Pentagonal and trigonal quasilattices and their approximants

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Abstract. The Strip Projection Method is used to generate pentagonal and trigonal 2-dimensional quasilattices as rational approximants to the icosahedral quasilattice. Further, 3-dimensionally periodic structures are generated as rational approximants to these 2-dimensional quasilattices. A simplified approach based on uniform distortions of perpendicular space is put forth for the analyses.

1. Introduction

The Strip Projection Method has proved to be a powerful method to generate quasilattices and their approximants (Kramer and Neri, 1984; Duneau and Katz, 1985; Bak, 1986, Elser, 1986; Katz and Duneau, 1986). Generation and investigation of various types of 3-D quasilattices have been limited essentially to icosahedral quasilattices (IQL) (Conway and Knowles, 1986; Kupke and Trebin, 1993) and hitherto no detailed studies of their approximant lattices have been done.

In the current work approximations to $\tau(= (1+\sqrt{5})/2)$ are made in the perpendicular space ($E_\perp$) (Ishii, 1991), thus retaining the original 'fat' and 'thin' rhombohedrons to tile 3-dimensional space ($E_\parallel$). The approximants generated here are thus *QC approximants.*

In this investigation, for generation of quasiperiodic and rational approximant tilings, the software developed by Ramakrishnan (1999) is used. The program is based on the modified version of the algorithm by Vogg and Ryder (1996) and uses the powerful matrix equation solution method of Lord, Sen and Venkaiah (1990).

In section 2 we generate a quasilattice with pentagonal symmetry and derive its approximants. Section 3 is devoted to the trigonal quasilattice (TQL) and its approximants. The periods are calculated by the methods introduced by Ishii (1991).

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2. Pentagonal quasilattice (PQL) as rational approximant to the icosahedral lattice

2.1 Matrix formulation

For the icosahedral quasilattice (IQL) the orthogonal 6×6 matrix that generates the hypercubic lattice in 6D can be split into a 3×6 matrix **A** (corresponding to the projection on $E_\parallel$) and a 3×6 matrix **B** (corresponding to the projection on $E_\perp$):

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & \tau & 0 & \tau & -1 \\ \tau & 1 & 0 & -1 & 0 & \tau \\ 0 & \tau & 1 & \tau & -1 & 0 \end{bmatrix}, \quad (1)$$

$$\mathbf{B} = \begin{bmatrix} -\tau & 0 & 1 & 0 & 1 & \tau \\ 1 & -\tau & 0 & \tau & 0 & 1 \\ 0 & 1 & -\tau & 1 & \tau & 0 \end{bmatrix}. \quad (2)$$

Where, $\tau = (1+\sqrt{5})/2$ is the Golden mean (number).

In the current form, the six columns in the **B** matrix refer to the six fivefold axes of an icosahedron and approximation of τ by p/q will produce a cubic approximant (the term approximant and rational approximant are interchangeably used in this work). It is to be noted at the outset that an overall factor in any row of the **B** matrix will not affect the pattern produced in $E_\parallel$ by the strip projection method, since any such factor corresponds to a scaling of the coordinate axis in $E_\perp$. Accordingly, it is permissible and convenient to work with an integer matrix **B**. In the work of Ishii (1991) this point was not recognized. The six columns in the **A** matrix refer to the six fivefold axes of an icosahedron in $E_\parallel$; these remain unchanged, so that the rhombohedral units of the approximants will be the same as those for the IQL.

2.2 Construction of a quasilattice with pentagonal symmetry

To construct a quasilattice with pentagonal symmetry the 2-fold axes are chosen perpendicular to $\mathbf{a}_1$ ($\mathbf{a}_i$ and $\mathbf{b}_i$ refer to the column vectors of the **A** and **B** matrices respectively):

$$\mathbf{a}_2 + \mathbf{a}_4,\ \mathbf{a}_3 - \mathbf{a}_6,\ -\mathbf{a}_2 + \mathbf{a}_5,\ -\mathbf{a}_3 - \mathbf{a}_4 \text{ and } -\mathbf{a}_5 + \mathbf{a}_6 .$$

Three orthogonal vectors can be chosen according to one of the following two alternatives:

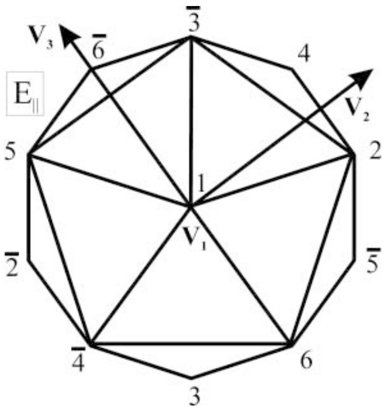

Fig. 1. Choice of basis vectors ($\mathbf{V}_i$) in $E_\parallel$ for the pentagonal quasilattice and its approximants. The numbers refer to the vertex vectors. This corresponds to scheme II described in the text.

Scheme I:
$\mathbf{a}_1, \mathbf{a}_4+\mathbf{a}_2, (\mathbf{a}_3+\mathbf{a}_6)-(-\mathbf{a}_5+\mathbf{a}_6)=\mathbf{a}_3+\mathbf{a}_5-2\mathbf{a}_6$
Scheme II:
$\mathbf{a}_1, \mathbf{a}_4+\mathbf{a}_2, (\mathbf{a}_5-\mathbf{a}_2)-(-\mathbf{a}_3-\mathbf{a}_4)=-\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5$.
The choice of mutually orthogonal basis vectors according scheme II is shown in Fig. 1.

The transformation matrices ($\mathbf{T}$ or $\mathbf{T}^{-1}$) which act upon $\mathbf{B}$ for the two schemes are:

$$[\mathbf{b}_1 \quad \mathbf{b}_4+\mathbf{b}_2 \quad \mathbf{b}_3+\mathbf{b}_5-2\mathbf{b}_6]=\mathbf{T}_{\mathrm{I}}=\begin{bmatrix}-\tau & 0 & -2\tau^{-1}\\ 1 & 0 & -2\\ 0 & 2 & 0\end{bmatrix},$$

$$[\mathbf{b}_1 \quad \mathbf{b}_4+\mathbf{b}_2 \quad -\mathbf{b}_2+\mathbf{b}_3+\mathbf{b}_4+\mathbf{b}_5]=\mathbf{T}_{\mathrm{II}}=\begin{bmatrix}-\tau & 0 & 2\\ 1 & 0 & 2\tau\\ 0 & 2 & 0\end{bmatrix}.$$

Eliminating overall factors from the rows of the inverses,

$$\mathbf{T}_{\mathrm{I}}^{-1}=\begin{bmatrix}\tau & -1 & 0\\ 0 & 0 & 1\\ 1 & \tau & 0\end{bmatrix} \quad\text{and}\quad \mathbf{T}_{\mathrm{II}}^{-1}=\begin{bmatrix}\tau & 1 & 0\\ 0 & 0 & 1\\ 1 & \tau & 0\end{bmatrix}. \tag{3}$$

Giving,

$$\mathbf{T}^{-1}\mathbf{B}_{\mathrm{I}}=\mathbf{B}'_{\mathrm{I}}=\begin{bmatrix}-\sqrt{5} & 1 & 1 & -1 & 1 & 1\\ 0 & 1 & -\tau & 1 & \tau & 0\\ 0 & -\tau & \tau^{-1} & \tau & \tau^{-1} & 2\end{bmatrix}$$

$$\text{and } \mathbf{T}^{-1}\mathbf{B}_{\mathrm{II}}=\mathbf{B}'_{\mathrm{II}}=\begin{bmatrix}-1 & -1 & 1 & 1 & 1 & \sqrt{5}\\ 0 & 1 & -\tau & 1 & \tau & 0\\ 0 & -\tau & \tau^{-1} & \tau & \tau^{-1} & 2\end{bmatrix}.$$

