Ph.D. Comprehensive Exam: Mathematics, 2016-17 Ist Semester.

(100 Marks)

Problem 1 (Vector Calculus): (10+10 Marks)

(i) Consider the vector field $\mathbf{h} = \alpha R^{-2}\mathbf{e}_R$, where α is a constant, R is the distance from origin, and $\mathbf{e}_R$ is the spherical coordinate basis vector pointing out from the origin. Determine the gradient, the divergence, and the curl of this vector field.

(ii) Let $\mathbf{v}$ be a smooth vector field defined on a simply-connected region Ω of the three-dimensional Euclidean space. A region is simply-connected if any closed curve contained in it can be continuously shrunk to one of its point without leaving the region. Show that the curl of $\mathbf{v}$ is zero if and only if there exists a smooth scalar field ϕ on Ω such that $\mathbf{v} = \nabla\phi$, where ∇ represents the gradient.

Problem 2 (Linear Algebra): (10+10 Marks)

(i) Apply Gauss elimination to the following real system of equations, decide if it is solvable, and if so determine its solution set:

$$\begin{aligned} x_1 - x_2 + 2x_3 - 3x_4 &= 7 \\ 4x_1 + 3x_3 + x_4 &= 9 \\ 2x_1 - 5x_2 + x_3 &= -2 \\ 3x_1 - x_2 - x_3 + 2x_4 &= -2 \end{aligned}$$

(ii) Let V be the real vector space of twice differential functions $f : \mathbb{R} \to \mathbb{R}$ defined on the domain $[-1, 1]$ such that $f(1) = f(-1) = 0$. Determine all the eigenvalues and the eigenvectors of the second derivative

$$\frac{d^2}{dx^2} : V \to V.$$

Problem 3 (ODE): (8+8+6 Marks)

(i) The integrating factor of the following first order ODE

$$2xydx + 3x^2dy = 0, \quad (y > 0),$$

is y^2. Using this integrating factor, find the solution of the above ODE.

(ii) The general solution of the homogeneous part of the following second order non-homogeneous ODE with constant coefficients

$$y'' + 2y' - 35y = 12e^{5x} + 37\sin 5x$$

is given by

$$y_h = C_1 e^{5x} + C_2 e^{-7x}$$

where C_1 and C_2 are constants. Find the *particular* solution of the ODE.

(iii) Find the solution of the following first order ODE:

$$e^x y' = 2(x+1)y^2.$$

Problem 4 (PDE): (20 Marks)

For a string of length π units fixed at both the ends and excited only by some initial displacement, the *hyperbolic* PDE+BC+IC governing the displacement $u(x,t)$ are:

PDE: $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{\partial^2 u}{\partial t^2} = 0, \quad (c_w = 1).$

BC: $(a)\ u(0,t) = 0, \quad (b)\ u(\pi,t) = 0, \quad \text{for } t \geq 0.$

IC: $(c)\ u(x,0) = \delta(\sin x - 0.5\sin 2x), \quad (d)\ \dfrac{\partial u}{\partial t}(x,0) = 0, \quad \text{for } 0 \leq x \leq \pi.$

Obtain the solution of the problem, i.e., the expression for $u(x,t)$.

Problem 5 (Numerical Methods): (8 Marks)

In an experiment, two non-dimensional parameters (x, y) were measured at five instances. The data is as follows:

Instance i	x_i	y_i
1	0	1.8
2	1	1.6
3	2	1.1
4	3	1.5
5	4	2.3

The investigator believes that the variation of y with x is linear, i.e. $y = a + bx$. Applying the *least square method* to the above data, find the constants a and b.

Problem 6 (Statistics): (10 Marks)

If the probability of producing a defective screw is $p = 0.01$, what is the probability that a lot of 100 screws will contain more than 2 defectives? [Hint: Use Poisson distribution]