a square bracket in the subscript is used to denote anti-symmetrization with respect to the enclosed indices while the two vertical bars are used to contain the indices which are to be exempted from anti-symmetrization. This conservation law ensures that the inner product is preserved under parallel transport along loops in

[1] The incompatibility measures written here are corrected version of the Eqs. (59)–(62) in [13].

the material space with respect to the material connection $\mathfrak{L}$. It can be shown, by direct substitution, that a non-trivial solution of (8) is given by

$$\tilde{Q}_{kij} = -2\tilde{q}_{ij;k}, \tag{9}$$

where $\tilde{q}_{ij} = \tilde{q}_{ji}$ are arbitrary, sufficiently differentiable, functions over $\mathscr{B}$ [12]. It is a consequence of the fundamental existence theorem of linear systems of first-order partial differential equations that if the matrix field $g_{ij} - 2\tilde{q}_{ij}$ is positive-definite for symmetric $\tilde{q}_{ij}$, then (9) is the only solution to (8) [12]. As a corollary of this result, we can introduce a symmetric positive-definite tensor with components $\bar{g}_{ij} = g_{ij} - 2\tilde{q}_{ij}$, which can serve as a metric, termed the *auxiliary material metric*, compatible with the material connection $\mathfrak{L}$, i.e., $\bar{g}_{ij;k} = 0$.

In the absence of torsional and curvature anomalies, the auxiliary material metric $\bar{g}_{ij}$ naturally introduces an *auxiliary material space* ($\mathscr{B}$, $\mathfrak{L}$, $\bar{\boldsymbol{g}}$) equipped with the material connection $\mathfrak{L}$ compatible with the auxiliary material metric $\bar{\boldsymbol{g}}$. As a consequence of the fundamental theorem of Riemannian geometry, the space ($\mathscr{B}$, $\mathfrak{L}$, $\bar{\boldsymbol{g}}$) can be realized as an isometric embedding $\bar{\boldsymbol{\chi}} : \mathscr{B} \to \mathbb{R}^3$ in $\mathbb{R}^3$, with an inner product $\cdot$, such that

$$\bar{g}_{\alpha\beta} = \bar{\boldsymbol{\chi}}_{,\alpha} \cdot \bar{\boldsymbol{\chi}}_{,\beta} = \bar{a}_{\alpha\beta} - 2\zeta \bar{b}_{\alpha\beta} + \zeta^2 \bar{a}^{\mu\nu} \bar{b}_{\alpha\mu} \bar{b}_{\beta\nu}, \tag{10a}$$

$$\bar{g}_{\alpha 3} = \bar{g}_{3\alpha} = \bar{\boldsymbol{\chi}}_{,\alpha} \cdot \hat{\boldsymbol{\chi}}_{,3} = 0, \quad \text{and} \tag{10b}$$

$$\bar{g}_{33} = \bar{\boldsymbol{\chi}}_{,3} \cdot \bar{\boldsymbol{\chi}}_{,3} = 1, \tag{10c}$$

where $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ are the first and the second fundamental form of the embedded image $\bar{\boldsymbol{\chi}}(\omega) \subset \mathbb{R}^3$. The *auxiliary fundamental forms* $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$, by construction, satisfy the Gauss and Codazzi-Mainardi equations such that we can write $\bar{\boldsymbol{\chi}}(\theta^\alpha, \zeta) = \bar{\boldsymbol{r}}(\theta^\alpha) + \zeta \bar{\boldsymbol{n}}(\theta^\alpha)$, where $\bar{\boldsymbol{r}}$ represents the embedded image of the auxiliary material space in $\mathbb{R}^3$, the *auxiliary configuration of the shell*, with unit normal $\bar{\boldsymbol{n}}$; and $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta$, $\bar{b}_{\alpha\beta} = -\bar{\mathbf{n}}_{,\alpha} \cdot \bar{\mathbf{a}}_\beta$, where $\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}}_{,\alpha}$, $\bar{\boldsymbol{n}} = (\bar{\boldsymbol{a}}_1 \times \bar{\boldsymbol{a}}_2)/|\bar{\boldsymbol{a}}_1 \times \bar{\boldsymbol{a}}_2|$.

