Semester II, 2017-18
Department of Physics, IIT Kanpur

PHY103A: Lecture # 3

(Text Book: Intro to Electrodynamics by Griffiths, 3rd Ed.)

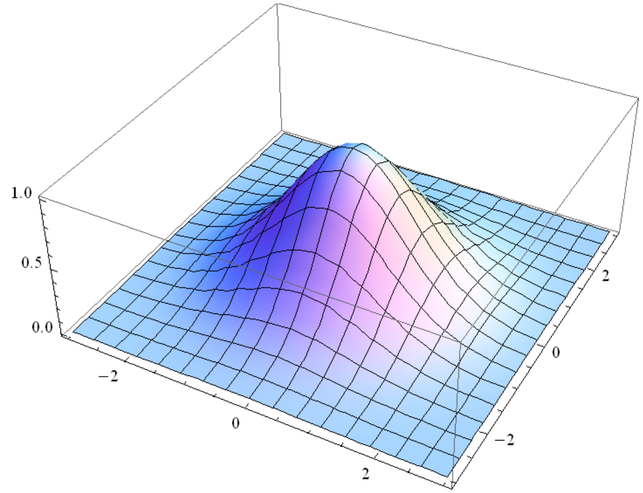
Anand Kumar Jha
08-Jan-2018

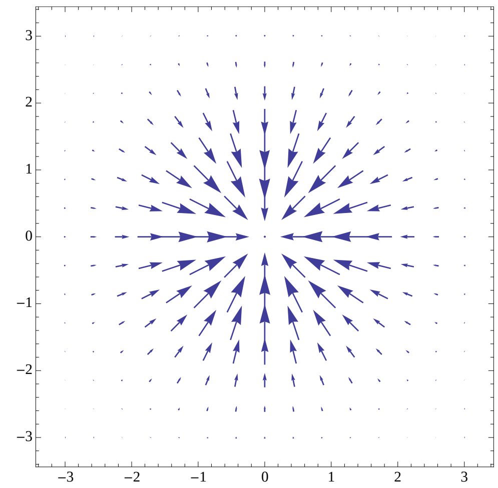
Notes

- The first tutorial is tomorrow (Tuesday).
- Updated lecture notes will be uploaded right after the class.
- Office Hour – Friday 2:30-3:30 pm
- Phone: 7014(Off); 962-142-3993(Mobile)
  akjha@iitk.ac.in; akjha9@gmail.com
- Tutorial Sections have been finalized and put up on the webpage.
- Course Webpage: http://home.iitk.ac.in/~akjha/PHY103.htm

Summary of Lecture # 2:

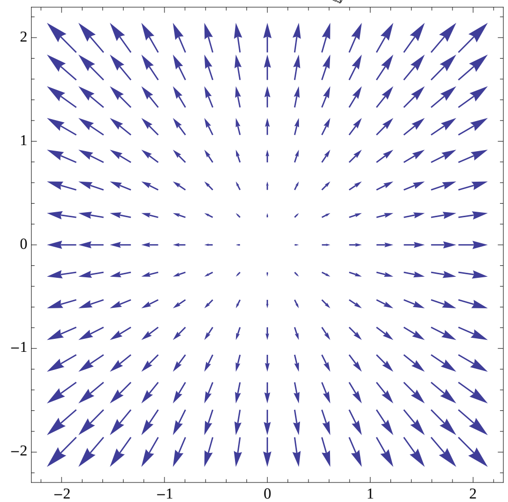
Gradient of a scalar $\boldsymbol{\nabla} T$





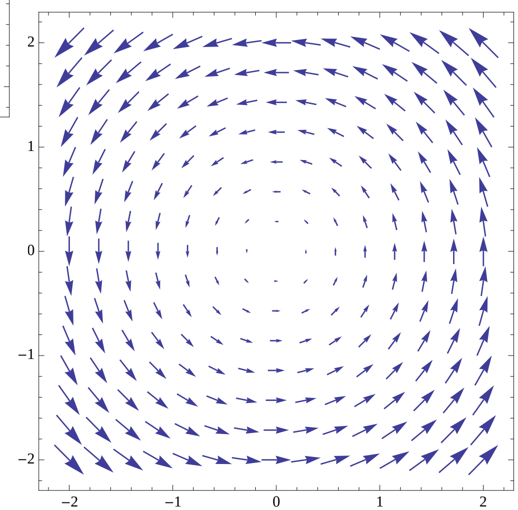
$$\boldsymbol{\nabla} T \equiv \frac{\partial T}{\partial x}\hat{\boldsymbol{x}} + \frac{\partial T}{\partial y}\hat{\boldsymbol{y}} + \frac{\partial T}{\partial z}\hat{\boldsymbol{z}}$$

