Coherence Properties of the Entangled Two-Photon Field Produced by Parametric Down-Conversion

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Introduction and Outline

Quantum Entanglement

- EPR paradox and non-locality, Hidden variable theories, Bell inequalities …
- Quantum cryptography, Quantum dense coding, Quantum lithography…

Parametric down-conversion provides a source of entangled photons
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Parametric down-conversion provides a source of entangled photons

\[ \omega_p = \omega_s + \omega_i \]  
Entanglement in time and energy

"Temporal" two-photon coherence

\[ q_p = q_s + q_i \]  
Entanglement in position and momentum

"Spatial" two-photon coherence

\[ l_p = l_s + l_i \]  
Entanglement in angular position and orbital angular momentum

"Angular" two-photon coherence
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\[ l_p = l_s + l_i \]  
**Entanglement in angular position and orbital angular momentum**  
“**Angular**” two-photon coherence
One-Photon Interference: “A photon interferes with itself” - Dirac

\[ I_A \propto 1 + \gamma (\Delta l) \cos (k_0 \Delta l) \]

Necessary condition for interference: \[ \Delta l < l_{\text{coh}} \]
Two-Photon Interference

• Hong-Ou-Mandel effect
  C. K. Hong et al., PRL 59, 2044 (1987)

• Induced Coherence
  X. Y. Zou et al., PRL 67, 318 (1991)

• Postponed Compensation Experiment

• Frustrated two-photon Creation
  T. J. Herzog et al., PRL 72, 629 (1994)
Two-Photon Interference

Herzog et al., PRL 72, 629 (1994)
Two-Photon Interference

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Two-Photon Interference

\[ \Delta L \equiv l_1 - l_2 \]

two-photon path-length

Jha, O'Sullivan, Chan, and Boyd, PRA 77, 021801(R) (2008)
Two-Photon Interference

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Two-Photon Interference

Necessary conditions for two-photon interference:

\[ R_{si} = C \left[ 1 + \gamma' (\Delta L') \gamma (\Delta L) \cos(k_0 \Delta L) \right] \]

Jha, O'Sullivan, Chan, and Boyd, PRA 77, 021801(R) (2008)

R. J. Glauber, Phys. Rev. 130, 2529 (1963)

\[ \Delta L \equiv l_1 - l_2 \]

two-photon path-length

\[ \Delta L' \equiv l'_1 - l'_2 \]

two-photon path-asymmetry length

\[ \Delta L < l_{coh}^P \sim 10 \text{ cm} \]

\[ \Delta L' < l_{coh} = \frac{c}{\Delta \omega} \sim 100 \mu \text{m} \]
Experimental Verification

\[ \Delta L = x_s + x_i \quad \Delta L' = 2x_s - 2x_i \quad \gamma(\Delta L) \sim 1 \]

\[ R_{si} = C\left[1 - \gamma'(2x_s - 2x_i) \cos[k_0(x_s + x_i)]\right] \]
Experimental Verification

\[ \Delta L = x_s + x_i \quad \Delta L' = 2x_s - 2x_i \quad \gamma(\Delta L) \sim 1 \]

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Some One-Photon Interference Effects

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\[ l_{coh} = 100 \mu m \]

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Some One-Photon Interference Effects

\[ \Delta L = x_s + x_i \quad \Delta L' = 2x_s - 2x_i \quad \gamma(\Delta L) \sim 1 \]

\[ R_{si} = C\{ 1 - \gamma'(2x_s - 2x_i) \cos[k_0(x_s + x_i)] \} \]

One-photon interference profile at a given detection point is the sum of the two-photon interference profiles

\[ R_X = \sum_i R_{XY_i} \]

\[ R_s = R_i = R_{si} \]

Jha, O'Sullivan, Chan, and Boyd, PRA 77, 021801(R) (2008)
Exploring time-energy entanglement using geometric phase

**Franson Interferometer**  J. D. Franson, PRL 62, 2205 (1989)

\[ R_{si} = C[1 + \cos(\Phi_s + \Phi_i)] \]

**Violation of CHSH Bell Inequality using dynamic phase**

Brendel et al., PRL 66, 1142 (1991)
Kwiat et al., PRA 47, R2472 (1993)
Strekalov et al., PRA 54, R1 (1996)
Barreiro et al., PRL 95, 260501 (2005)
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**Violation of CHSH Bell Inequality using geometric (Pancharatnam, Berry) phase**

\[ R_{si} = C\{1 - \cos[k_0(x_s + x_i) + 2\beta_s + 2\beta_i]\} \]