The geometric nature of $\tilde{q}_{ij}$ is such that it corrects the incompatibility of the material metric g_{ij} with respect to the material connection $\mathfrak{L}$ so as to yield a compatible metric field $\bar{g}_{ij}$. Accordingly, we specify the non-metricity Q_{kij} by prescribing both the compatible auxiliary configuration $\bar{\boldsymbol{r}}(\omega)$ and the field $\tilde{q}_{ij}$ defined on $\bar{\boldsymbol{r}}(\omega)$. The field $-\tilde{q}_{ij}$ is fundamentally a *strain-like* object defined with respect to a known configuration, in this case, an arbitrary compatible configuration $\bar{\boldsymbol{r}}(\omega)$. On the other hand, Q_{kij} is a *strain-gradient*-like quantity, cf. (9), $\tilde{Q}_{kij} = (g_{ij} - \bar{g}_{ij})_{;k}$. In order to fix the inherent arbitrariness in defining the non-metricity fields, without any loss of generality, *we assume the auxiliary configuration* $\bar{\boldsymbol{r}}(\omega)$ *to be identical with the reference configuration*, i.e., $\bar{\boldsymbol{r}}(\omega) = \boldsymbol{R}(\omega)$. Therefore, $\bar{a}_{\alpha\beta} = A_{\alpha\beta}$ and $\bar{b}_{\alpha\beta} = B_{\alpha\beta}$. This choice coincides with the conventional idea in thermoelastic problems of always specifying the temperature change with respect to a reference temperature which is identical to the temperature of the reference configuration.

4 The Föppl-von Kármán Approximation

The Föppl-von Kármán equations are obtained from a general nonlinear shell under the following kinematic assumptions:

1. The total and the elastic extensional (membrane) strains, and their first gradient, are *small*, i.e., of order $O(\epsilon)$ [13].

2. The total and the elastic curvature strains, and their first gradient, are *moderately large*, i.e., of order $O(\epsilon^{\frac{1}{2}})$ [13].
3. The reference configuration of the *shallow* shell has the Monge representation $\boldsymbol{R}(\theta^\alpha) = \theta^\alpha \mathbf{e}_\alpha + \mathrm{w}^0(\theta^\alpha)\mathbf{e}_3$, where $\mathbf{e}_i$ form a Cartesian bases of $\mathbb{R}^3$ [2], such that (a) the *height* $\mathrm{w}^0(\theta^\alpha)$ with respect to a flat surface in $\mathbb{R}^2$ is $O(\epsilon^{\frac{1}{2}})$ and (b) the reference second fundamental form $B_{\alpha\beta} = \mathrm{w}^0_{,\alpha\beta} + o(\epsilon^{\frac{1}{2}})$, implying $|LB_{\alpha\beta}| = O(\epsilon^{\frac{1}{2}})$ [10]. As a result, the Gaussian curvature $K = \det(B_{\alpha\beta})/\det(A_{\alpha\beta})$ of the reference surface is $O(\epsilon)$, and we have an approximate commutation of the repeated covariant derivative, i.e., $\partial_{\alpha\beta}u^\mu = \partial_{\beta\alpha}u^\mu + O(\epsilon)$, etc. [4].

In accordance with these, we assume the following form of the field $\tilde{q}_{\alpha\beta}(\theta^\alpha, \zeta)$:

$$\tilde{q}_{\alpha\beta} = q^0_{\alpha\beta} + \zeta q'_{\alpha\beta} + \zeta^2 q''_{\alpha\beta}, \quad \tilde{q}_{i3} = 0, \tag{11}$$

where the symmetric $q^0_{\alpha\beta}(\theta^\alpha)$, and their first spatial derivatives, are assumed to be of order $O(\epsilon)$; symmetric $q'_{\alpha\beta}(\theta^\alpha)$, and their first spatial derivatives, are assumed to be of order $O(\epsilon^{\frac{1}{2}})$; and symmetric $q''_{\alpha\beta}(\theta^\alpha)$ are assumed to be of order $O(1)$. Consequently, we obtain

$$Q_{\mu\alpha\beta} = -2\tilde{q}_{\alpha\beta;\mu}|_{\zeta=0} = -2q^0_{\alpha\beta,\mu} \quad \text{upto } O(\epsilon), \tag{12a}$$

