Plane Strain Problem in Elastically Rigid Finite Plasticity

Anurag Gupta* David. J. Steigmann†

February 4, 2014

Abstract

A theory of elastically rigid finite deformation plasticity emphasizing the role of material symmetry is developed. The fields describing lattice rotation, dislocation density, and plastic spin, irrelevant in the case of isotropy, are found to be central to the present framework. A plane strain characteristic theory for anisotropic plasticity is formulated wherein the solutions, as well as the nature of their discontinuities, show remarkable deviation from the classical isotropic slipline theory.

keywords. Finite deformation plasticity, elastically rigid deformation, plane strain, anisotropy, slipline theory, strong discontinuities.

1. Introduction. In the theory of elastically rigid plasticity elastic strains are neglected and stresses are determinate only during plastic flow. It is suitable for problems involving large plastic deformation and offers greater analytical tractability in comparison to elasto-plastic theories. A systematic development of the theory can be pursued either by considering an elasto-plastic theory, under the limiting case of extreme elastic moduli, or by *a priori* neglecting elastic strains [3]. The latter viewpoint is more transparent for it avoids any assumption on the nature of elasticity, inadvertently present in the former, while bringing out the truly plastic nature of stress and strain fields; it is the choice for the paper at hand. In the present formulation of a general theory the

*Department of Mechanical Engineering, Indian Institute of Technology, Kanpur, UP, India 208016, email: ag@iitk.ac.in (communicating author)

†Department of Mechanical Engineering, University of California, Berkeley, CA, USA 94720, email: dsteigmann@berkeley.edu

main objective is to bring out the role of material symmetry in posing a complete boundary value problem. Such a consideration has been mostly ignored in an otherwise classical discipline.

Rigid plasticity theory is best studied under the plane strain assumption where it displays an elegant mathematical structure allowing for rigorous construction of analytical solutions. In fact the isotropic plane strain problem remains one of the most well established subject in classical plasticity [18, 10]. It has found successful applications in diverse areas such as metal forming [18], soil mechanics [6], glacier mechanics [25], and tectonics [34], to name a few. The incompressible plane strain problem is, in particular, unique in that it remains hyperbolic for all the stress values in the plastic regime. This is neither true for the other two-dimensional problems of axial symmetry, plane stress, and plate theory [18, 10] nor for the problems in three-dimension [35]. The hyperbolic nature of the problem, which naturally allows for discontinuities in the solution, is to be contrasted with the problems in quasi-static linear elasticity which are always elliptic.

The anisotropic plane strain problem, on the other hand, has remained mostly unexplored with only a few analytical studies [2, 29, 20] and fewer applications [30]. Unlike the pressure-insensitive incompressible isotropic case, where the radius of the circular yield locus is the only constitutive input, the anisotropic problem involves a non-circular yield locus and a plastic spin tensor field; the locus, which can possibly have corners and flat segments, in fact rotates with the rate of change in the lattice rotation. All of the previous work, whether for a specific shape of the yield locus [17, 31] or for arbitrary shapes [2, 29], neglects the influence of lattice rotation and plastic spin. This is justifiable for incipient plastic flow problems but not for those involving continued plastic deformation, or those with dislocated domains. The inclusion of these features bring additional richness into the plane problem without disturbing its hyperbolicity. Moreover, the nature of discontinuities in the solution, which for anisotropy is significantly different from isotropy, is discussed rarely (cf. [31, 29]) and only in insufficient detail. The present work is an attempt to fill these gaps and develop a general framework to solve problems of anisotropic plane strain plastic flows.

After fixing the notation and summarizing the background theory in the rest of this section, a theory of anisotropic elastically rigid plastic solids is developed in Section 2. Several conceptual issues related to material symmetry, intermediate configuration, flow rules, and plastic spin are discussed. In Section 3 the boundary value problem is reduced under the plane strain assumption.

Subsequently, the theory of characteristics is used in Section 4 to develop solutions for the isotropic and the anisotropic problem. The latter is discussed first by neglecting, and then incorporating, the lattice rotation field. The nature of discontinuities in stress, velocity, and lattice rotation fields is studied in detail. The considerations of the present work are limited to a purely mechanical and rate-independent theory. The conceptual framework is developed along the lines of a recent work on finite anisotropic plasticity by the present authors [13, 14, 33].

1.1. Notation. Let $\mathcal{E}$ and $\mathcal{V}$ be the three-dimensional Euclidean point space and its translation space, respectively. The space of linear transformations from $\mathcal{V}$ to $\mathcal{V}$ (second order tensors) is denoted by Lin, the group of rotation tensors by $Orth^+$, and the linear subspaces of symmetric and skew tensors by Sym and Skw, respectively. For $\mathbf{A} \in Lin$, $\mathbf{A}^t$, $\mathbf{A}^{-1}$, $\mathbf{A}^*$, $Sym\mathbf{A}$, $Skw\mathbf{A}$, $\operatorname{tr}\mathbf{A}$, and J_A stand for the transpose, the inverse, the cofactor, the symmetric part, the skew part, the trace, and the determinant of $\mathbf{A}$, respectively. The tensor product of $\mathbf{a} \in \mathcal{V}$ and $\mathbf{b} \in \mathcal{V}$, written as $\mathbf{a} \otimes \mathbf{b}$, is a tensor such that $(\mathbf{a} \otimes \mathbf{b})\mathbf{c} = (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$ for any arbitrary $\mathbf{c} \in \mathcal{V}$. The inner product in Lin is given by $\mathbf{A} \cdot \mathbf{B} = \operatorname{tr}(\mathbf{A}\mathbf{B}^t)$, where $\mathbf{A}, \mathbf{B} \in Lin$; the associated norm is $|\mathbf{A}| = \sqrt{\mathbf{A} \cdot \mathbf{A}}$. The identity tensor is denoted by $\mathbf{I}$. The derivative of a scalar valued differentiable function $G(\mathbf{A})$, denoted by $\partial_{\mathbf{A}} G$, is a second order tensor defined by

$$G(\mathbf{A} + \mathbf{B}) = G(\mathbf{A}) + \partial_{\mathbf{A}} G \cdot \mathbf{B} + o(|\mathbf{B}|), \tag{1.1}$$

where $o(|\mathbf{B}|) \to 0$ as $|\mathbf{B}| \to 0$. Similar definitions hold for derivatives of vector and tensor valued functions. The set of real numbers is called $\mathbb{R}$.

1.2. Basic theory. Let $\kappa_r \subset \mathcal{E}$ and $\kappa_t \subset \mathcal{E}$ be the fixed reference placement and the current placement of the body, respectively, such that there exists a bijective map $\boldsymbol{\chi}$, the motion, between them; i.e. for every $\mathbf{X} \in \kappa_r$ and time t there is a unique $\mathbf{x} \in \kappa_t$ given by $\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t)$. The motion is assumed to be continuous but piecewise differentiable over κ_r and continuously differentiable with respect to t. The deformation gradient, $\mathbf{F} = \nabla\boldsymbol{\chi}$, is well-defined and invertible whenever $\boldsymbol{\chi}$ is differentiable. It is assumed that $J_F > 0$ for κ_r to be a kinematically possible configuration of the body. The particle velocity $\mathbf{v} \in \mathcal{V}$ is related to the motion by $\mathbf{v} = \dot{\boldsymbol{\chi}}$, where the superposed dot represents the material time derivative (at fixed $\mathbf{X}$). Let $s \subset \kappa_t$ be an evolving surface across which various fields, which otherwise are continuous in $\kappa_t \backslash s$, may be discontinuous. The surface s

is assumed to be oriented with a well defined unit normal $\mathbf{n}_s \in \mathcal{V}$ and normal velocity (with respect to a fixed reference frame) $V \in \mathbb{R}$.

The equations of mass balance are (cf. p. 71 in [32])

$$\partial_t \rho + \operatorname{div}(\rho \mathbf{v}) = 0 \text{ in } \kappa_t \backslash s \text{ and} \tag{1.2}$$

$$V[\![\rho]\!] - [\![\rho \mathbf{v}]\!] \cdot \mathbf{n}_s = 0 \text{ on } s, \tag{1.3}$$

where $\rho \in \mathbb{R}$ is the mass density in κ_t, ∂_t is the time derivative at fixed $\mathbf{x}$, and div is the divergence with respect to $\mathbf{x}$. Here, $[\![\cdot]\!] \equiv (\cdot)^+ - (\cdot)^-$ is the discontinuity on s, with subscripts $\pm$ denoting the limits of the argument as s is approached from the regions into which $\mathbf{n}_s$ and $-\mathbf{n}_s$ are directed, respectively. The equations of momentum balance are (cf. p. 73 in [32])

$$\operatorname{div} \mathbf{T} + \rho \mathbf{b} = \rho \partial_t \mathbf{v} + \rho \mathbf{L} \mathbf{v}, \;\; \mathbf{T} = \mathbf{T}^t \text{ in } \kappa_t \backslash s \text{ and} \tag{1.4}$$

$$[\![\mathbf{T}]\!] \mathbf{n}_s + j_s [\![\mathbf{v}]\!] = \mathbf{0} \text{ on } s, \tag{1.5}$$

where $\mathbf{T} \in Lin$ is the Cauchy stress, $\mathbf{b} \in \mathcal{V}$ is the specific body force density, $\mathbf{L} \equiv \operatorname{grad} \mathbf{v}$ is the velocity gradient (grad denotes the gradient with respect $\mathbf{x}$), and $j_s \equiv \langle \rho \rangle V - \langle \rho \mathbf{v} \rangle \cdot \mathbf{n}_s$ is the flux of mass through the singular surface, where $\langle \cdot \rangle \equiv \frac{(\cdot)^+ + (\cdot)^-}{2}$.

Assume the body to be a materially uniform simple solid [24]. For a simple body the material response at a point is given in terms of the local value of fields, whereas material uniformity requires all the points in the body to consist of the same material. For a materially uniform simple solid there always exists a uniform *undistorted* configuration with respect to which the material symmetry group of every material point is a subset of the orthogonal group and is uniform (i.e. the symmetry group is identical for all material points). The body is said to be dislocated (or inhomogeneous) if there is no globally continuous and piecewise differentiable map from κ_t to any undistorted configuration [24]. It is then useful to work with the local tangent space of the latter denoted by $\kappa_i \subset \mathcal{V}$ (the intermediate configuration) in our subsequent discussion. For an elasto-plastic body κ_i can be obtained as a local stress-free (or natural) configuration [13]. For an elastically rigid plastic isotropic body the current configuration κ_t can be identified as a uniform undistorted configuration and hence it remains homogeneous; that this is not so for anisotropic bodies is argued in Subsection 2.1 below. The lattice distortion $\mathbf{H} \in Lin$ is defined as the invertible map from κ_i to the translation space of κ_t. The plastic distortion $\mathbf{K} \in Lin$, which maps κ_i to the translation space of κ_r, is thus given by $\mathbf{H} = \mathbf{FK}$. Writing $\mathbf{K}^{-1}$ as $\mathbf{G}$, often denoted by $\mathbf{F}^p$ in the literature, this multiplicative decomposition becomes

$$\mathbf{F} = \mathbf{HG}. \tag{1.6}$$

We impose $J_H > 0$ and obtain $J_G > 0$. Unlike $\mathbf{F}$, neither $\mathbf{H}$ nor $\mathbf{G}$ are, in general, gradients of vector fields. A measure of their incompatibility, away from the singular surface, is furnished by the true dislocation density [4]

$$\boldsymbol{\alpha} \equiv J_H \mathbf{H}^{-1} \operatorname{curl} \mathbf{H}^{-1} \tag{1.7}$$

equivalently given by $J_G^{-1} \mathbf{G} \operatorname{Curl} \mathbf{G}$, where curl and Curl are the curl operators with respect to $\mathbf{x}$ and $\mathbf{X}$, respectively. On surface s the incompatibility is characterized by the surface dislocation density $\boldsymbol{\beta}$, defined as [1, 12]

$$\boldsymbol{\beta}^t \equiv [\![\mathbf{H}^{-1}]\!] \boldsymbol{\epsilon}_{(\mathbf{n})}, \tag{1.8}$$

where $\boldsymbol{\epsilon}_{(\mathbf{n})}$ is the two-dimensional permutation tensor density on $T_s(\mathbf{x})$, the tangent space of s at $\mathbf{x} \in s$. For any two unit vectors in $T_s(\mathbf{x})$, say $\mathbf{t}_1$ and $\mathbf{t}_2$, which with $\mathbf{n}_s$ form a positively oriented orthonormal basis at $\mathbf{x} \in s$, we have $\boldsymbol{\epsilon}_{(\mathbf{n})} = \mathbf{t}_1 \otimes \mathbf{t}_2 - \mathbf{t}_2 \otimes \mathbf{t}_1$. The two dislocation densities are related to each other by the compatibility condition [12]

$$[\![J_H^{-1} \boldsymbol{\alpha}^t \mathbf{H}^t]\!] \mathbf{n}_s = \operatorname{div}^s \boldsymbol{\beta}^t \text{ on } s, \tag{1.9}$$

where div^s is the surface divergence on s. It represents the fact that dislocation lines along the singular surface can end arbitrarily only to be continued within the neighboring bulk.

