Now, define S_1 as $S_1 = \frac{1}{t_1!}\sum_{i=1}^{t_1!} \Pi_{1,i}\tilde{S}_1\Pi_{1,i}^T$. Then,

$$S_{1,mm} = \frac{1}{t_1!}\sum_{i=1}^{t_1!} \tilde{S}_{1\,\pi_{1,i}(m),\pi_{1,i}(m)} = \frac{1}{t_1}\mathrm{tr}(\tilde{S}_1) \tag{33}$$

and

$$S_{1,mn} = \frac{1}{t_1!}\sum_{i=1}^{t_1!} \tilde{S}_{1\,\pi_{1,i}(m),\pi_{1,i}(n)} = \frac{1}{t_1}\sum_{m\neq n} \tilde{S}_{1,mn}. \tag{34}$$

Note that $S_{1,mn}$ is real because matrix $\tilde{S}_1$ is Hermitian. By applying the same derivation to other users, the result is obtained.

ACKNOWLEDGMENT

The authors would like to express their gratitude to the Associate Editor, Prof. Amos Lapidoth, and to the anonymous reviewers for their constructive comments, which contributed to enhance the quality of this correspondence.

REFERENCES

[1] P. Billingsley, *Probability and Measure*. New York: Wiley-Interscience, 1995.
[2] T. M. Cover and J. A. Thomas, *Elements of Information Theory*. New York: Wiley, 1991.
[3] A. J. Goldsmith, S. A. Jafar, N. Jindal, and S. Vishwanath, "Capacity limits of MIMO channels," *IEEE J. Select. Areas Commun.*, vol. 21, no. 5, pp. 684–702, June 2003.
[4] A. Grant, "Rayleigh fading multiple antenna channels," *EURASIP J. Appl. Signal Process., Special Issue on Space-Time Coding and its Applications—Part I*, vol. 2002, no. 3, pp. 316–329, Mar. 2002.
[5] R. M. Gray, *Entropy and Information Theory*. New York: Springer-Verlag, 1990.
[6] D. Hösli, "On the Role of the Line-of-Sight Component in Coherent MIMO Ricean Channels," Ph.D. dissertation, Swiss Federal Institute of Technology (ETH), Zurich, Switzerland, to be published.
[7] D. Hösli and A. Lapidoth, "The capacity of a MIMO Ricean channel is monotonic in the singular values of the mean," in *Proc. 5th Int. ITG Conf. Source and Channel Coding*, Erlangen, Germany, Jan. 2004, pp. 381–385.
[8] R. Horn and C. Johnson, *Matrix Analysis*. New York: Cambridge University Press, 1985.
[9] S. A. Jafar and A. J. Goldsmith, "Channel capacity and beamforming for multiple transmit and receive antennas with covariance feedback," *IEEE Trans. Wireless Commun.*, vol. 3, no. 4, pp. 1165–1175, Jul. 2004.
[10] N. Jindal, W. Rhee, S. Vishwanath, S. A. Jafar, and A. Goldsmith, "Sum power iterative water-filling for multi-antenna Gaussian broadcast channels," *IEEE Trans. Inform. Theory*, vol. 51, no. 4, pp. 1570–1580, Apr. 2005.
[11] E. Jorswieck and H. Boche, "Channel capacity and capacity-range of beamforming in MIMO wireless systems under correlated fading with covariance feedback," *IEEE Trans. Wireless Commun.*, vol. 3, no. 5, pp. 1543–1553, Sept. 2004.
[12] J. Kotecha and A. Sayeed, "On the capacity of correlated MIMO channels," in *Proc. Int. Symp. Inf. Theory*, Jul. 2003.
[13] Y. Liang and V. Veeravalli, "Correlated MIMO rayleigh fading channels: Capacity and optimal signalling," in *Proc. 37th Asilomar Conf. Signals, Syst. Comput.*, Pacific Grove, CA, Nov. 2003.
[14] W. Rhee and J. M. Cioffi, "On the capacity of multiuser wireless channels with multiple antennas," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2580–2595, Oct. 2003.
[15] S. Simon and A. Moustakas, "Optimizing MIMO antenna systems with channel covariance feedback," *IEEE J. Select. Areas Commun.*, vol. 21, pp. 406–417, Apr. 2003.
[16] E. Telatar, "Capacity of multi-antenna Gaussian channels," *European Trans. Telecomm.*, vol. 10, no. 6, pp. 585–595, Nov.–Dec. 1999.
[17] W. Yu and J. M. Cioffi, "Sum capacity of Gaussian vector broadcast channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 9, pp. 1875–1892, Sep. 2004.
[18] S. Venkatesan, S. H. Simon, and R. A. Valenzuela, "Capacity of a Gaussian MIMO channel with nonzero mean," in *Proc. of IEEE Veh. Technol. Conf.*, Oct. 2003.
[19] E. Visotsky and U. Madhow, "Space-time transmit precoding with imperfect feedback," *IEEE Trans. Inf. Theory*, vol. 47, no. 6, pp. 2632–2639, Sep. 2001.
[20] A. M. Tulino, A. Lozano, and S. Verdú, "Capacity-achieving input covariance for single-user multi-antenna channels," *IEEE Trans. Wireless Commun.*, vol. 5, no. 3, pp. 662–671, Mar. 2006.
[21] W. Yu, W. Rhee, S. P. Boyd, and J. M. Cioffi, "Iterative water-filling for Gaussian vector multiple access channels," *IEEE Trans. Inf. Theory*, vol. 50, no. 1, pp. 145–151, Jan. 2004.
[22] L. Zheng and D. N. C. Tse, "Communication on the Grassman manifold: A geometric approach to the noncoherent multiple-antenna channel," *IEEE Trans. Inf. Theory*, vol. 48, no. 2, pp. 359–383, Feb. 2002.

A New Framework for Constructing Mutually Orthogonal Complementary Sets and ZCZ Sequences

Rathinakumar Appuswamy, *Student Member, IEEE*, and
Ajit Kumar Chaturvedi, *Senior Member, IEEE*

***Abstract*—In this correspondence, new characterizations for the construction of zero correlation zone (ZCZ) sequences from mutually orthogonal Golay complementary sets (MOGCS) is presented. It is shown that the recursive construction of MOGCS is inherent in these characterizations. Previously known constructions of ZCZ sequences and MOGCS are shown to be special cases of this characterization. The notion of mutually orthogonal ZCZ sequence sets is also introduced.**

***Index Terms*—Complementary set, interference free window (IFW), large area synchronous CDMA (LAS-CDMA), mutually orthogonal Golay complementary set (MOGCS), mutually orthogonal zero correlation zone (ZCZ) sequence sets, zero correlation zone (ZCZ) sequences.**

I. INTRODUCTION

In a synchronous CDMA system, orthogonal spreading sequences can be employed to eliminate multiple access interference. However in a wireless channel, orthogonality among different users tends to diminish because of the inter-path interference. In order to reduce the interference among users in a multipath environment or in an approximately synchronized CDMA [16] system, the concept of generalized orthogonality was introduced and a recursive construction of zero correlation zone (ZCZ) sequences was presented in [12] and [14]. In [1], analytical and simulation results show that the ZCZ sequences (also known as LS or Loosely Synchronous sequences) are indeed more robust in multipath propagation channels compared to the orthogonal sequences. Study of these sequences is further motivated by the recently proposed large area synchronous CDMA (LAS-CDMA) for 4G systems [10] of which ZCZ sequences are an integral part.

A recursive construction of ZCZ sequences starting from a Golay complementary pair [15] was proposed in [12]. The construction in [12] was extended in [11] by applying the same recursive method on

Manuscript received July 7, 2004; revised January 29, 2006.
R. Appuswamy was with the Department of Electrical Engineering, Indian Institute of Technology, Kanpur UP 208016, India. He is now with the Department of Electrical and Computer Engineering, University of California, San Diego, La Jolla, CA 92093 USA (e-mail: rathnam@ucsd.edu).
A. K. Chaturvedi is with the Department of Electrical Engineering, Indian Institute of Technology, Kanpur UP 208016, India (e-mail: akc@iitk.ac.in).
Communicated by K. G. Paterson, Associate Editor for Sequences.
Digital Object Identifier 10.1109/TIT.2006.878171

a matrix whose rows form mutually orthogonal Golay complementary sets (MOGCS). In a recent work [4], two new methods for constructing ZCZ sequence sets based on the perfect sequences and some unitary matrices have been proposed. In this correspondence, we take the former approach of constructing ZCZ sequences from the MOGCS. Recursive construction of the binary MOGCS was discussed in [13]. Polyphase complementary sequences were studied in [3]. Related results on the complementary sequences and their constructions were reviewed in [2]. Construction of binary array sets of ZCZ sequences was studied in [5]. Reference [17] presented a general construction of multilevel complementary pairs. Transition from iterative methods to noniterative construction of complementary sequences was made in [6], where complementary sequences were constructed as second order cosets of Reed Muller codes. QAM constellation based complementary sequences were constructed in [7]. In [2], DFT matrices were used to demonstrate the existence of complementary sequences of different lengths. Such techniques cannot be used to construct sequences over any arbitrary constellation as the size of the signal set is based on the DFT matrix dimension. DFT matrix based sequences have also been reported in [18], [19].

In this paper, we present new characterizations providing a framework for constructing the ZCZ sequences from the MOGCS. We show that enlarging a given MOGCS is implicit in this framework. We draw out the inherent recursion in the characterization and present illustrative examples. Previously known construction of ZCZ sequences [11] and expansion of MOGCS [13] are shown to be special cases of the above characterization. Using the above framework, we show that, for a given Interference Free Window (IFW), it is possible to construct multiple sets of ZCZ sequences which are mutually orthogonal. In multiuser systems, orthogonal sets of ZCZ sequences can provide better protection against interpath interference compared to traditional orthogonal sequences.

Further, characterizations which are useful in constructing new ZCZ sequence lengths from MOGCS are presented. We show from these characterizations that construction of new polyphase MOGCS is possible for several cardinalities. Additionally, the proposed characterizations allow construction of ZCZ sequences for previously unobtainable window lengths. Using recursive constructions of [12], the window lengths that were possible are $2^k N, k = 0, 1$, etc., where N denotes the kernel length used. For example, to design a Interference Free Window (IFW) size of 3 (no length 3 kernel is known over a binary constellation), a code with larger window size has to be designed followed by rotations of the sequences as discussed in [16]. However in practical systems, it is usually not desirable to employ sequence sets where one sequence in the set is a shifted version of another sequence. The characterizations presented in this paper allow direct design of sequences for several new window lengths, including 3. Furthermore, we are free to choose the constellation over which sequences need to be designed. Several identities have also been developed to simplify the design problem.

The remaining part of the paper is organized as follows. Section II establishes notations and definitions. In Section III, we characterize a recursive construction. After discussing a few examples, we compare our results with previously known constructions. Section IV presents new characterizations for constructing MOGCS. Section IV-B gives a generalized framework for constructing ZCZ sequences. Some issues related to the presented characterizations are discussed in Section V. Section VI discusses construction of mutually orthogonal sets of ZCZ sequences with an example. Section VII discusses the possibility of constructing sequences with better correlation properties while keeping the number of sequences intact. We conclude our work in Section VIII.