$$Q_{3\alpha\beta} = -2\tilde{q}_{\alpha\beta;3}|_{\zeta=0} = -2q'_{\alpha\beta} \quad \text{upto } O\big(\epsilon^{\frac{1}{2}}\big), \quad \text{and} \tag{12b}$$

$$Q_{ij3} \equiv 0. \tag{12c}$$

Hence, we can have six distinct types of in-surface non-metricity components, represented by their densities $Q_{\mu\alpha\beta}$, and three distinct types of curvature non-metricity components, represented by $Q_{3\alpha\beta}$. We further obtain

$$M_{33i} = 0, \quad M_{3\alpha3} = M_{\alpha33} = 0, \tag{13a}$$

$$M_{3\alpha\beta} = M_{\alpha3\beta} = \frac{1}{2}Q_{3\beta\alpha} = -q'_{\alpha\beta}, \tag{13b}$$

$$M_{\alpha\beta3} = -\frac{1}{2}Q_{3\beta\alpha} = q'_{\alpha\beta}, \quad \text{and} \tag{13c}$$

$$M_{\alpha\beta\mu} = \frac{1}{2}(Q_{\alpha\mu\beta} - Q_{\mu\beta\alpha} + Q_{\beta\alpha\mu}) = -\big(q^0_{\mu\beta,\alpha} - q^0_{\beta\alpha,\mu} + q^0_{\alpha\mu,\beta}\big). \tag{13d}$$

Using (10a)–(10c) and (11), and comparing the terms of orders $O(\epsilon)$, $O(\epsilon^{\frac{1}{2}})$ and $O(1)$, respectively, we can identify

$$a_{\alpha\beta} = A_{\alpha\beta} + 2q^0_{\alpha\beta}, \quad b_{\alpha\beta} = B_{\alpha\beta} - q'_{\alpha\beta}, \quad \text{and} \quad q''_{\alpha\beta} = q^{0\mu\nu}q'_{\alpha\mu}q'_{\beta\nu}. \tag{14}$$

Accordingly, $Q_{\mu\alpha\beta} = -(a_{\alpha\beta} - A_{\alpha\beta})_{,\mu}$, and $Q_{3\alpha\beta} = 2(b_{\alpha\beta} - B_{\alpha\beta})$. With the Föppl-von Kármán approximations at hand, we can write the components of the elastic strain fields as (see Fig. 1) $E^e_{\alpha\beta} = (1/2)(\hat{A}_{\alpha\beta} - a_{\alpha\beta}) = E_{\alpha\beta} - q^0_{\alpha\beta}$, where we use $\hat{A}_{\alpha\beta} = A_{\alpha\beta} + 2E_{\alpha\beta}$ and $(14)_1$, and $\Lambda^e_{\alpha\beta} = -\hat{B}_{\alpha\beta} + b_{\alpha\beta} = \Lambda_{\alpha\beta} - q'_{\alpha\beta}$, using $\Lambda_{\alpha\beta} = -\mathrm{w}_{,\alpha\beta} + \mathrm{w}^0_{,\alpha\beta}$ and $(14)_2$, upto $O(\epsilon)$ and $O(\epsilon^{\frac{1}{2}})$, respectively. Here, $E_{\alpha\beta}$ is the total extensional strain and $\Lambda_{\alpha\beta}$ is the total curvature strain (from reference configuration to the current configuration).

We recall, cf. Koiter [4], the approximated equilibrium equations, consistent with the shallow reference configuration approximations, $A_{\alpha\beta} = \delta_{\alpha\beta} + O(\epsilon)$ and $B_{\alpha\beta} = \mathrm{w}^0_{,\alpha\beta} + O(\epsilon^{\frac{1}{2}})$,

$$\sigma^{\beta\alpha}_{,\beta} = 0 \quad \text{and} \quad M^{\alpha\beta}_{,\alpha\beta} - (B_{\alpha\beta} + \Lambda_{\alpha\beta})\sigma^{\alpha\beta} = 0; \tag{15}$$

