Closed Form BER Expressions for BPSK OFDM Systems with Frequency Offset

R. Uma Mahesh and A. K. Chaturvedi, *Senior Member, IEEE*

***Abstract*—This letter addresses the performance degradation caused by the presence of carrier frequency offset (CFO) in orthogonal frequency division multiplexing (OFDM) systems. Accurate closed form bit error rate (BER) expressions for BPSK-OFDM systems impaired by frequency offset are derived. The analysis is carried out for flat and frequency selective Rayleigh fading channels. Simulation results have been used to cross-check the accuracy of the theoretical analysis.**

***Index Terms*—Carrier frequency offset, orthogonal frequency division multiplexing, bit error rate, Rayleigh fading.**

I. Introduction

OFDM systems are receiving significant attention because of their robustness against frequency selective fading, low equalization complexity, efficient use of spectrum and reduced cost of implementation using FFT techniques. However, OFDM systems are more sensitive to frequency synchronization errors than single carrier systems [1]. The presence of CFO disturbs the orthogonality between the carriers thereby causing inter carrier interference (ICI). Hence there is a need to analyze the performance of such systems.

Several works discussing the error probability of OFDM systems with CFO can be found in the literature [1]-[3]. However, closed form analytical expressions were given for the first time in [4] for BPSK OFDM systems with CFO in AWGN, flat and frequency selective Rayleigh fading channels. Although the BER expression derived therein is correct for all values of CFO for AWGN channels, the same is not true for flat and frequency selective channels. This is because the expressions have been derived assuming the argument of Q function in the expressions (7) and (32) of [4] to be positive. This is not true for higher values of CFO, leading to a mismatch between the theoretical and actual BER in such cases. In this paper we derive accurate BER expressions valid for all values of CFO for BPSK OFDM systems in flat and frequency selective Rayleigh fading channels.

Notations: $\bar{\mathbf{A}}$, $\mathbf{A}^T$, $\mathbf{A}^H$ and $\mathbf{A}(:, 1:L)$ denote the conjugate, transpose, hermitian and the first L columns of a matrix $\mathbf{A}$ respectively, $|a|$ denotes the modulus of a and $diag(\mathbf{a})$ denotes a diagonal matrix with $\mathbf{a}$ along its diagonal.

The remaining sections of this letter are organized as follows. Section II contains the OFDM system model with CFO. Section III gives the accurate BER expressions for flat and frequency selective Rayleigh fading channels. Simulation results are compared with the derived expressions in Section IV. The paper is concluded in Section V.

Manuscript received January 17, 2010. The associate editor coordinating the review of this letter and approving it for publication was J. van de Beek.

R. U. Mahesh and A. K. Chaturvedi are with the Department of Electrical Engineering, Indian Institute of Technology Kanpur, India (e-mail:rumahesh@bel.co.in, akc@iitk.ac.in).

Digital Object Identifier 10.1109/LCOMM.2010.08.100088

II. System Model

We consider an OFDM system with N subcarriers. Frequency domain signal of an OFDM block is denoted as $\mathbf{d} = [d(0)\ d(1)\ \ldots\ d(N-1)]^T$. The corresponding time domain signal is given by $\mathbf{F}^H\mathbf{d}$, where $\mathbf{F}$ is the FFT matrix given by $\mathbf{F}(k,l) = \exp(-j2\pi kl/N)$ for $0 \leq k, l \leq N-1$. Suitable cyclic prefix (L_{cp}) is added to the time domain OFDM block to eliminate inter block interference. Let the discrete time channel impulse response (CIR) be $\mathbf{h} = [h(0)\ h(1)\ \ldots\ h(L-1)]^T$, where $\mathbf{h}$ is assumed to have no more than L taps. CIR is assumed to be static over an OFDM symbol.

The sampled signal for the kth subcarrier after the receiver fast Fourier transform processing can be written as [4]

$$\mathrm{x}_k = \alpha_k d_k S_1 + \sum_{l=1, l\neq k}^{N} \alpha_l d_l S_{l-k+1} + n_k, \quad k = 1, 2, \ldots, N \tag{1}$$

where α_k denotes the frequency domain channel response on subcarrier k and n_k is white Gaussian noise modelled as a zero mean Gaussian random variable with variance σ^2 per dimension. The ICI coefficients S_l depend on the CFO and are given by [4]

$$S_l = \frac{\sin(\pi[l-1+f])}{N\sin\left(\frac{\pi[l-1+f]}{N}\right)} \exp\left\{j\pi\left(1-\frac{1}{N}\right)(l-1+f)\right\} \tag{2}$$

where f is the normalized CFO.