2. Elastically rigid plasticity. Elastic rigidity requires $\mathbf{H} \in Orth^+$; hence elastic strain and elastic strain energy both vanish. The lattice distortion $\mathbf{H}$ is then identified as the lattice rotation field. The stress field however is non-zero but indeterminate, except within the plastic region. The velocity field in the rigid region is obtained from the boundary conditions, while within the plastic region it is solved using in addition the equations of plastic flow. It is discontinuous at the boundary of rigid and plastic region and possibly across surfaces within the plastic region. Before moving to a more detailed discussion on the nature of these solutions, several general considerations regarding the intermediate configuration, the plastic spin, and the flow rule will be taken up in the following.

2.1. Intermediate configuration. The non-uniqueness in determining the intermediate configuration κ_i for a given motion and material response is now investigated. At fixed t and for a given motion $\boldsymbol{\chi}$ consider lattice distortions $\mathbf{H}_1$ and $\mathbf{H}_2$ from two local configurations κ_{i_1} and κ_{i_2} to κ_t, respectively. The mapping from κ_{i_1} to κ_{i_2} is a tensor $\mathbf{A}$ defined by

$$\mathbf{H}_1 = \mathbf{H}_2\mathbf{A}. \tag{2.1}$$

$\mathbf{A}$ is a rotation since elastic rigidity requires both $\mathbf{H}_1$ and $\mathbf{H}_2$ to be rotations. The local configurations are required to yield the same motion, thereby requiring $\mathbf{F}_1 = \mathbf{F}_2$ and $\mathbf{G}_2 = \mathbf{A}\mathbf{G}_1$. Let $\mathcal{G}_1 \subset Orth^+$ be the symmetry group at a material point with respect to κ_{i_1} and a given material response function.

We show that the decomposition

$$\mathbf{A} = \mathbf{P}\mathbf{R} \tag{2.2}$$

holds, where $\mathbf{P} \in Orth^+$ is uniform[1]; if $\mathcal{G}_1$ is continuous (i.e. isotropy and transverse isotropy) then $\mathbf{R} \in \mathcal{G}_1$ is a piecewise-continuous field (possibly discontinuous across s) whereas if $\mathcal{G}_1$ is discrete then $\mathbf{R} \in \mathcal{G}_1$ is a piecewise-uniform field. Similar results, within the context of elastic bodies, were first given by Noll (Theorem 8 in [24]). To verify this proposition, consider a material response function given in terms of a smooth scalar-valued function $F = \hat{F}(\mathbf{S})$, where $\mathbf{S}$ is the second Piola-Kirchhoff stress (relative to κ_i) defined by

$$\mathbf{T} = \mathbf{H}\mathbf{S}\mathbf{H}^t, \tag{2.3}$$

and denote it by $\hat{F}_1(\mathbf{S}_1)$ or $\hat{F}_2(\mathbf{S}_2)$ depending on the choice of local intermediate configuration. This function is assumed to be invariant under superposed rigid body motions. The Cauchy stress tensor, being defined on κ_t, is invariant with respect to both (2.1) and material symmetry transformations. The former invariance, with the help of (2.1) and (2.3), yields $\mathbf{S}_2 = \mathbf{A}\mathbf{S}_1\mathbf{A}^t$, with obvious notation. Invariance of the material response with respect to the choice of local configuration requires $\hat{F}_2(\mathbf{S}_2) = \hat{F}_1(\mathbf{S}_1) = \hat{F}_1(\mathbf{A}^t\mathbf{S}_2\mathbf{A})$. On the other hand, for $\mathbf{R} \in \mathcal{G}_1$, invariance of F_1 and $\mathbf{T}$ under $\mathcal{G}_1$ implies $\hat{F}_1(\mathbf{S}_1) = \hat{F}_1(\mathbf{R}\mathbf{S}_1\mathbf{R}^t)$. Combine these two results to obtain

$$\hat{F}_2(\mathbf{S}_2) = \hat{F}_1(\mathbf{P}^t\mathbf{S}_2\mathbf{P}). \tag{2.4}$$

[1] A tensor field is uniform if it is continuous and independent of the position. It is piecewise-uniform if it is independent of the position and continuous everywhere except across s.

Material uniformity requires F_2 to depend on position only implicitly through $\mathbf{S}_2$; the response will be otherwise different at different material points for the same value of $\mathbf{S}_2$. This is possible if and only if $\mathbf{P}$ is uniform. If $\mathcal{G}_1$ is a discrete group then, by the uniformity of $\mathbf{P}$, $\mathbf{R}$ has to be piecewise-uniform to ensure that $\mathbf{H}_1$, given by (2.1), retains the piecewise-continuity from $\mathbf{H}_2$; whereas if $\mathcal{G}_1$ is a continuous group, then $\mathbf{R}$ can be any piecewise-continuous field with values drawn from $\mathcal{G}_1$. For isotropic response, i.e. $\mathcal{G}_1 = Orth^+$, $\mathbf{A}$ (which now belongs to $\mathcal{G}_1$) is non-uniform and hence there always exists an intermediate configuration with respect to which the lattice distortion $\mathbf{H}$ is equal to $\mathbf{I}$ at all material points. A materially uniform isotropic elastically-rigid body is hence homogeneous and κ_t is a uniform undistorted configuration (cf. Theorem 9 in [24]). Of course, if $\mathbf{H}$ is uniform then it can be reduced to $\mathbf{I}$ even in the anisotropic case; it is therefore truly relevant only for a dislocated anisotropic solid. It is also clear that, under symmetry transformations within a continuous group, the magnitude of dislocation density tensor will change implying that it can no longer be taken as a definitive measure of inhomogeneity. Otherwise, whenever the symmetry group is discrete, $\boldsymbol{\alpha}$ is uniquely determined modulo a rigid-body rotation and $\boldsymbol{\beta}$ is uniquely determined modulo a rigid-body rotation and a relative rigid rotation across s, with the latter restricted to be an element of the symmetry group.

Remark 2.1. (Noll's rule) The relation between the symmetry groups with respect to two local configurations can be derived by starting with (2.4) to obtain $\hat{F}_2(\mathbf{S}_2) = \hat{F}_2(\hat{\mathbf{R}}\mathbf{S}_2\hat{\mathbf{R}}^t)$, where $\hat{\mathbf{R}} = \mathbf{A}\mathbf{R}\mathbf{A}^t$; thus $\mathcal{G}_2 = \mathbf{A}\mathcal{G}_1\mathbf{A}^t$.

2.2. Dissipation and plastic spin. For an arbitrary part ω of κ_t, the dissipation $\mathcal{D}$ is defined as the difference between the power supplied to ω and the rate of change of the total energy in ω, the latter being equal to the change in the total kinetic energy of ω. Thus,

$$\mathcal{D} = \int_{\partial\omega} \mathbf{Tn}\cdot\mathbf{v}da + \int_{\omega} \mathbf{b}\cdot\mathbf{v}dv - \frac{d}{dt}\int_{\omega}\frac{1}{2}\rho|\mathbf{v}|^2 dv. \tag{2.5}$$

According to the mechanical version of the second law of thermodynamics we have $\mathcal{D} \geq 0$ for all $\omega \subset \kappa_t$, or equivalently [14]

$$D \equiv \mathbf{S}\cdot\dot{\mathbf{G}}\mathbf{G}^{-1} \geq 0 \text{ in } \kappa_t\backslash s \text{ and} \tag{2.6}$$

$$D_s \equiv \langle\mathbf{T}\rangle\mathbf{n}_s\cdot[\![\mathbf{v}]\!] \geq 0 \text{ on } s, \tag{2.7}$$

provided balances of mass and momentum are satisfied. These inequalities furnish restrictions on the evolution of plastic flow and the evolution of the discontinuity, respectively. It is straightforward to see that the local configurations, considered in Subsection 2.1, yield equal dissipation. Additional postulates, involving D, will be introduced in the next section to derive the equations of plastic flow while requiring inequality (2.6) to be satisfied everywhere in the plastic region. Inequality (2.7) will subsequently used to restrict the nature of stress and velocity discontinuities.

The skew part of $\dot{\mathbf{G}}\mathbf{G}^{-1}$, identified as plastic spin, makes no contribution to D in (2.6). This leads to the question whether or not plastic spin can be suppressed without affecting the initial-boundary-value problem. The answer is in affirmative for the case of isotropy (cf. [16]). Indeed, with the notation introduced in Subsection 2.1, we have $\dot{\mathbf{G}}_2\mathbf{G}_2^{-1} = \mathbf{A}(\dot{\mathbf{G}}_1\mathbf{G}_1^{-1} + \mathbf{A}^t\dot{\mathbf{A}})\mathbf{A}^t$, where $\mathbf{A}$ is a non-uniform rotation field. For a fixed material point define $\hat{\mathbf{\Omega}}(t) = Skw(\dot{\mathbf{G}}_1\mathbf{G}_1^{-1})$. The existence and uniqueness of the solution to $\dot{\mathbf{B}} = \hat{\mathbf{\Omega}}\mathbf{B}$, with $\mathbf{B}_0 \in Orth^+$ as the initial condition, is guaranteed by the standard theory of ordinary differential equations. To verify that the solution is a rotation (cf. p. 228 in [15]), let $\mathbf{Z}(t) = \mathbf{B}\mathbf{B}^t$ and obtain $\dot{\mathbf{Z}} = \hat{\mathbf{\Omega}}\mathbf{Z} - \mathbf{Z}\hat{\mathbf{\Omega}}$ with $\mathbf{Z}(0) = \mathbf{I}$. The foregoing equation has $\mathbf{Z} = \mathbf{I}$ as the unique solution; hence $\mathbf{B}$ is an orthogonal tensor. That $\mathbf{B} \in Orth^+$ follows from $\dot{J}_B = J_B \operatorname{tr} \hat{\mathbf{\Omega}} = 0$ and $J_{B_0} = 1$. The desired rotation which nullifies the plastic spin is $\mathbf{A} = \mathbf{B}^t$. Thus, there always exists an intermediate configuration with respect to which the plastic spin vanishes identically. This is not true for the anisotropic response, where $\mathbf{A}$ is uniform and hence cannot nullify plastic spin at every material point, unless $\hat{\mathbf{\Omega}}$ is also uniform. Moreover, as argued in [33], the assumption of vanishing plastic spin is incompatible with the notion of fixed lattice vectors in κ_i. It is shown in the next subsection that specifying plastic spin entails the prescription of additional constitutive restrictions (cf. [8]).

2.3. Constitutive framework for plastic flow. For a plastic deformation process, assumed to be rate-independent, the constitutive assumptions include a yield criterion, the maximum dissipation hypothesis, and a prescription for plastic spin, all adhering to material symmetry restrictions. The yield criterion restricts the stress field, during plastic flow, to a manifold (in the space of symmetric tensors) parameterized by other variables. It is assumed to be given by

$$F(\mathbf{S}, \boldsymbol{\alpha}) = 0, \tag{2.8}$$

where F is a continuously differentiable scalar function of its arguments. It is invariant with respect to superposed rigid-body motions and compatible changes in the reference configuration [13]. As noted earlier, $\boldsymbol{\alpha}$ is well-defined only for materials with discrete symmetry group and therefore so is the criterion given above; for isotropic materials F is necessarily independent of $\boldsymbol{\alpha}$. Let $\mathcal{G}$ be the material symmetry group relative to the undistorted configuration κ_i and with respect to the response function given by (2.8). Then [13]

$$F(\mathbf{S}, \boldsymbol{\alpha}) = F(\mathbf{R}^t\mathbf{S}\mathbf{R}, \mathbf{R}^t\boldsymbol{\alpha}\mathbf{R}), \tag{2.9}$$

where $\mathbf{R}$ is an arbitrary element of $\mathcal{G}$. This relation is used to decide the form of F for a given symmetry group, see for example Remarks 2.2 and 2.3 at the end of present subsection.