II. Notations and Definitions

A. Correlation Parameters

Aperiodic crosscorrelation $\psi_{\mathbf{ab}}(\tau)$ between two sequences $\mathbf{a}$ and $\mathbf{b}$ of length L is defined as

$$\psi_{\mathbf{ab}}(\tau) = \sum_{l=0}^{L-\tau-1} a_l\, b^*_{l+\tau}, \quad 0 \le \tau \le L-1.$$

Even or periodic crosscorrelation between the sequences $\mathbf{a}$ and $\mathbf{b}$ is defined as

$$\phi_{\mathbf{ab}}(\tau) = \psi_{\mathbf{ab}}(\tau) + \psi^*_{\mathbf{ba}}(L-\tau)$$

while odd correlation between the sequences $\mathbf{a}$ and $\mathbf{b}$ is given by

$$\phi^{\text{odd}}_{\mathbf{ab}}(\tau) = \psi_{\mathbf{ab}}(\tau) - \psi^*_{\mathbf{ba}}(L-\tau).$$

B. Orthogonal Sets of ZCZ Sequences

Let $\{\mathbf{b}_n\}_{n=1}^M$ be a set of M sequences, each of length L. The zero periodic autocorrelation zone T_{ACZ} and the zero periodic crosscorrelation zone T_{CCZ} of this sequence set are defined to be [12],

$$\begin{aligned} T_{\mathrm{ACZ}} &= \max\{T \mid \phi_{\mathbf{b}_n\mathbf{b}_n}(\tau) = 0, \forall n, \mid \tau \mid \le T, \tau \ne 0\} \\ T_{\mathrm{CCZ}} &= \max\{T \mid \phi_{\mathbf{b}_m\mathbf{b}_n}(\tau) = 0, \forall m \ne n, \mid \tau \mid \le T\} \end{aligned}$$

where, $\phi_{\mathbf{b}_m\mathbf{b}_n}(\tau)$ denotes the periodic crosscorrelation between the sequences $\mathbf{b}_m$ and $\mathbf{b}_n$.

Then, the interference free window (IFW) of the sequence set $\{\mathbf{b}_n\}_{n=1}^M$, denoted by T, is defined to be the minimum of the zero autocorrelation zone and the zero crosscorrelation zone values, i.e.

$$T = \min\{T_{\mathrm{ACZ}}, T_{\mathrm{CCZ}}\}.$$

The set $\{\mathbf{b}_n\}_{n=1}^M$ with IFW of T is said to constitute a Zero Correlation Zone sequence set of M sequences of length L and is denoted by ZCZ-(L, M, T).

Two distinct sets of sequences $\{\mathbf{b}^1_m\}_{m=1}^M$ and $\{\mathbf{b}^2_n\}_{n=1}^M$ are said to be mutually orthogonal, if

$$\phi_{\mathbf{b}^1_m\mathbf{b}^2_n}(0) = 0 \quad \forall\ m \text{ and } n.$$

This notion of orthogonality can easily be extended to more than two sets.

C. Complementary Sets

We denote the ith sequence in the nth sequence set as $\mathbf{a}^i_n$. A set $\{\mathbf{a}^i_n\}_{i=1}^M$ is said to be complementary, if,

$$\sum_{i=1}^{M} \psi_{\mathbf{a}^i_n\mathbf{a}^i_n}(\tau) = 0, \qquad \forall\ \tau \ne 0 \tag{1}$$

where $\psi_{\mathbf{a}^i_n\mathbf{a}^i_n}(\tau)$ is the aperiodic autocorrelation function of the sequence $\mathbf{a}^i_n$. Two sets $\{\mathbf{a}^i_m\}_{i=1}^M$ and $\{\mathbf{a}^i_n\}_{i=1}^M$ are said to be mutually orthogonal complementary (or mates) if,

$$\sum_{i=1}^{M} \psi_{\mathbf{a}^i_m\mathbf{a}^i_n}(\tau) = 0, \qquad \forall\ \tau \tag{2}$$

where $\psi_{\mathbf{a}_m^i \mathbf{a}_n^i}(\tau)$ is the aperiodic crosscorrelation function of the sequences $\mathbf{a}_m^i$ and $\mathbf{a}_n^i$.

D. $\varnothing_K$ Operation

A collection of sets

$$\{\mathbf{a}_j^i\}_{i=1}^M, \quad 1 \le j \le M$$

is called MOGCS if, $\{\mathbf{a}_j^i\}_{i=1}^M$ is a complementary set for each j, and any two such sets are mutually orthogonal.

Let $\mathbf{A}$ be a $M \times MN$ matrix whose rows form a collection of MOGCS. Each complementary set has M sequences of length N and the ith row of $\mathbf{A}$ constitutes the ith complementary set, such a matrix is called a MOGCS matrix. Let it be partitioned as $\mathbf{A} = [\mathbf{A}_1 \quad \mathbf{A}_2 \quad \cdots \mathbf{A}_M]$, where each $\mathbf{A}_i$ is an $M \times N$ matrix. We will denote the ith row of $\mathbf{A}$ by $\mathbf{a}_i$, the ith row of $\mathbf{A}_j$ by $\mathbf{a}_i^j$ and the (i,j)th scalar entry of $\mathbf{A}$ by $a_{i,j}$. Let $\mathbf{H}$ be a $P \times KQ$ matrix with complex valued entries having the same magnitude. For convenience, we will assume that this magnitude is 1, although the results in Sections IV and IV-B hold with slight modification even when the magnitude is any other constant. We define the matrix operation "$\varnothing_K$" as

$$\mathbf{A}\varnothing_K\mathbf{H} = \begin{bmatrix} \mathbf{C}(h_{11},\ldots,h_{1K}) & \cdots & \mathbf{C}(h_{1(QK-K+1)},\ldots,h_{1(QK)}) \\ \mathbf{C}(h_{21},\ldots,h_{2K}) & \cdots & \mathbf{C}(h_{2(QK-K+1)},\ldots,h_{2(QK)}) \\ \cdots & \ddots & \cdots \\ \mathbf{C}(h_{P1},\ldots,h_{PK}) & \cdots & \mathbf{C}(h_{P(QK-K+1)},\ldots,h_{P(QK)}) \end{bmatrix} \tag{3}$$

(viewed as a $PM \times KQMN$ scalar matrix) where

$$\mathbf{C}(h_{ij},h_{i(j+1)},\ldots,h_{i(j+K)}) = \Big[(h_{ij}\mathbf{A}_1,h_{i(j+1)}\mathbf{A}_1,\ldots,h_{i(j+K)}\mathbf{A}_1)\cdots (h_{ij}\mathbf{A}_M,h_{i(j+1)}\mathbf{A}_M,\ldots,h_{i(j+K)}\mathbf{A}_M)\Big]_{M\times KMN}$$

with $(h_{ij}\mathbf{A}_1,h_{i(j+1)}\mathbf{A}_1,\ldots,h_{i(j+K)}\mathbf{A}_1)$ denoting a sequence of length KN. It is easy to see that $\varnothing_K$ defined by (3) reduces to the operation introduced in [15] when $K=2$ and $P=Q=2$.

Each element of a column vector is cyclically moved one row up, upon premultiplication by $\mathbf{S}$, a shift operator. We will denote the conjugate transpose of the matrix $\mathbf{H}$ by $\mathbf{H}^*$. Binary vector $d_{11} = (1,0,1,0,\ldots,1,0)$ and $d_{21} = (1,1,0,1,1,0,\ldots,1,1,0)$. In general, d_{ij} is defined by a repeated pattern of i ones and j zeros. Matrix $\mathbf{D}_{ij} = \mathrm{diag}(d_{ij})$ and $\overline{\mathbf{D}}_{ij} = \mathrm{diag}(\overline{d}_{ij})$, where $\overline{d}_{ij}$ is the binary complement of d_{ij}.

III. Construction of ZCZ Sequences: A New Framework

In this section, we will characterize the $\mathbf{H}$ matrix in (3) such that the rows of the matrix $\mathbf{B} = \mathbf{A}\varnothing_2\mathbf{H}$ form a ZCZ sequence set (each row of $\mathbf{B}$ forms a single ZCZ sequence) in addition to constituting a collection of MOGCS (each row of $\mathbf{B}$ is a complementary set with QM sequences and two distinct rows are orthogonal complementary). In Section III-A we list a set of conditions on the $\mathbf{H}$ matrix to obtain the ZCZ property of the sequences constructed using $\varnothing_2$. Section III-B characterizes the $\mathbf{H}$ matrix so that the resulting sequences form MOGCS. Section III-C brings out the recursion inherent in the characterization. In Section III-D, we give examples and compare our results with previously known constructions [4], [11].

A. ZCZ Property

Theorem 1: If $\mathbf{A}$ is a MOGCS matrix, then the rows of the matrix $\mathbf{B} = \mathbf{A}\varnothing_2\mathbf{H}$ form a ZCZ-$(2MNQ, MP, N)$ if $\mathbf{H}$ satisfies the following:

1) $\mathbf{HH}^* = (MN)\mathbf{I}$ where $\mathbf{I}$ is the identity matrix.
2) $\mathbf{HD}_{11}\mathbf{SH}^* = 0 = \mathbf{HD}_{11}\mathbf{S}^*\mathbf{H}^*$

Proof: Let $\mathbf{b}_i$ denote the ith row of $\mathbf{B}$. It is enough to show that $\mathbf{b}_m\mathbf{b}_n^* = 0$ whenever $m \ne n$ and $\mathbf{b}_m\mathbf{S}^\tau\mathbf{b}_n^* = 0 = \mathbf{b}_m\mathbf{S}^{\tau *}\mathbf{b}_n^*$, $\forall m,n$, and $1 \le \tau \le N$. The nth row of $\mathbf{A}_k$ is denoted as $\mathbf{a}_\mathbf{n}^\mathbf{k}$, so $a_{n,Nk+\tau}$ will denote the τth element of the sequence $\mathbf{a}_\mathbf{n}^\mathbf{k}$. We will group the rows of $\mathbf{B}$ into P different sets, each set having M rows. For any m and n, we set r, l, s, and k such that $m = lM + r$ and $n = kM + s$. The proof is easy to obtain once we show that

$$\mathbf{H}\overline{\mathbf{D}}_{11}\mathbf{SH}^* = 0 = \mathbf{H}\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{H}^* \tag{4}$$

which can be shown to follow from condition 2 of the theorem using the results in Appendices B and C. The following four cases cover all possibilities:

1) m and n are from the same set (say lth set), and $m \ne n$. In this case, $r \ne s$

$$\mathbf{b}_m\mathbf{b}_n^* = \Big\langle \Big(h_{l1}\mathbf{a}_r^1, h_{l2}\mathbf{a}_r^1,\ldots,h_{l1}\mathbf{a}_r^M,h_{l2}\mathbf{a}_r^M,\ldots, h_{l(2Q-1)}\mathbf{a}_r^1,h_{l(2Q)}\mathbf{a}_r^1,\ldots,h_{l(2Q-1)}\mathbf{a}_r^M,h_{l(2Q)}\mathbf{a}_r^M\Big) \Big(h_{l1}\mathbf{a}_s^1, h_{l2}\mathbf{a}_s^1,\ldots,h_{l1}\mathbf{a}_s^M,h_{l2}\mathbf{a}_s^M,\ldots, h_{l(2Q-1)}\mathbf{a}_s^1,h_{l(2Q)}\mathbf{a}_s^1,\ldots,h_{l(2Q-1)}\mathbf{a}_s^M,h_{l(2Q)}\mathbf{a}_s^M\Big)\Big\rangle$$

from the definition of $\psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(\tau)$, it follows that:

$$\mathbf{b}_m\mathbf{b}_n^* = 2Q\sum_{i=1}^M \psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(0) = 2QMN\delta(r,s) \tag{5}$$

where $\delta(r,s)$ is the two-dimensional (2-D) Kronecker's delta function. The last line follows directly from the properties of complementary sets [13]. But $r \ne s$ when $m \ne n$ and m and n are from the same set, hence the desired result

$$\begin{aligned}\mathbf{b}_m\mathbf{Sb}_n^* &= \Big\langle \Big(h_{l1}\mathbf{a}_r^1, h_{l2}\mathbf{a}_r^1,\ldots,h_{l1}\mathbf{a}_r^M,h_{l2}\mathbf{a}_r^M,\ldots, h_{l(2Q-1)}\mathbf{a}_r^1,h_{l(2Q)}\mathbf{a}_r^1,\ldots,h_{l(2Q-1)}\mathbf{a}_r^M,h_{l(2Q)}\mathbf{a}_r^M\Big) \\ &\quad \Big(h_{l1}\mathbf{a}_s^1, h_{l2}\mathbf{a}_s^1,\ldots,h_{l1}\mathbf{a}_s^M,h_{l2}\mathbf{a}_s^M,\ldots, h_{l(2Q-1)}\mathbf{a}_s^1,h_{l(2Q)}\mathbf{a}_s^1,\ldots,h_{l(2Q-1)}\mathbf{a}_s^M,h_{l(2Q)}\mathbf{a}_s^M\Big)\mathbf{S}^*\Big\rangle \\ &= 2Q\sum_{i=1}^M \psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(1) \\ &\quad + \sum_{j=1}^M (a_{r,Nj}a_{s,Nj-(N-1)}^*)(\mathbf{h}_l\mathbf{D}_{11}\mathbf{Sh}_l^*) \\ &\quad + \sum_{j=1}^{M-1} (a_{r,Nj}a_{s,Nj+1}^*)(\mathbf{h}_l\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{h}_l^*) \\ &\quad + a_{r,NM}a_{s,1}^*(\mathbf{h}_l\overline{\mathbf{D}}_{11}\mathbf{Sh}_l^*)\end{aligned} \tag{6}$$

where $\mathbf{h}_l$ denotes the lth row of $\mathbf{H}$. Now, $\sum_i \psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(1) = 0$ and from conditions 2 and (4), terms involving inner products of rows of $\mathbf{H}$ vanish to give $\mathbf{b}_m\mathbf{S}\mathbf{b}_n^* = 0$. Similarly for $1 < \tau \leq (N-1)$ we can see

$$\begin{aligned}\mathbf{b}_m\mathbf{S}^\tau\mathbf{b}_n^* =& 2Q\sum_{i=1}^{M}\psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(\tau)\\ &+\sum_{j=1}^{M}(a_{r,(Nj-\tau+1)}a^*_{s,N(j-1)+1}\\ &+a_{r,(Nj-\tau+2)}a^*_{s,N(j-1)+2}\\ &+\cdots+a_{r,Nj}a^*_{s,N(j-1)+\tau})(\mathbf{h}_l\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)\\ &+\sum_{j=1}^{M-1}(a_{r,(Nj-\tau+1)}a^*_{s,Nj+1}+a_{r,(Nj-\tau+2)}a^*_{s,Nj+2}\\ &+\cdots+a_{r,Nj}a^*_{s,Nj+\tau})(\mathbf{h}_l\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{h}_l^*)\\ &+(a_{r,(NM-\tau+1)}a^*_{s,1}+a_{r,(NM-\tau+2)}a^*_{s,2}\\ &+\cdots+a_{r,NM}a^*_{s,\tau})(\mathbf{h}_l\overline{\mathbf{D}}_{11}\mathbf{S}\mathbf{h}_l^*)\end{aligned}\tag{7}$$

2) $m \neq n$, $r \neq s$, and m and n are from different sets k and l

$$\mathbf{b}_m\mathbf{b}_n^* = \Big(\sum_{i=1}^{M}\psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(0)\Big)\Big(\mathbf{h}_k\mathbf{h}_l^*\Big)\tag{8}$$

but from condition 1, $\mathbf{h}_k\mathbf{h}_l^* = 0$, hence the result follows. Similarly, for the present case, $\mathbf{b}_m\mathbf{S}^\tau\mathbf{b}_n^* = 0$ follows from conditions 2 and (4), when $\mathbf{h}_l\mathbf{X}\mathbf{h}_l^*$ is replaced by $\mathbf{h}_k\mathbf{X}\mathbf{h}_l^*$ in (7).

3) $m \neq n$ and $r = s$, it follows that $k \neq l$

$$\mathbf{b}_m\mathbf{b}_n^* = 2M\Big(\mathbf{h}_k\mathbf{h}_l^*\Big) = 0 \quad \text{(from condition 1)}\tag{9}$$

$$\begin{aligned}\mathbf{b}_m\mathbf{S}^\tau\mathbf{b}_n^* =& 2Q\sum_{i=1}^{M}\psi_{\mathbf{a}_r^i\mathbf{a}_r^i}(\tau)\\ &+\sum_{j=1}^{M}(a_{r,(Nj-\tau+1)}a^*_{r,N(j-1)+1}\\ &+a_{r,(Nj-\tau+2)}a^*_{r,N(j-1)+2}\\ &+\cdots+a_{r,Nj}a^*_{r,N(j-1)+\tau})(\mathbf{h}_k\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)\\ &+\sum_{j=1}^{M-1}(a_{r,(Nj-\tau+1)}a^*_{r,Nj+1}\\ &+a_{r,(Nj-\tau+2)}a^*_{r,Nj+2}\\ &+\cdots+a_{r,Nj}a^*_{r,Nj+\tau})(\mathbf{h}_k\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{h}_l^*)\\ &+(a_{r,(NM-\tau+1)}a^*_{s,1}+a_{r,(NM-\tau+2)}a^*_{s,2}\\ &+\cdots+a_{r,NM}a^*_{s,\tau})(\mathbf{h}_k\overline{\mathbf{D}}_{11}\mathbf{S}\mathbf{h}_l^*)\end{aligned}\tag{10}$$

for $1 \leq \tau \leq N-1$. The last line is a direct consequence of $\psi_{\mathbf{a}_r^i\mathbf{a}_r^i}(\tau)$ being null for this range of τ and conditions 2 and (4).

4) $m = n$, hence $r = s$ and $k = l$. For $1 \leq \tau < N$,

$$\begin{aligned}\mathbf{b}_m\mathbf{S}^\tau\mathbf{b}_n^* =& 2Q\sum_{i=1}^{M}\psi_{\mathbf{a}_r^i\mathbf{a}_r^i}(\tau)\\ &+\sum_{j=1}^{M}(a_{r,(Nj-\tau+1)}a^*_{r,N(j-1)+1}\\ &+a_{r,(Nj-\tau+2)}a^*_{r,N(j-1)+2}\\ &+\cdots+a_{r,Nj}a^*_{r,N(j-1)+\tau})(\mathbf{h}_l\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)\\ &+\sum_{j=1}^{M-1}(a_{r,(Nj-\tau+1)}a^*_{r,Nj+1}\\ &+a_{r,(Nj-\tau+2)}a^*_{r,Nj+2}\\ &+\cdots+a_{r,Nj}a^*_{r,Nj+\tau})(\mathbf{h}_l\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{h}_l^*)\\ &+(a_{r,(NM-\tau+1)}a^*_{s,1}+a_{r,(NM-\tau+2)}a^*_{s,2}\\ &+\cdots+a_{r,NM}a^*_{s,\tau})(\mathbf{h}_l\overline{\mathbf{D}}_{11}\mathbf{S}\mathbf{h}_l^*).\end{aligned}\tag{11}$$

The first term in (11) vanishes because of the complementary property of $\mathbf{a}_r^i$s, the second term is zero by condition 2, and the third and final term vanishes by (4).

Finally for all combinations of m and n, when $\tau = N$

$$\begin{aligned}\mathbf{b}_m\mathbf{S}^N\mathbf{b}_n^* =& \alpha(m,n)(\mathbf{h}_k\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)+\beta(m,n)(\mathbf{h}_k\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{h}_l^*)\\ &+\gamma(m,n)(\mathbf{h}_k\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)^*\end{aligned}$$

where α, β, and γ are m,n dependent constants and $\mathbf{h}_k\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^* = \mathbf{h}_k\overline{\mathbf{D}}_{11}\mathbf{S}^*\mathbf{h}_l^* = \mathbf{h}_k\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^* = 0$, hence the desired result. Proof for the case when $\mathbf{S}$ is replaced by $\mathbf{S}^*$ is similar. □

We call a solution for the characterization given in a theorem, a characteristic matrix of that theorem. It is easy to verify that the matrix used in the recursive construction of ZCZ sequences in [11] given by

$$\mathbf{H} = \begin{bmatrix}1 & 1 & 1 & -1\\ 1 & -1 & 1 & 1\end{bmatrix}\tag{12}$$

satisfies the conditions given in Theorem 1, hence, this is a characteristic matrix of Theorem 1. We note that the recursions given in [12] and [11] can be obtained from a characteristic matrix with $Q = 2$ when its entries are restricted to be from $\{1, -1\}$.

