Entangled Photons

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December 13th, 2014
Einstein objected to this kind of phenomenon
**EPR Paradox** [A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. 47, 777 (1935)]

**One photon system:**

- Plane-wave
- Diffracting wave
- Momentum is the Physical Reality
- Position is the Physical Reality

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \]

“When the operators corresponding to two physical quantities do not commute the two quantities cannot have simultaneous reality.” -- EPR rephrasing the uncertainty relation.

**Two-photon system (Entangled):**

\[ \Delta x_{\text{cond}}^{(1)} \Delta p_{\text{cond}}^{(1)} < \frac{\hbar}{2} \]

Non-local correlation ???

**EPR’s Questions:**

1. Is Quantum mechanics incomplete??
2. Does it require additional “hidden variables” to explain the measurement results.
Sources of Entangled Photons

Parametric down-conversion (PDC)

\[ |\psi_{tp}\rangle \neq |\psi\rangle_s \otimes |\psi\rangle_i \]

\[ q_p = q_s + q_i \] Conservation of momentum

\[ \omega_p = \omega_s + \omega_i \] Conservation of Energy

\[ l_p = l_s + l_i \] Conservation of Orbital Angular Momentum

Other method: Four-wave Mixing

Burnham and Weinberg, PRL 25, 85 (1970)

Robert W. Boyd, Nonlinear Optics, 2nd ed.
Orbital Angular momentum of a photon

Angular position

Laguerre-Gauss basis \( LG^l_p \)

\[
A_l = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \Psi(\phi) \exp(-il\phi)
\]

\[
\Psi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{+\infty} A_l \exp(il\phi)
\]

\[
J_z = \frac{\int\int \rho d\rho d\phi (\mathbf{\rho} \times \langle \mathbf{E} \times \mathbf{B} \rangle_z)}{c \int\int \rho d\rho d\phi \langle \mathbf{E} \times \mathbf{B} \rangle_z} = \frac{\hbar l}{\hbar \omega}
\]

Barnett and Pegg, PRA 41, 3427 (1990)

Allen et al., PRA 45, 8185 (1992)
Types of Entanglement

- **Entanglement in position and momentum**
  \[ \Delta x_{\text{cond}}^{(1)} \Delta p_{\text{cond}}^{(1)} < \frac{\hbar}{2} \]

- **Entanglement in time and energy**
  \[ \Delta t_{\text{cond}}^{(1)} \Delta E_{\text{cond}}^{(1)} < \frac{\hbar}{2} \]

- **Entanglement in angular position and orbital angular momentum**
  \[ \Delta \phi_{\text{cond}}^{(1)} \Delta L_{\text{cond}}^{(1)} < \frac{\hbar}{2} \]

- **Entanglement in Polarization**

- **Parametric down-conversion (PDC)**

- **Continuous-variable entanglement**


- **Two-dimensional entanglement**
(1) If signal photon has horizontal (vertical) polarization, idler photon is guaranteed to have horizontal (vertical) polarization

--- Is this entanglement ?? NO

--- Two independent classical sources can also produce such correlations
What is Polarization Entanglement?

(1) If signal photon has horizontal (vertical) polarization, idler photon is guaranteed to have horizontal (vertical) polarization
   --- Is this entanglement ??  NO
   --- Two independent classical sources can also produce such correlations

(2) If signal photon has 45° (-45°) polarization, idler photon is guaranteed to have 45° (-45°) polarization
   --- Is this entanglement ??  NO
   --- Two independent classical sources can also produce such correlations
What is Polarization Entanglement?

(1) If signal photon has horizontal (vertical) polarization, idler photon is guaranteed to have horizontal (vertical) polarization
   --- Is this entanglement ?? NO
   --- Two independent classical sources can also produce such correlations

(2) If signal photon has 45° (-45°) polarization, idler photon is guaranteed to have 45° (-45°) polarization
   --- Is this entanglement ?? NO
   --- Two independent classical sources can also produce such correlations

If correlations (1) and (2) exist simultaneously, then that is entanglement
Quantum Entanglement and hidden variables

- **1950s: hidden variable quantum mechanics by David Bohm**
  D. Bohm, Phys. Rev. 85, 166 (1952);

- **1964: Bell’s Inequality--- A proposed test for quantum entanglement**
  J. S. Bell, Physics 1, 195 (1964).