The evolution of plastic deformation is assumed to follow from the maximum dissipation hypothesis: for a given plastic distortion rate $\dot{\mathbf{G}}\mathbf{G}^{-1}$, at a fixed material point (away from the singular surface), the associated stress value is the one which maximizes dissipation D, defined in (2.6), while restricting the stress field to lie on or within the yield manifold. The hypothesis is a consequence of Ilyushin's postulate for finite elasto-plasticity [14]. This is not so within the present theory of *a priori* elastically-rigid plasticity, where Ilyushin's postulate reduces to the dissipation inequality (2.6), thereby furnishing no additional restriction. The required flow rule is therefore obtained by

$$\max(\mathbf{S}\cdot\dot{\mathbf{G}}\mathbf{G}^{-1}) \text{ subject to } G(\mathbf{S}, \boldsymbol{\alpha}) \leq 0 \text{ and } \mathbf{M} = \mathbf{0}, \tag{2.10}$$

where $\mathbf{M} = Skw(\mathbf{S})$ and $G(\cdot, \boldsymbol{\alpha})$ is a smooth extension of $F(\cdot, \boldsymbol{\alpha})$ from Sym to Lin satisfying the same material symmetry rule [33]. This is an optimization problem with both equality and inequality constraints; the relevant Kuhn-Tucker necessary condition, assuming $G(\cdot, \boldsymbol{\alpha})$ to be differentiable, is [37]

$$\dot{\mathbf{G}}\mathbf{G}^{-1} = \lambda\partial_{\mathbf{S}}G + (\partial_{\mathbf{S}}\mathbf{M})^t\bar{\boldsymbol{\Omega}}, \tag{2.11}$$

where $\lambda \in \mathbb{R}^+$ and $\bar{\boldsymbol{\Omega}} \in Skw$ are Lagrange multipliers. The derivative $\partial_{\mathbf{S}}G \in Lin$ is evaluated on Sym. The transpose of the fourth-order tensor $\partial_{\mathbf{S}}\mathbf{M}$ is defined by $(\partial_{\mathbf{S}}\mathbf{M})\mathbf{A}\cdot\mathbf{B} = \mathbf{A}\cdot(\partial_{\mathbf{S}}\mathbf{M})^t\mathbf{B}$, where $\mathbf{A} \in Lin$ and $\mathbf{B} \in Lin$ are arbitrary. With $(\partial_{\mathbf{S}}\mathbf{M})^t[\bar{\boldsymbol{\Omega}}] = \bar{\boldsymbol{\Omega}}$ (2.11) reduces to

$$\dot{\mathbf{G}}\mathbf{G}^{-1} = \lambda\partial_{\mathbf{S}}G + \bar{\boldsymbol{\Omega}}, \tag{2.12}$$

and hence

$$Sym(\dot{\mathbf{G}}\mathbf{G}^{-1}) = \lambda\partial_{\mathbf{S}}G \text{ and } Skw(\dot{\mathbf{G}}\mathbf{G}^{-1}) = \bar{\boldsymbol{\Omega}}. \tag{2.13}$$

The material time derivative of (1.6), with $\mathbf{H} \in Orth^+$, yields

$$\dot{\mathbf{F}}\mathbf{F}^{-1} = \dot{\mathbf{H}}\mathbf{H}^t + \mathbf{H}\dot{\mathbf{G}}\mathbf{G}^{-1}\mathbf{H}^t, \tag{2.14}$$

and consequently, using (2.13),

$$\mathbf{D} = Sym(\dot{\mathbf{F}}\mathbf{F}^{-1}) = \lambda\mathbf{H}(\partial_{\mathbf{S}}G)\mathbf{H}^t \text{ and} \tag{2.15}$$

$$\mathbf{W} = Skw(\dot{\mathbf{F}}\mathbf{F}^{-1}) = \dot{\mathbf{H}}\mathbf{H}^t + \mathbf{H}\bar{\boldsymbol{\Omega}}\mathbf{H}^t, \tag{2.16}$$

where $\mathbf{D}$ is the rate of deformation tensor and $\mathbf{W}$ is the vorticity tensor. Furthermore, we use (2.3) to define $H(\mathbf{T}, \mathbf{H}, \boldsymbol{\alpha}) = G(\mathbf{H}^t\mathbf{T}\mathbf{H}, \boldsymbol{\alpha})$ and obtain

$$\partial_{\mathbf{T}}H = \mathbf{H}(\partial_{\mathbf{S}}G)\mathbf{H}^t \text{ and} \tag{2.17}$$

$$\partial_{\mathbf{H}}H = \mathbf{H}\left(\mathbf{S}\partial_{\mathbf{S}}G - (\partial_{\mathbf{S}}G)\mathbf{S}\right). \tag{2.18}$$

Combining (2.15) and (2.17), we obtain

$$\mathbf{D} = \lambda\partial_{\mathbf{T}}H. \tag{2.19}$$

The rate of deformation tensor vanishes at some $\mathbf{x}$ if and only if $\mathbf{v}(\mathbf{x}, t) = \mathbf{W}(t)\mathbf{x} + \mathbf{d}(t)$, where $\mathbf{d}$ and $\mathbf{W}$ are uniform (cf. [15], p. 69). This will always happen in the absence of plastic flow, i.e. when $\dot{\mathbf{G}} = \mathbf{0}$.

Under material symmetry transformations $\partial_{\mathbf{S}}G \to \mathbf{R}^t(\partial_{\mathbf{S}}G)\mathbf{R}$ and $\dot{\mathbf{G}}\mathbf{G}^{-1} \to \mathbf{R}^t\dot{\mathbf{G}}\mathbf{G}^{-1}\mathbf{R}$, where $\mathbf{R} \in \mathcal{G}$; these follow from (2.9) and $\mathbf{G} \to \mathbf{R}^t\mathbf{G}$, respectively. Relation (2.12) therefore requires $\bar{\boldsymbol{\Omega}} \to \mathbf{R}^t\bar{\boldsymbol{\Omega}}\mathbf{R}$. At this stage the constitutive specification for $\bar{\boldsymbol{\Omega}}$ is restricted only by material symmetry and the usual invariance with respect to superposed rigid-body motion and compatible changes in κ_r. Additional restrictions are however imposed on $\bar{\boldsymbol{\Omega}}$ within a finite elasto-plasticity theory [33]. For an isotropic response $\bar{\boldsymbol{\Omega}}$ vanishes identically and the multiplier λ is calculated as a part of the boundary value problem. For anisotropic responses $\bar{\boldsymbol{\Omega}}$ is prescribed constitutively while λ is obtained by solving a partial differential equation generated from the consistency condition [14]. Finally it is clear that in the absence of lattice spin, i.e. $\dot{\mathbf{H}}\mathbf{H}^t = \mathbf{0}$, no constitutive rule is required for the plastic spin. The latter is then in one-one correspondence with the material spin $\mathbf{W}$.

Attention is confined to the rules of the form $\bar{\boldsymbol{\Omega}} = \hat{\boldsymbol{\Omega}}\left(Sym(\dot{\mathbf{G}}\mathbf{G}^{-1}), \mathbf{S}, \boldsymbol{\alpha}\right)$, with $\hat{\boldsymbol{\Omega}}$ a smooth function of its arguments; these are invariant under superposed rigid body motions and compatible changes in the reference configuration [13]. Rate-independence, in conjugation with $(2.13)_1$, requires $\hat{\mathbf{\Omega}}$ to be homogeneous of degree one in λ, thereby furnishing the necessary and sufficient representation

$$\bar{\mathbf{\Omega}} = \lambda \mathbf{\Omega}(\mathbf{S}, \boldsymbol{\alpha}), \tag{2.20}$$

where $\mathbf{\Omega} \in Skw$. Here we have used the fact that $\partial_{\mathbf{S}} G$ is a function of $\mathbf{S}$ and $\boldsymbol{\alpha}$.

Considerations of material symmetry, which require that

$$\mathbf{\Omega}(\mathbf{R}^t \mathbf{S} \mathbf{R}, \mathbf{R}^t \boldsymbol{\alpha} \mathbf{R}) = \mathbf{R}^t \mathbf{\Omega}(\mathbf{S}, \boldsymbol{\alpha}) \mathbf{R} \tag{2.21}$$

for all $\mathbf{R} \in \mathcal{G}$, are used to derive representations for $\mathbf{\Omega}$. Here it is assumed that the symmetry group obtained with respect to plastic spin function is identical to the group $\mathcal{G}$ defined with respect to the yield function. Under isotropic symmetry $\bar{\mathbf{\Omega}}$ vanishes identically. Indeed, for the aforementioned reasons, $\mathbf{\Omega}$ is then independent of $\boldsymbol{\alpha}$ and, according to a representation theorem (cf. [38]), a skew tensor function of (only) a symmetric tensor field necessarily vanishes.

Remark 2.2. (Isotropic symmetry) Under isotropy G is independent of $\boldsymbol{\alpha}$, and $\mathbf{S}$ and $\partial_{\mathbf{S}} G$ are coaxial. To see the latter, recall (2.9) to note that $G(\mathbf{S}) = G(\mathbf{R}^t \mathbf{S} \mathbf{R})$ for all $\mathbf{R} \in Orth^+$. Consider a one parameter family of rotations $\mathbf{R}(s)$ such that $\mathbf{R}(0) = \mathbf{I}$ ($s \in \mathbb{R}$ is the parameter). Differentiate the aforementioned relation with respect to s, and evaluate it at $s = 0$, to get $((\partial_{\mathbf{S}} G)\mathbf{S} - \mathbf{S}(\partial_{\mathbf{S}} G)) \cdot \mathbf{R}'(0) = 0$, where $\mathbf{R}'(0) \in Skw$ is arbitrary. Hence $(\partial_{\mathbf{S}} G)\mathbf{S} = \mathbf{S}(\partial_{\mathbf{S}} G)$. This combined with (2.17) and (2.18) to imply $H(\mathbf{T}) = G(\mathbf{S})$. Moreover, H and G depend on their arguments necessarily through their invariants. The flow rule is given in (2.19). For a pressure insensitive isotropic von Mises type yield criterion $\partial_{\mathbf{T}} H = \mathbf{T}^d$, where $\mathbf{T}^d$ is the deviatoric part of $\mathbf{T}$; consequently $\mathbf{D} = \lambda \mathbf{T}^d$, which is the classical Lévy–von Mises flow rule.

Remark 2.3. (Cubic symmetry) Assume F to be independent of $\boldsymbol{\alpha}$ and $\operatorname{tr} \mathbf{S}$ (i.e. pressure insensitive); and that it depends on $\mathbf{S}$ through a homogeneous function of degree two. Under the cubic symmetry group the simplest representation for F is

$$F = \frac{A_1}{2} \left((S^d_{11})^2 + (S^d_{22})^2 + (S^d_{33})^2 \right) + A_2 \left(S^2_{12} + S^2_{13} + S^2_{23} \right) - d^2, \tag{2.22}$$

where A_1, A_2, and k are constants and $S^d_{ij} = \mathbf{S}^d \cdot Sym(\mathbf{e}_i \otimes \mathbf{e}_j)$ ($\mathbf{S}^d$ is the deviatoric part of $\mathbf{S}$) are the components of $\mathbf{S}^d$ with respect to an orthonormal basis, given by $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, aligned with the cube axes (which are fixed and assumed to be known). The above representation is obtained using invariant theorems for a scalar function dependent on a symmetric tensor, cf. [11]. With the above expression for the yield function we obtain

$$\mathbf{S}\cdot(\partial_{\mathbf{S}}F) = A_1\left((S^d_{11})^2+(S^d_{22})^2+(S^d_{33})^2\right)+2A_2\left(S^2_{12}+S^2_{13}+S^2_{23}\right). \tag{2.23}$$

The dissipation inequality (2.6) hence yields $A_1 \geq 0$ and $A_2 \geq 0$. To derive a simple relation for plastic spin under the cubic symmetry group it is assumed that $\mathbf{\Omega}$ is independent of $\boldsymbol{\alpha}$. It is also required, based on phenomenological considerations, that $\mathbf{\Omega}$ is an odd function of $\mathbf{S}$. The simplest representation of $\mathbf{\Omega}$ is a third order polynomial in $\mathbf{S}$ given by [9]

$$\begin{aligned}
2\Omega_{32} &= B_0 p S_{23}(S_{22}-S_{33}) + B_1 S_{11}S_{23}(S_{22}-S_{33}) + B_2 S_{23}\left(S^2_{12}-S^2_{13}\right) + B_3 S_{12}S_{13}(S_{33}-S_{22}),\\
2\Omega_{13} &= B_0 p S_{13}(S_{33}-S_{11}) + B_1 S_{22}S_{13}(S_{33}-S_{11}) + B_2 S_{13}\left(S^2_{23}-S^2_{12}\right) + B_3 S_{12}S_{23}(S_{11}-S_{33}), \quad \text{and}\\
2\Omega_{21} &= B_0 p S_{12}(S_{11}-S_{22}) + B_1 S_{33}S_{12}(S_{11}-S_{22}) + B_2 S_{12}\left(S^2_{13}-S^2_{23}\right) + B_3 S_{13}S_{23}(S_{22}-S_{11}),
\end{aligned} \tag{2.24}$$

where $\Omega_{ij} = \mathbf{\Omega}\cdot Skw(\mathbf{e}_i\otimes\mathbf{e}_j)$ and $3p = -\operatorname{tr}\mathbf{S}$; the scalar coefficients B_0, B_1, B_2, and B_3 are constant parameters. The plastic spin components given above vanish if $\mathbf{S}$ is coaxial with the cubic axes.