Jha, Malik, and Boyd, PRL 101, 180405 (2008)
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Jha, Malik, and Boyd, PRL 101, 180405 (2008)

**Visibility:** \[ V = 77\% \ (> 70.7\%) \]

**Bell Parameter:** \[ |S| = 2.18 \pm 0.04 \ (> 2.0) \]

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**Franson Interferometer**

J. D. Franson, PRL 62, 2205 (1989)

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**Diagram:**

- Pump
- PDC
- D_s
- D_i
- ID
- F
- Q_{s1}, Q_{s2}, Q_{11}, Q_{12}
- M_s, M_p, M_i
- \Phi_s, \Phi_i
- f = 1 m
- Coincidence

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**Graphs:**

- Graphs showing coincidence counts for 5 seconds with different values of \(2\beta_i\).
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“Angular” two-photon coherence
Spatial One-Photon Interference: review

Intensity at the detector:
\[ I_A(r_p) \propto S(\rho_{p1}, z) + S(\rho_{p2}, z) + W(\rho_{p1}, \rho_{p2}, z)e^{-ik_0(d_1-d_2)} + c.c. \]

Cross-spectral density:
\[ |W(\rho_{p1}, \rho_{p2}, z)| = \sqrt{S(\rho_{p1}, z)S(\rho_{p2}, z)}\mu(\Delta \rho_p, z) \]

\[ \Delta \rho_p = \rho_{p1} - \rho_{p2} \]

(Gaussian Schell-model beam)

Mandel and Wolf, *Optical Coherence and Quantum Optics*
Spatial One-Photon Interference: review

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Cross-spectral density:
\[ |W(\rho_{p1}, \rho_{p2}, z)| = \sqrt{S(\rho_{p1}, z)S(\rho_{p2}, z)} \mu(\Delta \rho_p, z) \]

Spectral density:
\[ S(\rho_{p1}, z) = C \exp \left\{ -(1/2) \left[ \rho_{p1}/\sigma_s(z) \right]^2 \right\} \]
\[ \sigma_s(z) = z \sqrt{\sigma_{\mu}^2 + 4\sigma_s^2/2k_0\sigma_s\sigma_{\mu}} \]

Degree of coherence:
\[ \mu(\Delta \rho_p, z) = \exp \left\{ -(1/2) \left[ \Delta \rho_p/\sigma_\mu(z) \right]^2 \right\} \]
\[ \sigma_\mu(z) = z \sqrt{\sigma_\mu^2 + 4\sigma_s^2/2k_0\sigma_s^2} \]
Spatial Two-photon Interference

(Gaussian Schell-model pump)
Spatial Two-photon Interference

Two-photon transverse position vector: \[ \rho_1 \equiv \frac{\rho_{s1} + \rho_{i1}}{2}, \quad \rho_2 \equiv \frac{\rho_{s2} + \rho_{i2}}{2}; \quad \Delta \rho = \rho_1 - \rho_2 \]

Two-photon position-asymmetry vector: \[ \rho'_1 \equiv \rho_{s1} - \rho_{i1}, \quad \rho'_2 \equiv \rho_{s2} - \rho_{i2}; \quad \Delta \rho' = \rho'_1 - \rho'_2 \]
Spatial Two-photon Interference

Two-photon transverse position vector: \( \rho_1 \equiv \frac{\rho_{s1} + \rho_{i1}}{2}, \quad \rho_2 \equiv \frac{\rho_{s2} + \rho_{i2}}{2}; \quad \Delta \rho = \rho_1 - \rho_2 \)

Two-photon position-asymmetry vector: \( \rho'_1 \equiv \rho_{s1} - \rho_{i1}, \quad \rho'_2 \equiv \rho_{s2} - \rho_{i2}; \quad \Delta \rho' = \rho'_1 - \rho'_2 \)

Coincidence count rate:
\[
R_{si}(r_s, r_i) \propto S^{(2)}(\rho_1, z) + S^{(2)}(\rho_2, z) + W^{(2)}(\rho_1, \rho_2, z)e^{ik_0[(d_{s1} + d_{i1})/2 - (d_{s2} + d_{i2})/2]} + \text{c.c.}
\]

Two-photon cross-spectral density:
\[
|W^{(2)}(\rho_1, \rho_2, z)| = \sqrt{S^{(2)}(\rho_1, z)S^{(2)}(\rho_2, z)\mu^{(2)}(\Delta \rho, z)}
\]