- **1980s -90s --- Experimental violations of Bell’s inequality**
Bell’s Inequality for Polarization-Entangled Photons

\[ |45^0\rangle = |H\rangle + |V\rangle \]

H-polarized cone from 1

V-polarized cone from 2

\[ |\psi\rangle = |H_s\rangle |H_i\rangle + |V_s\rangle |V_i\rangle \]

\[ |\psi\rangle = |45\rangle_s |45\rangle_i + |45\rangle_s |45\rangle_i \]
Bell’s Inequality for Polarization-Entangled Photons

\[ |45^0\rangle = |H\rangle + |V\rangle \]

\[ |\psi\rangle = |H_s\rangle |H_i\rangle + |V_s\rangle |V_i\rangle \]

\[ |\psi\rangle = |45\rangle_s |45\rangle_i + |-45\rangle_s |-45\rangle_i \]

**Bell Parameter:** 
\[ S = E(a,b) - E(a,b') + (a',b) + E(a',b') \]

\[ E(\alpha, \beta) = \frac{N(\alpha, \beta) + N(\alpha_\perp, \beta_\perp) - N(\alpha, \beta_\perp) - N(\alpha_\perp, \beta)}{N(\alpha, \beta) + N(\alpha_\perp, \beta_\perp) + N(\alpha, \beta_\perp) + N(\alpha_\perp, \beta_\perp)} \]

\[ \alpha = -45^0; \quad \alpha' = 0^0; \quad \alpha_\perp = 45^0; \quad \alpha'_\perp = 90^0 \]

\[ \beta = -22.5^0; \quad \beta' = 22.5^0; \quad \beta_\perp = 67.5^0; \quad \beta'_\perp = 112.5^0 \]

\[ |S| \leq 2 \quad \text{For hidden variable theories} \]

\[ |S| \leq 2\sqrt{2} \quad \text{For quantum correlations} \]
Types of Entanglement

Parametric down-conversion (PDC)

Entanglement in position and momentum
\[ \Delta x^{(1)}_{\text{cond}} \Delta p^{(1)}_{\text{cond}} < \frac{\hbar}{2} \]
Continuous-variable entanglement

Entanglement in time and energy
\[ \Delta t^{(1)}_{\text{cond}} \Delta E^{(1)}_{\text{cond}} < \frac{\hbar}{2} \]

Entanglement in angular position and orbital angular momentum
\[ \Delta \phi^{(1)}_{\text{cond}} \Delta L^{(1)}_{\text{cond}} < \frac{\hbar}{2} \]
Two-dimensional entanglement

Burnham and Weinberg, PRL 25, 85 (1970)
Robert W. Boyd, Nonlinear Optics, 2nd ed.
Verifying continuous variable entanglement


\[
\Delta x_{\text{cond}}^{(1)} \Delta p_{\text{cond}}^{(1)} < 0.06\hbar
\]

Time-energy Entanglement

Phys. Rev. A 73, 031801(R), 2006
Nature Physics 9, 19 (2013)

Angular-position Orbital-angular-momentum Entanglement [Science 329, 662 (2010).]

\[
\Delta \phi_{\text{cond}}^{(1)} \Delta L_{\text{cond}}^{(1)} < 0.15\hbar
\]
Bell inequality violation in 2D state space of continuous variables

**Position-momentum Entanglement** [Phys. Rev. Lett. 64, 2495 (1990)]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [ |p_1\rangle_s |p_2\rangle_i + |p_2\rangle_s |p_1\rangle_i ] \]

**Time-energy Entanglement** [Phys. Rev. Lett. 103, 253601 (2009)]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [ |\omega_1\rangle_s |\omega_2\rangle_i + |\omega_2\rangle_s |\omega_1\rangle_i ] \]