Remark 2.4. (Multiple yield surfaces) The flow rule (2.12) is easily modified when there are more than one inequality constraints in the optimization problem (2.10). For p continuously differentiable functions $G_a(\mathbf{S},\boldsymbol{\alpha})$ on $Lin\times Lin$, the problem is to find

$$\max(\mathbf{S}\cdot\dot{\mathbf{G}}\mathbf{G}^{-1}) \text{ subject to } G_a(\mathbf{S},\boldsymbol{\alpha})\leq 0 \text{ and } \mathbf{M}=\mathbf{0},\ a=1,\ldots,p. \tag{2.25}$$

The Kuhn-Tucker conditions, which replace (2.12), are given by (cf. [21])

$$\dot{\mathbf{G}}\mathbf{G}^{-1} = \sum_{a=1}^{p}\lambda_a\left(\partial_{\mathbf{S}}G_a + \mathbf{\Omega}_a(\mathbf{S},\boldsymbol{\alpha})\right), \tag{2.26}$$

where $\lambda_a\in\mathbb{R}^+$ and $\mathbf{\Omega}_a\in Skw$ are Lagrange multipliers (the derivatives $\partial_{\mathbf{S}}G_a$ are evaluated on Sym), leading to

$$\mathbf{D} = \sum_{a=1}^{p}\lambda_a\partial_{\mathbf{T}}H_a, \tag{2.27}$$

where $H_a(\mathbf{T},\mathbf{H},\boldsymbol{\alpha}) = G_a(\mathbf{H}^t\mathbf{T}\mathbf{H},\boldsymbol{\alpha})$.

2.4. Governing equations. The complete set of equations for an initial-boundary-value problem for elastically-rigid plastic deformations is now collected. Away from the discontinuity surface the problem constitutes of equation of motion $(1.4)_1$, the yield criterion (2.8) with $\mathbf{S} = \mathbf{H}^t\mathbf{T}\mathbf{H}$ and $\boldsymbol{\alpha} = \mathbf{H}^t\operatorname{curl}\mathbf{H}^t$ ($\mathbf{H}\in Orth^+$), and flow rules (2.15) and (2.16) with $\mathbf{D} = Sym(\operatorname{grad}\mathbf{v})$,

$\mathbf{W} = Skw(\operatorname{grad}\mathbf{v})$, and $\mathbf{\Omega}$ given by (2.20). The dislocation density tensor is non-trivial if and only if $\mathbf{H}$ is non-uniform. Thus, we have a total of thirteen equations for thirteen variables: $\mathbf{T} \in Sym$, $\mathbf{H} \in Orth^+$, $\mathbf{v} \in \mathcal{V}$, and $\lambda \in \mathbb{R}^+$. These are to be supplemented with boundary conditions for traction, velocity, and lattice rotation, and initial conditions for velocity and lattice rotation. Imposing plastic incompressibility, i.e. $J_G = 1$, requires $J_F = 1$ or equivalently

$$\operatorname{div}\mathbf{v} = 0 \text{ in } \kappa_t \backslash s. \tag{2.28}$$

Discontinuities in stress, velocity, and lattice rotation fields (and their gradients) occur either at the boundary of the rigid and the plastic regions or within the plastic region. The jumps in velocity and stress fields are restricted by (1.3) and (1.5), respectively. The former of these reduces to

$$[\![\mathbf{v}]\!] \cdot \mathbf{n}_s = 0 \text{ on } s \tag{2.29}$$

under the assumption of plastic incompressibility and continuity of mass density in κ_r. Taking a dot product of (1.5) with $\mathbf{n}_s$ and $\mathbf{t} \in T_s(\mathbf{x})$ yields $[\![\mathbf{T}]\!]\mathbf{n}_s \cdot \mathbf{n}_s = 0$ and $[\![\mathbf{T}]\!]\mathbf{n}_s \cdot \mathbf{t} + j_s[\![\mathbf{v}]\!] \cdot \mathbf{t} = 0$, respectively. The normal stress is therefore always continuous across l while the shear stress is balanced by the inertial term. Due to the impossibility of non-trivial rank-one connections between two different rotation tensors any discontinuity in lattice rotation $\mathbf{H}$ always leads to a non-trivial surface dislocation density. The definition of the latter as well as the compatibility relation connecting the bulk and the surface dislocation density were provided in Equations (1.8) and (1.9).

3. Plane strain problem. The plane strain problem is introduced by assuming the velocity field to be of the form

$$\mathbf{v} = u(x, y, t)\mathbf{e}_1 + v(x, y, t)\mathbf{e}_2, \tag{3.1}$$

where x and y are the Cartesian coordinates of $\mathbf{x} \in \kappa_t$ along $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively, such that unit vectors $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form a fixed right-handed orthonormal basis. Define $\varepsilon_{ij} = \mathbf{D} \cdot Sym(\mathbf{e}_i \otimes \mathbf{e}_j)$ and $\omega_{ij} = \mathbf{W} \cdot Skw(\mathbf{e}_i \otimes \mathbf{e}_j)$ as the components of $\mathbf{D}$ and $\mathbf{W}$ with respect to the fixed basis, respectively, where the subscripts vary from one to three. The gradient of (3.1) then yields $\varepsilon_{11} = \partial_x u$, $\varepsilon_{22} = \partial_y v$, $\varepsilon_{12} = \varepsilon_{21} = \frac{1}{2}(\partial_y u + \partial_x v)$, and $\omega_{12} = -\omega_{21} = \frac{1}{2}(\partial_y u - \partial_x v)$, the rest being zero. Hence $\mathbf{D} = \varepsilon_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$ and $\mathbf{W} = \omega_{\alpha\beta}\mathbf{e}_\alpha \otimes \mathbf{e}_\beta$, where the Greek indices vary between one and two and summation is implied for repeated indices.

Additionally, assume $\mathbf{e}_3$ to be the axis of lattice rotation field, i.e. $\mathbf{He}_3 = \mathbf{e}_3$, and let $\mathbf{H}$ be independent of the out-of-plane coordinate. This immediately furnishes the representation

$$\mathbf{H} = \mathbf{h}_1 \otimes \mathbf{e}_1 + \mathbf{h}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3, \tag{3.2}$$

where

$$\mathbf{h}_1 = \cos\gamma \mathbf{e}_1 + \sin\gamma \mathbf{e}_2, \text{ and } \mathbf{h}_2 = -\sin\gamma \mathbf{e}_1 + \cos\gamma \mathbf{e}_2 \tag{3.3}$$

Here $\gamma = \hat{\gamma}(x, y, t)$ represents the angular change in the lattice orientation measured anticlockwise with respect to the $\mathbf{e}_1$-axis. The function $\hat{\gamma}$ is piecewise continuously differentiable with respect to the space variables but continuously differentiable with respect to time. The bulk dislocation density, defined in (1.7), reduces to

$$\boldsymbol{\alpha} = (\mathbf{h}_1 \cdot \operatorname{grad}\gamma)\mathbf{e}_3 \otimes \mathbf{e}_1 + (\mathbf{h}_2 \cdot \operatorname{grad}\gamma)\mathbf{e}_3 \otimes \mathbf{e}_2. \tag{3.4}$$

A convenient form of the dislocation density is obtained when $\boldsymbol{\alpha}$ is written with respect to κ_t, i.e. $\bar{\boldsymbol{\alpha}} = \mathbf{H}\boldsymbol{\alpha}\mathbf{H}^t$; with (3.2) this yields $\bar{\boldsymbol{\alpha}} = \mathbf{e}_3 \otimes \operatorname{grad}\gamma$ (cf. [4]). On the other hand the surface dislocation density, calculated using (3.2) and (1.8), takes the form

$$\boldsymbol{\beta} = 2\sin\frac{[\![\gamma]\!]}{2}\Big(\sin(\phi - \langle\gamma\rangle)\mathbf{e}_3 \otimes \mathbf{e}_1 - \cos(\phi - \langle\gamma\rangle)\mathbf{e}_3 \otimes \mathbf{e}_2\Big), \tag{3.5}$$

where ϕ is the angle between the tangent to the line of discontinuity in the plane and $\mathbf{e}_1$, measured anticlockwise with respect to the latter. In (3.5), and rest of the paper, it is assumed that the surface of discontinuity is such that $\mathbf{n}_s$ is spanned by $\{\mathbf{e}_1, \mathbf{e}_2\}$; thus $\boldsymbol{\epsilon}_{(\mathbf{n})} = \mathbf{t} \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{t}$, where $\mathbf{t} = \cos\phi \mathbf{e}_1 + \sin\phi \mathbf{e}_2$. Thus it will always intersect the plane in a line. If this line bisects $\mathbf{h}_\alpha^+$ and $\mathbf{h}_\alpha^-$ then $\phi = \langle\gamma\rangle$, reducing (3.5) to $\boldsymbol{\beta} = -2\sin\frac{[\![\gamma]\!]}{2}\mathbf{e}_3 \otimes \mathbf{e}_2$. As expected, dislocation densities $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ vanish if and only if $\operatorname{grad}\gamma = \mathbf{0}$ and $[\![\gamma]\!] = 0$, respectively, i.e. if γ is uniform. According to (3.4) and (3.5), $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are densities of edge dislocations with Burgers vector in the $\{\mathbf{e}_1, \mathbf{e}_2\}$ plane and dislocation lines along the $\mathbf{e}_3$-axis.

Let $S_{ij} = \mathbf{S} \cdot Sym(\mathbf{e}_i \otimes \mathbf{e}_j)$ and $\sigma_{ij} = \mathbf{T} \cdot Sym(\mathbf{e}_i \otimes \mathbf{e}_j)$. The flow rule (2.19) in conjunction with the plane strain assumptions imply that the yield function H is independent of σ_{13}, σ_{23}, and σ_{33} (or equivalently that G is independent of S_{13}, S_{23}, and S_{33}). With the assumption of plastic incompressibility, i.e. $\operatorname{tr}\mathbf{D} = 0$, the yield functions involve the stress through its deviatoric part and hence are of the form

$$H(\sigma_{11} - \sigma_{22}, \sigma_{12}, \gamma, \boldsymbol{\alpha}) = G(S_{11} - S_{22}, S_{12}, \boldsymbol{\alpha}) \tag{3.6}$$

keeping (2.3) and (3.2) in mind.[2] The flow rule (2.19) then reduces to two independent equations, which upon eliminating λ yields

$$\varepsilon_{11}\partial_{\sigma_{12}}H - 2\varepsilon_{12}\partial_{(\sigma_{11}-\sigma_{22})}H = 0. \tag{3.7}$$

It will be useful to write the above relations in an alternative form. Let σ_1 and σ_2 be the in-plane principal stresses and $\mathbf{u}_1$ and $\mathbf{u}_2$ be the corresponding principal directions. Let θ be the angle between $\mathbf{e}_1$ and $\mathbf{u}_1$ measured anticlockwise from the former. Hence obtain $\sigma_{11}-\sigma_{22} = (\sigma_1 - \sigma_2)\cos 2\theta$, $2\sigma_{12} = (\sigma_1-\sigma_2)\sin 2\theta$, $S_{11}-S_{22} = (\sigma_1-\sigma_2)\cos(2\theta-2\gamma)$, and $2S_{12} = (\sigma_1-\sigma_2)\sin(2\theta-2\gamma)$. Accordingly, the yield criterion can be expressed in the form $\hat{H}(\sigma_1-\sigma_2, \theta-\gamma, \boldsymbol{\alpha}) = 0$, where $\hat{H}$ is a continuously differentiable function. Use of the implicit function theorem allows us to solve this for $\sigma_1 - \sigma_2$ and hence furnishes the following form of the criterion

$$\sigma_1 - \sigma_2 = 2k(\theta-\gamma, \boldsymbol{\alpha}), \tag{3.8}$$

where k is a (non-zero) continuously differentiable function usually interpreted as the maximum shear stress (the coefficient 2 is for conventional reasons). The flow rule (3.7) consequently reduces to

$$2\partial_x u \cos(2\theta + 2\psi) + (\partial_y u + \partial_x v)\sin(2\theta+2\psi) = 0, \tag{3.9}$$

where ψ is defined from

$$k' = 2k\cot 2\psi. \tag{3.10}$$

Here k' denotes the partial derivative of k with respect to its first argument. For a physical interpretation of ψ it is convenient to visualize the yield criterion (3.8) as a polar plot $(k(\theta - \gamma, \cdot), 2\theta - 2\gamma)$ for fixed $\boldsymbol{\alpha}$. The horizontal and the vertical projections of a polar ray are given by $\frac{1}{2}(S_{11} - S_{22})$ and S_{12}, respectively. According to (3.10), 2ψ is the anticlockwise inclination of the tangent to the polar plot with respect to the radial direction. For an isotropic yield locus, the polar plot is circular (k constant) with $\psi = \pm\pi/4$.