B. Orthogonal Complementariness of Resulting Sequences

Theorem 2: If $\mathbf{A}$ is a MOGCS matrix, then the rows of $\mathbf{B} = \mathbf{A}\varnothing_2\mathbf{H}$ form a collection of MP mutually orthogonal complementary sets with each set having MQ sequences of length $2N$ if the $\mathbf{H}$ matrix satisfies the following conditions:

1) $\mathbf{H}\mathbf{H}^* = (MN)\mathbf{I}$ where $\mathbf{I}$ is the identity matrix.
2) $\mathbf{H}\mathbf{D}_{11}\mathbf{S}\mathbf{H}^* = 0 = \mathbf{H}\mathbf{D}_{11}\mathbf{S}^*\mathbf{H}^*$

Proof: As in the proof of Theorem 1, rows of $\mathbf{B}$ are grouped into P different sets in their natural order. For any m and n, we set $r, l, s,$ and k such that $m = lM + r$ and $n = kM + s$. In the following, α, β, and γ are constants. We consider the following cases:

(1) Complementariness of the rows: For any m, the sum of the autocorrelation function $\psi_{\mathbf{b}_m^j\mathbf{b}_m^j}(\tau)$ for $1 \leq \tau \leq N$ given by

$$\begin{aligned}\sum_{j=1}^{2M}\psi_{\mathbf{b}_m^j\mathbf{b}_m^j}(\tau) =& Q\sum_{j=1}^{M}\psi_{\mathbf{a}_r^j\mathbf{a}_r^j}(\tau)\\ &+\sum_{j=1}^{M}\alpha_j(\tau)(\mathbf{h}_l\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)\end{aligned}\tag{13}$$

is zero by complementariness of $\mathbf{a}_r^i$'s and condition 2. For $N \leq \tau \leq 2N$,

$$\sum_{j=1}^{2M}\psi_{\mathbf{b}_m^j\mathbf{b}_m^j}(\tau) = \sum_{j=1}^{M}\beta_j(\tau)(\mathbf{h}_l\mathbf{D}_{11}\mathbf{S}\mathbf{h}_l^*)\tag{14}$$

vanishes by condition 2.

(2) Orthogonal complementariness of any two rows m and n: The sum of the crosscorrelation function $\psi_{\mathbf{b}_m^j \mathbf{b}_n^j}(\tau)$ for $r \neq s$ with $0 \leq \tau \leq N$ is

$$\sum_{j=1}^{2M} \psi_{\mathbf{b}_m^j \mathbf{b}_n^j}(\tau) = Q \sum_{j=1}^{M} \psi_{\mathbf{a}_r^j \mathbf{a}_s^j}(\tau) + \sum_{j=1}^{M} \gamma_j(\tau)(\mathbf{h}_l \mathbf{D}_{11} \mathbf{S} \mathbf{h}_k^*) \quad (15)$$

and when $N \leq \tau \leq 2N$,

$$\sum_{j=1}^{2M} \psi_{\mathbf{b}_m^j \mathbf{b}_n^j}(\tau) = \sum_{j=1}^{M} \gamma_j(\tau)(\mathbf{h}_l \mathbf{D}_{11} \mathbf{S} \mathbf{h}_k^*). \quad (16)$$

Right hand side of (15) and (16) are zero by mutual complementariness of $\mathbf{a}_r^i$s and $\mathbf{a}_s^i$s and condition 2. When $r = s$, $m \neq n$, and $\tau = 0$

$$\sum_{j=1}^{2M} \psi_{\mathbf{b}_m^j \mathbf{b}_n^j}(0) = (\mathbf{h}_l \mathbf{h}_k^*) \sum_{j=1}^{M} \psi_{\mathbf{a}_r^j \mathbf{a}_s^j}(0) \quad (17)$$

is zero by condition 1. For $\tau \neq 0$, (15)(16) hold with $r = s$ and the expressions vanishes by applying the complementary property of $\mathbf{a}_r^i$s and condition 2.

□

The characteristic matrix in (12) was used in [13] to recursively expand a given set of MOGCS. Theorem 1 with $P = Q = 2$ generalizes Theorem 13 of [13] to polyphase sequences.

C. Recursive Property

By noting that a characteristic matrix of Theorem 1 is also a characteristic matrix of Theorem 2, we can obtain a recursive relation to construct a larger number of ZCZ sequences of increased length as well as to increase the number of orthogonal complementary sets, the number of sequences in each complementary set and their lengths. By Theorem 2, the matrix $\mathbf{B} = \mathbf{A} \varnothing_2 \mathbf{H}$ is a MOGCS matrix, hence, $\mathbf{B}$ can also be used for constructing ZCZ sequences. Using this argument repeatedly, we obtain the following recursion:

$$\mathbf{B}^{(n)} = \mathbf{B}^{(n-1)} \varnothing_2 \mathbf{H}, \quad n = 1, 2, \ldots \quad (18)$$

with $\mathbf{B}^{(0)} = \mathbf{A}$. Theorem 1 implies that the rows of $\mathbf{B}^{(n)}$ form a ZCZ sequence set and Theorem 2 ensures that the rows of $\mathbf{B}^{(n)}$ form a collection of MOGCS. We emphasize that the recursion formulated here generalizes the techniques reported in [2], [3], [13] to enlarge the given number of MOGCS. Note that although (18) uses the same $\mathbf{H}$ matrix in every recursion, one can use different characteristic matrices as the recursions progress.

D. Construction of ZCZ Sequences

As with earlier recursive methods [11], [12], we start with a known kernel[1] and its mates [2], [13]. The matrix $\mathbf{A}$ in the beginning of the recursion has this kernel and its mates as rows. For the special case of $N = 1$, it can be any orthogonal matrix. [2] has an extensive[2] (though not exhaustive) list of all known kernel lengths and examples. The dimension of the $\mathbf{H}$ matrix can be chosen according to the required sequence length. In binary sequence construction, for the matrix $\mathbf{H}$ used in the construction to have a solution, the number of columns in $\mathbf{H}$ has to be an even multiple of 2. If a characteristic matrix such that Q is not a power of 2 is obtained, the recursive construction proposed here generates ZCZ sequences for length L other than known earlier [1]. Design over polyphase constellations will also generate sequences of new lengths if a characteristic matrix over that space can be found. We now present example ZCZ sequence constructions based on these observations. Let F_n denote the sequence set (collection of the rows of $\mathbf{B}^{(n)}$) generated after the nth recursion with the convention that $F_0 = \mathbf{A}$.

Example 1: Let $\mathbf{A}$ be a $M_0 \times M_0$ orthogonal matrix ($\mathbf{A}\mathbf{A}^* = M_0 \mathbf{I}$), $\mathbf{H}$ be a 2×4 binary characteristic matrix ($P = Q = 2, N = 1$) of Theorem 1. Then $F_0 =$ ZCZ-$(M_0, M_0, 0)$ and the rows of $\mathbf{A} \varnothing_2 \mathbf{H}$ form the set $F_1 =$ ZCZ-$(4M_0, 2M_0, 1)$, and so on. In general, $F_n =$ ZCZ-$(4^n M_0, 2^n M_0, 2^{n-1})$.

Example 2: Let $\mathbf{A}$ be a $M_0 \times M_0 N$ matrix with orthogonal complementary sequence sets as rows, each sequence length being N and $\mathbf{H}$ be a $P \times 2Q$ characteristic matrix of Theorem 1. Then the rows of $\mathbf{B}^{(1)}$ form the set $F_1 =$ ZCZ-$(2QM_0N, PM_0, N)$, the second recursion produces the set $F_2 =$ ZCZ-$(2^2Q^2M_0N, P^2M_0, 2N)$ and so on. In general, $F_n =$ ZCZ-$(2^nQ^nM_0N, P^nM_0, 2^{n-1}N)$.

Let us try to look at these results from the perspective of some known bounds. In binary sequence construction, the following bound relates the length of the sequence L, cardinality of the ZCZ set M and IFW length T_b [9]:

$$T_b \leq \frac{L}{2M}. \quad (19)$$

Existing binary constructions satisfy this bound with equality. In general, the following bound holds in ZCZ sequence design:

$$T \leq \frac{L}{M} - 1. \quad (20)$$

However no known construction exists which can satisfy this bound with equality. Sequence constructions in [4] approach this bound asymptotically provided longer perfect sequences[3] can be found. In their constructions, the relation between these quantities is given by,

$$T = \frac{(l-2)L}{lM} \quad (21)$$

where l is the period of the perfect sequence used to generate the ZCZ sequence sets. As noted in [4], only length 4 binary perfect sequences are known and in the quadriphase case, l can be $2, 4, 8$, and 16. So the following upper bound is obtained for the quadriphase sequences:

$$T_Q \leq \frac{7L}{8M}. \quad (22)$$

However their constructions are useful only when a small number of sequences with large IFW are needed. The first construction in [4] cannot construct sets with more than 16 sequences. In their second construction, based on the length of the perfect sequence used, a lower bound on the length L exists. For example, when a perfect sequence of length 16 is used, smallest possible L is 512 and the set can have a maximum of 32 sequences with IFW given by (21). The minimum sequence length

[1]A Complementary set which is not obtainable by transforming others of the same length or from known sets of smaller length is called a kernel.

[2]It is still an open problem to give a compilation of all known kernel lengths.

[3]A sequence $\mathbf{a}$ is said to be perfect if the periodic autocorrelation function of the sequence $\phi_{\mathbf{aa}}(\tau)$ is zero for all $\tau \neq 0$.

possible with a perfect sequence of length 8 is 128. To construct quadriphase sequences of L less than 128, l can be either 2 or 4, accordingly, the bound for this case is given by

$$T_Q \leq \frac{L}{2M}$$

which is same as the binary case. On the contrary, in our constructions, as with earlier recursive methods [1], [11] M can take large values as can be seen from Examples 1 and 2. From Example 2, we obtain

$$T = \frac{L}{2M}\left(\frac{P}{Q}\right)^n \tag{23}$$

after the nth recursion. From Theorem 1, we note that the rows of a characteristic matrix constitute a ZCZ-$(2Q, P, 1)$ set. Hence for the binary case, $P \leq Q$ and the bound in (23) reduces to (19).

IV. Construction of Other Complementary and ZCZ Sequences

Motivated by the results in Section III, new characterizations using the operations $\varnothing_3$ and $\varnothing_4$ are presented here. We summarize results for a general K in the following section.

A. Construction of New Complementary Sequences

Theorem 3: For a MOGCS matrix $\mathbf{A}$ and a $P \times 3Q$ matrix $\mathbf{H}$, the rows of the matrix $\mathbf{B} = \mathbf{A}\varnothing_3\mathbf{H}$ form a collection of PM MOGCS with each set containing QM sequences of length $3N$ if the matrix $\mathbf{H}$ is such that as follows.

1) $\mathbf{HH}^* = (3Q)\mathbf{I}$ where $\mathbf{I}$ is the identity matrix;
2) $\mathbf{HD}_{21}\mathbf{SH}^* = 0$;
3) $\mathbf{HD}_{12}\mathbf{S}^2\mathbf{H}^* = 0$.

Proof: Proof is given in Appendix A. □

Theorem 4: For a MOGCS matrix $\mathbf{A}$ and a $P \times 4Q$ matrix $\mathbf{H}$, the rows of the matrix $\mathbf{B} = \mathbf{A}\varnothing_4\mathbf{H}$ form a collection of PM MOGCS with each set containing QM sequences of length $4N$ if the matrix $\mathbf{H}$ satisfies the following:

1) $\mathbf{HH}^* = (4Q)\mathbf{I}$;
2) $\mathbf{HD}_{31}\mathbf{SH}^* = 0$;
3) $\mathbf{HD}_{22}\mathbf{S}^2\mathbf{H}^* = 0$;
4) $\mathbf{HD}_{13}\mathbf{S}^3\mathbf{H}^* = 0$.