the approximated strain compatibility conditions,

$$e^{\alpha\beta} e^{\mu\lambda}\left[E_{\alpha\mu,\beta\lambda} - B_{\alpha\mu}\Lambda_{\beta\lambda} + \frac{1}{2}\Lambda_{\alpha\mu}\Lambda_{\beta\lambda}\right] = 0 \quad \text{and} \quad e^{\alpha\beta} e^{\mu\lambda}\Lambda_{\beta\mu,\lambda} = 0; \tag{16}$$

and the linear elastic constitutive relations

$$\sigma^{\alpha\beta} = \frac{Eh}{1-\nu^2}\big((1-\nu)\delta^{\alpha\mu}\delta^{\beta\nu} + \nu\delta^{\mu\nu}\big)E^e_{\mu\nu} \quad \text{and} \tag{17a}$$

$$M^{\alpha\beta} = \frac{Eh^3}{12(1-\nu^2)}\big((1-\nu)\delta^{\alpha\mu}\delta^{\beta\nu} + \nu\delta^{\mu\nu}\big)\Lambda^e_{\mu\nu}. \tag{17b}$$

Combining these with the relations derived in the previous paragraph, we arrive at the Föppl-von Kármán equations (1a)–(1b) with

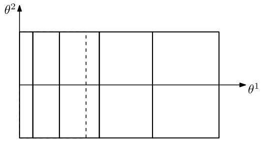
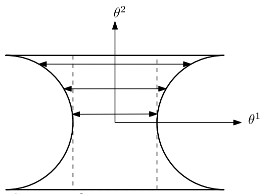
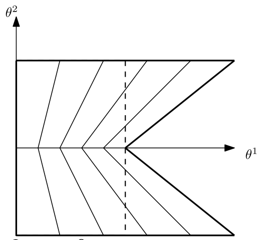
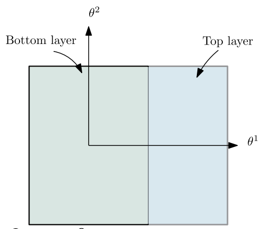
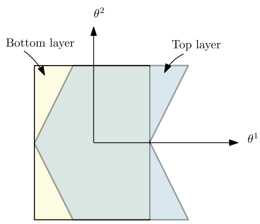
$$\lambda^g = \frac{1}{2} e^{\alpha\mu} e^{\beta\nu} Q_{\alpha\mu\beta,\nu} \quad \text{and} \tag{18a}$$

$$\Omega^g = -\frac{1}{2}\big(\nu Q_{3\alpha\alpha,\beta\beta} + (1-\nu)Q_{3\alpha\beta,\alpha\beta}\big). \tag{18b}$$

The right hand side of Eq. (18a) can be equivalently expressed as $e^{\alpha\mu} e^{\beta\nu} q^0_{\alpha\beta,\mu\nu} = q^0_{22,11} + q^0_{11,22} - 2q^0_{12,12}$ or as $(1/2)((Q_{121,2} - Q_{122,1}) - (Q_{211,2} - Q_{212,1}))$. The right hand side of Eq. (18b) can be equivalently expressed as $\nu q'_{\alpha\alpha,\beta\beta} + (1-\nu)q'_{\alpha\beta,\alpha\beta}$. Recalling Eqs. (2a)–(2b), we observe that our geometric formulation coincides with the conventional formulation of specifying growth in terms of strain fields $E^g_{\alpha\beta}$ and $\Lambda^g_{\alpha\beta}$ by imposing the identifications $E^g_{\alpha\beta} = q^0_{\alpha\beta}$ and $\Lambda^g_{\alpha\beta} = q'_{\alpha\beta}$. Combining this with the expressions for the elastic strains we can immediately recover the additive decompositions of the total strains into their growth and elastic counterparts. It is pertinent to remember that this exact correspondence is an outcome of our identification of the auxiliary configuration with the reference configuration, without any loss of generality. Note that, in the absence of curvature growth, i.e., $Q_{3\alpha\beta} = 0$ (or $\Omega^g = 0$), the Föppl-von Kármán equations remain invariant if one replaces w with −w, implying that the sign of the deflection is undetermined. A non-zero value of Ω^g (or equivalently, $Q_{3\alpha\beta}$) therefore fixes the preferred direction of the deflection.