Flow rules for anisotropic plastic evolution additionally involve the spin relation (2.16) which is now simplified under plane strain. To this end the material time derivative of (3.2) is substituted into (2.16) to obtain

$$\mathbf{W} = \dot{\gamma}(\mathbf{h}_2 \otimes \mathbf{h}_1 - \mathbf{h}_1 \otimes \mathbf{h}_2) + \mathbf{H}\bar{\boldsymbol{\Omega}}\mathbf{H}^t, \tag{3.11}$$

[2] According to (2.3) and (3.2) $S_{11}-S_{22} = (\sigma_{11}-\sigma_{22})\cos 2\gamma + 2\sigma_{12}\sin 2\gamma$ and $2S_{12} = -(\sigma_{11}-\sigma_{22})\sin 2\gamma + 2\sigma_{12}\cos 2\gamma$.

or equivalently

$$\mathbf{h}_2 \cdot \mathbf{W}\mathbf{h}_1 = \dot{\gamma} + \mathbf{e}_2 \cdot \bar{\mathbf{\Omega}}\mathbf{e}_1 \text{ and} \tag{3.12}$$

$$\bar{\mathbf{\Omega}}\mathbf{e}_3 = \mathbf{0}. \tag{3.13}$$

As shown below, (3.13) can be used to impose restrictions on the (out-of-plane) stress state of the plastic body. Relation (3.12) can be written in a succinct form

$$\omega_{21} = \dot{\gamma} + \bar{\Omega}_{21}, \tag{3.14}$$

where $\bar{\Omega}_{21} = -\bar{\Omega}_{12} = \bar{\mathbf{\Omega}} \cdot (\mathbf{e}_2 \otimes \mathbf{e}_1)$ are the only non-zero components of $\bar{\mathbf{\Omega}}$ with respect to the fixed basis. The above reduction requires $\mathbf{h}_2 \cdot \mathbf{W}\mathbf{h}_1 = \mathbf{e}_2 \cdot \mathbf{W}\mathbf{e}_1$ which follows from $\mathbf{WH} = \mathbf{HW}$. To verify the latter equality observe that both $\mathbf{W}$ and $\mathbf{H}$ have identical axial vector $\mathbf{e}_3$. As a result, for an arbitrary $\mathbf{a} \in \mathcal{V}$, $\mathbf{HWa} = \mathbf{H}(\mathbf{e}_3 \times \mathbf{a})$ which, on using the definition of the cofactor, can be rewritten as $\mathbf{He}_3 \times \mathbf{Ha}$ or equivalently as $\mathbf{WHa}$. Constitutive prescription for plastic spin therefore entails finding a representation for the two-dimensional skew tensor $\bar{\mathbf{\Omega}} = \bar{\Omega}_{21}(\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2)$, such that $\bar{\mathbf{\Omega}} = \lambda \mathbf{\Omega}(\mathbf{S}, \boldsymbol{\alpha})$, cf. (2.20).

3.1. Material symmetry and constitutive rules. The constitutive input for the plane strain anisotropic problem is provided by prescribing maximum shear stress k, see (3.8), and plastic spin $\mathbf{\Omega}(\mathbf{S}, \boldsymbol{\alpha})$. Under a given symmetry group the constitutive representations can be derived from the three-dimensional theory. As an illustration consider the yield function and plastic spin tensor, obtained for cubic symmetry group, from Remark 2.3. Assume the orthonormal triad $\mathbf{e}_i$ to be aligned with the cube axes. The yield function in (2.22) in conjunction with the flow rule (2.17) and the plane strain assumptions imply $S_{13} = 0$, $S_{23} = 0$, and $S^d_{33} = 0$. As a result $S^d_{11} = -S^d_{22} = \frac{1}{2}(S_{11} - S_{22})$, and $S_{33} = \frac{1}{2}(S_{11} + S_{22}) = -p$. Substituting these equalities into (2.22) and (2.24), and some straightforward manipulation, furnishes a yield criterion in the form (3.8) with

$$k(\theta - \gamma) = \frac{d}{\sqrt{a_1 \cos^2(2\theta - 2\gamma) + a_2 \sin^2(2\theta - 2\gamma)}}, \tag{3.15}$$

where $a_1 = \frac{A_1}{2}$, $a_2 = A_2$, and d are material constants, and components of the plastic spin tensor as

$$\Omega_{13} = \Omega_{23} = 0, \text{ and } 2\Omega_{21} = b_0 p k^2 \sin(4\theta - 4\gamma), \tag{3.16}$$

where $b_0 = (B_0 - B_1)$ is a material constant and k is as given in (3.15).

4. Slipline theory. The purpose of this section is to solve the plane strain problem using the method of characteristics. The emphasis is to obtain characteristic curves, with the associated normal forms, and discuss the nature of velocity, stress, and rotation discontinuities. Before proceeding further the set of governing equations is collected. The equations of equilibrium (1.4), after neglecting inertial terms and body force, are reduced to

$$\partial_x \sigma_{11} + \partial_y \sigma_{12} = 0 \text{ and } \partial_x \sigma_{12} + \partial_y \sigma_{22} = 0. \tag{4.1}$$

The incompressibility equation (2.28) reduces to

$$\partial_x u + \partial_y v = 0. \tag{4.2}$$

The yield criterion and the flow rules are given by Equations (3.8), (3.9), and (3.14). All together these are six equations for six unknowns $\sigma_1,\ \sigma_2,\ \theta,\ u,\ v$, and γ, with three independent variables $x,\ y$, and t. The equations are valid everywhere on the plane except at the line of discontinuity. The stress equations are decoupled from the velocity equations as long as k is independent of γ. The solution should satisfy appropriate boundary conditions, initial values, and jump conditions across the line of discontinuity. The latter include (2.29) and $[\![\mathbf{T}]\!]\mathbf{n}_s = \mathbf{0}$ (from (1.5) ignoring inertia), which with $\mathbf{n}_s = \sin\phi \mathbf{e}_1 - \cos\phi \mathbf{e}_2$ can be rewritten as

$$[\![u]\!] \sin\phi - [\![v]\!] \cos\phi = 0, \tag{4.3}$$

$$[\![\sigma_{11}]\!] \sin\phi - [\![\sigma_{12}]\!] \cos\phi = 0, \text{ and} \tag{4.4}$$

$$[\![\sigma_{12}]\!] \sin\phi - [\![\sigma_{22}]\!] \cos\phi = 0. \tag{4.5}$$

The last two relations together with (3.8) yield

$$\cot(2\phi - 2\langle\theta\rangle) = \frac{[\![k]\!] \cot[\![\theta]\!]}{2\langle k\rangle}, \tag{4.6}$$

which is an equation for the slope of stress discontinuity curve, and

$$[\![p]\!] = -2\langle k\rangle \sin[\![\theta]\!] \frac{\cos(4\phi - 4\langle\theta\rangle)}{\sin(2\phi - 2\langle\theta\rangle)}, \tag{4.7}$$

where $2p = -(\sigma_1 + \sigma_2)$ is the hydrostatic pressure. Hence the in-plane stress components are continuous if and only if p is continuous. In fact the jump in stress tensor, projected onto the plane, can be written as

$$[\![\mathbb{P}\mathbf{T}\mathbb{P}]\!] = -2[\![p]\!]\mathbf{t} \otimes \mathbf{t}, \tag{4.8}$$

where $\mathbf{t} = \cos\phi\mathbf{e}_1 + \sin\phi\mathbf{e}_2$ is the unit tangent to the discontinuity line and $\mathbb{P} = \mathbf{I} - \mathbf{e}_3 \otimes \mathbf{e}_3$ is the projection tensor.

We now show that, for a strictly convex yield contour, the jump in the velocity field cannot coincide with the jump in the stress field and that the strain-rate tensor always vanishes at such a stress discontinuity (cf. pp. 271-273 in [19]). Recall the surface dissipation inequality (2.7) which, in the present situation, takes the form

$$\mathbf{T}^{\pm} \cdot [\![\mathbf{v}]\!] \otimes \mathbf{n}_s \geq 0, \tag{4.9}$$

where the superscript $\pm$ implies that either of the signs can be chosen. Following Subsection 2.3, the postulate of maximum dissipation can be invoked to obtain

$$Sym([\![\mathbf{v}]\!] \otimes \mathbf{n}_s) = \chi^{\pm}(\partial_{\mathbf{T}} H)^{\pm} = \hat{\chi}^{\pm}\mathbf{D}^{\pm}, \tag{4.10}$$

where $\chi^{+} \in \mathbb{R}^{+}$ and $\chi^{-} \in \mathbb{R}^{+}$ are plastic multipliers and $\hat{\chi}^{\pm} = \chi^{\pm}/\lambda^{\pm}$. The second equality in (4.10) has been obtained using (2.19). According to the stress equilibrium at the surface, i.e. $[\![\mathbf{T}]\!]\mathbf{n}_s = \mathbf{0}$, and (4.10), the stress jump and the strain-rate tensor at the discontinuity are orthogonal to each other, i.e.

$$[\![\mathbf{T}]\!] \cdot \mathbf{D}^{\pm} = 0, \text{ and consequently } [\![\mathbf{T}]\!] \cdot [\![\mathbf{D}]\!] = 0. \tag{4.11}$$

However, for a strictly convex yield surface, with an associated flow rule given by (2.19),

$$[\![\mathbf{T}]\!] \cdot \mathbf{D}^{+} > 0 \text{ and } [\![\mathbf{T}]\!] \cdot \mathbf{D}^{-} < 0. \tag{4.12}$$

The contradictory result proves the non-coincidence of a velocity discontinuity with that of stress, unless the yield contour is no longer strictly convex. The latter is true, for instance, when the contour has linear segments (as in Figure 1). Furthermore, for a continuous velocity field, $[\![\mathbf{D}]\!] = \mathbf{k} \otimes \mathbf{n}_s + \mathbf{n}_s \otimes \mathbf{k}$ (where $\mathbf{k} \in \mathcal{V}$ is arbitrary), which when combined with $[\![\mathbf{T}]\!]\mathbf{n}_s = \mathbf{0}$ again leads to $(4.11)_2$. The ensuing contradiction with (4.12) implies the absence of any plastic deformation at the interface, i.e. $\mathbf{D}^{\pm} = \mathbf{0}$. Note that the above results hold only for the stress states away from the vertex on the yield contour; the relevant modification is straightforward and will not be discussed here.

The theory of characteristics is now used to develop a slipline field theory for isotropic and anisotropic flows. The latter is dealt with first by neglecting lattice rotation and then subsequently incorporating it. The characteristic directions as well as the normal forms have been obtained using the standard procedure from the theory of quasilinear hyperbolic equations. A brief account of the mathematical theory is given below before moving on to its application.