Divergence of a vector $\boldsymbol{\nabla} \cdot \mathbf{V}$



$$\boldsymbol{\nabla} \cdot \mathbf{V} = \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)$$

Curl of a vector $\boldsymbol{\nabla} \times \mathbf{V}$



$$\boldsymbol{\nabla} \times \mathbf{V} = \begin{vmatrix} \hat{\boldsymbol{x}} & \hat{\boldsymbol{y}} & \hat{\boldsymbol{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix}$$

$$= \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\hat{\boldsymbol{x}} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\hat{\boldsymbol{y}} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\hat{\boldsymbol{z}}$$

Integral Calculus:

The ordinary integral that we know of is of the form: $\int_a^b f(x)dx$

In vector calculus we encounter many other types of integrals.

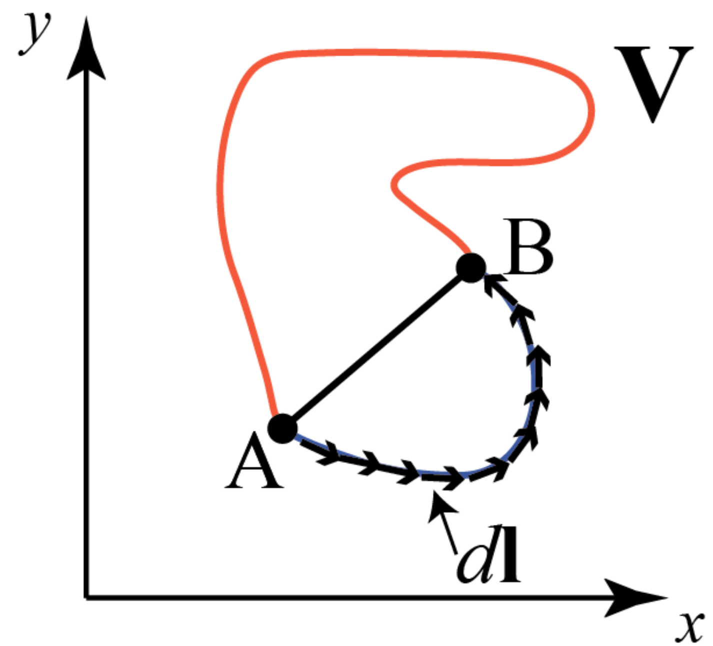
Line Integral:

$$\int_a^b \mathbf{V} \cdot d\mathbf{l}$$

Vector field

Infinitesimal
Displacement vector
Or Line element

$$d\mathbf{l} = dx\,\hat{\boldsymbol{x}} + dy\hat{\mathbf{y}} + dz\hat{\mathbf{z}}$$



If the path is a closed loop then the line integral is written as

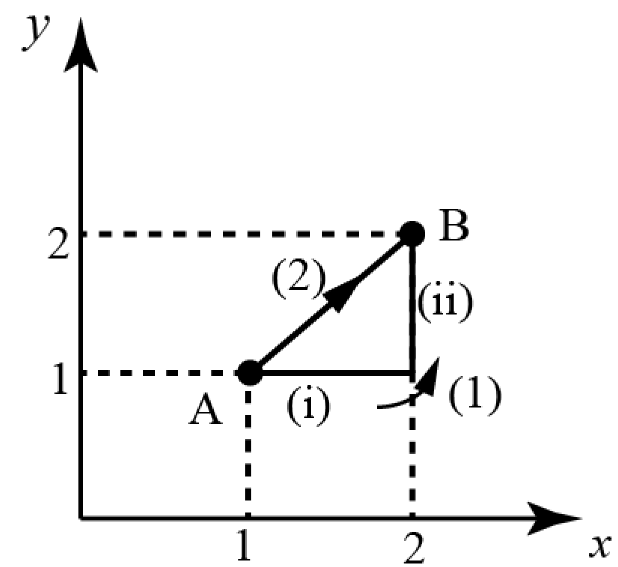
$$\oint \mathbf{V} \cdot d\mathbf{l}$$

- When do we need line integrals?