III. BER Expressions

Let $\mathbf{E}_M$ be an $M \times 2^{M-1}$ matrix having columns $\mathbf{e}_k$ as the binary representation of the number $2^M - k$ and zeros replaced by -1s.

A. Rayleigh Flat Fading Channel

Here, $(\alpha_1, \alpha_2, \ldots, \alpha_N)$ are all equal to a value, say c where $|c|$ has a Rayleigh distribution given by $p(|c|) = \frac{|c|}{\sigma_R^2}\exp\{-\frac{|c|^2}{2\sigma_R^2}\}$. Conditional bit error probability is given as [4]

$$P_{e|c} = \frac{1}{2^{N-1}} \sum_{l=1}^{2^{N-2}} \left\{ Q(\sqrt{2\gamma}\theta_l|c|) + Q(\sqrt{2\gamma}\beta_l|c|) \right\} \tag{3}$$

where $\theta_l = \Re(S_1 + \mathbf{S}^T\mathbf{e}_l)$, $\beta_l = \Re(S_1 - \mathbf{S}^T\mathbf{e}_l)$, $\mathbf{S} = [S_2\ S_3\ \ldots\ S_N]^T$, $\mathbf{E}_{N-1}$ is of dimension $(N-1)\times\ 2^{N-2}$,

$\gamma = \frac{E_b}{N_o}$, $E_b = 1$ and $\sigma^2 = N_o/2$, N_o being the noise power spectral density and $Q(x)$ is the Gaussian Q function.

The bit error probability P_e can be obtained by averaging (3) over the Rayleigh distributed random variable $|c|$. Hence P_e is given by

$$P_e = \int_0^\infty P_{e|c}\ p(|c|)\ d|c| \tag{4}$$

For finding P_e, Craig's formula has been used in [4] assuming that the arguments of the Q function are positive. However, the arguments of the Q function in (3) are positive only for small values of CFO. For high CFO values, they are negative and hence the expressions derived in [4] are not suitable for all CFO values.

As shown in the Appendix, we proceed by changing the order of integration in (4) and obtain P_e as

$$P_e = \frac{1}{2} - \frac{1}{2^N} \sum_{l=1}^{2^{N-2}} sgn(\theta_l) \sqrt{\frac{2\gamma\theta_l^2\sigma_R^2}{1+2\gamma\theta_l^2\sigma_R^2}} + sgn(\beta_l) \sqrt{\frac{2\gamma\beta_l^2\sigma_R^2}{1+2\gamma\beta_l^2\sigma_R^2}} \tag{5}$$

where $sgn()$ is the signum function. As expected, it can be noted that if θ_l and β_l are positive, our expression reduces to the one given in [4] for flat fading channels.

B. Frequency Selective Rayleigh Fading Channel

In this case $(\alpha_1, \alpha_2, \ldots, \alpha_N)^T = \mathbf{F}_L\mathbf{h}$. Multipath Rayleigh fading channel with an exponential power delay profile (normalized to unit power) is considered. For BPSK modulation, $d_n \in \{-1, 1\}$ and without loss of generality, we consider the analysis for the first subcarrier with a +1 transmitted. Conditional error probability is given by [4]

$$P_{e|\alpha_1} = \frac{1}{2^{N-1}} \sum_{l=1}^{2^{N-2}} Q\left(\frac{|\alpha_1|[\Re(S_1)+a_l]}{\sigma\sqrt{1+\frac{b_l}{2\sigma^2}}}\right) + Q\left(\frac{|\alpha_1|[\Re(S_1)-a_l]}{\sigma\sqrt{1+\frac{b_l}{2\sigma^2}}}\right) \tag{6}$$