The pentagonal quasilattice is periodic along $\mathbf{b}_1$ and hence, replacing $\sqrt{5}$ by a rational p/q (the details of the approximation scheme are worked out in appendix I) but leaving the second and third row in $\mathbf{B}$ unchanged, we get:

$$\mathbf{B}'_{\mathrm{I}}[p/q,\,\tau,\,\tau]=\begin{bmatrix}-p & q & q & -q & q & q\\ 0 & 1 & -\tau & 1 & \tau & 0\\ 0 & -\tau & \tau^{-1} & \tau & \tau^{-1} & 2\end{bmatrix} \quad\text{and}$$

$$\mathbf{B}'_{\mathrm{II}}[p/q,\,\tau,\,\tau]=\begin{bmatrix}-q & -q & q & q & q & p\\ 0 & 1 & -\tau & 1 & \tau & 0\\ 0 & -\tau & \tau^{-1} & \tau & \tau^{-1} & 2\end{bmatrix}. \tag{4}$$

In scheme I:

$$\mathbf{b}_2-\mathbf{b}_4=\begin{bmatrix}2q\\0\\-2\tau\end{bmatrix},\quad \mathbf{b}_3+\mathbf{b}_5=\begin{bmatrix}2q\\0\\2\tau^{-1}\end{bmatrix},\quad \mathbf{b}_6=\begin{bmatrix}q\\0\\2\end{bmatrix}.$$

Therefore,

$$(\mathbf{b}_2-\mathbf{b}_4)+(\mathbf{b}_3+\mathbf{b}_5)+\mathbf{b}_6=\begin{bmatrix}5q\\0\\0\end{bmatrix}=-\frac{5q}{p}\,\mathbf{b}_1.$$

Hence,

$$5q\mathbf{b}_1+p(\mathbf{b}_2+\mathbf{b}_3-\mathbf{b}_4+\mathbf{b}_5+\mathbf{b}_6)=0.$$

If p not divisible by 5, the period is $[5q\mathbf{a}_1+p(\mathbf{a}_2-\mathbf{a}_4+\mathbf{a}_3+\mathbf{a}_5+\mathbf{a}_6)]$

$$\Rightarrow \text{period (I)}=[(5q+p\sqrt{5})\,\mathbf{a}_1]. \tag{5}$$

If p is divisible by 5, the period is:

$$\text{period (I)}=[(q+p/\sqrt{5})\,\mathbf{a}_1]. \tag{6}$$

A three-dimensionally periodic structure is obtained by approximating the surds in the second and third rows of equation 4 with p_2/q_2 and p_3/q_3 respectively; i.e., using $\mathbf{B}'_{\mathrm{I}}$ $[p/q, p_2/q_2, p_3/q_3]$.

To determine the period along the $\mathbf{V}_2$ direction (i.e., along $\mathbf{a}_2+\mathbf{a}_4$), using scheme I we get:

$$p_2(\mathbf{b}_2+\mathbf{b}_4)+q_2(\mathbf{b}_3-\mathbf{b}_5)=0.$$

Hence, the period along $\mathbf{V}_2$ is:

$$p_2(\mathbf{a}_2+\mathbf{a}_4)+q_2(\mathbf{a}_3-\mathbf{a}_5)=(p_2+\tau^1 q_2)\,(\mathbf{a}_2+\mathbf{a}_4). \tag{7}$$

For the period along $\mathbf{V}_3=\mathbf{a}_3+\mathbf{a}_5-2\mathbf{a}_6=\tau(-\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5)$, τ and τ^1 the third row of $\mathbf{B}'_{\mathrm{I}}$ are approximated with p_3/q_3 and p_3/q_3-1 respectively, to get:

$$2(2p_3-q_3)\,(\mathbf{b}_3+\mathbf{b}_5-2\mathbf{b}_6)+5q_3(-\mathbf{b}_2+\mathbf{b}_3+\mathbf{b}_4+\mathbf{b}_5)=0.$$

Hence, the period along $\mathbf{V}_3$ is:

$$2(2p_3-q_3)\,(\mathbf{a}_3+\mathbf{a}_5-2\mathbf{a}_6)+5q_3(-\mathbf{a}_2+\mathbf{a}_3+\mathbf{a}_4+\mathbf{a}_5)$$
$$=[4p_3+(5\tau-7)\,q_3]\,(\mathbf{a}_3+\mathbf{a}_5-2\mathbf{a}_6). \tag{8}$$

It is to be noted that the period is halved if q_3 is even.

