On the Connectivity of Circularly Distributed Nodes in Ad Hoc Wireless Networks

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***Abstract*—This paper examines the probability of connectivity of ad hoc wireless networks in which nodes are uniformly distributed on the circumference of a circle. We derive the exact probability that the network is composed of at most C clusters. The probability of connectivity is obtained by considering the case of C equal to unity. We also consider the distribution of nodes over a circular disk. The probability of connectivity for this case is found by fitting a log-logistic function using Minimum Mean Square Error (MMSE) criterion.**

***Index Terms*—Ad hoc networks, connectivity, wireless networks.**

I. Introduction

THE probability of connectivity (P_{con}) of wireless adhoc networks has been an area of keen interest. Although many definitions of connectivity have been given in the literature [3], the most popular is the one in which a network is called connected if every pair of nodes in the network is connected. An exact formula for the probability of connectivity for a uniform distribution of nodes for a one-dimensional network is given in [4]. More recently, an exact expression for the probability that a uniformly distributed one-dimensional network is composed of at most C clusters was derived in [1]. The probability of connectivity corresponds to the special case when C is unity. The probability of connectivity of non-identically distributed nodes is investigated in [2] using a regression based approach. All these studies have assumed a random distribution of nodes on a straight line. Exact results for probability of connectivity are not known for any other setting. In the case of two-dimensional networks, only asymptotic connectivity for infinitely many nodes has been analyzed in [5]-[7].

In this paper we consider the case when the nodes are uniformly distributed on the circumference of a circle and derive an exact expression for the probability that such a network is composed of at most C clusters. This is the first exact analysis for a setting beyond the straight line. Further, we derive a regression based expression for the probability of connectivity when the nodes are uniformly distributed over a circular disk. The expression has been found to be in agreement with the asymptotic analysis of [5].

The paper is organized as follows. In Section II we state the main result for the case of nodes distributed on a circle while in Section VI we provide its proof. Monte Carlo simulation results are given in Section III. Nodes distributed on a disk are considered in Section IV where we give a minimum mean square error (MMSE) based expression for the probability of connectivity. The paper is concluded in Section V.

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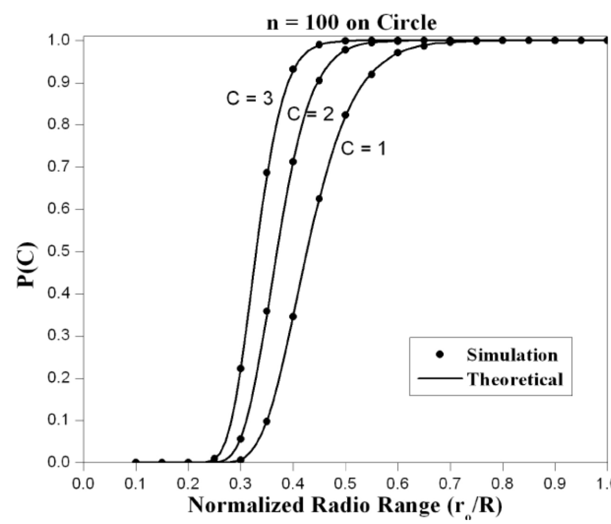

Fig. 1. Probability of atmost C clusters when 100 nodes are distributed on the circumference.

II. P_{con} for Nodes on the Circumference

Theorem 1: Assume that n nodes with a radio range of r_o are distributed uniformly and independently on the circumference of a circle of radius R. From simple geometry, the maximum angular separation between any two nodes on this circle for them to be in range of each other is $\theta_c = 2\sin^{-1}\frac{r_o}{2R}$. Then the probability $P(C)$, that the network is composed of at most C clusters is given by:

$$P(C) = 1 - \sum_{i=C}^{m}(-1)^{i-C}\binom{i-1}{C-1}\binom{n-1}{i}\Bigg[(1+\frac{(i+1)\theta_c}{2\pi})^n + \frac{n\theta_c}{2\pi}(1-\frac{(i+1)\theta_c}{2\pi})^{n-1}\Bigg] + (-1)^{i-C}\binom{i-1}{C}\binom{n-1}{i}p_{\Delta_i}$$

where $m = \min(n-1, [\frac{2\pi}{\theta_c}]-1)$, $\binom{i}{C} = 0$ when $i < C$, $[x]$ represents the greatest integer that is less than x and

$$p_{\Delta_i} = \begin{cases} (1-i\frac{\theta_c}{2\pi})^n & \frac{\theta_c}{2\pi} > \frac{1}{i+1} \\ [(1-i\frac{\theta_c}{2\pi})^n - n\frac{\theta_c}{2\pi}(1-(i+1)\frac{\theta_c}{2\pi})^{n-1} & \frac{\theta_c}{2\pi} \le \frac{1}{i+1} \\ -(1-(i+1)\frac{\theta_c}{2\pi})^n] & \end{cases}$$

As already mentioned, P_{con} corresponds to the special case when C is equal to unity. The theorem is proved in Section VI.