Jha and Boyd, Accepted in PRA
Spatial Two-photon Interference

Two-photon transverse position vector: \( \rho_1 \equiv \frac{\rho_{s1} + \rho_{i1}}{2}, \quad \rho_2 \equiv \frac{\rho_{s2} + \rho_{i2}}{2}; \quad \Delta \rho = \rho_1 - \rho_2 \)

Two-photon position-asymmetry vector: \( \rho_1' \equiv \rho_{s1} - \rho_{i1}, \quad \rho_2' \equiv \rho_{s2} - \rho_{i2}; \quad \Delta \rho' = \rho_1' - \rho_2' \)

Coincidence count rate:
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R_{si}(r_s, r_i) \propto S^{(2)}(\rho_1, z) + S^{(2)}(\rho_2, z) + W^{(2)}(\rho_1, \rho_2, z)e^{ik_0[(d_{s1}+d_{i1})/2-(d_{s2}+d_{i2})/2]} + \text{c.c.}
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Two-photon cross-spectral density:
\[
|W^{(2)}(\rho_1, \rho_2, z)| = \sqrt{S^{(2)}(\rho_1, z)S^{(2)}(\rho_2, z)\mu^{(2)}(\Delta \rho, z)}
\]

Two-photon spectral density:
\[
S^{(2)}(\rho_1, z) = C \exp\left\{-\frac{1}{2} \left[ \rho_1/\sigma_s^{(2)}(z) \right]^2 \right\}
\]
\[
\sigma_s^{(2)}(z) = z \sqrt{\sigma_{\mu}^2 + 4\sigma_s^2/2k_0\sigma_s\sigma_{\mu}}
\]

Degree of spatial-two-photon coherence:
\[
\mu^{(2)}(\Delta \rho, z) = \exp\left\{-\frac{1}{2} \left[ \Delta \rho/\sigma_{\mu}^{(2)}(z) \right]^2 \right\}
\]
\[
\sigma_{\mu}^{(2)}(z) = z \sqrt{\sigma_{\mu}^2 + 4\sigma_s^2/2k_0\sigma_{\mu}^2}
\]

Jha and Boyd, Accepted in PRA
Two-Photon Coherence and Entanglement

(Gaussian Schell-model pump)

Jha and Boyd, Accepted in PRA
Two-Photon Coherence and Entanglement

Entangled two-qubit state

\[ \rho_{\text{qubit}} = \begin{pmatrix} a & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ d & 0 & 0 & b \end{pmatrix} \]

\[ a = \eta S^{(2)}(\rho_1, z) \]
\[ b = \eta S^{(2)}(\rho_2, z) \]
\[ c = d^* = \eta W^{(2)}(\rho_1, \rho_2, z) \]
\[ \eta = 1 /[S^{(2)}(\rho_1, z) + S^{(2)}(\rho_2, z)] \]

O’Sullivan et al., PRL 94, 220501 (2005)
Neves et al., PRA 76, 032314 (2007)
Walborn et al., PRA 76, 062305 (2007)
Taguchi et al., PRA 78, 012307 (2008)

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\[ \eta = 1/[S^{(2)}(\rho_1, z) + S^{(2)}(\rho_2, z)] \]

Entanglement of the state (Concurrence):

Concurrence

\[ \zeta = \rho_{\text{qubit}}(\sigma_y \otimes \sigma_y)\rho_{\text{qubit}}^*(\sigma_y \otimes \sigma_y) \]
\[ C(\rho_{\text{qubit}}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\} \]

O’Sullivan et al., PRL 94, 220501 (2005)
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W. K. Wootters, PRL 80, 2245 (1998)

Jha and Boyd, Accepted in PRA
Two-Photon Coherence and Entanglement

Entangled two-qubit state

\[
\rho_{\text{qubit}} = \begin{pmatrix}
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\end{pmatrix}
\]

\[
a = \eta S^{(2)}(\rho_1, z)
\]

\[
b = \eta S^{(2)}(\rho_2, z)
\]

\[
c = d^* = \eta W^{(2)}(\rho_1, \rho_2, z)
\]

\[
\eta = 1/[S^{(2)}(\rho_1, z) + S^{(2)}(\rho_2, z)]
\]

Entanglement of the state (Concurrence):

\[
C(\rho_{\text{qubit}}) = 2|c| = 2\eta|W^{(2)}(\rho_1, \rho_2, z)|
\]

\[
C(\rho_{\text{qubit}}) = \mu^{(2)}(\Delta \rho, z) \quad \text{(with } a = b)\]