**Angular-position Orbital-angular-momentum Entanglement** [Optics Express 17, 8287 (2009)]

\[ |\psi\rangle = \frac{1}{\sqrt{2}} [ |l_1\rangle_s |l_2\rangle_i + |l_2\rangle_s |l_1\rangle_i ] \]
Quantum Cryptography (Quantum Key Distribution)

**Older Method** (scylate)

- **Message:** OPTICS
- **Encrypted message:** OQTJDS
- **Encrypt with Key:** 010110
- **Decrypt with Key:** 010110
- **Decrypted Message:** OPTICS

**Modern Method**

- **Message:** OPTICS
- **Encrypted message:** OQTJDS
- **Encrypt with Key:** 010110
- **Decrypt with Key:** 010110
- **Decrypted Message:** OPTICS

Image source: Wikipedia and google images
Quantum Cryptography (Quantum Key Distribution)

**Older Method**  
(scylate)

**Modern Method**

Message: **OPTICS**  
Encrypted message: **OQTJDS**  
Encrypt with Key: **010110**  
Decrypt with Key: **010110**  
Encrypted message: **OQTJDS**  
Decrypted Message: **OPTICS**

**Main issue: Security**

**Future?**

**Quantum Key Distribution**

Image source: Wikipedia and google images

- Alice sends a bit to Bob by measuring her bit; whatever bit she measures becomes the incoming bit for Bob.

\[
|\psi_{ab}\rangle = |H\rangle_a |H\rangle_b + |V\rangle_a |V\rangle_b \\
|\psi_{ab}\rangle = |D\rangle_a |D\rangle_b + |A\rangle_a |A\rangle_b
\]

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perfectly secure because of the laws of quantum mechanics
Quantum Superposition: Application (Quantum Cryptography)

What are those laws?

1. Measurement in an incompatible basis changes the quantum state

2. No Cloning Theorem:
\[ \hat{U} |S\rangle |H\rangle \rightarrow |0\rangle |HH\rangle \]
\[ \hat{U} |S\rangle |V\rangle \rightarrow |0\rangle |VV\rangle \]
\[ \hat{U} |S\rangle (|H\rangle + |V\rangle) \rightarrow |0\rangle (|HH\rangle + |VV\rangle) \]
\[ \not\equiv |0\rangle (H + V)(H + V)\]

• C cannot clone an arbitrary quantum state sent out by A
Quantum Computation:


The basic building block for quantum computation:

two-qubit state, or more generally N-qudit state

Polarization Two-qubit state:  
\[ |\psi\rangle = |H_s\rangle |H_i\rangle + |V_s\rangle |V_i\rangle \]

OAM Two-qubit state:  
\[ |\psi\rangle = \frac{1}{\sqrt{2}} (|l_1\rangle_s |l_2\rangle_i + |l_2\rangle_s |l_1\rangle_i) \]
Entanglement Quantification

Most general Two-qubit state:

\[
\rho_{\text{qubit}} = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{21} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{31} & \rho_{32} & \rho_{33} & \rho_{34} \\
\rho_{41} & \rho_{42} & \rho_{43} & \rho_{44}
\end{pmatrix}
\]

What is the entanglement of such a two-qubit state:
The most widely accepted quantifier is Wootter’s Concurrence, which ranges from 0 to 1.

\[
\zeta = \rho_{\text{qubit}}(\sigma_y \otimes \sigma_y)\rho^*_{\text{qubit}}(\sigma_y \otimes \sigma_y)
\]

\[
C(\rho_{\text{qubit}}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}
\]

Entanglement quantifier for a general N-qudit state is yet to be found
Quantum Entanglement (Current Status of the Field)

Questions related to Foundations
• Non-locality and physical reality
• Physical origin of correlations between entangled particles
• Decay of correlation between entangled photons
• Quantification of entanglement in a quantum states

Applications
• Quantum Information, Quantum Cryptography, Quantum Teleportation
• Preparation of entangled states: Two-Qubit state, N-Qudit state
• Improved ways of making entangled quantum states
• Quantum Metrology, Quantum remote sensing
Two-photon Coherence: an alternative approach to Studying Entanglement

What is one-photon coherence?