Proof: Using the identities in the Appendices B and C, we can show that

$$\mathbf{H}\overline{\mathbf{D}}_{13}\mathbf{S}^*\mathbf{H}^* = \mathbf{H}\overline{\mathbf{D}}_{22}\mathbf{S}^{2*}\mathbf{H}^* = \mathbf{H}\overline{\mathbf{D}}_{31}\mathbf{S}^{3*}\mathbf{H}^* = 0.$$

The rest of the proof is similar to the proof of Theorem 3 as given in Appendix A. □

B. A Generalized Framework for Constructing ZCZ Sequences

The following theorems summarize the construction of ZCZ sequences using $\varnothing_3$ and $\varnothing_4$ operations.

Theorem 5: Let $\mathbf{A}$ be a MOGCS matrix, and $\mathbf{H}$ be a $P \times 3Q$ matrix. Then, the rows of $\mathbf{B} = \mathbf{A}\varnothing_3\mathbf{H}$ form a ZCZ-$(3MNQ, MP, 2N)$ if the matrix $\mathbf{H}$ satisfies the following:

1) $\mathbf{HH}^* = (3Q)\mathbf{I}$
2) $\mathbf{HD}_{12}\mathbf{S}^2\mathbf{H}^* = \mathbf{HD}_{12}\mathbf{S}^*\mathbf{H}^* = 0$
3) $\mathbf{H}\overline{\mathbf{D}}_{12}\mathbf{S}^2\mathbf{H}^* = \mathbf{HD}_{21}\mathbf{S}^*\mathbf{H}^* = \mathbf{HD}_{21}\mathbf{SH}^* = 0$

Proof: In Appendix A, we extend the proof of Theorem 2 to $\mathrm{K} = 3$. Proof of this theorem is a similar extension of proof of Theorem 1 to $\mathrm{K} = 3$. □

Theorem 6: Let $\mathbf{A}$ be a MOGCS matrix and $\mathbf{H}$ be a $P \times 4Q$ matrix, the rows of $\mathbf{B} = \mathbf{A}\varnothing_4\mathbf{H}$ form a ZCZ-$(4MNQ, MP, 3N)$ if

1) $\mathbf{HH}^* = (4Q)\mathbf{I}$;
2) $\mathbf{HD}_{13}\mathbf{S}^*\mathbf{H}^* = \mathbf{HD}_{13}\mathbf{S}^3\mathbf{H}^* = \mathbf{HD}_{31}\mathbf{SH}^* = 0$;
3) $\mathbf{HD}_{22}\mathbf{S}^2\mathbf{H}^* = \mathbf{HD}_{22}\mathbf{S}^{2*}\mathbf{H}^* = 0$;
4) $\mathbf{H}\overline{\mathbf{D}}_{13}\mathbf{S}^3\mathbf{H}^* = 0$.

Proof: The following conditions are necessary to set the proof:

$$\mathbf{H}\overline{\mathbf{D}}_{31}\mathbf{S}^{3*}\mathbf{H}^* = \mathbf{H}\overline{\mathbf{D}}_{13}\mathbf{S}^*\mathbf{H}^* = \mathbf{H}\overline{\mathbf{D}}_{31}\mathbf{SH}^* = 0 \tag{24}$$

$$\mathbf{H}\overline{\mathbf{D}}_{22}\mathbf{S}^{2*}\mathbf{H}^* = \mathbf{H}\overline{\mathbf{D}}_{22}\mathbf{S}^2\mathbf{H}^* = 0 \tag{25}$$

$$\mathbf{H}\overline{\mathbf{D}}_{13}\mathbf{S}^*\mathbf{H}^* = \mathbf{HD}_{31}\mathbf{S}^{3*}\mathbf{H}^* = 0 \tag{26}$$

which are easily derived from the above conditions. The rest of the proof is obvious once the proof of Theorem 1 is extended to $\mathrm{K} = 3$. □

We note that Theorem 3 (or 4) implies that the matrix $\mathbf{B}$ constructed in Theorem 5 (or 6) will be a MOGCS matrix. By arguments paralleling those given to obtain (18), for any K, we obtain the following recursion:

$$\mathbf{B}^{(n)} = \mathbf{B}^{(n-1)}\varnothing_K\mathbf{H}, \quad n = 1, 2, \ldots \tag{27}$$

with $\mathbf{B}^{(0)} = \mathbf{A}$. Note that it is not necessary to use the same $\mathbf{H}$ matrix in every recursion as will be shown in Corollary 2, Section VI. We now present example ZCZ sequence constructions based on the above results. Let F_n denote the sequence set (collection of the rows of $\mathbf{B}^{(n)}$) generated after the nth recursion.

Example 3: Let $\mathbf{A}$ be a $M_0 \times M_0N$ MOGCS matrix and $\mathbf{H}$ be a $P \times 3Q$ characteristic matrix of Theorem 5. Then the rows of $\mathbf{B}$ form the set $F_1 =$ ZCZ-$(3QM_0N, M_0P, 2N)$, the second recursion produces the set $F_2 =$ ZCZ-$(3^2Q^2M_0N, M_0P^2, 2 \cdot 3N)$, and so on. In general, $F_n =$ ZCZ-$(3^nQ^nM_0N, M_0P^n, 2 \cdot 3^{n-1}N)$.

In the above example, sequences with lengths 3^nQ^nN for IFW lengths of $2 \cdot 3^{n-1}N$ can be constructed. The following example uses Theorem 6 to construct sequences with lengths 4^nQ^nN for IFW lengths of $3 \cdot 4^{n-1}N$.

Example 4: Let $\mathbf{A}$ be a $M_0 \times M_0N$ MOGCS and $\mathbf{H}$ be a $P \times 4Q$ characteristic matrix of Theorem 6. Then the rows of $\mathbf{B}$ form the set $F_1 =$ ZCZ-$(4QM_0N, M_0P, 3N)$, the second recursion produces the set $F_2 =$ ZCZ-$(4^2Q^2M_0N, M_0P^2, 3 \cdot 4N)$ and so on. In general, $F_n =$ ZCZ-$(4^nQ^nM_0N, M_0P^n, 3 \cdot 4^{n-1}N)$.

Example 5: The following is a characteristic matrix of Theorem 5 ($\mathrm{K} = 3$) constructed over binary space with $P = 2$ and $Q = 4$:

$$\mathbf{H} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Setting $\mathbf{A} = [1]$, we obtain the following binary quad of length 3 and its mate from the rows of $\mathbf{B}^{(1)}$:

$$\Big\{\Big\{(010), (110), (000), (100)\Big\}\Big\{(011), (101), (001), (111)\Big\}\Big\}.$$

Rows of the matrix $\mathbf{B}^{(2)}$ form 4 MOGCS of cardinality 16 with sequence length 9. To save space, we list the sequences using the octal notation

$$\Big\{\Big\{252,552,222,522,616,116,666,166,070,770,000,700,$$
$$434,334,444,344\Big\}$$
$$\Big\{343,443,333,433,522,225,555,255,161,661,111,611,$$
$$707,007,777,077\Big\}$$
$$\Big\{244,525,225,555,611,161,661,111,077,707,007,777,$$
$$433,343,443,333\Big\}$$

$$\{344,434,334,444,522,252,552,222,166,616,116,666, 700,070,770,000\}\}.$$

Since the above $\mathbf{H}$ is a characteristic matrix of Theorem 5, concatenating each complementary set results in four ZCZ sequences. No construction procedure was known earlier to generate binary or quadriphase complementary sets and their mates of length 9.

We observe from the above examples that the operations $\varnothing_3$ and $\varnothing_4$ can be used to obtain new MOGCS sets and IFWs unobtainable in earlier construction methods [1], [4], [11], [12]. In all earlier known recursive constructions of MOGCS, sequence length and the cardinality of the set double with each recursion. The general recursive construction in [17] applies only for constructing multi-level Golay complementary pairs. When using the $\varnothing_K$ operation at each step, sequence length increases K fold and the cardinality of the set increases Q fold for a $\mathbf{H}$ matrix of dimension $P \times QK$. Thus, constructing complementary sequence sets of several cardinalities for the same sequence length is possible by merely choosing different Qs. Similarly, choosing different Ks for a given Q will generate different sequence lengths for the same cardinality. Thus, the proposed characterizations can generate several new MOGCS sets, however, individual complementary sets generated by the characterizations presented here are all obtainable by known constructions in [2], [3], [6], [15], [23].

Recursions in [1], [11] can generate sequences of lengths 2^n and additionally $2^n N$, provided a kernel of length N and its mates are known. In their constructions, IFW lengths possible with a length N kernel are $2^{n-1}N, n = 1, 2$, etc. It is known that quadriphase complementary pairs exist for all lengths below 100 except $7, 9, 15$, and 17 [14], [22]. But when the recursions in [1], [11] are applied on a length 3 kernel, for example, longer sequences can be produced only for large IFW lengths. In fact, IFW will be 3 only for L = 12 using recursion in [11] with a kernel of length 3. On the contrary, using the proposed constructions, by applying $\varnothing_4$ on a large orthogonal matrix $\mathbf{A}$, IFW of 3 can be obtained for large sequence lengths. Constructions in [4] can generate sequences which are closest to known bounds [8] for sequence lengths which are multiples of 64 for IFW length 6 or multiples of 48 and sequence lengths which are multiples of 256 for IFW length 14 or multiples of 224. From the characterizations developed using the $\varnothing_K$ operation, and Corollary 3 in Section VII, sequences of lengths $2^n Q^n N$ and $3^n Q^n N$ for IFW lengths of $2^{n-1}N, 2 \cdot 3^{n-1}N, 3^{n-1}N$ and $3 \cdot 4^{n-1}N$ can be constructed. Moreover, by alternatively using different $\varnothing_K$s in the construction, combinations of these lengths are also possible.

V. Discussion

A. On Generalizing for Arbitrary K

We have characterized the $\mathbf{H}$ matrices to be used with $\varnothing_2$, $\varnothing_3$, and $\varnothing_4$. Such characterizations can be obtained for any integer K. Hence using the $\varnothing_K$ operator it is possible to obtain for any sequence length, a characterization for constructing a complementary set and its mates. It is easy to list a set of sufficient conditions on $\mathbf{H}$. For any K, if the matrix $\mathbf{H}$ is such as follows.

1) Rows of $\mathbf{H}$ are mutually orthogonal;
2) $\mathbf{H}\mathbf{D}_{ij}\mathbf{S}^j\mathbf{H}^* = \mathbf{H}\overline{\mathbf{D}}_{ij}\mathbf{S}^{i*}\mathbf{H}^* = 0$ for all positive integers i and j such that $i + j = K$.

then it can be used to construct complementary sequences and their mates using the operator $\varnothing_K$. To construct ZCZ sequences, in addition to the above, the terms $\mathbf{H}\overline{\mathbf{D}}_{ij}\mathbf{S}^{j*}\mathbf{H}^*$, $\mathbf{H}\mathbf{D}_{ij}\mathbf{S}^{j*}\mathbf{H}^*$, $\mathbf{H}\overline{\mathbf{D}}_{ij}\mathbf{S}^{i}\mathbf{H}^*$ and $\mathbf{H}\overline{\mathbf{D}}_{ij}\mathbf{S}^{j}\mathbf{H}^*$ are also required to be zero, resulting in $12(K-1)+1$ conditions. However, many of these conditions would be redundant and can be eliminated using identities similar to those developed in Appendices B and C.