III. Simulations

In Fig. 1. we plot the Monte Carlo simulation results and the exact expression for $P(C)$ as a function of the normalized range $\frac{r_o}{R}$ for 100 nodes on the circumference of a

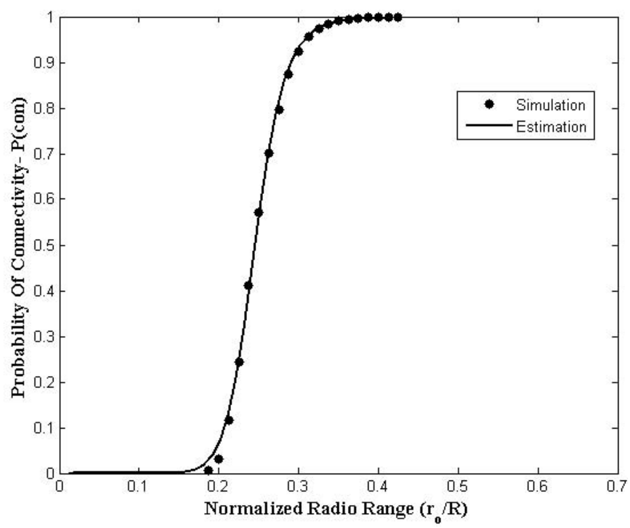

Fig. 2. MMSE estimation of P_{con} for 200 nodes distributed on a disk.

circle. Each simulation point was obtained by simulating 5000 random topologies. Further, the probability that the network is composed of at most C clusters is depicted for $C = 1, 2, 3$. It may be noted that there is complete agreement between the simulation and theoretical results.

Similar exercise was also done for $10, 20, \ldots 300$ nodes and in all the cases the simulation results matched the theoretical results.

IV. P_{con} FOR NODES ON A CIRCULAR DISK

The case of nodes distributed on the circumference of a circle is fundamentally different from that of a circular disk. As shown in the previous section, the probability of connectivity for the former can be computed by using the angular separation between adjacent nodes as a sufficient statistic when they are ordered according to their locations. In the case of a disk this approach seems intractable because in the absence of a natural ordering, angular separation does not suffice as a sufficient statistic.

The probability of connectivity results reported in the literature for two dimensional networks [5]-[7] are mainly asymptotic in nature as the nodes increase without bound. In this section we consider the same problem but for the practical case when the number of nodes is finite. We aim to find the probability of connectivity of nodes distributed uniformly in both R and θ. Since an exact analysis is difficult we take recourse to regression using a log-logistic function (1). The log-logistic function has been earlier used in [2] to fit a probability of connectivity (P_{con}) curve:

$$P_{con} = \frac{1}{1 + \exp^{-\left(\frac{\log(r_o/R)-\alpha}{\beta}\right)}} \tag{1}$$

Firstly, we obtained a P_{con} versus $\frac{r_o}{R}$ curve for a disk by doing sufficiently large number of Monte Carlo simulations for n nodes uniformly distributed in both R and θ.

We fit the obtained curve to a log-logistic function by estimating the parameters α and β in (1) using the MMSE method. The estimated curve using these parameters as well as the simulated points for $n = 200$ are plotted in Fig. 2.

