ME Dept., Ph.D. Comprehensive Exam: Mathematics, 2015-16 II Semester.

(1 hr 30 mins)

Problem 1 (Linear Algebra): (20 Marks) Show that the equations

$$\omega x_1 + 3x_2 + x_3 = 5$$
$$2x_1 - x_2 + 2\omega x_3 = 3$$
$$x_1 + 4x_2 + \omega x_3 = 6$$

possess a unique solution when $\omega \neq \pm 1$, that no solution exists when $\omega = -1$, and that infinitely many solutions exist when $\omega = 1$.

Problem 2 (Statistics):

Problem 3 (Vector Analysis): (10+10 Marks) (i) Let $\mathbf{u}$ be a smooth vector field over $\Omega \in \mathbb{R}^3$. Show that $\operatorname{grad}\mathbf{u} - (\operatorname{grad}\mathbf{u})^T$ is a skew-symmetric tensor (superscript T denotes the transpose) with axial vector $\operatorname{curl}\mathbf{u}$. Hence, or otherwise, show that

$$\int_{\partial\Omega} \mathbf{n} \times \mathbf{u} \, da = \int_{\Omega} \operatorname{curl}\mathbf{u} \, dv, \tag{1}$$

where $\mathbf{n}$ the outward unit vector normal to the boundary $\partial\Omega$ of Ω.

(ii) A smooth scalar field ζ is harmonic if $\triangle\zeta = 0$, where $\triangle$ denotes the Laplacian. If ζ is harmonic, then show that (for some $\Omega \in \mathbb{R}^3$)

$$\int_{\Omega} |\nabla\zeta|^2 \, dv = \int_{\partial\Omega} \zeta(\nabla\zeta \cdot \mathbf{n}) \, da, \tag{2}$$

where $\mathbf{n}$ is the normal to $\partial\Omega$. As a consequence, prove that ζ will be constant in Ω if it is harmonic and if its normal derivative on $\partial\Omega$ vanishes.

Problem 4 (ODE): (8+6+6 Marks) (I) Show that the following 1^{st} order ODE is *exact* and solve it:

$$ye^x dx + (2y + e^x)dy = 0.$$

(II) Find the integrating factor of the following 1^{st} order ODE:

$$(y+1)dx - (x+1)dy = 0, \ (x > 0).$$

(III) Find the general solution of the following *Euler-Cauchy equation*:

$$x^2y'' - 4xy' + 6y = 0.$$

Problem 5 (PDE): (20 Marks) The *parabolic* PDE governing the temperature $u(x,t)$ is

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0, \quad (c_T = 1).$$

For a rod of length 1 unit *insulated at both the ends* and having some initial temperature distribution, the BC and IC are:

$$(i)\ \frac{\partial u}{\partial x}(0,t) = 0, \quad (ii)\ \frac{\partial u}{\partial x}(1,t) = 0, \text{ for } t \geq 0;$$

$$u(x,0) = a\cos(\pi x), \text{ for } 0 \leq x \leq 1.$$

Obtain the solution of the problem, i.e., the expression for $u(x,t)$.

Problem 6 (Numerical Methods): (10 Marks) Perform three Newton-Raphson iterations to find a root of the following non-linear algebraic equation:

$$f(x) = (4-x)e^{-x} - 2.$$

Use the initial guess of 0.5. Retain at least 5 decimal digits in all your calculations.