where $a_l = c_{\alpha_1\alpha_1}^{-1}\Re(\mathbf{P}_l^T\mathbf{C}_{\boldsymbol{\alpha}\alpha_1})$, $c_{\alpha_l\alpha_m} = E(\alpha_l\bar{\alpha}_m)$, $\boldsymbol{\alpha} = (\alpha_2\ \alpha_3\ \ldots\ \alpha_N)^T$, $\mathbf{C}_{\boldsymbol{\alpha}\alpha_1} = (c_{\alpha_2\alpha_1}\ c_{\alpha_3\alpha_1}\ \ldots\ c_{\alpha_N\alpha_1})^T$, $\mathbf{P}_l = diag([S_2\ S_3\ \ldots\ S_N])\mathbf{e}_l$, $b_l = \mathbf{P}_l^T\mathbf{C}_{\boldsymbol{\alpha}|\alpha_1}\bar{\mathbf{P}}_l$, $\mathbf{C}_{\boldsymbol{\alpha}|\alpha_1} = \mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} - c_{\alpha_1\alpha_1}^{-1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}^H$ and $\mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}}$ is obtained from the result

$$\begin{aligned}\mathbf{C} &= E\{(\alpha_1\boldsymbol{\alpha}^T)^T(\bar{\alpha_1}\boldsymbol{\alpha}^H)\} = \begin{pmatrix} c_{\alpha_1\alpha_1} & \mathbf{C}_{\boldsymbol{\alpha}\alpha_1}^H \\ \mathbf{C}_{\boldsymbol{\alpha}\alpha_1} & \mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} \end{pmatrix} \\ &= \mathbf{F}_L\mathbf{C}_h\mathbf{F}_L^H \end{aligned} \tag{7}$$

where $\mathbf{C}_h$ is the time domain channel covariance matrix. The BER is given by averaging (6) over the random variable $|\alpha_1|$ having a pdf given by $p(|\alpha_1|) = \frac{2|\alpha_1|}{c_{\alpha_1\alpha_1}}\exp\left(\frac{-|\alpha_1|^2}{c_{\alpha_1\alpha_1}}\right)$. Proceeding again in the way shown in the Appendix, the BER is given by

$$P_e = \frac{1}{2} - \frac{1}{2^N} \sum_{l=1}^{2^{N-2}} f_1 \sqrt{\frac{\gamma c_{\alpha_1\alpha_1}[\Re(S_1)+a_l]^2}{1+\gamma(c_{\alpha_1\alpha_1}[\Re(S_1)+a_l]^2+b_l)}} + f_2 \sqrt{\frac{\gamma c_{\alpha_1\alpha_1}[\Re(S_1)-a_l]^2}{1+\gamma(c_{\alpha_1\alpha_1}[\Re(S_1)-a_l]^2+b_l)}} \tag{8}$$

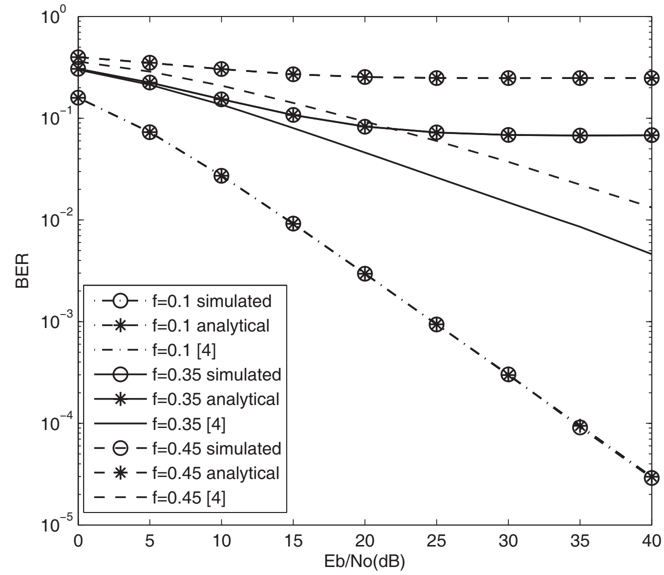

Fig. 1. Probability of error for BPSK OFDM over Rayleigh flat fading channel with N=8, σ_R^2=0.5.