In Table 1, we list all the non-trivial components of the non-metricity tensor Q and explain their physical meaning in terms of the specific growth type that they represent in the context of a Föppl-von Kármán shell. The non-metricity components in the first block of two rows do not contribute to either λ^g or Ω^g. The components in the second block of four rows contribute only to λ^g (see Eq. (18a)) and those in the last block of three rows contribute only to Ω^g (see Eq. (18b)). In third column of the table, we provide some illustrations of symmetric growth components about the θ^1-axis in a Föppl-von Kármán plate. The dashed domains represent the rectangular flat reference configuration. The figure in the first row depicts uniaxial membrane longitudinal growth $Q_{111} > 0$, non-uniform elongation along θ^1, that does not affect the flatness of the domain. In the fourth row, the figure illustrates membrane growth $Q_{211} > 0$ with elongation along θ^1 while increasing along θ^2. In the fifth row,

Table 1 Growth represented by different components of the non-metricity tensor

Components of the non-metricity tensor	The type of growth it represents	Illustration
Q_{111}	Extensional growth along θ^1 varying in θ^1 direction	 $Q_{111} > 0$
Q_{222}	Extensional growth along θ^2 varying in θ^2 direction	
Q_{122}	Extensional growth along θ^2 varying in θ^1 direction	
Q_{211}	Extensional growth along θ^1 varying in θ^2 direction	 $Q_{211} > 0$
$Q_{121} (= Q_{112})$	In-plane shear growth along θ^1 direction	 $Q_{121} < 0$
$Q_{212} (= Q_{221})$	In-plane shear growth along θ^2 direction	
Q_{311}	Curvature (differential) growth along θ^1	 $Q_{311} > 0$
Q_{322}	Curvature (differential) growth along θ^2	
$Q_{312} (= Q_{321})$	Curvature (differential) shear growth	 $Q_{312} < 0$

the figure corresponds to $Q_{121} < 0$ with in-plane membrane growth of shear type, the initial angle $\pi/2$ decreasing along θ^1. The figures in row seven and nine correspond to the differential growth of a layered surface, where the top and the bottom layer experience different relative growths. In the former the top layer is non-uniformly expanded by $Q_{311} > 0$ along θ^1 with respect to the bottom layer whereas, in the latter, the top layer is uniformly sheared by $Q_{312} < 0$ along θ^1 with respect to the bottom layer.

Example 1 (Non-metricity as a Power Law) For a rectangular plate of length A, width B, and thickness $2h$, let $Q_{111} = Q_{121} = Q_{122} = Q_{311} = Q_{312} = (\theta^1/A)^n$ and $Q_{222} = Q_{221} = Q_{211} = Q_{322} = (\theta^2/B)^n$, where n is the growth exponent. The even and odd values of n represent, respectively, symmetrical and asymmetrical growth about the respective coordinate axes. A form of Q_{211}, proportional to the one given above, was used in [7] (with $n = 10$) to explain the edge ripples observed in growing long leaves, where the proportionality constant stood for the maximum growth at the edges.

Example 2 (Isotropic Growth) In case of isotropic growth, $\tilde{q}_{ij}(\theta^\alpha, \zeta) = \tilde{q}(\theta^\alpha, \zeta) g_{ij}(\theta^\alpha, \zeta)$, i.e., $\tilde{Q}_{kij}(\theta^\alpha, \zeta) = \tilde{Q}_{,k}(\theta^\alpha, \zeta) g_{ij}(\theta^\alpha, \zeta)$, where $\tilde{Q} = \ln(2\tilde{q} - 1)$ [1]. We expand the field $\tilde{q}(\theta^\alpha, \zeta)$ about $\zeta = 0$, keeping in mind the Föppl-von Kármán assumptions, as

$$\tilde{q}\big(\theta^\alpha, \zeta\big) = q^0\big(\theta^\alpha\big) + \zeta q'\big(\theta^\alpha\big) + \zeta^2 q''\big(\theta^\alpha\big), \tag{19}$$