Work done by a force along a given path involves line integral.

Example (G: Ex. 1.6)

Q: $\mathbf{V} = y^2\,\hat{\boldsymbol{x}} + 2x(y+1)\hat{\boldsymbol{y}}$? What is the line integral from A to B along path (1) and (2)?



Along path (1) We have $\boldsymbol{dl} = dx\,\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}}$.

(i) $\boldsymbol{dl} = dx\,\hat{\boldsymbol{x}}$; *y=1;* $\int \mathbf{V}\cdot\boldsymbol{dl} = \int y^2 dx = 1$

(ii) $\boldsymbol{dl} = dy\,\hat{\boldsymbol{y}}$; *x=2;* $\int \mathbf{V}\cdot\boldsymbol{dl} = \int_1^2 4(y+1)dy = 10$

Along path (2): $\boldsymbol{dl} = dx\,\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}}$; $x = y$; $dx = dy$

$\int \mathbf{V}\cdot\boldsymbol{dl} = \int_1^2 (x^2 + 2x^2 + 2x)dx = 10$

$\oint \mathbf{V}\cdot\boldsymbol{dl}$=11-10=1

This means that if **V** represented the force vector, it would be a non-conservative force

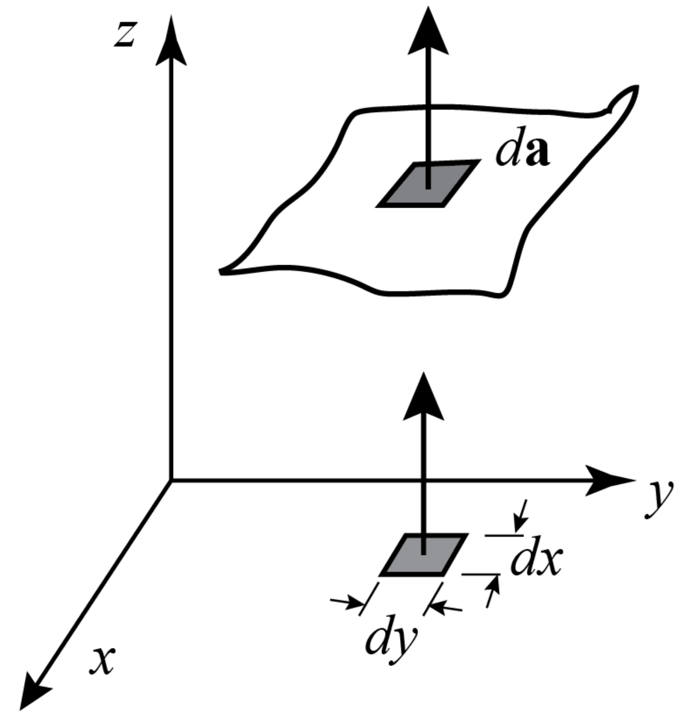
Surface Integral:

$$\int \mathbf{V} \cdot d\mathbf{a} \quad \text{or} \quad \oint \mathbf{V} \cdot d\mathbf{a}$$

Vector field

Infinitesimal area vector

Closed loop



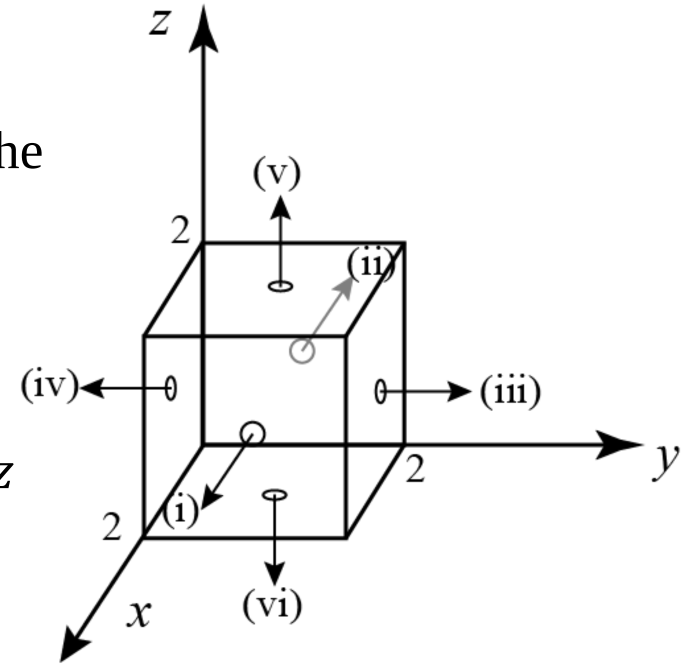
- For a closed surface, the area vector points outwards.