If the channel taps are independent, the above BER expression reduces to

$$P_e = \frac{1}{2} - \frac{1}{2^N} \sum_{l=1}^{2^{N-2}} f_1 \sqrt{\frac{\gamma[\Re(S_1)+a_l]^2}{1+\gamma([\Re(S_1)+a_l]^2+b_l)}} + f_2 \sqrt{\frac{\gamma[\Re(S_1)-a_l]^2}{1+\gamma([\Re(S_1)-a_l]^2+b_l)}} \tag{9}$$

where $f_1 = sgn(\Re(S_1)+a_l)$, $f_2 = sgn(\Re(S_1)-a_l)$ and the modified parameters a_l and b_l are given by $a_l = \Re(\mathbf{P}_l^T\mathbf{C}_{\boldsymbol{\alpha}\alpha_1})$, $b_l = \mathbf{P}_l^T(\mathbf{C}_{\boldsymbol{\alpha}\boldsymbol{\alpha}} - \mathbf{C}_{\boldsymbol{\alpha}\alpha_1}\mathbf{C}_{\boldsymbol{\alpha}\alpha_1}^H)\bar{\mathbf{P}}_l$.

Here also, for positive $\Re(S_1)+a_l$, $\Re(S_1)-a_l$, our error probability expression reduces to the one given in [4] for frequency selective fading channels.

IV. Simulation Results

In this section, the BER performance is evaluated using simulations and compared with the analytical expressions derived. The simulated and analytical BER performance of BPSK OFDM systems in the presence of CFO is shown in Figs. 1 and 2 for flat and frequency selective Rayleigh fading channels respectively. The corresponding analytical expressions of [4] are also plotted for comparison.

As expected, the expressions given in [4] are matching with the simulation results at low values of CFO. However, at high values of CFO, this is not true. Thus in Fig. 1, for a CFO of 0.1, the expression in [4] exactly matches our analytical and simulated BER. However, for CFO values 0.35 and 0.45, there is a huge mismatch. For frequency selective channels, the expression in [4] matches with the simulations for CFO of 0.1 and 0.35 as can be seen from Fig. 2. However, for a CFO of 0.45, there is a deviation from the simulated BER. In both figures, our analytical expressions coincide exactly with their simulation counterparts for all the CFO values considered. It is also clear from the simulations that if we detect the symbols without compensating for the frequency offset, the performance of the system degrades drastically.

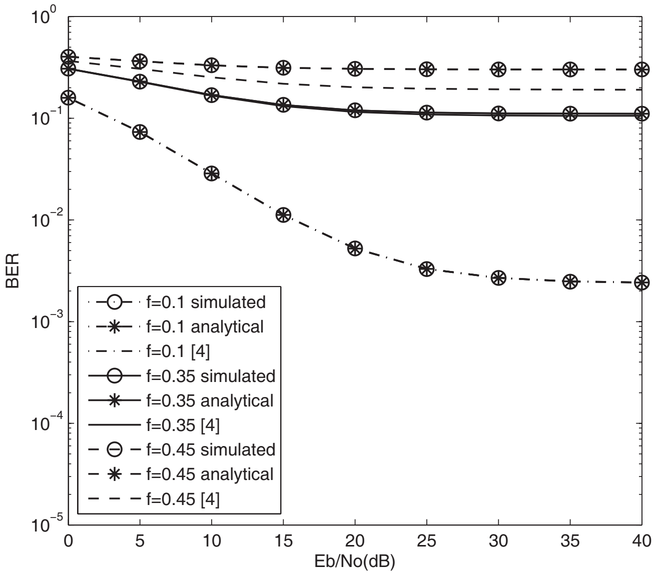

Fig. 2. Probability of error for BPSK OFDM over frequency selective Rayleigh fading channel with N=8, L=2.

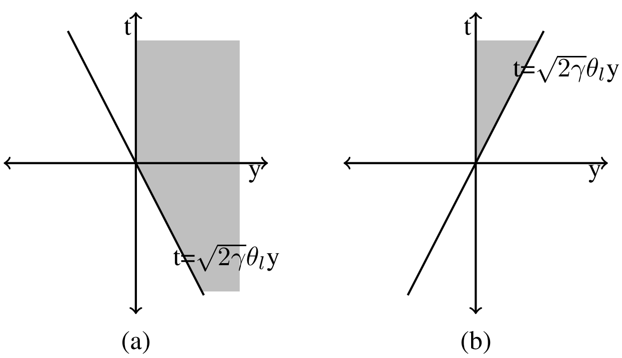

Fig. 3. Area of integration in expression (11): (a) for negative θ_l; (b) for positive θ_l.