Consider a system of n quasilinear first-order equations

$$A\partial_x U + B\partial_y U + C = 0, \tag{4.13}$$

defined over some region of a two-dimensional Euclidean point space, where U is a $n \times 1$ matrix of dependent variables, A, B (both $n \times n$) and C ($n \times 1$) are known matrix functions of independent variables x, y, and U; $\partial_x U$ and $\partial_y U$ are partial derivatives of U with respect to x and y, respectively. Assume A to be non-singular and let $D = A^{-1}B$ have real eigenvalues. The system of equations (4.13) is called *strictly hyperbolic* if all the eigenvalues of D are distinct. It is *hyperbolic* if there are repeated eigenvalues but D is diagonalizable, i.e. there exists n linearly independent eigenvectors ξ_a ($1 \times n$ matrix functions) such that $\xi_a D = \lambda_a \xi_a$. If the n linearly independent eigenvectors do not exist then the system (4.13) is called *degenerate hyperbolic.* The matrix D is no more diagonalizable and it is not possible to reduce (4.13) to a set of ordinary differential equations. The stress and velocity equations in Subsections 4.1 and 4.2, as well for the unsteady flow case in Subsection 4.3, are decoupled from each other. Individually they form a system of strictly hyperbolic equations. Taken together they represent a system of hyperbolic equations with two eigenvalues, each of multiplicity two. The steady flow case in Subsection 4.3, on the other hand, consist of a coupled system of stress, velocity, and lattice orientation equations. It is degenerate hyperbolic with two eigenvalues of multiplicity two and one eigenvalue of multiplicity one.

If coefficients A, B and the solution U are discontinuous across a curve, say $\mathcal{C}$, but smooth elsewhere then there exists a (generalized) solution restricted by the jump condition (see [7], pp. 486-488 for details)

$$\left[\!\!\left[\left(A - \frac{dx}{dy} B \right) U \right]\!\!\right] = 0 \text{ on } \mathcal{C}, \tag{4.14}$$

which, for continuous coefficients, reduces to

$$\left(A - \frac{dx}{dy} B \right) [\![U]\!] = 0. \tag{4.15}$$

Hence a non-trivial discontinuity in U is permissible only when $\mathcal{C}$ is characteristic. The motivation for constructing a generalized solution should always come from some physical principle. For example, while considering a generalized solution for the velocity field, relations such as (4.10) are derived from the principle of maximum dissipation.

4.1. Isotropy. Under isotropy the yield criterion (3.8) is independent of $\theta-\gamma$ and $\boldsymbol{\alpha}$. It is reduced to

$$\sigma_1 - \sigma_2 = 2k \tag{4.16}$$

with constant k. Equations (4.1) and (4.16) form a system of strictly hyperbolic quasilinear equations for stresses. The two (mutually orthogonal) characteristic directions (sliplines) are identical with the maximum shear stress directions, given by $\theta \mp \pi/4$. The two family of curves are labeled as α-lines, with slope $\tan(\theta-\pi/4)$, and β-lines, with slope $\tan(\theta+\pi/4)$. The partial differential equations when written in the normal form lead to the Hencky relations, $p+2k\theta =$ constant (on an α-line) and $p-2k\theta =$ constant (on a β-line) [18, 10]. The velocity equations (3.9) and (4.2), the former of which is reduced to (with $2\psi = \pi/2$)

$$2\partial_x u - (\partial_y u + \partial_x v)\cot 2\theta = 0, \tag{4.17}$$

are a pair of linear first-order strictly hyperbolic equations with characteristic curves identical to those of stress equations. Let s_1 and s_2 be the arc-length parameterizations along α and β-lines, respectively. The corresponding normal form, which requires vanishing of extension rate along sliplines, is given by Geiringer's equations [18, 10]

$$\partial_{s_1} v_1 - v_2 \partial_{s_1}\theta = 0 \text{ and } \partial_{s_2} v_2 + v_1 \partial_{s_2}\theta = 0, \tag{4.18}$$

where v_1 and v_2 are the components of the velocity along α- and β-lines, respectively.

According to (4.6), the curves with stress discontinuity are inclined at anticlockwise angle of $\phi = \langle\theta\rangle + \pi/4 \pm n\pi$ (n integer) from $\mathbf{e}_1$, i.e. bisecting the discontinuous β-lines [27]; also, $[\![p]\!] = 2k\sin[\![\theta]\!]$. The yield surface is strictly convex and therefore discontinuities in stress and velocity never coincide; and strain-rate necessarily vanishes at every stress discontinuity. Discontinuities in the velocity field, as well as its gradient, are allowed only across the sliplines, cf. (4.10) and (4.15). The velocity jumps are tangential and constant along a slipline, as can be deduced by subtracting (4.18) for the two limiting sliplines across the discontinuity.

The boundary between the plastically deforming region and the (non-deforming) rigid region, say $\mathcal{C}$, is a characteristic (or an envelope of characteristics). The strain-rate field is necessarily discontinuous across $\mathcal{C}$, with $\mathbf{D}$ vanishing in the rigid part but not otherwise. The velocity in the rigid region can be taken as zero by superimposing a suitable rigid-body motion. If the velocity field is continuous at $\mathcal{C}$ then the boundary has to be characteristic since otherwise the zero-velocity solution will be extended to the deforming region. If the velocity is discontinuous then we need to consider a generalized solution which vanishes in the non-deforming part and satisfies a jump condition of the type (4.14). The latter in turn requires $\mathcal{C}$ to be a characteristic for a non-trivial velocity field in the deforming region. This result was first proved in [22], although within a less rigorous framework.

4.2. Anisotropy without lattice rotation. Attention is now confined to yield criteria (3.8) which are independent of γ and $\boldsymbol{\alpha}$; hence given by

$$\sigma_1 - \sigma_2 = 2k(\theta), \tag{4.19}$$

where the maximum shear stress k depends on the inclination θ of the first principal direction of stress. This is justified during incipient flow when γ can be assumed to be *uniform*. A latitude in choosing the intermediate configuration then exists such that γ can be assumed to vanish, without loss of generality, and consequently the plastic spin field be identified with the vorticity tensor. The Cauchy stress $\mathbf{T}$ is identical to the second Piola Kirchhoff stress $\mathbf{S}$. A general slipline theory in this context was developed in [2, 29, 20]; earlier studies [17, 31] were restricted to specific choices of the anisotropic yield surface. The function $k(\theta)$ is assumed to be differentiable for all θ. The stress equations, obtained from (4.1) and (4.19), are a pair of quasilinear strictly hyperbolic equations with characteristic directions $\theta + \psi_\alpha$ (α-lines) and $\theta + \psi_\beta$ (β-lines), where ψ_α and ψ_β are roots of (3.10) (with $\gamma = 0$) such that $\psi_\beta > \psi_\alpha$ and $\psi_\alpha, \psi_\beta \in [-\pi/2, \pi/2]$. According to (3.10) $\psi_\beta = \psi_\alpha + \pi/2$ and $2\psi_\beta$ is the anticlockwise angle from the radial direction to the tangential direction of the polar plot $(k(\theta), 2\theta)$. The anticlockwise inclination of the outward normal to the polar plot with respect to the $\theta = \pi/4$ line (or equivalently the positive σ_{12}-axis) is given by $2\theta + 2\psi_\beta + \pi$, i.e. twice the inclination for α-lines. The normal form is given by

$$-\sin(2\psi_\alpha)\partial_{s_1} p + 2k\partial_{s_1}\theta = 0 \text{ and } \sin(2\psi_\beta)\partial_{s_2} p - 2k\partial_{s_2}\theta = 0, \tag{4.20}$$

which can be written equivalently as $p + l =$ constant (along an $\alpha-$line) and $p - l =$ constant (along a $\beta-$line), where l, the arc-length parameter measured anticlockwise from the $\theta = 0$ line on the polar plot, satisfy $\sin(2\psi_\beta)dl = 2kd\theta$ [20]. The velocity equations, given by (3.9) and (4.2), are a pair of linear strictly hyperbolic equations whose characteristics are identical to those of the stress equations with the associated normal form given by

$$\partial_{s_1} v_1 - v_2 \partial_{s_1}(\theta + \psi) = 0 \text{ and } \partial_{s_2} v_2 + v_1 \partial_{s_2}(\theta + \psi) = 0, \tag{4.21}$$

which also characterizes vanishing of the extension rates along the sliplines.

Anisotropy allows for the yield locus to have vertices as well as linear segments; the contours in Figure 1, for example, are constructed entirely of such elements. Studying the nature of solutions in regions where stress states are restricted entirely to an edge or a vertex can provide insights unique to an anisotropic theory, not to mention of their practical importance [30]. Consider a region whose stress states lie on a single linear segment of the yield locus, such that the outward normal to the edge is inclined at a constant anticlockwise angle 2η with respect to the $\theta = \pi/4$ line on the polar plot. The sliplines within the region are two families of straight lines inclined at anticlockwise angles η (α-lines) and $\eta + \pi/2$ (β-lines) with respect to $\mathbf{e}_1$. As is clear from (4.21), the respective velocity components are constant along the characteristic direction, i.e. $v_1 = v_1(s_2)$ and $v_2 = v_2(s_1)$. Furthermore it is interesting to note that, for the considered stress states, $k(\theta) = C|\sin(2\eta - 2\theta)|^{-1}$, where C is a material constant. Indeed, for η constant, $d\eta = d\theta + d\psi = 0$, where $d(\cdot)$ denotes the differential of $(\cdot)$. This can be used in (3.10), with $\gamma = 0$ and k independent of $\boldsymbol{\alpha}$, to integrate the equation and obtain the desired result.

On the other hand, for a region with the stress state corresponding to a vertex on the yield locus, the stress is uniform within the region and the stress equations are trivially satisfied. The characteristic curves for the velocity equations are derived by starting from the modified flow rule given in Remark 2.4. To this end consider two smooth loci, $H_1(\sigma_{11} - \sigma_{22}, \sigma_{12}) = 0$ and $H_2(\sigma_{11} - \sigma_{22}, \sigma_{12}) = 0$, such that they intersect at the vertex. Their respective outward normals at the vertex are assumed to be inclined at angles $2\eta_1$ and $2\eta_2$, measured anticlockwise with respect to the $\theta = \pi/4$ line on the polar plot. However, unlike previous cases where the plastic multiplier was eliminated, the velocity equations retain an undetermined variable (cf. [28], pp. 15-18). The velocity equations are given by

$$2(\cos 2\eta_1 + \hat{\lambda}\cos 2\eta_2)\partial_x u + (\sin 2\eta_1 + \hat{\lambda}\sin 2\eta_2)(\partial_y u + \partial_x v) = 0 \tag{4.22}$$

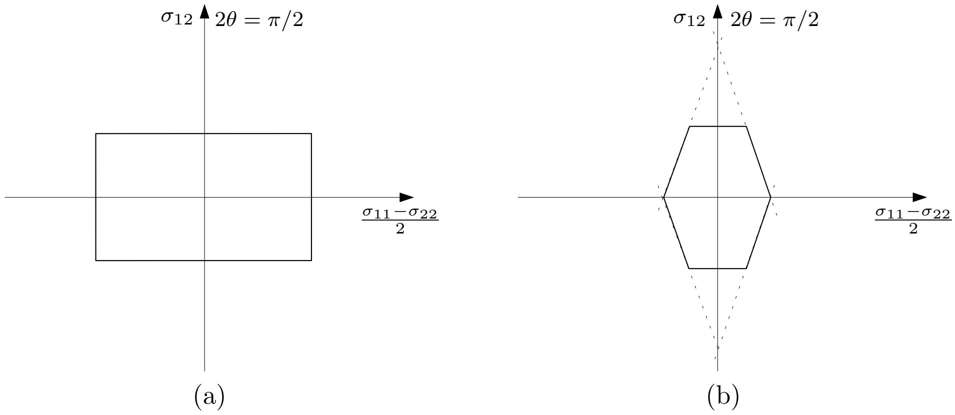

Figure 1: Polar plots of two convex polygonal yield contours in the Mohr plane with $\gamma = 0$. Here $\dfrac{\sigma_{11}-\sigma_{22}}{2} = \dfrac{S_{11}-S_{22}}{2}$ and $\sigma_{12} = S_{12}$. (a) The rectangular yield locus was used for NaCl-type ionic single crystal to study indentation [29]. (b) The hexagonal yield locus was used for a fcc-type single crystal to obtain stress fields in the neighborhood of a crack tip [30].

and (4.2), where

$$\hat{\lambda} = \frac{\lambda_2}{\lambda_1}\sqrt{\frac{4(\partial_{(\sigma_{11}-\sigma_{22})}H_2)^2 + (\partial_{\sigma_{12}}H_2)^2}{4(\partial_{(\sigma_{11}-\sigma_{22})}H_1)^2 + (\partial_{\sigma_{12}}H_1)^2}}. \tag{4.23}$$

Relation (4.22), derived from (2.26), can be read as a one parameter family (with respect to $\hat{\lambda}$) of velocity equations which reduces to (3.9) when $\eta_1 = \eta_2$. The characteristic curves are mutually orthogonal and still given as those along which the extension rates vanish; their slope ϕ satisfy

$$\frac{\sin(2\phi - 2\eta_1)}{\sin(2\phi - 2\eta_2)} = -\hat{\lambda}. \tag{4.24}$$

With both λ_1 and λ_2 positive, it is clear that $\eta_1 < \phi < \eta_2$ or $\eta_2 < \phi < \eta_1$ (depending on whether $\eta_1 < \eta_2$ or $\eta_1 > \eta_2$) for α-lines and $\eta_1 + \pi/2 < \phi < \eta_2 + \pi/2$ or $\eta_2 + \pi/2 < \phi < \eta_1 + \pi/2$ for β-lines.