For scheme II:

$$\mathbf{b}_2-\mathbf{b}_4=\begin{bmatrix}-2q\\0\\-2\tau\end{bmatrix},\quad \mathbf{b}_3+\mathbf{b}_5=\begin{bmatrix}2q\\0\\2\tau^{-1}\end{bmatrix},\quad \mathbf{b}_6=\begin{bmatrix}p\\0\\2\end{bmatrix}.$$

Therefore,

$$(\mathbf{b}_2-\mathbf{b}_4)+(\mathbf{b}_3+\mathbf{b}_5)+\mathbf{b}_6=\begin{bmatrix}p\\0\\0\end{bmatrix}=-\frac{p}{q}\,\mathbf{b}_1.$$

Therefore,

$$p\mathbf{b}_1+q(\mathbf{b}_2-\mathbf{b}_4+\mathbf{b}_3+\mathbf{b}_5+\mathbf{b}_6)=0$$

$$\Rightarrow \text{period (II)}=[(p+q\sqrt{5})\,\mathbf{a}_1]. \tag{9}$$

As for scheme I, periods along $\mathbf{V}_2$ and $\mathbf{V}_3$ can be derived. The pentagonal quasilattice arising from scheme II is illustrated here. The symmetry of these pentagonal quasilattices is $\bar{5}m$. Three indices in curly brackets are used to represent the rational approximant. Subscript 'P' is used

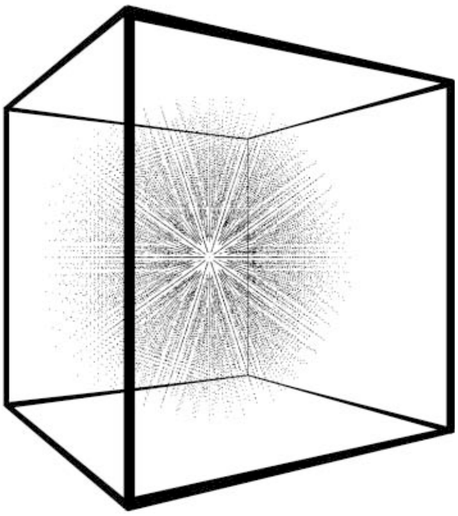

Fig. 2. The $\{^1/_1\ \tau\ \tau\}_P$ Pentagonal quasilattice showing a perspective view along the $[1\ \tau\ 0]$ direction. The quasilattice has a $\bar{5}m$ point group symmetry.

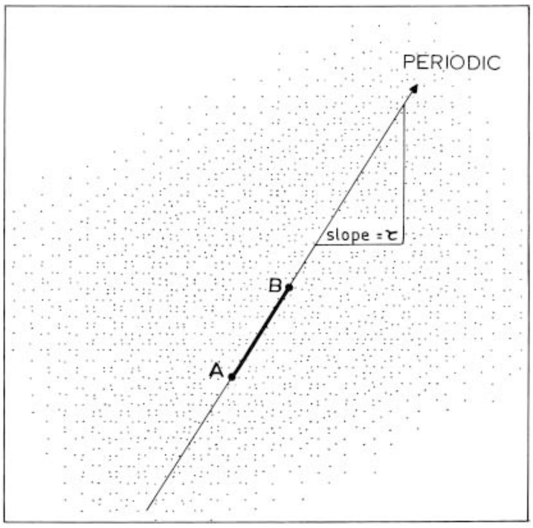

Fig. 3. The $\{^1/_1\ \tau\ \tau\}_P$ Pentagonal quasilattice showing the [0 0 1] projection with periodic direction at an inclination of τ. The structure consists of quasiperiodic planes stacked periodically.

outside the curly bracket to emphasize that the pentagonal approximant is being referred to. Figs. 2, 3 show the various aspects of the pentagonal quasilattice. The pentagonal symmetry in perspective is seen in Fig. 2. The periodic direction is inclined at an angle of $\tan^{-1}(\tau)$, when projected along [0 0 1] ($\perp$ $[1\ \tau\ 0]$) (Fig. 3). The periodicity along $[1\ \tau\ 0]$ increases with increasing order of the approximant, approaching the IQL in the limit. For the case of the $\{^1/_1\ \tau\ \tau\}_P$ and $\{^2/_1\ \tau\ \tau\}_P$ approximants, using equation 9:

$$\frac{\text{Period}\ \{2\ \tau\ \tau\}}{\text{Period}\ \{1\ \tau\ \tau\}} = \frac{2+\sqrt{5}}{1+\sqrt{5}} \approx 1.3\,.$$