TABLE I
Number of Distinct Characteristic Matrices of Size $P \times KQ$ for $K = 2$

PSK order	Q	$P = 1$	$P = 2$	$P = 3$	$P = 4$
2	2	8	8	0	0
	4	72	136	128	64
	6	800	7712	-	-
3	3	54	0	0	0
	6	7722	55152	-	-
4	2	32	64	0	0
6	2	72	216	0	0

B. Comments Regarding Search for Characteristic Matrices

While searching for characteristic matrices of a particular theorem, it is useful to note that each row of the characteristic matrix is a distinct solution for the theorem with $P = 1$. Hence, a general strategy in the search is to obtain a list of solutions with $P = 1$ and then search for larger Ps within this reduced set. In Table I, we list the results of a computer search for characteristic matrices of Theorem 1 over different PSK constellations (a "-" implies that at least one characteristic matrix was found but a comprehensive search was not carried out owing to prohibitive complexity). Table I does not take account of the symmetry classes for these characteristic matrices. Such symmetries would condense the enumeration so that, for example, there is only one characteristic matrix for $P = 1, Q = 2, K = 2$, and BPSK, given by

$$H = [1 \;\; 1 \;\; 1 - 1].$$

Proposition 1: A characteristic matrix does not exist for $P > Q$.

Proof: Note that for N = 1, a characteristic matrix of any of the theorems would produce PM mutually orthogonal sets of QM sequences each. Proof of the proposition now follows from the result that the number of mutually orthogonal sets cannot exceed the cardinality of the complementary set itself [2]. □

But matrices over some constellations could not be found even when $P \leq Q$ (over constellation size 3, for example). It remains open to conduct more thorough searches for characteristic matrices for $K > 2$. It would be interesting to devise analytical tools to construct characteristic matrices for the different theorems.

C. Choosing a MOGCS Matrix

As noted earlier, any unitary matrix is a MOGCS matrix with $N = 1$. For other values of N, known kernel pairs, triads, quads and in some cases their mates are given in [2]. Any of these kernels and their mates can be used as an initial $\mathbf{A}$ matrix. However in many cases, all mates of the kernels are not known, hence, one may not be able to construct the maximum possible number of MOGCS when starting from some N other than 1.

VI. Mutually Orthogonal ZCZ Sequence Sets

As an immediate consequence of the above characterization, we show the construction of orthogonal ZCZ sequence sets. The following Theorem forms the basis for constructing mutually orthogonal ZCZ sequence sets.

Theorem 7: If both $\mathbf{H}_1$ and $\mathbf{H}_2$ are characteristic matrices of Theorem 1 and $\mathbf{H}_1\mathbf{H}_2^* = 0$, then the rows of $\mathbf{B}_1^{(1)} = \mathbf{A}\varnothing_2\mathbf{H}_1$ and

$\mathbf{B}_2^{(1)} = \mathbf{A}\oslash_2\mathbf{H}_2$ are mutually orthogonal and hence the rows of $\mathbf{B}_1^{(1)}$ and $\mathbf{B}_2^{(1)}$ form mutually orthogonal ZCZ sequence sets.

Proof: With $r = m \pmod M$, $s = n \pmod M$, $m = kM + r$ and $n = lM + s$, $\forall\, \mathbf{b}_m^1 \in \mathbf{B}_1^{(1)}$ and $\mathbf{b}_n^2 \in \mathbf{B}_2^{(1)}$

$$\mathbf{b}_m^1\mathbf{b}_n^{2*} = \sum_{i=1}^{M}\psi_{\mathbf{a}_r^i\mathbf{a}_s^i}(0)(\mathbf{h}_k^1\mathbf{h}_l^{2*}) = 0.$$

□

We state the generalization as a corollary.

Corollary 1: If $\mathbf{H}_1, \mathbf{H}_2, \ldots, \mathbf{H}_n$ are characteristic matrices of Theorem 1 and $\mathbf{H}_i\mathbf{H}_j^* = 0 \quad \forall i \neq j$, then the sets $\{\mathbf{B}_m^{(1)} = \mathbf{A}\oslash_2\mathbf{H}_m\}_{m=1}^n$ are mutually orthogonal and hence form a collection of mutually orthogonal ZCZ sequence sets.

Proof of this corollary is clear from the proof of the theorem. Alternatively, these results can be obtained by observing that the proposed construction is equivalent to a tensor product followed by a dimension dependent permutation. The following result is also obvious.

Corollary 2: For $\mathbf{H}_1$ and $\mathbf{H}_2$ as in Theorem 7, let $\mathbf{B}_{11}^{(2)} = \mathbf{A}\oslash_2\mathbf{H}_1\oslash_2\mathbf{H}_1$ and $\mathbf{B}_{12}^{(2)} = \mathbf{A}\oslash_2\mathbf{H}_1\oslash_2\mathbf{H}_2$. Then $\mathbf{B}_{11}^{(2)}$ and $\mathbf{B}_{12}^{(2)}$ form orthogonal ZCZ sequence sets.

Similarly, with $\mathbf{B}_{mn} = \mathbf{A}\oslash_2\mathbf{H}_m\oslash_2\mathbf{H}_n$ for $1 \leq m, n \leq 2$, four orthogonal sets can be constructed. Starting with $\{\mathbf{H}_i\}_{i=1}^R$, we can construct R orthogonal sets at the first stage (R can be at most $\frac{2Q}{P}$). After the nth recursion, R^n orthogonal sets will be obtained.

We contend that a multiuser system using two mutually orthogonal ZCZ sets, each having $\frac{L}{2}$ sequences of length L with IFW 1 would perform better in a wireless (inter-path interference present) situation than a system using L traditional orthogonal sequences. This is because, when using mutually orthogonal ZCZ sets, each user can cancel at least half the number of interferers who are within one chip duration and the other half of the interferers would contribute as much interference as they would in a traditional orthogonal system. Hence, on the average, this user would face less interference than in a traditional system. The following example from [21] illustrates this construction.

Example 6: For the case when the matrix $\mathbf{H}$ is over a binary constellation and $Q = 2$, it can be easily verified that

$$\mathbf{H}_1 = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \tag{28}$$

$$\mathbf{H}_2 = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & -1 & -1 \end{bmatrix} \tag{29}$$

satisfy the conditions in Theorem 1 and $\mathbf{H}_1\mathbf{H}_2^* = 0$, so they can be used to construct orthogonal ZCZ sequence sets. The matrices $\mathbf{H}_1$ and $\mathbf{H}_2$ can be used to construct ZCZ$_1$-(L, M, T) and ZCZ$_2$-(L, M, T) with L, M and T depending on the initial MOGCS matrix. Let the matrix $\mathbf{A}$ be a 4×4 Hadamard matrix and let $\mathbf{B}_1^{(1)} = \mathbf{A}\oslash_2\mathbf{H}_1$ and $\mathbf{B}_2^{(1)} = \mathbf{A}\oslash_2\mathbf{H}_2$. Then, $\mathbf{B}_1^{(1)}$ forms a ZCZ-(16, 8, 1), so does $\mathbf{B}_2^{(1)}$ and $\mathbf{B}^{(1)} = [\mathbf{B}_1^{(1)*}\mathbf{B}_2^{(1)*}]^*$ is an orthogonal sequence set. The resulting sequences are listed below. Sequences in $\mathbf{B}_1$

$$\begin{aligned}
\mathbf{b}_1 &= (0000000001010101)\\
\mathbf{b}_2 &= (0011001101100110)\\
\mathbf{b}_3 &= (0000111101011010)\\
\mathbf{b}_4 &= (0011110001101001)\\
\mathbf{b}_5 &= (0101010100000000)\\
\mathbf{b}_6 &= (0110011000110011)\\
\mathbf{b}_7 &= (0101101000001111)\\
\mathbf{b}_8 &= (0110100100111100).
\end{aligned}$$

Sequences in $\mathbf{B}_2$

$$\begin{aligned}
\mathbf{b}_1 &= (0000000010101010)\\
\mathbf{b}_2 &= (0011001110011001)\\
\mathbf{b}_3 &= (0000111110100101)\\
\mathbf{b}_4 &= (0011110010010110)\\
\mathbf{b}_5 &= (0101010111111111)\\
\mathbf{b}_6 &= (0110011011001100)\\
\mathbf{b}_7 &= (0101101011110000)\\
\mathbf{b}_8 &= (0110100111000011).
\end{aligned}$$

The two orthogonal $\mathbf{H}$ matrices will generate a large collection of orthogonal ZCZ sequence sets in the subsequent recursions as discussed immediately after Corollary 2 of Theorem 7. In the present example, letting $\mathbf{B}_{11}^{(2)} = \mathbf{B}_1^{(1)}\oslash_2\mathbf{H}_1$, $\mathbf{B}_{12}^{(2)} = \mathbf{B}_1^{(1)}\oslash_2\mathbf{H}_2$, $\mathbf{B}_{21}^{(2)} = \mathbf{B}_2^{(1)}\oslash_2\mathbf{H}_1$ and $\mathbf{B}_{22}^{(2)} = \mathbf{B}_2^{(1)}\oslash_2\mathbf{H}_2$, each $\mathbf{B}_{ij}^{(2)}$ forms a ZCZ-(64, 16, 2) in addition to the fact that each pair is mutually orthogonal.

In Example 6, the four sets $\mathbf{B}_{ij}^{(2)}$ are mutually orthogonal, each with 16 sequences of length 64 and IFW 2. Combining the sets $\mathbf{B}_{11}^{(2)}$ and $\mathbf{B}_{12}^{(2)}$ results in an orthogonal set of 32 sequences with an overall IFW of 0, although it is possible to construct the same number of sequences with IFW 1. But once a set of 32 sequences of IFW 1 are constructed, the present framework does not provide any means to improve the IFW length of its subsets without increasing the length of the constituent sequences. In other words, we can design a collection of orthogonal ZCZ sequence sets with large IFW, but combining even two of the subsets immediately reduces the IFW of the combined set to zero. It is desirable to have constructions where, when more than one ZCZ set is combined, the resulting set forms a larger ZCZ set with reduced but nonzero IFW. We discuss this problem in the next section.

VII. ZCZ Sequences With Large IFW Subsets

In this section we discuss a possible approach to construct ZCZ sequence sets with a nonzero IFW whose subsets form ZCZ sets of larger IFW. The following corollary is fundamental to the construction of such sequences. In the following, we observe that when only first few conditions in Theorem 6 are satisfied, the resulting $\{\mathbf{b}_m\}$s still produce a ZCZ sequence set, but with smaller IFW.