We repeated this exercise for $n = 10, 20, \ldots 300$ and found that $\alpha(n)$ closely fits a $\log n$ curve whereas $\beta(n)$ closely fits a $\frac{\log n}{n}$ curve. The functions $\alpha(n)$ and $\beta(n)$ were estimated using the MMSE criterion again. The obtained $\alpha(n)$ and $\beta(n)$ are as follows:

$$\alpha(n) = a_1 + a_2 \log n \quad a_1 = 2.043 \quad a_2 = -0.7424$$
$$\beta(n) = b_1 + b_2 \frac{\log n}{n} \quad b_1 = 0.07879 \quad b_2 = 0.1964$$

These functions can be directly substituted in (1), thus giving us a relation between P_{con}, n and $\frac{r_o}{R}$. This relation can also be expressed in terms of radio range $r_o(n)$ required to meet a desired probability of connectivity in a network with n nodes:

$$r_o(n) = Re^{\alpha(n)} e^{\beta(n) log \frac{P_{con}}{1-P_{con}}} \tag{2}$$

For a disk of unit area, i.e. $\pi R^2 = 1$, it is shown in [5] that if:

$$\pi r_o^2(n) = \frac{\log n + c(n)}{n} \tag{3}$$

then the network is asymptotically connected with probability 1 if and only if $c(n) \to \infty$. Interestingly, the expressions for P_{con} and $r_o(n)$ obtained for a finite number of nodes are in agreement with the asymptotic analysis of [5] i.e. if $r_o(n)$ from (3) is substituted in (1), $P_{con} \to 1$ as $c(n) \to \infty$ when $n \to \infty$. Also for any P_{con} arbitrarily close to 1, $c(n) \to \infty$ as $n \to \infty$.

V. CONCLUSIONS

We derived an exact expression for the probability of at most C clusters for nodes uniformly distributed on the circumference of a circle. The probability of connectivity corresponds to the case when C is equal to unity. The probability of connectivity for nodes on a circular disk was also estimated using a log-logistic function. The estimated expression was found to be in agreement with the asymptotic analysis of [5].

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VI. Proof for Theorem 1

The location of i^{th} node on the circumference is specified by θ_i where θ_i's are i.i.d $U[0, 2\pi]$. Let us define random variables (RVs) $\{\Delta_i\}_{i=1}^{n}$ as a linear combination of the ordered RVs $\{\theta_{(i)}\}_{i=1}^{n}$:

$$\Delta_i = \theta_{(i+1)} - \theta_{(i)} \quad i = 0, 1 \ldots n-1, \quad where\ \theta_{(0)} = 0 \tag{4}$$

We obtain the joint pdf of the RVs $\{\Delta_i\}_{i=0}^{n-1}$ by performing a Jacobian transformation and then integrating over Δ_0 to get the joint pdf of the RVs $\{\Delta_i\}_{i=1}^{n-1}$.

$$\begin{aligned}
f_{\Delta_1\ldots\Delta_{n-1}}(\delta_1\ldots\delta_{n-1}) &= \tfrac{n!(2\pi-\delta_1-\ldots\delta_{n-1})}{(2\pi)^n} \quad if \quad \sum_{i=1}^{n-1}\delta_i \le 2\pi \\
f_{\Delta_1\ldots\Delta_{n-1}|A}(\delta_1\ldots\delta_{n-1}) &= \tfrac{n!(2\pi-\delta_1-\ldots\delta_{n-1})}{(2\pi)^n P(A)} \quad if \quad \sum_{i=1}^{n-1}\delta_i \le 2\pi-\theta_c \\
f_{\Delta_1\ldots\Delta_{n-1}|\bar{A}}(\delta_1\ldots\delta_{n-1}) &= \tfrac{n!(2\pi-\delta_1-\ldots\delta_{n-1})}{(2\pi)^n(1-P(A))} \quad if \quad 2\pi-\theta_c \le \sum_{i=1}^{n-1}\delta_i \le 2\pi \quad ,\theta_c = 2\sin^{-1}\tfrac{r_o}{2R}
\end{aligned} \tag{5}$$

where A denotes the event that the first node and the last node are not within radio range of each other i.e. $\theta_{(n)} - \theta_{(1)} \le 2\pi - \theta_c$ and θ_c is the maximum angular distance between two nodes for them to be in radio range of each other. $P(C)$ can be obtained by using the total probability formula and the principle of inclusion and exclusion [8]. Thus:

$$\begin{aligned}
P(C) &= P(C|A)P(A) + P(C|\bar{A})P(\bar{A}) \\
P(C|A) &= 1 - \sum_{r=C}^{m}(-1)^{r-C}\binom{r-1}{C-1}\binom{n-1}{r}P(\Delta_1 \ge \theta_c \ldots \Delta_r \ge \theta_c|A) \\
P(C|\bar{A}) &= 1 - \sum_{r=C+1}^{m}(-1)^{r-C}\binom{r-1}{C-1}\binom{n-1}{r}P(\Delta_1 \ge \theta_c \ldots \Delta_r \ge \theta_c|\bar{A})) \quad ,m = \min(n-1, [\tfrac{2\pi}{\theta_c}]-1)
\end{aligned} \tag{6}$$