2.3 Approximants to the PQL

A series of RA to the pentagonal quasilattice can be generated by approximations along the remaining two quasiperiodic basis directions. These RA include: $\{^1/_1\ ^1/_1\ \tau\}_P$, $\{^1/_1\ ^2/_1\ \tau\}_P$, $\{^2/_1\ ^2/_1\ ^2/_1\}_P$, $\{^3/_2\ ^2/_1\ ^1/_1\}_P$, etc. It is important to note that the second ($\mathbf{V}_2$) and third ($\mathbf{V}_3$) directions/indices are equivalent (but not identical); the first index has a different role and approximations are to be made keeping this in view. Fig. 4 shows the $\{^1/_1\ ^1/_1\ ^1/_1\}_P$ RA to the pentagonal quasilattice. The space group of this approximant is $C2/m$. The detailed consideration of the symmetry of the approximants will be published elsewhere. The pseudo-pentagonal symmetry is seen in the figure. Similarly, pseudo-trigonal symmetry is revealed in the [1 1 1] perspective. Unlike the cubic and orthorhombic RA to the IQC, where square or rectangular cells are observed in [0 0 1] projection; here a parallelogram unit cell is observed in the [0 0 1] projection. As an interesting exercise 'inverse' approximants like $\{\tau\ ^1/_1\ ^1/_1\}_P$ can be envisaged, where periodic planes with approximate 5-fold symmetry are stacked aperiodically.

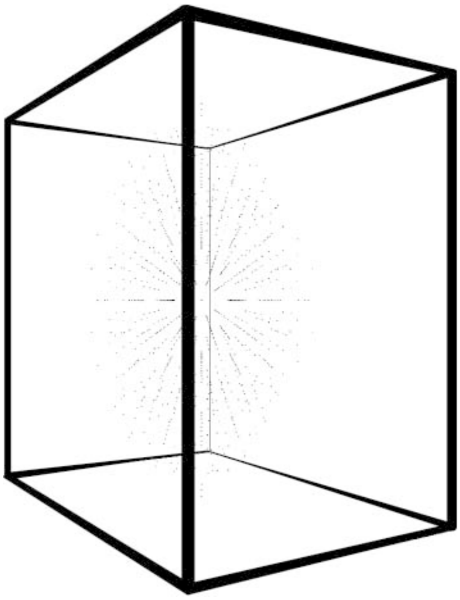

Fig. 4. The $\{^1/_1\ ^1/_1\ ^1/_1\}_P$ RA to the pentagonal quasilattice showing a perspective view along the $[1\ \tau\ 0]$ direction. Pseudo-pentagonal symmetry is seen in the approximant. The approximant has a $C2/m$ symmetry.

3. Trigonal quasilattice (TQL)

3.1 Construction of the trigonal quasilattice

First, the following identities amongst the $\mathbf{a}_i$ vectors are noted:

$$-(\mathbf{a}_4+\mathbf{a}_5+\mathbf{a}_6) = \tau^3(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)\,,$$

$$\mathbf{a}_6-\mathbf{a}_5 = \tau(\mathbf{a}_2-\mathbf{a}_3)\,,$$

$$\mathbf{a}_6-\mathbf{a}_4 = \tau(\mathbf{a}_1-\mathbf{a}_3)\,.$$

The choice of basis vectors for generation of the trigonal quasilattice and its approximants is shown in Fig. 5.