- For open surfaces, the direction of the area vector is decided based on a given problem.

- When do we need area integrals?

  Flux through a given area involves surface integral.

Example (Griffiths: Ex. 1.7)



Q: $\mathbf{V} = 2xz\,\hat{\boldsymbol{x}} + (x+2)\hat{\boldsymbol{y}} + y(z^2-3)\hat{\mathbf{z}}$? Calculate the Surface integral.

"upward and outward" is the positive direction

(i) $x{=}2,\ d\boldsymbol{a} = dydz\,\hat{\boldsymbol{x}};\ \ V\cdot d\boldsymbol{a} = 2xzdydz = 4zdydz$

$\int \boldsymbol{V}\cdot d\boldsymbol{a} = \int_0^2\int_0^2 4zdydz{=}16$

(ii) $\int \mathbf{V}\cdot d\mathbf{a}{=}0$

(iii) $\int \mathbf{V}\cdot d\mathbf{a}{=}12$

(iv) $\int \mathbf{V}\cdot d\mathbf{a}{=}{-}12$

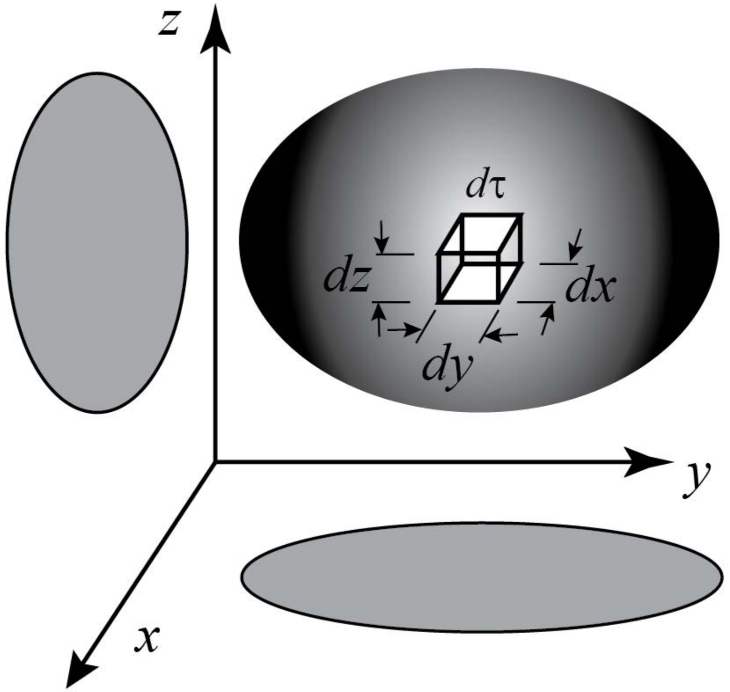
(v) $\int \mathbf{V}\cdot d\mathbf{a}{=}4$

(vi) $\int \mathbf{V}\cdot d\mathbf{a}{=}{-}12$

$$\oint \mathbf{V}\cdot d\mathbf{a} = 16+0+12-12+4-12 = 8$$

Volume Integral:

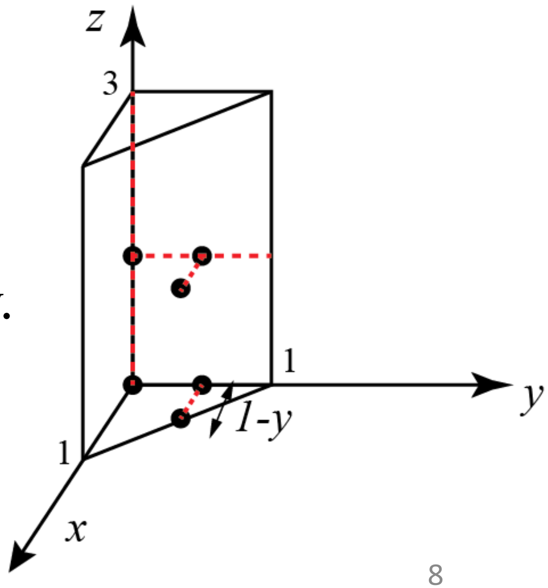
$$\int T(x, y, z)d\tau$$



- In Cartesian coordinate system the volume element is given by $d\tau = dxdydz$.
- One can have the volume integral of a vector function which $\mathbf{V}$ as $\int \mathbf{V}d\tau = \int \mathbf{V}_x d\tau\ \hat{\boldsymbol{x}} + \int \mathbf{V}_y d\tau \hat{\boldsymbol{y}} + \int \mathbf{V}_z d\tau\ \hat{\mathbf{z}}$

<u>**Example (Griffiths: Ex. 1.8)**</u>

Q: Calculate the volume integral of $T = xyz^2$ over the volume of the prism



We find that z integral runs from 0 to 3. The y integral runs from 0 to 1, but the x integral runs from 0 to $1 - y$ only. Therefore, the volume integral is given by

$$\int_0^{1-y} \int_0^1 \int_0^3 xyz^2 dxdydz = \frac{3}{8}$$

The fundamental Theorem of Calculus:

$$\int_a^b F(x)dx = \boxed{\int_a^b \frac{df}{dx}dx = f(b) - f(a)}$$

The integral of a **derivative** over a **region** is given by the value of the function at the **boundaries**

Example

$$\int_a^b x dx = \int_a^b \frac{d\left(\frac{x^2}{2}\right)}{dx}dx = \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2}$$

- In vector calculus, we have three different types of derivative (gradient, divergence, and curl) and correspondingly three different types of regions and end points.

The fundamental Theorem of Calculus:

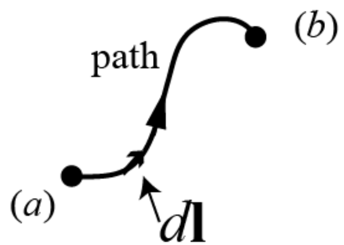
$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

The integral of a **derivative** over a **region** is given by the value of the function at the **boundaries**

The fundamental Theorem for Gradient:

$$\int_{a\ Path}^{b} \nabla T \cdot d\mathbf{l} = T(b) - T(a)$$

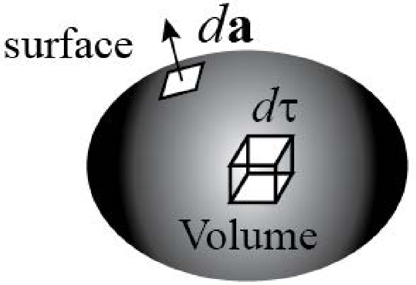
The integral of a **derivative (gradient)** over a **region (path)** is given by the value of the function at the **boundaries (end-points)**



The fundamental Theorem for Divergence **(Gauss's theorem):**

$$\int_{Vol} (\nabla \cdot \mathbf{V})\, d\tau = \oint_{Surf} \mathbf{V} \cdot d\mathbf{a}$$

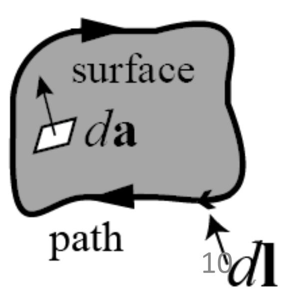
The integral of a **derivative (divergence)** over a **region (volume)** is given by the value of the function at the **boundaries (bounding surface)**



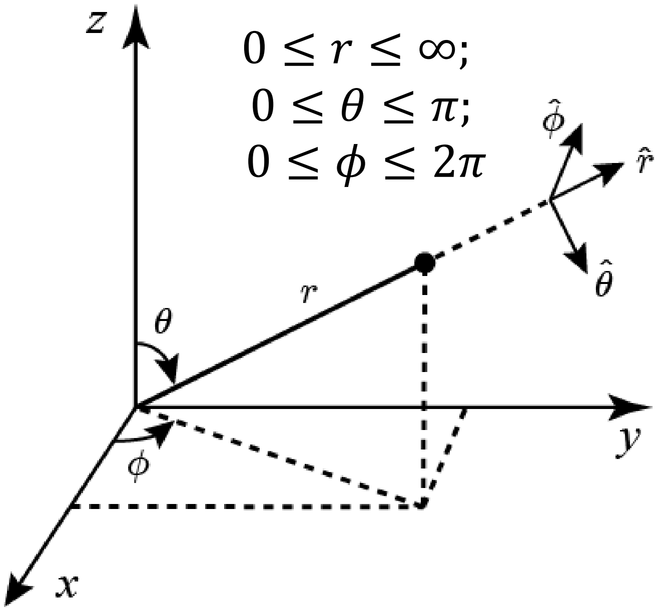
The fundamental Theorem for Curl **(Stokes' theorem):**