What is two-photon coherence?

How is two-photon coherence connected to two-photon entanglement?
One-Photon Interference: “A photon interferes with itself” - Dirac

\[ I_A \propto \langle V_A^* (t) V_A (t) \rangle_t \]

\[ I_A \propto 1 + \gamma (\Delta l) \cos (k_0 \Delta l) \]

\[ \gamma (\Delta l) = \frac{\langle V_1^* (t) V_2 (t - \Delta l/c) \rangle_t}{\sqrt{|V_1 (t)|^2 |V_2 (t)|^2}} \]

\[ \Delta l = l_1 - l_2 \]

Necessary condition for interference:

\[ \Delta l < l_{coh} \]

Mandel and Wolf,
*Optical Coherence and Quantum Optics*
One-Photon Interference: “A photon interferes with itself” - Dirac

\[ \Delta l = l_1 - l_2 \]

Necessary condition for interference:

\[ I_A \propto 1 + \gamma(\Delta l) \cos(k_0 \Delta l) \]

\[ \Delta l < l_{\text{coh}} \]
A photon interferes with itself: Spatial

\[ I_A(x) = k_1^2 S(x_1, z) + k_2^2 S(x_2, z) + 2k_1 k_2 \sqrt{S(x_1, z)S(x_2, z)} \mu(\Delta x, z) \cos(k_0 \Delta l) \]

Necessary condition for interference:

\[ |\Delta \rho_p| < \sigma_\mu(z) \]
A photon interferes with itself: Angular
A photon interferes with itself: Angular

$\theta = 0$

$s_2 \angle \alpha \beta$

$s_1$

$D_A$

OAM-mode detector

Computer generated hologram ($l = -1$)

$LG_1$

$LG_0$

$LG_1$

$LG_2$

Single mode fiber
A photon interferes with itself: Angular
A photon interferes with itself: Angular

\[ \psi_{1l} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \Psi_1(\phi) e^{-il\phi} \]
\[ = \frac{\alpha}{\sqrt{2\pi}} \text{sinc} \left( \frac{l\alpha}{2} \right) \]

\[ \psi_{2l} = \frac{\alpha}{\sqrt{2\pi}} \text{sinc} \left( \frac{l\alpha}{2} \right) e^{-il\beta} \]

A photon interferes with itself: Angular

\[ \psi_{1l} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \Psi_1(\phi) e^{-il\phi} \]

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\[ \psi_{2l} = \frac{\alpha}{\sqrt{2\pi}} \text{sinc} \left( \frac{l\alpha}{2} \right) e^{-il\beta} \]

OAM-mode distribution:

\[ I_A = C \frac{\alpha^2}{\pi} \text{sinc}^2 \left( \frac{l\alpha}{2} \right) [1 \pm \cos (l\beta)] \]

Two-photon interference (an example)

Hong-Ou-Mandel Effect  C. K. Hong et al., PRL 59, 2044 (1987)

Image source: Wikipedia and google images
Two-photon interference (an example)

Hong-Ou-Mandel Effect  

C. K. Hong et al., PRL 59, 2044 (1987)

Bunching of photons at a beam splitter
Two-photon interference (an example)

Hong-Ou-Mandel Effect  C. K. Hong et al., PRL 59, 2044 (1987)

Is this interference between two different photons?
Is Dirac’s statement incorrect? - NO

Here, a two-photon is interfering with itself
Two-photon interference (an example)

Hong-Ou-Mandel Effect  
C. K. Hong et al., PRL 59, 2044 (1987)

Applications in quantum metrology

1 + cos2\(\varphi\)

1 + cos4\(\varphi\)

Two-Photon Interference (Other examples)

• **Hong-Ou-Mandel effect**  
  C. K. Hong et al., PRL 59, 2044 (1987)

- Quantum Optical Lithography  

• **Induced Coherence**  
  X. Y. Zou et al., PRL 67, 318 (1991)

• **Postponed Compensation Experiment**  

• **Frustrated two-photon Creation**  
  T. J. Herzog et al., PRL 72, 629 (1994)