The corresponding relations amongst the $\mathbf{b}_i$ are:

$$\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3 = \begin{bmatrix}\sigma\\ \sigma\\ \sigma\end{bmatrix}, \qquad \mathbf{b}_6-\mathbf{b}_5 = \begin{bmatrix}-\sigma\\ 1\\ -\tau\end{bmatrix},$$

$$\mathbf{b}_6-\mathbf{b}_4 = \begin{bmatrix}\tau\\ -\tau^{-1}\\ -1\end{bmatrix}.$$

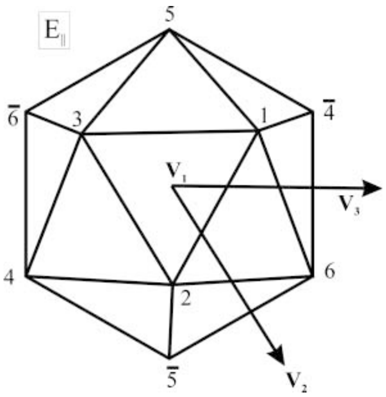

Fig. 5. Choice of basis vectors ($\mathbf{V}_i$) in $E_\parallel$ for the trigonal quasilattice and its approximants. The numbers refer to the vertex vectors.

Where, $\sigma = (1-\sqrt{5})/2 = -\tau^{1}$.

The transformation matrix $\mathbf{T}$ is written as:

$$\mathbf{T} = \begin{bmatrix} 1 & -\sigma & \tau \\ 1 & 1 & \sigma \\ 1 & -\tau & -1 \end{bmatrix}, \quad \mathbf{T}^{-1} = \frac{1}{6}\begin{bmatrix} 2 & 2 & 2 \\ -\tau^{-2} & \tau^2 & -\sqrt{5} \\ \tau^2 & -\sqrt{5} & -\tau^{-2} \end{bmatrix}. \tag{10}$$

The modified projection matrix $\mathbf{B}'$ is:

$$\mathbf{T}^{-1}\mathbf{B} = \mathbf{B}' = \begin{bmatrix} \sigma & \sigma & \sigma & \tau^2 & \tau^2 & \tau^2 \\ -\tau & 2\tau & -\tau & -1 & 2 & -1 \\ 2\tau & -\tau & -\tau & 2 & -1 & -1 \end{bmatrix}. \tag{11}$$

In general for different approximations p_i/q_i $(i = 1, 2, 3)$ along the basis directions:

$$\mathbf{B}'\left[\frac{p_1}{q_1}, \frac{p_2}{q_2}, \frac{p_3}{q_3}\right] = \begin{bmatrix} q_1-p_1 & q_1-p_1 & q_1-p_1 & p_1+q_1 & p_1+q_1 & p_1+q_1 \\ -p_2 & 2p_2 & -p_2 & -q_2 & 2q_2 & -q_2 \\ 2p_3 & -p_3 & -p_3 & 2q_3 & -q_3 & -q_3 \end{bmatrix} \tag{12}$$

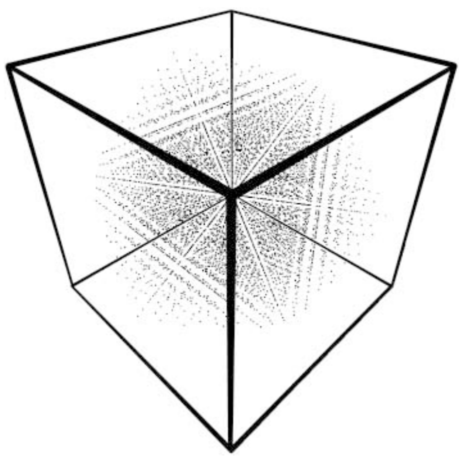

Fig. 6. The trigonal quasilattice $\{^1/_1\ \tau\ \tau\}_T$ showing the perspective along [1 1 1]. Three-fold symmetry is observed in the figure. The point group symmetry of the quasilattice is $\bar{3}m$.

Assuming $p_i/q_i = p/q$ for the present:

$$\mathbf{b}_1+\mathbf{b}_2+\mathbf{b}_3 = \begin{bmatrix} 3(q-p) \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_6-\mathbf{b}_5 = \begin{bmatrix} 0 \\ -3q \\ 0 \end{bmatrix},$$

$$\mathbf{b}_6-\mathbf{b}_4 = \begin{bmatrix} 0 \\ 0 \\ 3q \end{bmatrix}$$

$$\mathbf{b}_4+\mathbf{b}_5+\mathbf{b}_6 = \begin{bmatrix} 3(q+p) \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{b}_2-\mathbf{b}_3 = \begin{bmatrix} 0 \\ 3p \\ 0 \end{bmatrix},$$