$$\int_{Surf} (\nabla \times \mathbf{V}) \cdot d\mathbf{a} = \oint_{Path} \mathbf{V} \cdot d\mathbf{l}$$

The integral of a **derivative (curl)** over a **region (surface)** is given by the value of the function at the **boundaries (closed-path)**



Spherical Polar Coordinates:



$0 \leq r \leq \infty;$
$0 \leq \theta \leq \pi;$
$0 \leq \phi \leq 2\pi$

$x = r\,\sin\theta\,\cos\phi, \qquad y = r\sin\theta\,\sin\phi, \qquad z = r\,\cos\theta$

A vector in the spherical polar coordinate is given by

$$\mathbf{A} = \mathrm{A}_r\hat{\boldsymbol{r}} + \mathrm{A}_{\boldsymbol{\theta}}\widehat{\boldsymbol{\theta}} + \mathrm{A}_{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}$$

$$\hat{\boldsymbol{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\widehat{\boldsymbol{\theta}} = \cos\theta\cos\phi\,\hat{\mathbf{x}} + \cos\theta\sin\phi\,\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

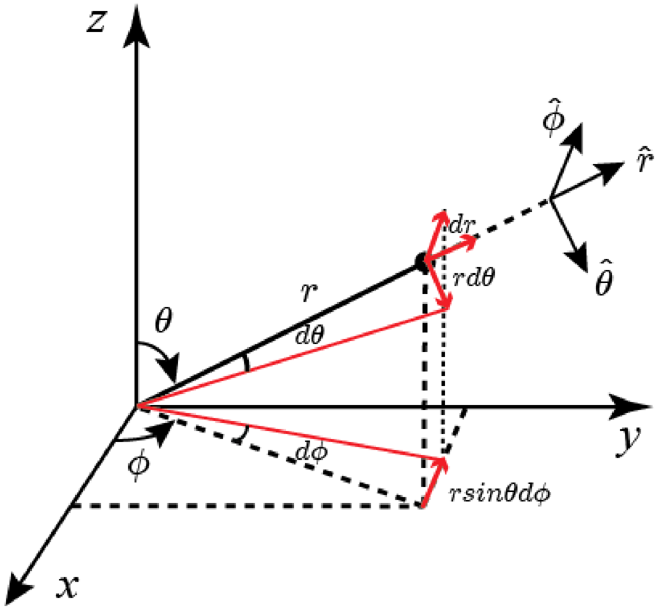
$$\widehat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

Griffiths: Prob 1.37

The infinitesimal displacement vector in the spherical polar coordinate is given by

$d\mathbf{l} = dl_r\hat{\boldsymbol{r}} + dl_{\boldsymbol{\theta}}\widehat{\boldsymbol{\theta}} + dl_{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}$ (In Cartesian system we have $d\mathbf{l} = dx\,\widehat{\boldsymbol{x}} + dy\widehat{\boldsymbol{y}} + dz\hat{\mathbf{z}}$)

Spherical Polar Coordinates:



$x = r\,\sin\theta\,\cos\phi, \qquad y = r\sin\theta\,\sin\phi, \qquad z = r\,\cos\theta$

A vector in the spherical polar coordinate is given by

$$\mathbf{A} = A_r\hat{\boldsymbol{r}} + A_{\boldsymbol{\theta}}\widehat{\boldsymbol{\theta}} + A_{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}$$

$$\hat{\boldsymbol{r}} = \sin\theta\,\cos\phi\,\hat{\mathbf{x}} + \sin\theta\,\sin\phi\,\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

$$\widehat{\boldsymbol{\theta}} = \cos\theta\,\cos\phi\,\hat{\mathbf{x}} + \cos\theta\,\sin\phi\,\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}}$$

$$\widehat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

Griffiths: Prob 1.37

The infinitesimal displacement vector in the spherical polar coordinate is given by