- Quantum Imaging with undetected photons  
  Nature 512, 409 (2014)
Two-Photon Interference: A two-photon interferes with itself

Necessary conditions for two-photon interference:

\[ \Delta L \equiv l_1 - l_2 \]

two-photon path-length difference

\[ \Delta L' \equiv l_1' - l_2' \]

two-photon path-asymmetry length difference

\[ \Delta \phi \equiv (\phi_{s1} + \phi_{i1} + \phi_{p1}) - (\phi_{s2} + \phi_{i2} + \phi_{p2}) \]

\[ R_{si} = C[1 + \gamma' (\Delta L') \gamma (\Delta L) \cos(k_0 \Delta L + \Delta \phi)] \]

\[ \gamma (\Delta L) = \frac{\langle v_1(t)v_2^*(t + \Delta L/c) \rangle_t}{\sqrt{|v_1|^2|v_2|^2}} \]

\[ \gamma' (\Delta L') = \frac{\langle g_1^*(\tau)g_2(\tau - \Delta L'/c) \rangle_\tau}{\sqrt{|g_1|^2|g_2|^2}} \]

Jha, O’Sullivan, Chan, and Boyd et al., PRA 77, 021801(R) (2008)
Two-Photon Coherence and Entanglement

(Gaussian Schell-model pump)

Coincidence Rate $R_{si}(r_s, r_i) = k_1^2 S^{(2)}(\rho_{s1}, \rho_{i1}, z) + k_2^2 S^{(2)}(\rho_{s2}, \rho_{i2}, z) + k_1 k_2 W^{(2)}(\rho_{s1}, \rho_{i1}, \rho_{s2}, \rho_{i2}, z) e^{i(\omega_s(t_s-t_s) + \omega_i(t_i-t_i))} + \text{c.c.}$
A photon interferes with itself: Spatial

\[ \Delta \rho_p = \rho_{p1} - \rho_{p2} \]

\[ I_A(x) = k_1^2 S(x_1, z) + k_2^2 S(x_2, z) + 2k_1k_2 \sqrt{S(x_1, z)S(x_2, z)} \mu(\Delta x, z) \cos(k_0 \Delta l) \]

Necessary condition for interference:

\[ |\Delta \rho_p| < \sigma_\mu(z) \]
Two-Photon Coherence and Entanglement

Entangled two-qubit state

\[ \rho_{\text{qubit}} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & b \end{pmatrix} \]

\[ a = \eta S^{(2)}(\rho_1, z) \]
\[ b = \eta S^{(2)}(\rho_2, z) \]
\[ c = d^* = \eta W^{(2)}(\rho_1, \rho_2, z) \]
\[ \eta = 1/[S^{(2)}(\rho_1, z) + S^{(2)}(\rho_2, z)] \]

Entanglement of the state (Concurrence):

\[ C(\rho_{\text{qubit}}) = 2|c| = 2\eta |W^{(2)}(\rho_1, \rho_2, z)| \]

\[ C(\rho_{\text{qubit}}) = \mu^{(2)}(\Delta\rho, z) \] (with \( a = b \))

Coincidence Rate

\[ R_{si}(r_s, r_i) = k_1^2 S^{(2)}(\rho_{s1}, \rho_{i1}, z) + k_2^2 S^{(2)}(\rho_{s2}, \rho_{i2}, z) + k_1 k_2 W^{(2)}(\rho_{s1}, \rho_{i1}, \rho_{s2}, \rho_{i2}, z) e^{i[\omega_s(t_{s1} - t_{s2}) + \omega_i(t_{i1} - t_{i2})]} + \text{c.c.} \]

O’Sullivan et al., PRL 94, 220501 (2005)
Neves et al., PRA 76, 032314 (2007)
Walborn et al., PRA 76, 062305 (2007)
Taguchi et al., PRA 78, 012307 (2008)

Concurrence

W. K. Wootters, PRL 80, 2245 (1998)

\[ \zeta = \rho_{\text{qubit}}(\sigma_y \otimes \sigma_y)\rho_{\text{qubit}}^* (\sigma_y \otimes \sigma_y) \]

\[ C(\rho_{\text{qubit}}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} \]