$$\mathbf{b}_1-\mathbf{b}_3 = \begin{bmatrix} 0 \\ 0 \\ 3p \end{bmatrix}.$$

Hence, the periods are:

$$\left.\begin{aligned} &(p+q)\,(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3) + (p-q)\,(\mathbf{a}_4+\mathbf{a}_5+\mathbf{a}_6) \\ &\quad = \{p+q-\tau^3(p-q)\}\,(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3) \\ &p(\mathbf{a}_6-\mathbf{a}_5) + q(\mathbf{a}_2-\mathbf{a}_3) = (p\tau+q)\,(\mathbf{a}_2-\mathbf{a}_3) \\ &p(\mathbf{a}_6-\mathbf{a}_4) + q(\mathbf{a}_1-\mathbf{a}_3) = (p\tau+q)\,(\mathbf{a}_1-\mathbf{a}_3) \end{aligned}\right\}. \tag{13}$$

The period along the axis $(\mathbf{a}_1+\mathbf{a}_2+\mathbf{a}_3)$ is halved if $(p+q)$ and $(p-q)$ are both even, i.e.: if p and q are both odd.

The c/a ratio is given by:

$$c/a = \frac{\tau^{2n-2}}{\sqrt{3}}. \tag{14}$$

Fig. 6 shows the [1 1 1] perspective of the trigonal quasilattice, designated as $\{^1/_1\ \tau\ \tau\}_T$. This structure, as in the case of the pentagonal quasilattice, can be visualized as quasiperiodic planes stacked in a periodic fashion. The symmetry of the trigonal quasilattices is $\bar{3}m$. Again, as in the pentagonal case a series of trigonal quasilattices can be obtained with increasing periodicity along [1 1 1]; ap-

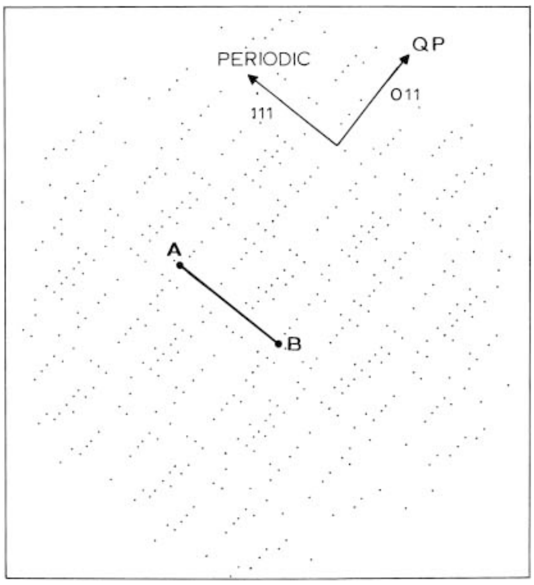

Fig. 7. Projection along [$\bar{2}$ 1 1] direction of the $\{^1/_1\ \tau\ \tau\}_T$ trigonal quasilattice with mutually perpendicular periodic and aperiodic directions. The structure consists of quasiperiodic planes stacked periodically.

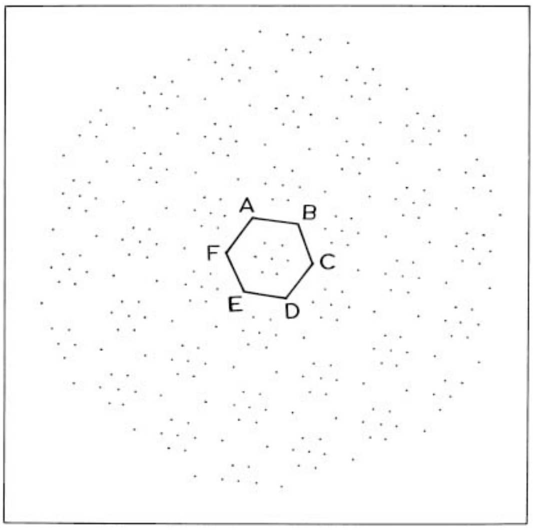

Fig. 8. The $\{^1/_1\,^1/_1\,^1/_1\}_T$ trigonal approximant showing a hexagonal cell ABCDEF in the [1 1 1] projection. The symmetry of the approximant is $P\bar{3}m1$.

proaching the icosahedral quasilattice in the limit. Fig. 7 shows the [2 1 1] projection with periodic and QP directions perpendicular to one another.