Concurrence

\[
\zeta = \rho_{\text{qubit}}(\sigma_y \otimes \sigma_y)\rho_{\text{qubit}}^\dagger(\sigma_y \otimes \sigma_y)
\]

\[
C(\rho_{\text{qubit}}) = \max\{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\}
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l_p = l_s + l_i \quad \text{Entanglement in angular position and orbital angular momentum}
\]

- “Angular” two-photon coherence
Angular Fourier Relationship

Angular position

\[
A_l = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \Psi(\phi) \exp(-il\phi)
\]

\[
\Psi(\phi) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{+\infty} A_l \exp(il\phi)
\]

Laguerre-Gauss basis

\[LG_{p}^{l} \quad \text{with } p=0\]

\[l=0 \quad l=1 \quad l=2\]

Allen et al., PRA 45, 8185 (1992)

Barnett and Pegg, PRA 41, 3427 (1990)
Angular One-Photon Interference

\( l=0 \)

\( s_1 \) \( \alpha \) \( s_2 \) \( \beta \) \( l \)

\( D_A \)

OAM-mode detector
Angular One-Photon Interference

\[ \psi_{1l} = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} d\phi \Psi_1(\phi)e^{-il\phi} = \frac{\alpha}{\sqrt{2\pi}} \text{sinc} \left( \frac{l\alpha}{2} \right) \]

\[ \psi_{2l} = \frac{\alpha}{\sqrt{2\pi}} \text{sinc} \left( \frac{l\alpha}{2} \right) e^{-il\beta} \]


Jha, Jack, Yao, Leach, Boyd, Buller, Barnett, Franke-Arnold, Padgett, PRA \textbf{78}, 043810 (2008)
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\[ \psi_{2l} = \frac{\alpha}{\sqrt{2\pi}} \text{sinc} \left( \frac{l\alpha}{2} \right) e^{-il\beta} \]

OAM-mode distribution:

\[ I_A = C \frac{\alpha^2}{\pi} \text{sinc}^2 \left( \frac{l\alpha}{2} \right) \left[ 1 + \cos \left( \frac{l\beta}{2} \right) \right] \]


Jha, Jack, Yao, Leach, Boyd, Buller, Barnett, Franke-Arnold, Padgett, PRA 78, 043810 (2008)
Angular Two-Photon Interference

\[ \alpha = \pi/10 \]
\[ \beta = \pi/4 \]

State of the two photons produced by PDC:

\[ |\psi_{tp}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s - l\rangle_i \]
Angular Two-Photon Interference

\[ \alpha = \pi/10 \]
\[ \beta = \pi/4 \]

State of the two photons produced by PDC:

\[ |\psi_{tp}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s - l\rangle_i \]

State of the two photons after the aperture:

\[ \rho_{\text{qubit}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix} \]
\[ \rho_{14} = \rho_{41}^* = \sqrt{\rho_{11}\rho_{44}} \mu e^{i\theta} \]
\[ \rho_{11} + \rho_{44} = 1 \]
Angular Two-Photon Interference

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$$\rho_{11} + \rho_{44} = 1$$

Coincidence count rate:

$$R_{si} = \frac{A^2\alpha^4}{4\pi^2} \left| \sum_l c_l \text{sinc} \left[ (l_s - l) \frac{\alpha}{2} \right] \text{sinc} \left[ (l_i + l) \frac{\alpha}{2} \right] \right|^2$$

$$\times \left\{ 1 + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos [(l_s + l_i)\beta + \theta] \right\}$$

Visibility:

$$V = 2\sqrt{\rho_{11}\rho_{44}} \mu$$
Angular Two-Photon Interference

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\times \left\{ 1 + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos [(l_s + l_i)\beta + \theta] \right\}
\]

Visibility:

\[
V = 2\sqrt{\rho_{11}\rho_{44}} \mu
\]

Concurrence of the two-qubit state:

\[
C(\rho_{\text{qubit}}) = 2|\rho_{14}| = 2\sqrt{\rho_{11}\rho_{44}} \mu = V
\]

Angular Two-Photon Interference

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\times \left[ 1 + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos \left( (l_s + l_i)\beta + \theta \right) \right]
\]

Visibility:

\[ V = 2\sqrt{\rho_{11}\rho_{44}} \mu \]