Angular Two-Photon Interference

\[ \alpha = \frac{\pi}{10} \]
\[ \beta = \frac{\pi}{4} \]

State of the two photons produced by PDC:

\[ |\psi_{tp}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s - l\rangle_i \]
Angular Two-Photon Interference

State of the two photons produced by PDC:

$$|\psi_{tp}\rangle = \sum_{l=\infty}^{\infty} c_{l} |l\rangle_{s} - l\rangle_{i}$$

State of the two photons after the aperture:

$$\rho_{\text{qubit}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix}$$

$$\rho_{14} = \rho_{41}^{*} = \sqrt{\rho_{11}\rho_{44}} \mu e^{i\theta}$$

$$\rho_{11} + \rho_{44} = 1$$
Angular Two-Photon Interference

\[ \alpha = \pi / 10 \]
\[ \beta = \pi / 4 \]

State of the two photons after the aperture:
\[ |\psi_{tp}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s |l\rangle_i \]

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\[ \rho_{14} = \rho_{41}^* = \sqrt{\rho_{11}\rho_{44}} \mu e^{i\theta} \]
\[ \rho_{11} + \rho_{44} = 1 \]

Coincidence count rate:
\[ R_{si} = \frac{A^2 \alpha^4}{4\pi^2} \left| \sum_l c_l \text{sinc} \left[ (l_s - l) \frac{\alpha}{2} \right] \text{sinc} \left[ (l_i + l) \frac{\alpha}{2} \right] \right|^2 \]
\[ \times \left\{ \rho_{11} + \rho_{44} + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos [(l_s + l_i)\beta + \theta] \right\} \]

Visibility:
\[ V = 2\sqrt{\rho_{11}\rho_{44}} \mu \]

Concurrence:
\[ \zeta = \rho_{\text{qubit}} (\sigma_y \otimes \sigma_y) \rho_{\text{qubit}}^* (\sigma_y \otimes \sigma_y) \]
\[ C(\rho_{\text{qubit}}) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \} \]

W. K. Wootters, PRL 80, 2245 (1998)
Angular Two-Photon Interference

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State of the two photons produced by PDC:
\[ |\psi_{tp}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s - l\rangle_i \]

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\[ \rho_{\text{qubit}} = \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
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\[ \rho_{11} + \rho_{44} = 1 \]

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\[ R_{si} = \frac{A^2 \alpha^4}{4\pi^2} \left| \sum_l c_l \text{sinc} \left[ (l_s - l) \frac{\alpha}{2} \right] \text{sinc} \left[ (l_i + l) \frac{\alpha}{2} \right] \right|^2 \]
\[ \times \left\{ \rho_{11} + \rho_{44} + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos \left[ (l_s + l_i) \beta + \theta \right] \right\} \]

Visibility:
\[ V = 2\sqrt{\rho_{11}\rho_{44}} \mu \]

Concurrence:
\[ \zeta = \rho_{\text{qubit}} (\sigma_y \otimes \sigma_y) \rho_{\text{qubit}}^* (\sigma_y \otimes \sigma_y) \]

\[ C(\rho_{\text{qubit}}) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \} \]

Concurrence of the two-qubit state:
\[ C(\rho_{\text{qubit}}) = 2|\rho_{14}| = 2\sqrt{\rho_{11}\rho_{44}} \mu = V \]

A. K. Jha et al., PRL 104, 010501 (2010)

W. K. Wootters, PRL 80, 2245 (1998)
Angular Two-Photon Interference

State of the two photons produced by PDC:

$$|\psi_{tp}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s |l\rangle_i$$

State of the two photons after the aperture:

$$\rho_{qubit} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix} = \sqrt{\rho_{11} \rho_{44}} \mu e^{i\theta} \quad \rho_{11} + \rho_{44} = 1$$