Corollary 3: (to Theorem 6) When the matrix $\mathbf{H}$ satisfies only the first p conditions in Theorem 6, the rows of $\mathbf{B} = \mathbf{A}\oslash_4\mathbf{H}$ form a ZCZ-$(4MNQ, MP, (p-1)N)$ for $p \leq 4$.

Proof of the corollary follows directly from the proof of Theorem 6. The following is an example quadriphase matrix which satisfies only the first two conditions of Theorem 6 with $P = 4, Q = 2$

$$\mathbf{H} = \begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 2 & 1 & 3 \\ 0 & 0 & 3 & 3 & 0 & 2 & 3 & 1 \\ 0 & 2 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 0 & 2 & 2 \end{pmatrix}. \tag{30}$$

It follows that when only the first two conditions are satisfied, a ZCZ sequence set with IFW size of N will result. The same observations hold for Theorem 5. It is obvious that for a given Q, as we relax the number of conditions, the number of rows of $\mathbf{H}$ i.e., P can only increase.

Suppose $\mathbf{H} = \left[\mathbf{H}_1^T \mid \mathbf{H}_2^T\right]^T$ is used in the recursive construction. Further, let the matrix $\mathbf{H}$ satisfy only the first two conditions of Theorem 6 and both $\mathbf{H}_1$ with P_1 rows and $\mathbf{H}_2$ with P_2 rows individually satisfy the first three conditions. Then the resulting sequences can be partitioned into two sets of cardinalities P_1M and P_2M, respectively, individually having IFW of $2N$ and overall IFW of N. Further, $\mathbf{H}_1^T$

and $\mathbf{H}_2^T$ can be individually partitioned into smaller matrices and so on. In multiuser systems, if it is possible to isolate the users who are severely interfering with each other (which may be due to geometric proximity and large synchronization mismatch) into groups, each such group can be assigned channelization sequences from a ZCZ set with large IFW. ZCZ sequence sets assigned to any two groups will be mutually orthogonal. However our search for characteristic matrices for such a construction have not been encouraging for $Q = 2$ and 4.

VIII. Conclusion

The problem of constructing ZCZ sequences was studied in a new framework and characterizations useful in their construction have been presented. A class of recursive methods for constructing increasingly larger ZCZ sequence sets and to enlarge an initial set of mutually orthogonal complementary sets have been given. The notion of mutually orthogonal ZCZ sequence sets was introduced and a construction method was presented. Construction of new ZCZ sequence sets and IFW lengths has also been possible in this framework. We also noted the possibility of constructing ZCZ sequence sets with subsets possessing large IFW lengths. It is left to further work to establish how many characteristic matrices exist for constellations other than BPSK and to devise analytical methods to construct such matrices.

Appendix A
Proof of Theorem 3

The following conditions are obtained from the theorem:

$$\mathbf{H}\overline{\mathbf{D}}_{12}\mathbf{S}^*\mathbf{H}^* = 0 \tag{31}$$

$$\mathbf{H}\overline{\mathbf{D}}_{21}\mathbf{S}^{2*}\mathbf{H}^* = 0. \tag{32}$$

Setting the notations as in the proof of Theorem 1, consider the following cases:

1) $m \neq n$, m and n are from the same set (say l), then $r \neq s$. For $0 \leq \tau \leq N$

$$\sum_{i=1}^{QM} \psi_{\mathbf{b}_m^i \mathbf{b}_n^i}(\tau) = 3Q \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(\tau) + \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(-(N-\tau)) \left(\mathbf{h}_l \mathbf{D}_{21} \mathbf{S} \mathbf{h}_l^*\right) \tag{33}$$

is zero by (2). For $N + 1 \leq \tau \leq 2N$,

$$\sum_{i=1}^{QM} \psi_{\mathbf{b}_m^i \mathbf{b}_n^i}(\tau) = \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(\tau - N) \left(\mathbf{h}_l \mathbf{D}_{21} \mathbf{S} \mathbf{h}_l^*\right) + \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(-(2N-\tau)) \left(\mathbf{h}_l \mathbf{D}_{12} \mathbf{S}^2 \mathbf{h}_l^*\right) \tag{34}$$

for $2N + 1 \leq \tau \leq 3N - 1$

$$\sum_{i=1}^{QM} \psi_{\mathbf{b}_m^i \mathbf{b}_n^i}(\tau) = \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(\tau - 2N) \left(\mathbf{h}_l \mathbf{D}_{12} \mathbf{S}^2 \mathbf{h}_l^*\right) \tag{35}$$

again by using (2) it can be seen that RHS of (34) and (35) vanish.

2) $m \neq n$, m and n are from different sets k and l, and $r \neq s$. When $0 \leq \tau \leq N$

$$\sum_{i=1}^{QM} \psi_{\mathbf{b}_m^i \mathbf{b}_n^i}(\tau) = \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(\tau) \left(\mathbf{h}_k \mathbf{h}_l^*\right) + \sum_{i=1}^{M} \psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(-(N-\tau)) \left(\mathbf{h}_k \mathbf{D}_{21} \mathbf{S} \mathbf{h}_l^*\right) \tag{36}$$

is zero by (2). For $N + 1 \leq \tau \leq 2N$, (34) applies with $(\mathbf{h}_l \mathbf{X} \mathbf{h}_l^*)$ terms replaced by $(\mathbf{h}_k \mathbf{X} \mathbf{h}_l^*)$. When $2N + 1 \leq \tau \leq 3N - 1$, (35) applies with the same substitution. Again, (2) is enough to obtain the desired result.

3) $m \neq n$, m and n are from different sets k and l, and $r = s$. Equations (36) and the other two expressions corresponding to $N + 1 \leq \tau \leq 2N$ and $2N + 1 \leq \tau \leq 3N - 1$ of Case 2 hold with cross-correlation function $\psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(\tau)$ replaced by autocorrelation function $\psi_{\mathbf{a}_r^i \mathbf{a}_r^i}(\tau)$ for $0 \leq \tau \leq N$. The terms corresponding to $0 \leq \tau \leq N$ vanish by conditions 1, 2, and (31). The terms for other ranges of τ vanish by conditions 2, 3, and (31).

4) $m = n$. So $k = l$ and $r = s$. For $0 < \tau \leq N$, (33) holds with crosscorrelation function $\psi_{\mathbf{a}_r^i \mathbf{a}_s^i}(\tau)$ replaced by autocorrelation function $\psi_{\mathbf{a}_r^i \mathbf{a}_r^i}(\tau)$ for $0 < \tau \leq N$ and is zero by (1), condition 2, and (31). For $N+1 \leq \tau \leq 2N$ and $2N+1 \leq \tau \leq 3N-1$, (34) and (35) hold correspondingly and vanish by conditions 2, 3, and (31).

Proof is complete for positive τ.

For negative τ, slight modification of the corresponding expressions for positive τ (by substituting $\overline{\mathbf{D}}_{12}\mathbf{S}^*$ in place of $\mathbf{D}_{21}\mathbf{S}$ and $\overline{\mathbf{D}}_{21}\mathbf{S}^{2*}$ in place of $\mathbf{D}_{12}\mathbf{S}^2$), and using the same set of conditions, all expressions can be shown to be zero. That completes the proof.

Appendix B
Auxiliary Results

Inherent structure in the definition of $\mathbf{D}_{ij}$ and $\mathbf{S}^\tau$ can be exploited to establish the following identities useful in reducing the number of conditions required in a characterization. We denote a $N \times N$ matrix $\mathbf{D}_{ij}$ by $\mathbf{D}_{ij}(N)$ and the $N \times N$ identity matrix by $\mathbf{I}_N$

Lemma 1: For any i and j $\mathbf{D}_{ij}\mathbf{S}^{i+j} = \mathbf{S}^{i+j}\mathbf{D}_{ij}$.

Proof: With $l = N-(i+j)$ and $k = i+j$, consider the following partition:

$$\mathbf{D}_{ij}(N) = \begin{bmatrix} \mathbf{D}_{ij}(l) & \mathbf{0}_{l\times k} \\ \mathbf{0}_{k\times l} & \mathbf{D}_{ij}(k) \end{bmatrix}, \quad \mathbf{S}^k = \begin{bmatrix} \mathbf{0}_{l\times k} & \mathbf{I}_l \\ \mathbf{I}_k & \mathbf{0}_{k\times l} \end{bmatrix}.$$

Then

$$\mathbf{D}_{ij}(N)\mathbf{S}^k = \begin{bmatrix} \mathbf{0}_{l\times k} & \mathbf{D}_{ij}(l) \\ \mathbf{D}_{ij}(k) & \mathbf{0}_{k\times l} \end{bmatrix}. \tag{37}$$

Partitioning $\mathbf{D}_{ij}(N)$ as

$$\mathbf{D}_{ij}(N) = \begin{bmatrix} \mathbf{D}_{ij}(k) & \mathbf{0}_{k\times l} \\ \mathbf{0}_{l\times k} & \mathbf{D}_{ij}(l) \end{bmatrix}$$

it is easily verified that $\mathbf{S}^k\mathbf{D}_{ij}(N)$ is equal to (37). By induction, it is easy to prove that for any positive integer n, $\mathbf{D}_{ij}\mathbf{S}^{n(i+j)} = \mathbf{S}^{n(i+j)}\mathbf{D}_{ij}$. □

Lemma 2: If $\mathbf{D}_{ij}\mathbf{S}^\tau = \mathbf{S}^\tau\mathbf{D}_{kl}$ for any i, j, k and l then $\overline{\mathbf{D}}_{ij}\mathbf{S}^\tau = \mathbf{S}^\tau\overline{\mathbf{D}}_{kl}$

Proof:

$$\begin{aligned}\mathbf{D}_{ij}\mathbf{S}^\tau + \overline{\mathbf{D}}_{ij}\mathbf{S}^\tau &= (\mathbf{D}_{ij} + \overline{\mathbf{D}}_{ij})\mathbf{S}^\tau \\ &= \mathbf{S}^\tau = \mathbf{S}^\tau(\mathbf{D}_{kl} + \overline{\mathbf{D}}_{kl}) \\ &= \mathbf{S}^\tau\mathbf{D}_{kl} + \mathbf{S}^\tau\overline{\mathbf{D}}_{kl}.\end{aligned}$$

□

Lemma 3: If $\mathbf{D}_{ij}\mathbf{S}^\tau = \mathbf{S}^\tau\mathbf{D}_{ij}$, then $\mathbf{D}_{ij}\mathbf{S}^{\tau*} = \mathbf{S}^{\tau*}\mathbf{D}_{ij}$

Proof

$$\mathbf{D}_{ij}\mathbf{S}^{\tau*} = (\mathbf{S}^\tau\mathbf{D}_{ij})^* = (\mathbf{D}_{ij}\mathbf{S}^\tau)^* = \mathbf{S}^{\tau*}\mathbf{D}_{ij}.$$

□

The diagonal matrix $\mathbf{D}_{ij}$ can be represented as,

$$\mathbf{D}_{ij} = \mathbf{S}^j\overline{\mathbf{D}}_{ji}\mathbf{S}^{j*} = \mathbf{S}^{i*}\overline{\mathbf{D}}_{ji}\mathbf{S}^i. \tag{38}$$

The following two results are nontrivial when $i \neq \tau$ and $j \neq \tau$, respectively.