Concurrence of the two-qubit state:

\[ C(\rho_{\text{qubit}}) = 2|\rho_{14}| = 2\sqrt{\rho_{11}\rho_{44}} \mu = V \]

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$$|\psi_{\text{tp}}\rangle = \sum_{l=-\infty}^{\infty} c_l |l\rangle_s - |l\rangle_i$$

State of the two photons after the aperture:

$$\rho_{\text{qubit}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix} = \sqrt{\rho_{11}\rho_{44}} \mu e^{i\theta}$$

$$\rho_{11} + \rho_{44} = 1$$

Coincidence count rate:

$$R_{si} = \frac{A^2\alpha^4}{4\pi^2} \left| \sum_{l} c_l \text{sinc} \left[ \frac{(l_s - l)\alpha}{2} \right] \text{sinc} \left[ \frac{(l_i + l)\alpha}{2} \right] \right|^2 \times \{1 + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos [(l_s + l_i)\beta + \theta]\}$$

Visibility:

$$V = 2\sqrt{\rho_{11}\rho_{44}} \mu$$

Concurrence of the two-qubit state:

$$C(\rho_{\text{qubit}}) = 2|\rho_{14}| = 2\sqrt{\rho_{11}\rho_{44}} \mu = V$$

Angular Two-Photon Interference

\[ \alpha = \pi / 10 \]
\[ \beta = \pi / 4 \]

Coincidence count rate:

\[ R_{si} = \frac{A^2 \alpha^4}{4\pi^2} \sum_l c_l \text{sinc} \left[ \left( l_s - l \right) \frac{\alpha}{2} \right] \text{sinc} \left[ \left( l_i + l \right) \frac{\alpha}{2} \right] \cdot \left\{ 1 + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos \left( \left( l_s + l_i \right) \beta + \theta \right) \right\} \]

Visibility:

\[ V = 2\sqrt{\rho_{11}\rho_{44}} \mu \]

Concurrence of the two-qubit state:

\[ C(\rho_{\text{qubit}}) = 2|\rho_{14}| = 2\sqrt{\rho_{11}\rho_{44}} \mu = V \]

Angular Two-Photon Interference

\[ \alpha = \pi/10 \]
\[ \beta = \pi/4 \]

Pump

\[ l=0 \]

PDC

\[ l_s \]

\[ l_i \]

\[ D_s \]

\[ D_i \]

OAM-mode order of signal and idler photons \((l_s, l_i)\)

\[
R_{si} = \frac{A^2 \alpha^4}{4\pi^2} \sum_l c_l \sin \left[ \left( l_s - l \right) \frac{\alpha}{2} \right] \sin \left[ \left( l_s + l_i \right) \frac{\alpha}{2} \right]^2 \times \left\{ 1 + 2\sqrt{\rho_{11}\rho_{44}} \mu \cos \left[ (l_s + l_i)\beta + \theta \right] \right\}
\]

Coincidence count rate:

\[ C(\rho_{\text{qubit}}) = 0.963 \]

Visibility:

\[ V = 2\sqrt{\rho_{11}\rho_{44}} \mu \]

Concurrence of the two-qubit state:

\[ C(\rho_{\text{qubit}}) = 2|\rho_{14}| = 2\sqrt{\rho_{11}\rho_{44}} \mu = V \]

Angular Two-Photon Interference

\[ \alpha = \pi / 10 \]
\[ \beta = \pi / 4 \]

\[ \text{Pump} \]
\[ l=0 \]
\[ \text{PDC} \]

\[ \text{Coincidence counts in 5 sec} \]

OAM-mode order of signal and idler photons \((l_i, l_f)\)

\[ \rho_{\text{qubit}} = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & 0 & 0 \\ 0 & 0 & \rho_{33} & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix} \]
\[ \rho_{14} = \rho_{41}^* \]
\[ = \sqrt{\rho_{11}\rho_{44}} \mu e^{i\theta} \]
\[ \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1 \]

Concurrence of the two-qubit state:

\[ C^{(c)}(\rho_{\text{qubit}}) = V^{(c)} - \sqrt{\rho_{22}\rho_{33}} \]

Angular Two-Photon Interference

\[ \alpha = \pi/10 \]
\[ \beta = \pi/4 \]

\[
\begin{align*}
\rho_{\text{qubit}} &= \begin{pmatrix}
\rho_{11} & 0 & 0 & \rho_{14} \\
0 & \rho_{22} & 0 & 0 \\
0 & 0 & \rho_{33} & 0 \\
\rho_{14} & 0 & 0 & \rho_{44}
\end{pmatrix} \\
\rho_{14} &= \rho_{41}^* \\
\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} &= 1
\end{align*}
\]