The nature of discontinuities for the case at hand is considerably different from that under isotropy. A stress discontinuity curve, whose inclination is obtained from (4.6), can possibly intersect with a slipline (this will entail a suitable modification in (4.21)). For instance, α-lines from either side can coincide with the discontinuity curve, i.e. $\phi = \theta^{\pm} + \psi^{\pm}$, if $[\![k\sin(2\psi)]\!] = 0$. Interestingly, stress discontinuities in a region, where the stress states belong to the same linear segment on the yield locus, always coincide with a slipline. This can be seen by substituting $k^{\pm} = C|\sin(2\eta - 2\theta^{\pm})|^{-1}$, obtained in a previous paragraph, into (4.6). Moreover, it is precisely in this region that the stress and the velocity discontinuities can coincide. This is proved in the discussion following (4.10), where a flow rule for the velocity jumps is also prescribed. In any other case the stress discontinuity curves necessarily allow for only continuous velocity field and vanishing strain-rates. On the other hand, the curves across which stress is continuous but velocity (and its gradients) discontinuous, are necessarily sliplines, cf. (4.10) and (4.15). The velocity jumps are constant across such curves. Further, the boundary between the plastically deforming region and the (non-deforming) rigid region is always a characteristic curve. The last two conclusions follow from the arguments made previously in the case of isotropy.

Remark 4.1. (Polygonal yield surface, cf. [31, 29, 30]) Consider polygonal yield loci such as those illustrated in Figure 1. The following conclusions can be made based on the above discussion. i) A stress discontinuity curve inside the plastic region, where all the stress states belong to one edge of the yield contour, coincides with a slipline. ii) The slope of a stress discontinuity curve between plastic regions of constant stress, each belonging to distinct vertices but sharing a common edge on the yield contour, is identical to one of the slipline families associated with the common edge. (iii) The velocity can be discontinuous across the curves considered in (i) and (ii). iv) A stress discontinuity curve can also exist between plastic regions with stress states on different edges or non-neighboring vertices of the yield contour. The velocity field, however, is always continuous across these curves.

4.3. Anisotropy with lattice rotation. Let the yield criterion (3.8) be independent of dislocation density such that

$$\sigma_1 - \sigma_2 = 2k(\theta - \gamma) \tag{4.25}$$

with k a smooth function of its argument. The finite and non-uniform lattice rotation field γ cannot be eliminated; hence we require non-trivial constitutive prescription for plastic spin. The six governing equations for three in-plane stress components (σ_1, σ_2, θ), two velocity components (u, v), and lattice rotation angle (γ) are given by (4.1), (4.25), (3.9), (4.2), and (3.14). The equation for lattice rotation, (3.14), can be rewritten as

$$\partial_x v - \partial_y u = 2\partial_t \gamma + 2u\partial_x \gamma + 2v\partial_y \gamma + 2\lambda \Omega_{21}, \tag{4.26}$$

where $\Omega_{21} = -\Omega_{12} = \mathbf{e}_2 \cdot \mathbf{\Omega} \mathbf{e}_1$. Assume $\mathbf{\Omega} \in Skw$ to be a constitutive function of p and $\theta - \gamma$ and therefore independent of the out-of-plane stress components and dislocation density. The plastic

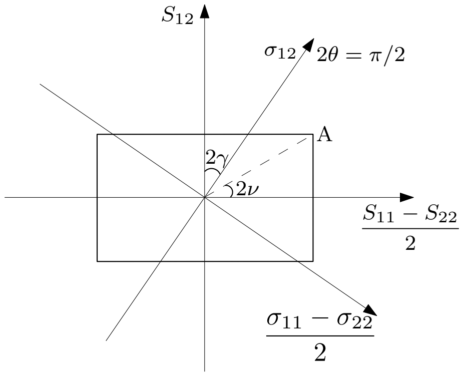

Figure 2: Polar plot of a rectangular yield contour with $\gamma \neq 0$ (Compare with Figure 1 (a)). A point on the yield locus can be given either in terms of polar coordinates $(k(\theta-\gamma), 2\theta-2\gamma)$ or Cartesian coordinates $(\frac{S_{11}-S_{22}}{2}, S_{12})$. The coordinate system $(\frac{\sigma_{11}-\sigma_{22}}{2}, \sigma_{12})$ is as shown above. Due to absence of any hardening the yield surface remains fixed in $(\frac{S_{11}-S_{22}}{2}, S_{12})$ space. However, the contour rotates with changing γ, when viewed with respect to $(\frac{\sigma_{11}-\sigma_{22}}{2}, \sigma_{12})$ axes.

multiplier λ can be eliminated from (4.26), using flow rule (2.19), to get

$$2\frac{\Omega_{21}\sin(2\psi)}{\sin(2\theta+2\psi)}\partial_x u + \partial_y u - \partial_x v + 2\partial_t\gamma + 2u\partial_x\gamma + 2v\partial_y\gamma = 0. \tag{4.27}$$

The system of equations is studied under the assumption of unsteady and steady flow. The former assumes γ to be known at a given time instant. The stress and the velocity field can then be solved for the given γ and substituted into (4.27) to calculate the evolution of lattice rotation. Steady flow, on the other hand, requires $\partial_t\gamma = 0$ for an otherwise unknown γ; the equations are coupled to each other and have to be solved simultaneously for stress, velocity, and lattice rotation.

First, consider the case of unsteady flow. The stress and the velocity equations are uncoupled to each other; both are strictly hyperbolic with coinciding characteristic directions given by $\theta+\psi_\alpha$ (α-lines) and $\theta+\psi_\beta$ (β-lines), where ψ_α and ψ_β are roots of (3.10) such that $\psi_\beta > \psi_\alpha$ and $\psi_\alpha, \psi_\beta \in [-\pi/2, \pi/2]$. According to (3.10) $\psi_\beta = \psi_\alpha + \pi/2$ and $2\psi_\beta$ is the anticlockwise angle from the radial direction to the tangential direction of the polar plot $(k(\theta-\gamma), 2\theta-2\gamma)$. The anticlockwise inclination of the outward normal to the polar plot with respect to the $\theta = \pi/4$ line (or equivalently the positive σ_{12}-axis, see Figure 2) is given by $2\theta + 2\psi_\beta + \pi$, i.e. twice the inclination for α-lines.

The stress equations can be equivalently written as

$$\sin(2\psi_\alpha)\partial_{s_1} p - 2k\partial_{s_1}(\theta - \gamma) = 2k\sin(2\psi_\alpha)\partial_{r_2}\gamma \text{ and} \tag{4.28}$$

$$\sin(2\psi_\beta)\partial_{s_2} p - 2k\partial_{s_2}(\theta - \gamma) = 2k\sin(2\psi_\beta)\partial_{r_1}\gamma, \tag{4.29}$$

where r_1 and r_2 are arc-length parameterizations along lines inclined at clockwise angles of $2\psi_\alpha$ with respect to the α- and β-line, respectively. The bulk dislocation density, given in terms of $\operatorname{grad}\gamma$ from (3.4), contributes as body force. The normal form associated with velocity equations, identified with the vanishing of extension rates along the sliplines, is as given in (4.21). All together the stress and the velocity equations, in this reduced form, are four ordinary differential equations along the characteristic curves. Their solution is substituted into (4.27) to obtain $\partial_t\gamma$ and hence to calculate γ for the next time instant.

Consider steady flow, i.e. $\partial_t\gamma = 0$. The equations for stress, velocity, and lattice rotation are all coupled. The characteristic curves corresponding to stress and velocity fields remain the same as above, while the characteristic direction for the lattice rotation is given by velocity streamlines inclined at an anticlockwise angle $\varphi = \arctan(v/u)$ with respect to $\mathbf{e}_1$-axis. It is clear that for non-vanishing (bulk) dislocation density, i.e for $\operatorname{grad}\gamma \neq \mathbf{0}$, the stress equations (reduced to (4.28) and (4.29)) are no longer given by ordinary differential equations along the characteristic curves. They are in fact coupled with the velocity equations, which remain in the form (4.21), and the equation for lattice rotation, as given below. In this sense, the structure of these equations is comparable to those obtained for plane strain problems of isotropic hardening and granular materials [5, 6]. The normal form for the spin relation is determined by assuming that neither the stress/velocity characteristics nor the stress discontinuity curves intersect with streamlines. On using velocity equations (3.9) and (4.2) in addition to (4.27) we obtain

$$w\partial_q(\gamma - \varphi) + \left(\frac{\Omega_{21}\sin(2\psi) + \cos(2\varphi - 2\theta - 2\psi)}{\sin(2\varphi - 2\theta - 2\psi)}\right)\partial_q w = 0, \tag{4.30}$$

where w and q denote the magnitude of velocity and the arc-length parametrization, respectively, along the streamline direction. On the other hand, if the streamline direction is identical to a characteristic curve (say an α-line), but is away from any stress discontinuity, then (4.30) is to be replaced by

$$2w\partial_{s_1}\gamma + (1 - \Omega_{21}\sin(2\psi_\alpha))\,\partial_{s_2} w - (1 + \Omega_{21}\sin(2\psi_\alpha))\,w\partial_{s_1}(\theta + \psi) = 0, \tag{4.31}$$

whence it is clear that the normal form is no longer an ordinary differential equation along the characteristic. Additionally, the speed w is constant along the streamline, i.e. $\partial_{s_1} w = 0$, and $\partial_{s_2}(\theta + \psi) = 0$; both of these follow from (4.21). For a streamline with stress discontinuity, the above equations can be suitably modified with the stress dependent coefficients obtainable from either side of the curve.

The sliplines in a region, with stress states lying on the same linear segment of the yield locus, are two families of curves inclined at η (α-lines) and $\eta + \pi/2$ (β-lines) with respect to $\mathbf{e}_1$. Here 2η is the anticlockwise inclination of the outward normal to the edge with respect to the positive σ_{12}-axis (see Figure 2) on the polar plot. Thus $\eta = \gamma + \eta_0$, where $2\eta_0$ is the constant anticlockwise inclination of the outward normal with respect to the positive S_{12}-axis. Hence the curvature of the slipline fields is identical to that of the *glide lines*, where the latter are curves along which the dislocations undergo pure gliding; they are given by a family of two mutually orthogonal curves each with a spatial distribution of edge-type dislocations of same sign [26] (see also [23, 36], where the two families of curves are *a priori* assumed to be identical). It should be noted that the sliplines and the glide lines are in general dissimilar (cf. §5 in [26]); the present case being, of course, an exception.

In the same region as considered above, the velocity equations (4.21) are reduced to

$$\partial_{s_1} v_1 - v_2 \partial_{s_1} \gamma = 0 \text{ and } \partial_{s_2} v_2 + v_1 \partial_{s_2} \gamma = 0. \tag{4.32}$$

Furthermore, for the considered stress states, $k(\theta - \gamma) = C\,|\sin(2\eta_0 - 2(\theta - \gamma))|^{-1}$, where C is a material constant. Indeed $d(\theta - \gamma) + d\psi = 0$, which can be used in (3.10), with k independent of $\boldsymbol{\alpha}$, to integrate the equation and obtain the desired result. If a streamline coincides with an α-line in the considered region then the lattice rotation field is a linear function in s_1 given by

$$\gamma = -\frac{w'}{w} s_1 + f(s_2), \tag{4.33}$$

where w' is the derivative of $w(s_2)$ and $f(s_2)$ is any smooth function such that $\partial_{s_2} \gamma = 0$. This follows from (4.31) with the assumption that $(1 - \Omega_{21} \sin(2\psi_\alpha)) \neq 0$.