Coincidence count rate:

$$R_{si} = \frac{A^2 \alpha^4}{4\pi^2} \left| \sum_l c_l \text{sinc} \left( (l_s - l) \frac{\alpha}{2} \right) \text{sinc} \left( (l_i + l) \frac{\alpha}{2} \right) \right|^2 \times \left\{ \rho_{11} + \rho_{44} + 2\sqrt{\rho_{11} \rho_{44}} \mu \cos \left[ (l_s + l_i) \beta + \theta \right] \right\}$$

Visibility:

$$V = 2\sqrt{\rho_{11} \rho_{44}} \mu$$

Concurrence of the two-qubit state:

$$C(\rho_{qubit}) = 2|\rho_{14}| = 2\sqrt{\rho_{11} \rho_{44}} \mu = V$$

A. K. Jha et al., PRL 104, 010501 (2010)
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\[ \alpha = \frac{\pi}{10} \]
\[ \beta = \frac{\pi}{4} \]

Pump

\( l = 0 \)

PDC

\( A_{sa} \)
\( A_{sb} \)

OAM-mode detector

\( D_s \)

\( l_s \)

\( D_i \)

\( l_i \)

OAM-mode detector

Coincidence count rate:

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Angular Two-Photon Interference

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\alpha = \frac{\pi}{10}
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\[
\frac{A_{sb}}{A_{sa}} \quad \frac{A_{ib}}{A_{ia}}
\]

OAM-mode detector

\[
D_s \quad D_i
\]

Pump

PDC

\[
l_i = \pm 2
\]

\[
\text{Coincidence count rate: } C(\rho_{\text{qubit}}) = 0.963
\]

\[
R_{si} = \frac{A^2 \alpha^4}{4\pi^2} \left| \sum_l c_l \text{sinc} \left[ \left(l_s - l \right) \frac{\alpha}{2} \right] \text{sinc} \left[ \left(l_i + l \right) \frac{\alpha}{2} \right] \right|^2
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A. K. Jha et al., PRL 104, 010501 (2010)
Jha, Agarwal, and Boyd, PRA84, 063847 (2011)
## Summary

### Parametric down-conversion (PDC)


### Table: Conservation laws and coherence

<table>
<thead>
<tr>
<th>variable</th>
<th>Conservation law</th>
<th>Entanglement</th>
<th>EPR Paradox</th>
<th>Two-photon coherence</th>
</tr>
</thead>
<tbody>
<tr>
<td>Energy</td>
<td>( \omega_p = \omega_s + \omega_i )</td>
<td>Time and energy</td>
<td>( \Delta t^{(1)}<em>{\text{cond}} \Delta E^{(1)}</em>{\text{cond}} &lt; \frac{\hbar}{2} )</td>
<td>Temporal</td>
</tr>
<tr>
<td>Transverse Momentum</td>
<td>( q_p = q_s + q_i )</td>
<td>Position and momentum</td>
<td>( \Delta x^{(1)}<em>{\text{cond}} \Delta p^{(1)}</em>{\text{cond}} &lt; \frac{\hbar}{2} )</td>
<td>Spatial</td>
</tr>
<tr>
<td>Orbital angular momentum</td>
<td>( l_p = l_s + l_i )</td>
<td>Angular position and orbital angular momentum</td>
<td>( \Delta \phi^{(1)}<em>{\text{cond}} \Delta L^{(1)}</em>{\text{cond}} &lt; \frac{\hbar}{2} )</td>
<td>Angular</td>
</tr>
</tbody>
</table>
Entangled Photons: Future directions

1. Foundations of Quantum Mechanics. (Theory + Experiment)
   - Questions related to non-locality and physical reality.
   - Complete description of two-photon entanglement in terms of coherence measures.
   - Extension of coherence-based measure for quantifying high-dimensional entanglement.
   - Photon-statistics of entangled photons.
   - Correlated-noise measurements of entangled photons.

2. Applications of Quantum Entanglement. (Theory + Experiment)
   - Developing sources of entangled photon based on parametric down-conversion.
   - Use of OAM-entangled photons for high-dimensional Quantum information processing.
   - Use of entangled photons for high-resolution imaging, remote sensing and communication through turbulent atmosphere.
Entangled Photons: Open Problems!

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PhD and Post-Doc positions available within the group

Thank you for your attention