Concurrence of the two-qubit state:

\[ C^{(c)}(\rho_{\text{qubit}}) = V^{(c)} - \sqrt{\rho_{22}\rho_{33}} \]

Summary and Conclusions

1. Temporal two-photon interference

   (i) Presented a description of temporal two-photon coherence in terms of $\Delta L$ and $\Delta L'$

   (ii) Showed that time-energy entanglement can also be explored using geometric phase

2. Spatial two-photon interference

   (i) The spatial coherence properties of the pump beam get entirely transferred to the spatial coherence properties of the entangled two-photon field.

   (ii) The entanglement of spatial two-qubit state is equal to the degree of spatial two-photon coherence

3. Angular two-photon interference

   (i) Verified angular Fourier relationship using entangled photons

   (ii) Studied angular two-photon interference effects and demonstrated an angular two-qubit state
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- Prof. Steve Barnett
- Dr. Cliff Chan, Malcolm O’Sullivan, Mehul Malik
Angular One-Photon Interference

\[ l=0 \]

\[ \begin{array}{c}
\text{s}_2 \\
\alpha \\
\beta \\
\text{s}_1
\end{array} \]

\[ l \]

\[ D_A \]

OAM-mode detector

\[ \begin{array}{c}
\text{Computer generated hologram (}l=-1)\end{array} \]

\[ \begin{array}{c}
\text{LG}_0 \\
\text{LG}_1 \\
\text{LG}_2
\end{array} \]

Single mode fiber
Angular One-Photon Interference

$l=0$

$s_2$ $\alpha$ $\beta$ $s_1$

$D_A$

OAM-mode detector

Computer generated hologram ($l = -2$)

Single mode fiber
Two-Photon Interference: A two-photon interferes with itself

Necessary conditions for two-photon interference:

\[ R_{si} = C[1 + \gamma' (\Delta L') \gamma(\Delta L) \cos(k_0 \Delta L + k_d \Delta L' + \Delta \phi)] \]

\[ \Delta L \approx \frac{l_p}{\text{coh}} \sim 10 \text{ cm} \]

\[ \Delta L' < \frac{l_{coh}}{\Delta \omega} \sim 100 \mu\text{m} \]

\[ k_d = (k_{s0} - k_{i0})/2 \]

Quantum Entanglement

Led to many foundational work in Quantum Mechanics

• EPR paradox and non-locality
• Hidden variable theories
  J. S. Bell, Physics 1, 195 (1964)
• Bell inequalities
  D. Bohm, Phys. Rev. 85, 166 (1952)

Has applications in Quantum Computation and Quantum Information

• Quantum cryptography
• Quantum dense coding
  C. H. Bennett et al., PRL 69, 2881 (1992)
• Quantum lithography
  A. N. Boto et al., PRL 85, 2733 (2000)

Entanglement can exist between Photons, Atoms, Ions,…

Parametric down-conversion provides a source of entangled photons
Outline

Parametric down-conversion (PDC)

\[ \omega_p = \omega_s + \omega_i \]

**Entanglement in time and energy**

"Temporal" two-photon coherence

\[ q_p = q_s + q_i \]

**Entanglement in position and momentum**

"Spatial" two-photon coherence

\[ l_p = l_s + l_i \]

**Entanglement in angular position and angular momentum**

"Angular" two-photon coherence

Burnham and Weinberg, PRL 25, 85 (1970)

Outline

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Entanglement in angular position and angular momentum

“Angular” two-photon coherence

Burnham and Weinberg, PRL 25, 85 (1970)

Robert W. Boyd, Nonlinear Optics, 2nd ed.
Entanglement in position and momentum

Entanglement in time and energy

Entanglement in angular position and angular momentum

**Outline**

**Parametric down-conversion (PDC)**

\[ \omega_p = \omega_s + \omega_i \]

\[ q_p = q_s + q_i \]

\[ l_p = l_s + l_i \]

Burnham and Weinberg, PRL 25, 85 (1970)


\[ \chi^{(2)} \]

Pump

Signal

Idler

D_s

Coincidence counting

D_i

Emission of two photons from the same source; they are correlated in various physical properties.

- **Angular** two-photon coherence
- **Spatial** two-photon coherence
- **Temporal** two-photon coherence
- **Angular** two-photon coherence