For a region whose (Second Piola) stress state belongs to a vertex on the yield contour, $S_{11} - S_{22}$ and S_{12} are fixed but $\sigma_{11} - \sigma_{22}$, σ_{12}, as well as p are variable due to non-uniformity of γ. The vertex state is also characterized by a given constant value of $(\theta - \gamma)$ (which, for example, is ν for the vertex A in Figure 2); hence $d\theta = d\gamma$. Let $H_1(\sigma_{11} - \sigma_{22}, \sigma_{12}, \gamma) = 0$ and $H_2(\sigma_{11} - \sigma_{22}, \sigma_{12}, \gamma) = 0$ be two yield loci intersecting at the vertex. The stress equations, decoupled from rest of the system, are strictly hyperbolic with characteristic curves inclined at $\theta \pm \pi/4$ with corresponding normal form as $p \pm 2k\theta =$ constant, where k can be obtained from either of the yield loci. The curvature of these characteristic curves is hence related to the distribution of dislocation density in the region. The velocity equations, derived from (2.26), consist of (4.2),

$$\partial_x u = \lambda_1 \partial_{(\sigma_{11}-\sigma_{22})} H_1 + \lambda_2 \partial_{(\sigma_{11}-\sigma_{22})} H_2, \quad 2(\partial_x v + \partial_y u) = \lambda_1 \partial_{\sigma_{12}} H_1 + \lambda_2 \partial_{\sigma_{12}} H_2, \tag{4.34}$$

where λ_1 and λ_2 are plastic multipliers, and

$$\partial_x v - \partial_y u = 2\partial_t \gamma + 2u\partial_x \gamma + 2v\partial_y \gamma + 2\lambda_1 \Omega^1_{21} + +2\lambda_2 \Omega^2_{21}, \tag{4.35}$$

where $\Omega^1_{21} = \mathbf{e}_2 \cdot \boldsymbol{\Omega}_1 \mathbf{e}_1$ and $\Omega^2_{21} = \mathbf{e}_2 \cdot \boldsymbol{\Omega}_2 \mathbf{e}_1$ are assumed to be constitutive functions of p. These are all together four equations for two velocity components and two plastic multipliers. The latter can be eliminated to get a pair of equations for two velocity fields. The slopes of characteristic directions as well as the normal forms can be obtained after a straightforward, although cumbersome, calculation. It is noted that the resultant characteristic curves do not coincide with those of stress. More interestingly it should be remarked that, unlike the vertex case discussed in the previous subsection, a completely determined system of velocity equations is now obtained. This adds to the relevance of prescribing plastic spin for anisotropic flows.

The stress discontinuity curves, if present, are inclined at angles calculated from (4.6) and can coincide with sliplines as well as streamlines. However within a region, whose stress states belong to one edge of the yield locus, they necessarily coincide with either sliplines or streamlines. The former situation arises when γ is continuous across the discontinuity curve (the proof is identical to the one provided in the last subsection). The latter is whenever γ jumps, which is possible only across streamlines (see below). The sliplines will have discontinuous slopes across such a streamline with the jump given by $[\![\gamma]\!]$. On the other hand, across stress discontinuity curves outside the considered region, velocity remains continuous and with vanishing strain-rates, cf. the discussion following (4.12). A possible exception could be within a region with stress state on a single vertex and where stress and γ discontinuities coincide. This can be seen by appropriately modifying (4.10) and the ensuing discussion.

The velocity discontinuity curves, with continuous stress and lattice rotation, are necessarily sliplines; this is verified by constructing a generalized solution of velocity equations (3.9) and (4.2) and using (4.15). The curves with discontinuous lattice rotation fields are necessarily streamlines. Indeed for a generalized solution of (4.27),

$$[\![(u - v \cot\phi)\gamma]\!] = 0, \tag{4.36}$$

where $\tan\phi$ is the inclination of the discontinuity, cf. (4.14). According to (4.3) $[\![(u - v\cot\phi)]\!] = 0$ at every singular curve, thus reducing (4.36) to $(u - v\cot\phi)[\![\gamma]\!] = 0$, which furnishes the required result. Finally, it should be noted that discontinuities in γ characterize the presence of surface dislocation density (or dislocation walls), cf. (3.5).

Remark 4.2. (Polygonal yield surface) The curve separating two regions with stress states belonging to a linear segment on the yield contour, or to vertices sharing a common edge, is either a slipline (when $[\![\gamma]\!] = 0$) or a streamline (when $[\![\gamma]\!] \neq 0$) associated with the edge state. The velocity field can be discontinuous across the curve.

5. Conclusion. A theory of elastically rigid finite plasticity is developed while emphasizing the role of material symmetry. In particular, the nature of intermediate configuration, plastic spin, and anisotropic flow rule have all been carefully studied. The appearance of dislocation density and lattice rotation tensors, otherwise absent in an isotropic theory, have also been emphasized. Subsequently attention is restricted to a plane strain version of the three-dimensional theory which is then used to develop a slipline theory for transient anisotropic plastic flows. Solutions are discussed separately under the assumption of isotropy, anisotropy without lattice rotation, and anisotropy with lattice rotation. Anisotropy brings in distinctive constitutive features in the theory such as vertices and linear segments on the yield locus. Including lattice rotation in the anisotropic theory brings forth additional richness in the form of plastic spin and dislocation density. The resulting solutions too demonstrate increasing sophistication. The difference is also evident in the nature of discontinuities in stress, velocity, and rotation fields. Whereas it is well known that the isotropic theory does not allow for stress and velocity discontinuity curves to coincide, the anisotropic theory places no such restriction. In fact if the stress state is restricted to one edge of a piecewise linear yield locus then the stress discontinuity necessarily coincides with either a slipline or a streamline. A velocity discontinuity curve, across which stress and lattice rotation are continuous, coincides with a slipline. On the other hand a curve with discontinuous rotation, which can also be interpreted as an array of dislocations, is necessarily a streamline.

References.

[1] B. A. Bilby. Types of dislocation source. In *Report of Bristol Conference on Defects in Crystalline Solids*, pages 124–133. London: Physical Society, 1955.

[2] J. R. Booker and E. H. Davis. A general treatment of plastic anisotropy under conditions of plane strain. *Journal of Mechanics and Physics of Solids*, 20:239–250, 1972.

[3] J. Casey. On finitely deforming rigid-plastic materials. *International Journal of Plasticity*, 2:247–277, 1986.

[4] P. Cermelli and M. E. Gurtin. On the characterization of geometrically necessary dislocations in finite plasticity. *Journal of Mechanics and Physics of Solids*, 49:1539–1568, 2001.

[5] I. F. Collins. Boundary value problems in plane strain plasticity. In H. G. Hopkins and M. J. Sewell, editors, *Mechanics of Solids*, pages 135–184. Pergamon Press, 1982.

[6] I. F. Collins. Plane strain characteristics theory for soil and granular materials with density dependent yield criteria. *Journal of Mechanics and Physics of Solids*, 38:1–25, 1990.

[7] R. Courant and D. Hilbert. *Methods of Mathematical Physics, Vol. II.* Interscience Publishers, 1962.

[8] Y. F. Dafalias. A missing link in the macroscopic constitutive formulation of large plastic deformations. In A. Sawczuk and G. Bianchi, editors, *Plasticity Today*, pages 135–151. Elsevier, 1985.

[9] J. Edmiston, D. J. Steigmann, G. J. Johnson, and N. Barton. A model for elastic-viscoplastic deformations of crystalline solids based on material symmetry: Theory and plane-strain simulations. *International Journal of Engineering Science*, 63:10–22, 2013.

[10] H. Geiringer. Some recent results in the theory of an ideal plastic body. *Advances in Applied Mechanics*, 3:197–294, 1953.

[11] A. E. Green and J. E. Adkins. *Large Elastic Deformations (2nd edition).* Oxford University Press, Oxford, 1970.

[12] A. Gupta and D. J. Steigmann. Plastic flow in solids with interfaces. *Mathematical Methods in the Applied Sciences*, 35:1799–1824, 2012.

[13] A. Gupta, D. J. Steigmann, and J. Stölken. On the evolution of plasticity and incompatibility. *Mathematics and Mechanics of Solids*, 12:583–610, 2007.

[14] A. Gupta, D. J. Steigmann, and J. Stölken. Aspects of the phenomenological theory of elastic-plastic deformation. *Journal of Elasticity*, 104:249–266, 2011.

[15] M. E. Gurtin. *An Introduction to Continuum Mechanics.* Academic Press, Orlando, 1981.

[16] M. E. Gurtin and L. Anand. The decomposition $\mathbf{F}^e\mathbf{F}^p$, material symmetry, and plastic irrotationality for solids that are isotropic-viscoplastic or amorphous. *International Journal of Plasticity*, 21:1686–1719, 2005.

[17] R. Hill. The theory of plane plastic strain for anisotropic metals. *Proceedings of the Royal Society at London A*, 198:428–437, 1949.

[18] R. Hill. *The Mathematical Theory of Plasticity.* Oxford University Press, 1950.

[19] R. Hill. Discontinuity relations in mechanics of solids. *Progress in Solid Mechanics*, 2:245–276, 1961.

[20] R. Hill. Basic stress analysis of hyperbolic regimes in plastic media. *Mathematical Proceedings of the Cambridge Philosophical Society*, 88:359–369, 1980.

[21] W. T. Koiter. Stress-strain relations, uniqueness and variational theorems for elastic-plastic materials with a singular yield surface. *Quarterly of Applied Mathematics*, 11:350–354, 1953.

[22] E. H. Lee. The theoretical analysis of metal-forming problems in plane strain. *ASME Journal of Applied Mechanics*, 19:97–103, 1952.

[23] T. Mura. Continuous distribution of dislocations and the mathematical theory of plasticity. *Physica Status Solidi*, 10:447–453, 1965.

[24] W. Noll. Materially uniform simple bodies with inhomogeneities. *Archive of Rational Mechanics and Analysis*, 27:1–32, 1967.

[25] J. F. Nye. The flow of glaciers and ice-sheets as a problem in plasticity. *Proceedings of the Royal Society at London A*, 207:554–572, 1951.

[26] J. F. Nye. Some geometrical relations in dislocated crystals. *Acta Metallurgica*, 1:153–162, 1953.

[27] W. Prager. Discontinuous solutions in the theory of plasticity. In K. O. Friedrichs, O. E. Neugebauer, and J. J. Stoker, editors, *Studies and Essays: Courant Anniversary Volume*, pages 289–300. Interscience, New York, 1948.

[28] W. Prager. *An Introduction to Plasticity.* Addison-Wesley Publishing Company, Inc., 1959.

[29] J. R. Rice. Plane strain slip line theory for anisotropic rigid/plastic materials. *Journal of Mechanics and Physics of Solids*, 21:63–74, 1973.

[30] J. R. Rice. Tensile crack tip fields in elastic-ideally plastic crystals. *Mechanics of Materials*, 6:317–335, 1987.

[31] E. M. Shoemaker. A theory of linear plasticity for plane strain. *Archive of Rational Mechanics and Analysis*, 14:283–300, 1963.

[32] M. Šilhavý. *The Mechanics and Thermodynamics of Continuous Media.* Springer–Verlag Berlin Heidelberg, 1997.

[33] D. J. Steigmann and A. Gupta. Mechanically equivalent elastic-plastic deformations and the problem of plastic spin. *Theoretical and Applied Mechanics*, 38:397–417, 2011.

[34] P. Tapponnier and P. Molnar. Slip-line field theory and large-scale continental tectonics. *Nature*, 264:319–324, 1976.

[35] T. Y. Thomas. *Plastic Flow and Fracture in Solids.* Academic Press, 1961.

[36] G. J. Weng and A. Phillips. On the kinematics of continuous distribution of dislocations in plasticity. *International Journal of Engineering Science*, 14:65–73, 1976.

[37] W. I. Zangwill. *Nonlinear Programming.* Prentice-Hall, Englewood Cliffs, N. J., 1969.

[38] Q. S. Zheng. Theory of representations for tensor functions – A unified approach to constitutive equations. *ASME Applied Mechanics reviews*, 47:545–586, 1994.