where $q''(= (q')^2/q^0)$, q^0, and its first spatial derivatives, are all of $O(\epsilon)$, and q' is of $O(\epsilon^{\frac{1}{2}})$. In thermal deformation problems, $\tilde{q}(\theta^\alpha, \zeta) = \alpha \tilde{T}(\theta^\alpha, \zeta)$, where α is the homogeneous thermal expansion coefficient and $\tilde{T}$ is the temperature change with respect to the temperature of the reference configuration. The fields $T^0(\theta^\alpha)$ and $T'(\theta^\alpha)$, corresponding to $q^0(\theta^\alpha) = \alpha T^0(\theta^\alpha)$ and $q'(\theta^\alpha) = \alpha T'(\theta^\alpha)$, respectively, represent the *first-order* and the *second-order* temperature in a thin layered shell [3]. The curvature longitudinal growth components for the isotropic case, with $Q_{311} = Q_{322}$ and $Q_{312} = 0$, appear in the literature of growing elastic bilayers [11, 15].

5 Nilpotency

5.1 Nilpotent Growth

Nilpotent growth is characterized by a non-trivial non-metricity tensor Q for which the right hand side incompatibility terms I_1, I_2, and I_3 in (5) vanish identically. Such *compatible* growth fields produce stress-free and moment-free (i.e., relaxed) current configurations (realizable in the physical Euclidean space). This nilpotency condition reduces to

$$e^{\alpha\beta} e^{\mu\lambda} \left[q^0_{\alpha\mu,\beta\lambda} - \mathrm{w}^0_{,\alpha\mu} q'_{\beta\lambda} + \frac{1}{2} q'_{\alpha\mu} q'_{\beta\lambda} \right] = 0 \quad \text{and} \tag{20a}$$

$$e^{\alpha\beta} e^{\mu\lambda} q'_{\beta\mu,\lambda} = 0. \tag{20b}$$

A general solution of these equations is given by [4]

$$\big(q^0_{\alpha\beta}\big)_{\text{nil}} = \frac{1}{2}\big(U_{\alpha,\beta} + U_{\beta,\alpha} - 2\mathrm{w}^0_{,\alpha\beta} W + W_{,\alpha} W_{,\beta}\big) \quad \text{and} \tag{21a}$$

$$\big(q'_{\alpha\beta}\big)_{\text{nil}} = W_{,\alpha\beta}, \tag{21b}$$

for arbitrary potential functions U_α and W defined over ω, once and twice continuously differentiable, respectively. The corresponding non-metricity tensor has components

$$(Q_{\mu\alpha\beta})_{\text{nil}} = -2U_{(\alpha,\beta)\mu} + 2\big(\mathrm{w}^0_{,\alpha\beta} W\big)_{,\mu} - (W_{,\alpha} W_{,\beta})_{,\mu} \quad \text{and} \tag{22a}$$

$$(Q_{3\alpha\beta})_{\text{nil}} = -2W_{,\alpha\beta}. \tag{22b}$$

Equations (22a)–(22b) characterize the nilpotent growth fields in Föppl-von Kármán shells. We assume $U_\alpha = 0$, so as to specify a general distribution of nilpotent growth through a single scalar field $W(\theta^\alpha)$. The nilpotent source terms, $(\lambda^g)_{\text{nil}} = [(\mathrm{w}^0 - (1/2)W), W]$ and $(\Omega^g)_{\text{nil}} = \Delta^2 W$, when substituted into the Föppl-von Kármán equations (1a)–(1b), yield the system of differential equations

$$\Delta^2\Phi + \frac{E}{2}[\mathrm{w},\mathrm{w}] = \frac{E}{2}\big[\mathrm{w}^0 - W, \mathrm{w}^0 - W\big] \quad \text{and} \tag{23a}$$

$$D\Delta^2\mathrm{w} - [\mathrm{w},\Phi] = D\Delta^2\big(\mathrm{w}^0 - W\big), \tag{23b}$$

whose solutions are given by $\Phi = 0$ and $\mathrm{w} = \mathrm{w}^0 - W$.