$d\mathbf{l} = dl_r\hat{\boldsymbol{r}} + dl_{\boldsymbol{\theta}}\widehat{\boldsymbol{\theta}} + dl_{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}}$ (In Cartesian system we have $d\mathbf{l} = dx\,\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$)

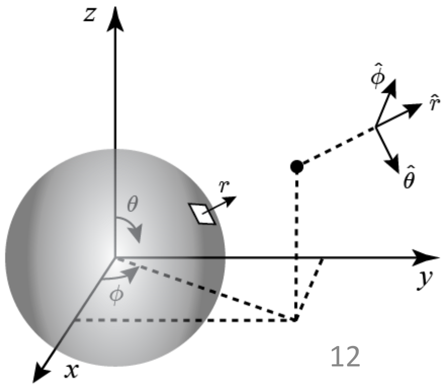
$$d\mathbf{l} = dr\hat{\boldsymbol{r}} + rd\theta\widehat{\boldsymbol{\theta}} + r\sin\theta d\phi\widehat{\boldsymbol{\phi}}$$

The infinitesimal volume element: $d\tau = dl_r dl_{\boldsymbol{\theta}} dl_{\boldsymbol{\phi}} = r^2\sin\theta dr d\theta d\phi$

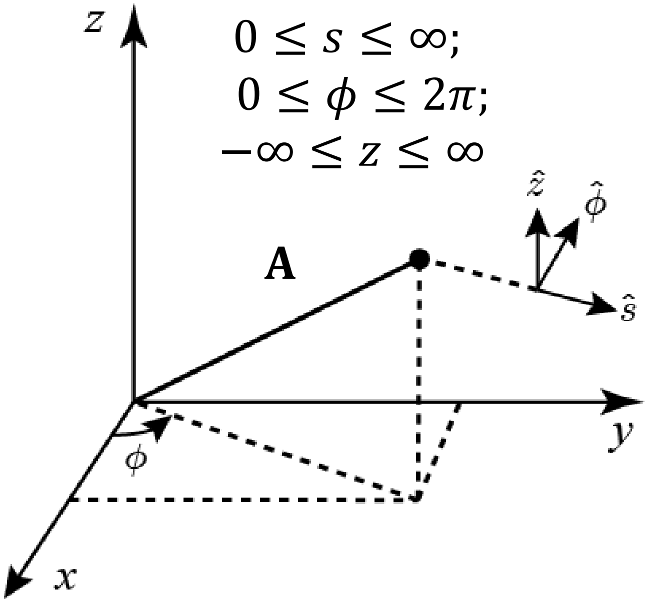
The infinitesimal area element (it depends):

$$d\mathbf{a} = dl_{\boldsymbol{\theta}} dl_{\boldsymbol{\phi}}\hat{\boldsymbol{r}} = r^2\sin\theta d\theta d\phi\hat{\boldsymbol{r}}$$

(over the surface of a sphere)



Cylindrical Coordinates:



$0 \leq s \leq \infty;$

$0 \leq \phi \leq 2\pi;$

$-\infty \leq z \leq \infty$

$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z$

A vector in the cylindrical coordinates is given by

$$\mathbf{A} = A_s\hat{\boldsymbol{s}} + A_{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}} + A_{\boldsymbol{z}}\hat{\boldsymbol{z}}$$