V. Conclusions

Accurate closed form BER expressions for BPSK OFDM systems with carrier frequency offset are derived for flat and frequency selective Rayleigh fading channels. It can be seen that the performance of the system is severely degraded with these offsets. The analytical expressions obtained closely match the simulation results.

Appendix

In this appendix we derive (5). For this, consider the expression

$$E = \int_0^\infty Q(\sqrt{2\gamma}\theta_l y)p(y)dy \tag{10}$$

$$= \int_0^\infty \frac{1}{\sqrt{2\pi}} \int_{\sqrt{2\gamma}\theta_l y}^\infty e^{-t^2/2}dt \ \frac{y}{\sigma_R^2} e^{-y^2/2\sigma_R^2} dy \tag{11}$$

We proceed by first finding the area to be integrated, followed by dividing the area into simpler parts, if possible, and then changing the order of integration. The shaded area in Fig. 3(a) and Fig. 3(b) shows the area of integration in (11) for negative and positive values of θ_l respectively. For negative θ_l case, splitting the area of integration and then changing the order of integration gives

$$\begin{aligned} E = &\frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^\infty \frac{y}{\sigma_R^2} e^{-y^2/2\sigma_R^2} dy \ e^{-t^2/2} dt \\ &+ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 \int_{t/\sqrt{2\gamma}\theta_l}^\infty \frac{y}{\sigma_R^2} e^{-y^2/2\sigma_R^2} dy \ e^{-t^2/2} dt \end{aligned} \tag{12}$$

A direct integration of the above expression after some substitutions of variables gives E as

$$E = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{2\gamma\theta_l^2\sigma_R^2}{1+2\gamma\theta_l^2\sigma_R^2}} \tag{13}$$

When θ_l is positive, changing the order of integration over the area shown in Fig. 3(b) gives E as

$$E = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_0^{t/\sqrt{2\gamma}\theta_l} \frac{y}{\sigma_R^2} e^{-y^2/2\sigma_R^2} dy \ e^{-t^2/2} dt \tag{14}$$

$$= \frac{1}{2} - \frac{1}{2}\sqrt{\frac{2\gamma\theta_l^2\sigma_R^2}{1+2\gamma\theta_l^2\sigma_R^2}} \tag{15}$$

Hence for any θ_l, E is given by

$$\int_0^\infty Q(\sqrt{2\gamma}\theta_l y)p(y)dy = \frac{1}{2} - \frac{1}{2} sgn(\theta_l)\sqrt{\frac{2\gamma\theta_l^2\sigma_R^2}{1+2\gamma\theta_l^2\sigma_R^2}} \tag{16}$$

Replacing θ_l by β_l gives $\int_0^\infty Q(\sqrt{2\gamma}\beta_l y)p(y)dy$, say denoted by $\tilde{E}$. Changing $y = |c|$ in E, $\tilde{E}$ and then substituting these E and $\tilde{E}$ back into (4) gives (5). It can be noted that (8) can also be obtained similarly, just by changing variables.

References

[1] T. Pollet, M. Van Bladel, and M. Moeneclaey, “BER sensitivity of OFDM systems to carrier frequency offset and Wiener phase noise,” *IEEE Trans. Commun.*, vol. 43, pp. 191-193, Feb./Mar./Apr. 1995.

[2] K. Sathananthan and C. Tellambura, “Probability of error calculation of OFDM systems with frequency offset,” *IEEE Trans. Commun.*, vol. 49, no. 11, pp. 1884-1888, Nov. 2001.

[3] N. C. Beaulieu and P. Tan, “On the effects of receiver windowing on OFDM performance in the presence of carrier frequency offset,” *IEEE Trans. Wireless Commun.*, vol. 6, no. 1, pp. 202-209, Jan. 2007.

[4] P. Dharmawansa, N. Rajatheva, and H. Minn, “An exact error probability analysis of OFDM systems with frequency offset,” *IEEE Trans. Commun.*, vol. 57, no. 1, pp. 26-31, Jan. 2009.