5.2 Designing Growth to Achieve a Relaxed Target Shape

We can use the solution obtained above to substitute $W = \mathrm{w}^0 - \mathrm{w}$ into Eqs. (22a)–(22b) so as to calculate the required growth field on a given reference shape w^0 which will produce a given target stress-free and moment-free, i.e., relaxed, shape w. These growth fields are

$$Q_{\mu\alpha\beta} = 2\big(\mathrm{w}^0_{,\alpha\beta}\big(\mathrm{w}^0 - \mathrm{w}\big)\big)_{,\mu} - \big(\big(\mathrm{w}^0 - \mathrm{w}\big)_{,\alpha}\big(\mathrm{w}^0 - \mathrm{w}\big)_{,\beta}\big)_{,\mu} \quad \text{and} \tag{24a}$$

$$Q_{3\alpha\beta} = -2\big(\mathrm{w}^0 - \mathrm{w}\big)_{,\alpha\beta}. \tag{24b}$$

For instance, if we assume the following shallow shapes

$$\mathrm{w}^0\big(\theta^1,\theta^2\big) = -\frac{1}{2}\kappa_1^0\big(\theta^1\big)^2 - \frac{1}{2}\kappa_2^0\big(\theta^2\big)^2 \quad \text{and} \quad \mathrm{w}\big(\theta^1,\theta^2\big) = -\frac{1}{2}\kappa_1\big(\theta^1\big)^2 - \frac{1}{2}\kappa_2\big(\theta^2\big)^2, \tag{25}$$

for known constants κ_α^0 and κ_α, the reference and target principal curvatures, respectively, then the design growth fields are

$$Q_{111} = -2\kappa_1\big(\kappa_1 - \kappa_1^0\big)\theta^1, \;\; Q_{222} = -2\kappa_2\big(\kappa_2 - \kappa_2^0\big)\theta^2, \tag{26a}$$

$$Q_{121} = -\big(\kappa_1 - \kappa_1^0\big)\big(\kappa_2 - \kappa_2^0\big)\theta^2, \quad Q_{212} = -\big(\kappa_1 - \kappa_1^0\big)\big(\kappa_2 - \kappa_2^0\big)\theta^1, \tag{26b}$$

$$Q_{122} = -2\kappa_2^0\big(\kappa_1 - \kappa_1^0\big)\theta^1, \quad Q_{211} = -2\kappa_1^0\big(\kappa_2 - \kappa_2^0\big)\theta^2, \tag{26c}$$

$$Q_{311} = -2\big(\kappa_1 - \kappa_1^0\big), \quad Q_{322} = -2\big(\kappa_2 - \kappa_2^0\big), \quad \text{and} \quad Q_{312} = 0. \tag{26d}$$

5.3 Nilpotent $\boldsymbol{\lambda}^g$ and $\boldsymbol{\Omega}^g$

An alternate approach towards nilpotency would be to evaluate the growth strain fields $q^0_{\alpha\beta}$, $q'_{\alpha\beta}$, and non-metricity fields $Q_{i\alpha\beta}$, which satisfy $\lambda^g = 0$ and $\Omega^g = 0$. These necessarily give rise to vanishing stresses and deflections (i.e., $\Phi = 0$ and $\mathrm{w} = \mathrm{w}^0$), unlike the weaker situation of vanishing stresses and moments in the previous section. These growth fields can be obtained as solutions to

$$e^{\alpha\mu}e^{\beta\nu}q^{0}_{\alpha\beta,\mu\nu}=0 \quad \text{and} \quad \nu q'_{\alpha\alpha,\beta\beta}+(1-\nu)q'_{\alpha\beta,\alpha\beta}=0 \tag{27}$$

for growth strains, and

$$e^{\alpha\mu}e^{\beta\nu}Q_{\alpha\mu\beta,\nu}=0 \quad \text{and} \quad \nu Q_{3\alpha\alpha,\beta\beta}+(1-\nu)Q_{3\alpha\beta,\alpha\beta}=0 \tag{28}$$

for the non-metricity components. Note the decoupling of the in-surface and the out-of-surface parts of growth. A general solution to Eq. (27) is

$$q^{0}_{\alpha\beta}=\phi_{(\alpha,\beta)} \quad \text{and} \quad q'_{\alpha\beta}=\big[(1+\nu)\delta_{\alpha\mu}\delta_{\beta\nu}-\nu\delta_{\alpha\beta}\delta_{\mu\nu}\big]e^{\mu\rho}e^{\nu\sigma}\psi_{(\rho,\sigma)}, \tag{29}$$

for arbitrary independent vector potentials $\phi_\alpha(\theta^\alpha)$ and $\psi_\alpha(\theta^\alpha)$ with sufficient smoothness.

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