$$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$$

$$\widehat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$$

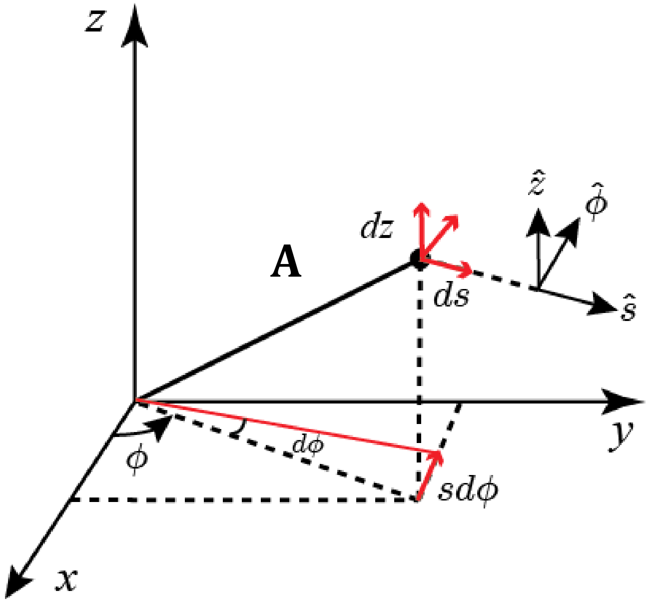
$$\hat{\mathbf{z}} = \hat{\mathbf{z}}$$

Griffiths: Prob 1.41

The infinitesimal displacement vector in the cylindrical coordinates is given by

$d\mathbf{l} = dl_{\boldsymbol{s}}\hat{\boldsymbol{s}} + dl_{\boldsymbol{\phi}}\widehat{\boldsymbol{\phi}} + dl_{\boldsymbol{z}}\hat{\boldsymbol{z}}$ (In Cartesian system we have $d\mathbf{l} = dx\,\widehat{\boldsymbol{x}} + dy\widehat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$)

Cylindrical Coordinates:



$x = s\cos\phi, \qquad y = s\sin\phi, \qquad z = z$

A vector in the cylindrical coordinates is given by

$\mathbf{A} = A_s\hat{\mathbf{s}} + A_\phi\hat{\boldsymbol{\phi}} + A_z\hat{\mathbf{z}}$

$\hat{\mathbf{s}} = \cos\phi\,\hat{\mathbf{x}} + \sin\phi\,\hat{\mathbf{y}}$

$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$

$\hat{\mathbf{z}} = \hat{\mathbf{z}}$

Griffiths: Prob 1.41

The infinitesimal displacement vector in the cylindrical coordinates is given by

$d\mathbf{l} = dl_s\hat{\mathbf{s}} + dl_\phi\hat{\boldsymbol{\phi}} + dl_z\hat{\mathbf{z}}$ (In Cartesian system we have $d\mathbf{l} = dx\,\hat{\boldsymbol{x}} + dy\hat{\boldsymbol{y}} + dz\hat{\boldsymbol{z}}$)

$d\mathbf{l} = ds\hat{\mathbf{s}} + sd\phi\hat{\boldsymbol{\phi}} + dz\hat{\mathbf{z}}$

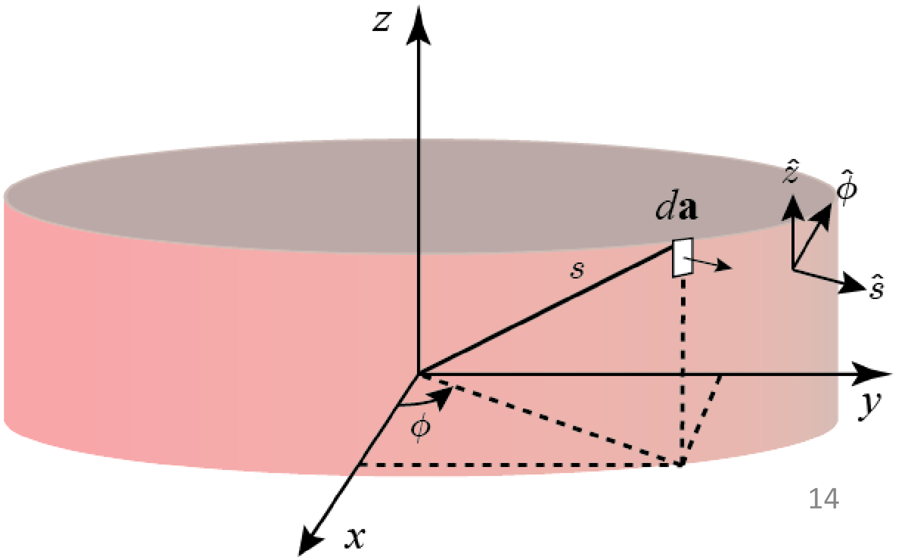
The infinitesimal volume element:

$d\tau = dl_s dl_\phi dl_z = s\, ds d\phi dz$

The infinitesimal area element (it depends):

$d\mathbf{a} = dl_\phi dl_z\hat{\mathbf{s}} = s\, d\phi dz\hat{\mathbf{s}}$

(over the surface of a cylinder)



Gradient, Divergence and Curl in Cartesian, Spherical-polar and Cylindrical Coordinate systems:

- See the formulas listed inside the front cover of Griffiths