3.2 Approximants to the TQL

Various kinds of RA to the trigonal quasilattice can be envisaged with varying order of approximation along the three basis directions keeping in mind the equivalence of second and third directions/indices. These include: $\{^1/_1\,^1/_1\,\tau\}_T$, $\{^1/_1\,^2/_1\,\tau\}_T$, $\{^1/_1\,^1/_1\,^2/_1\}_T$, $\{^1/_1\,^2/_1\,^3/_2\}_T$, etc. Pseudo-trigonal symmetry is seen in the [1 1 1] perspective of the $\{^1/_1\,^1/_1\,^1/_1\}_T$ approximant. The symmetry of this approximant is $P\bar{3}m1$. Fig. 8 shows the projection along this direction. The aperiodicity observed in the $[0\,\bar{1}\,1]$ projection of TQL is lost and a rectangular cell is observed in the case of the $\{^1/_1\,^1/_1\,^1/_1\}_T$ approximant.

4. Conclusions

Uniform distortions in $E_\perp$, in the form of removal of factors from the **B** matrix, do not affect the pattern in $E_\parallel$. Quasilattices with pentagonal and trigonal symmetry have been generated using the same prototiles as those in the IQL and their approximants have been obtained. As approximations have been done in $E_\perp$ the directions of pseudosymmetries in the rational approximant lattices are the same as those of the true symmetries in the corresponding quasilattices.

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Appendix I

Approximating $\sqrt{5}$ for the Generation of the PQL

The recursion relation for the general Fibonacci sequence with multipliers m and n can be written as:

$$F_n[m,\,n] = mF_{n-1} + nF_{n-2}\,.$$

For starting terms a and b the sequence is:

$$F[m,\,n;\,a,\,b] \equiv a,\,b,\,mb + na,\,(mn)a + (m^2 + n)\,b,\,(m^2n + n^2)\,a + (m^3 + 2mn)\,b,\,(m^3n + 2mn^2)\,a + (m^4 + 3m^2n + n^2)\,b\ldots.$$

Hence, the standard Fibonacci sequence is:

$$F[1,1;\,1,\,1] \equiv 1,\,1,\,2,\,3,\,5,\,8,\,13,\,21,\,34,\,55,\ldots$$

The sequence of ratio of successive terms (R) can be written as:

$$R[1,\,1;\,1,\,1] \equiv {}^1/_1,\,{}^2/_1,\,{}^3/_2,\,{}^5/_3,\,{}^8/_5,\,{}^{13}/_8,\,{}^{21}/_{13},\,{}^{34}/_{21},\,{}^{55}/_{34},\ldots\tau\ \text{(in the limit)}\,.$$

The usual rational approximants are considered in the $R[1,\,1;\,1,\,1]$ sequence. In the present case the irrational number $\sqrt{5}$ has to be approximated.

To approximate $\sqrt{5}$ there are two basic options:

(a) Fibonacci based approximation ($R_n(\sqrt{5})$)

$R_n(\sqrt{5}) = 2R_n(\tau) - 1$. Where $R_n(\tau)$ are the approximants to the Fibonacci sequence.

(b) Continuous fraction based approximation

Coefficients of continuous fraction expansion of $\sqrt{5} \equiv \{2, 4, 4, 4, 4, 4\ldots\}$

Successive approximants $\equiv$

$$\left\{2,\,\frac{9}{4},\,\frac{38}{17},\,\frac{161}{72},\,\frac{682}{305},\,\frac{2889}{1292},\ldots\right\}.$$

The rapid convergence of the continued fraction based approximants may make it look vastly different from Fibonacci based approximants. Actually, the sequence of continued fraction based approximants is obtained by taking every third term in the sequence of Fibonacci based approximants. Also, two Fibonacci type sequences can be constructed ($F[4,\,1;\,1,\,2]$ and $F[4,\,1;\,0,\,1]$), the ratios of whose corresponding terms give the sequence of approximants based on the continued fraction.

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