Lemma 4:

1) $\mathbf{D}_{ij}\mathbf{S}^\tau = \mathbf{S}^\tau\overline{\mathbf{D}}_{ji}$ implies $\mathbf{D}_{ij}\mathbf{S}^{\tau*} = \mathbf{S}^{j-\tau}\mathbf{D}_{ij}\mathbf{S}^{j*}$;
2) $\mathbf{D}_{ij}\mathbf{S}^{\tau*} = \mathbf{S}^{\tau*}\overline{\mathbf{D}}_{ji}$ implies $\mathbf{D}_{ij}\mathbf{S}^\tau = \mathbf{S}^{\tau-i}\mathbf{D}_{ij}\mathbf{S}^i$.

Proof

[Proof of 1]: Assuming $\mathbf{D}_{ij}\mathbf{S}^\tau = \mathbf{S}^\tau\overline{\mathbf{D}}_{ji}$ and using $\mathbf{D}_{ij} = \mathbf{S}^j\overline{\mathbf{D}}_{ji}\mathbf{S}^{j*}$,

$$\begin{aligned}\mathbf{D}_{ij}\mathbf{S}^{\tau*} &= (\mathbf{S}^\tau\mathbf{D}_{ij})^* = \left(\mathbf{S}^\tau\mathbf{S}^j\overline{\mathbf{D}}_{ji}\mathbf{S}^{j*}\right)^* = \left(\mathbf{S}^j\mathbf{S}^\tau\overline{\mathbf{D}}_{ji}\mathbf{S}^{j*}\right)^* \\ &= \left(\mathbf{S}^j\mathbf{D}_{ij}\mathbf{S}^\tau\mathbf{S}^{j*}\right)^* = \left(\mathbf{S}^j\mathbf{D}_{ij}\mathbf{S}^{\tau-j}\right)^* = \mathbf{S}^{j-\tau}\mathbf{D}_{ij}\mathbf{S}^{j*}.\end{aligned}$$

[Proof of 2]: Assuming $\mathbf{D}_{ij}\mathbf{S}^{\tau*} = \mathbf{S}^{\tau*}\overline{\mathbf{D}}_{ji}$ and using $\mathbf{D}_{ij} = \mathbf{S}^{i*}\overline{\mathbf{D}}_{ji}\mathbf{S}^i$

$$\begin{aligned}\mathbf{D}_{ij}\mathbf{S}^\tau &= (\mathbf{S}^{\tau*}\mathbf{D}_{ij})^* = \left(\mathbf{S}^{\tau*}\mathbf{S}^{i*}\overline{\mathbf{D}}_{ji}\mathbf{S}^i\right)^* = \left(\mathbf{S}^{i*}\mathbf{S}^{\tau*}\overline{\mathbf{D}}_{ji}\mathbf{S}^i\right)^* \\ &= \left(\mathbf{S}^{i*}\mathbf{D}_{ij}\mathbf{S}^{\tau*}\mathbf{S}^i\right)^* = \mathbf{S}^{\tau-i}\mathbf{D}_{ij}\mathbf{S}^i.\end{aligned}$$

□

Using appropriate matrix partitioning, many other identities listed in Appendix C are easily proved.

APPENDIX C
IDENTITIES DERIVED BY MATRIX PARTITIONING

1) $\overline{\mathbf{D}}_{11}\mathbf{S} = \mathbf{S}\mathbf{D}_{11}$;
2) $\mathbf{D}_{12}\mathbf{S}^* = \mathbf{S}^*\overline{\mathbf{D}}_{21}$;
3) $\mathbf{D}_{12}\mathbf{S}^2 = \mathbf{S}^2\overline{\mathbf{D}}_{21}$;
4) $\overline{\mathbf{D}}_{21}\mathbf{S} = \mathbf{S}\mathbf{D}_{12}$;
5) $\mathbf{D}_{21}\mathbf{S}^{2*} = \mathbf{S}^{2*}\overline{\mathbf{D}}_{12}$;
6) $\mathbf{D}_{13}\mathbf{S}^* = \mathbf{S}^*\overline{\mathbf{D}}_{31}$;
7) $\mathbf{D}_{13}\mathbf{S}^3 = \mathbf{S}^3\overline{\mathbf{D}}_{31}$;
8) $\mathbf{D}_{31}\mathbf{S} = \mathbf{S}\overline{\mathbf{D}}_{13}$;
9) $\mathbf{D}_{31}\mathbf{S}^{3*} = \mathbf{S}^{3*}\overline{\mathbf{D}}_{13}$;
10) $\mathbf{D}_{22}\mathbf{S}^2 = \mathbf{S}^2\overline{\mathbf{D}}_{22}$.

We demonstrate the use of the Appendices B and C in the following examples:

Example 7: If $\mathbf{H}$ is such that $\mathbf{H}\overline{\mathbf{D}}_{22}\mathbf{S}^{2*}\mathbf{H}^* = 0$, taking conjugate transpose gives $\mathbf{H}\mathbf{S}^2\overline{\mathbf{D}}_{22}\mathbf{H}^* = 0$. But from Appendix C, $\mathbf{S}^2\overline{\mathbf{D}}_{22} = \mathbf{D}_{22}\mathbf{S}^2$. Hence, $\mathbf{H}\mathbf{D}_{22}\mathbf{S}^2\mathbf{H}^* = 0$.

Example 8: Starting with $\mathbf{H}\mathbf{D}_{13}\mathbf{S}^3\mathbf{H}^* = 0$, it can be shown that $\mathbf{H}\overline{\mathbf{D}}_{31}\mathbf{S}^{3*}\mathbf{H}^* = 0$.

ACKNOWLEDGMENT

The authors would like to thank the anonymous reviewers and the associate editor for their comments and suggestions which have led to improvements in the paper. Reviewer A pointed out [22] and [23] and suggested the possibility of Proposition 1. His elaborate comments have substantially improved the readability and presentation of this correspondence.

REFERENCES

[1] P. Fan and L. Hao, "Generalized orthogonal sequences and their applications in synchronous CDMA systems," *IEICE Trans. Fund.*, vol. E89-A, no. 11, pp. 2054–2066, Nov. 2000.
[2] R. L. Frank, "Polyphase complementary codes," *IEEE Trans. Inf. Theory*, vol. IT-26, no. 6, pp. 641–647, Nov. 1980.
[3] R. Sivaswamy, "Multiphase complementary codes," *IEEE Trans. Inf. Theory*, vol. IT-24, no. 5, pp. 546–552, Sep. 1978.
[4] H. Torii, M. Nakamura, and N. Suehiro, "A new class of zero-correlation zone sequences," *IEEE Trans. Inf. Theory*, vol. 50, no. 3, pp. 559–565, Mar. 2004.
[5] X. H. Tang, P. Z. Fan, D. B. Li, and N. Suehiro, "Binary array set with zero correlation zone," *IEE Electron. Lett.*, vol. 37, no. 13, pp. 841–842, Jun. 2001.
[6] K. G. Paterson, "Generalized Reed-Muller codes and power control in OFDM modulation," *IEEE Trans. Inf. Theory*, vol. 46, pp. 104–120, Jan. 2000.
[7] C. V. Chong, R. Venkataramani, and V. Tarokh, "A new construction of 16-QAM Golay complementary sequences," *IEEE Trans. Inf. Theory*, vol. 49, pp. 2953–2959, Nov. 2003.
[8] X. H. Tang, P. Z. Fan, and S. Matsufuji, "Lower bounds on the maximum correlation of sequence set with low or zero correlation zone," *IEE Electron. Lett.*, vol. 36, no. 13, pp. 551–552, Mar. 2000.
[9] S. Matsufuji, N. Suehiro, N. Kuroyanagi, and P. Z. Fan, "Spreading sequence sets for approximately synchronized CDMA system with no co-channel interference and high data capacity," in *Proc. 2nd Int. Symp. Wireless Pers. Multimedia Commun. (WPMC'99)*, Amsterdam, The Netherlands, Sep. 1999, pp. 333–339.
[10] D. Li, "The perspectives of large area synchronous CDMA technology for the fourth-generation mobile radio," *IEEE Commun. Mag.*, vol. 19, no. 5, pp. 114–118, Mar. 2003.
[11] X. Deng and P. Fan, "Spreading sequence sets with zero correlation zone," *Inst. Elect. Eng. Electron. Lett.*, vol. 36, no. 11, pp. 993–994, May 2000.
[12] P. Z. Fan, N. Suehiro, and X. Deng, "Class of binary sequences with zero correlation zone," *Inst. Elect. Eng. Electron. Lett.*, vol. 35, no. 10, pp. 777–778, May 1999.
[13] C. C. Tseng and C. L. Liu, "Complementary sets of sequences," *IEEE Trans. Inf. Theory*, vol. 18, no. 5, pp. 644–652, Sep. 1972.
[14] P. Fan and M. Darnell, *Sequence Design for Communications Applications*. New York: Wiley, 1999.
[15] M. J. E. Golay, "Complementary series," *IRE Trans. Inf. Theory*, vol. IT-7, pp. 82–87, Apr. 1961.
[16] N. Suehiro, "A signal design without co-channel interference for approximately synchronized CDMA systems," *IEEE J. Sel. Areas Commun.*, vol. 12, pp. 837–841, Jun. 1994.
[17] S. Z. Budisin, "New complementary pairs of sequences," *Electron. Lett.*, vol. 26, no. 13, pp. 881–883, Jun. 1990.
[18] S. R. Park, I. Song, S. Yoon, and J. Lee, "A new polyphase sequence with perfect even and good odd cross-correlation functions for DS/CDMA systems," *IEEE Trans. Veh. Technol.*, vol. 51, no. 5, pp. 855–866, Sep. 2002.
[19] R. L. Frank, "Polyphase codes with good nonperiodic correlation properties," *IEEE Trans. Inf. Theory*, vol. IT-4, pp. 43–45, Jan. 1963.
[20] H. D. Schotten and H. H. Mahram, "Binary and quadriphase sequences with optimal autocorrelation properties: A survey," *IEEE Trans. Inf. Theory*, vol. 49, no. 12, pp. 3271–3288, Dec. 2003.
[21] A. Rathinakumar and A. K. Chaturvedi, "Mutually orthogonal sets of ZCZ sequences," *Electron. Lett.*, vol. 40, pp. 1133–1134, Sep. 2004.
[22] W. H. Holzmann and H. Kharaghani, "A computer search for complex Golay sequenes," *Australasian J. Combin.*, no. 10, pp. 251–258, Dec. 2003.
[23] M. G. Parker and C. Tellambura, Reports in Informatics University of Bergen, 242, 2003, A construction for binary sequence sets with low peak-to-average power ratio.