We need $f_{\Delta_1\ldots\Delta_r|A}(\delta_1\ldots\delta_r)$ for calculating (6), hence we integrate $f_{\Delta_1\ldots\Delta_{n-1}|A}(\delta_1\ldots\delta_{n-1})$ over $\delta_{n-1}\ldots\delta_{r+1}$

$$\int_0^{2\pi-\theta_c-\delta_1-\ldots\delta_r}\ldots\int_0^{2\pi-\theta_c-\delta_1-\ldots\delta_{n-2}} f_{\Delta_1\ldots\Delta_{n-1}|A}(\delta_1\ldots\delta_{n-1})d\delta_{n-1}\ldots d\delta_{r+1}$$

$$f_{\Delta_1\ldots\Delta_r|A}(\delta_1\ldots\delta_r) = \tfrac{n!(2\pi-\theta_c-\delta_1-\ldots\delta_{n-1})^{n-r}}{(2\pi)^n P(A)(n-r)!} + \tfrac{n!\theta_c(2\pi-\theta_c-\delta_1-\ldots\delta_{n-1})^{n-r-1}}{(2\pi)^n P(A)(n-r-1)!}, \quad \sum_{i=1}^{r-1}\delta_i \le 2\pi-\theta_c \tag{7}$$

Similarly we integrate $f_{\Delta_1\ldots\Delta_{n-1}|\bar{A}}(\delta_1\ldots\delta_{n-1})$ over $\delta_{n-1}\ldots\delta_{r+1}$

$$\int_{\max(2\pi-\theta_c-\delta_1-\ldots\delta_r,0)}^{2\pi-\delta_1-\ldots\delta_r}\ldots\int_{\max(2\pi-\theta_c-\delta_1-\ldots\delta_{n-2},0)}^{2\pi-\delta_1-\ldots\delta_{n-2}} f_{\Delta_1\ldots\Delta_{n-1}|\bar{A}}(\delta_1\ldots\delta_{n-1})d\delta_{n-1}\ldots d\delta_{r+1}$$

$$f_{\Delta_1\ldots\Delta_r|\bar{A}}(\delta_1\ldots\delta_r) = \begin{cases} \tfrac{n!(2\pi-\delta_1-\ldots\delta_{n-1})^{n-r}}{(2\pi)^n(1-P(A))(n-r)!} & 2\pi-\theta_c \le \sum_{i=1}^{r-1}\delta_i \le 2\pi\ ; \\ \tfrac{n![(2\pi-\delta_1-\ldots\delta_{n-1})^{n-r}-(n-r)\theta_c(2\pi-\theta_c-\delta_1-\ldots\delta_{n-1})^{n-r-1}-(2\pi-\theta_c-\delta_1-\ldots\delta_{n-1})^{n-r}]}{(2\pi)^n(1-P(A))(n-r)!} & 0 \le \sum_{i=1}^{r-1}\delta_i \le 2\pi-\theta_c \end{cases} \tag{8}$$

The expression for $P(\Delta_1 \ge \theta_c \ldots \Delta_r \ge \theta_c|A)$ is obtained by integrating (7) with these limits:

$$\int_{\theta_c}^{\min(2\pi-\theta_c,2\pi-(r-1)\theta_c)} - \ldots \int_{\theta_c}^{\min(2\pi-\theta_c,2\pi-\delta_1-\ldots\delta_{r-1})} f_{\Delta_1\ldots\Delta_r|A}(\delta_1\ldots\delta_r)d\delta_r\ldots d\delta_1 \tag{9}$$

and the expression for $P(\Delta_1 \ge \theta_c \ldots \Delta_r \ge \theta_c|\bar{A})$ is obtained by integrating (8) with the following limits:

$$\int_{\max(2\pi-\theta_c-\delta_1-\ldots\delta_{r-1},\theta_c)}^{2\pi-\delta_1-\ldots\delta_{r-1}}\ldots\int_{\max(2\pi-\theta_c-\delta_1-\ldots\delta_{n-2},\theta_c)}^{2\pi-\delta_1-\ldots\delta_{n-2}} f_{\Delta_1\ldots\Delta_r|\bar{A}}(\delta_1\ldots\delta_r)d\delta_r\ldots d\delta_1 \tag{10}$$

Substituting the values of above integrals in (6) we get the stated theorem.