Example Problems in Mathematics

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Calculators may be used for numerical problems

February 17, 2017

More problems may be posted from time to time; keep checking at `http://home.iitk.ac.in/~anindya/handouts.htm`

1 Some more problems

1. Diagonalize the matrix
$$[A] = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$
Illustrate the transformation geometrically by investigating its action on the unit vectors and on the eigenvectors. What can you say about the eigenvalues and eigenvectors of symmetric matrices?

2. Show that the equations
$$\omega x_1 + 3x_2 + x_3 = 5 \tag{1}$$
$$2x_1 - x_2 + 2\omega x_3 = 3 \tag{2}$$
$$x_1 + 4x_2 + \omega x_3 = 6 \tag{3}$$
possess a unique solution when $\omega \neq \pm 1$, that no solution exists when $\omega = -1$, and that infinitely many solutions exist when $\omega = 1$. Also investigate the corresponding situation when the right-hand members are replaced by zeros.

3. We consider 2- and 3-dimensional Euclidean spaces as appropriate. A linear transformation maps the vectors $[1,\ 2]^T$ and $[4,\ 5]^T$ to vectors $[2,\ 0,\ 1]^T$ and $[1,\ 2,\ 0]^T$, respectively.
   (a) What are the domain and co-domain of this transformation?
   (b) Determine the null space and the range of this transformation.
   (c) Develop a matrix representation for this transformation.
   (d) Find the image of the vector $[3,\ 4]^T$ under this transformation.

4. Write the second-order ordinary differential equation
$$\frac{d^2y}{dx^2} + \frac{dy}{dx} + x = 0$$
as a collection of first-order ordinary differential equations. Choose your favorite non-trivial initial conditions, and solve the differential equation. Draw y's evolution as a function of x, as well as showing the solution's trajectory in the phase plane.

5. Solve the ordinary differential equation of the previous problem using Laplace transforms. Retain the initial conditions that you have previously employed.

6. Let $\mathbf{u}$ be a vector field which is continuously differentiable in a domain D. Show that $\operatorname{grad}\mathbf{u} - (\operatorname{grad}\mathbf{u})^T$ is a skew-symmetric tensor field on D with axial vector $\operatorname{curl}\mathbf{u}$. Hence, or otherwise, show that
$$\int_{\partial R} \mathbf{n} \times \mathbf{u}\, da = \int_R \operatorname{curl}\mathbf{u}\, dv, \tag{4}$$
where R is a regular region contained in D and $\mathbf{n}$ is the outward unit vector normal to its boundary ∂R.

7. If β is an approximation to a root α of the equation $f(x) = 0$, and if $f(\beta) = \epsilon$, show that $\alpha - \beta = -\dfrac{\epsilon}{f'(\eta)}$, where η is between β and α if $f'(x)$ is continuous.

8. Let $f(\mathbf{x})$ be a scalar function of a vector variable $\mathbf{x} \in \mathcal{R}^n$. Let $\mathbf{Q} \in \mathcal{R}^{n\times n}$ be an orthogonal matrix, such that its columns $\mathbf{q}_1, \mathbf{q}_2, \cdots, \mathbf{q}_n$ form an orthonormal basis for $\mathcal{R}^n$.

   (a) For small α, find a first order (or linear) approximation for $f(\mathbf{x} + \alpha\mathbf{q}_j) - f(\mathbf{x})$.

   (b) Hence, show that the directional derivative $\dfrac{\partial f}{\partial \mathbf{q}_j} = \mathbf{q}_j^T \nabla f(\mathbf{x})$.

   (c) Now, compose the vector resultant $\displaystyle\sum_{j=1}^{n} \frac{\partial f}{\partial \mathbf{q}_j}\mathbf{q}_j$ and show that it equals to $\nabla f(\mathbf{x})$.

9. Show that for a solution $w(x, y)$ of Laplace's equation $\nabla^2 w = 0$ in a region $\mathcal{R}$ with boundary curve C and outer normal vector $\mathbf{n}$,

$$\int\int_{\mathcal{R}} \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy = \oint_C w \frac{\partial w}{\partial n} ds\,.$$

   (You might like to consider integration by parts.)

10. Consider the ODE $y' = y^{1/4}$, $y(0) = 0$. Does a solution exist? Is it unique?

11. A function $y_0(x)$ satisfies the equation $y'' + P(x)y' + Q(x)y = 0$. Find a function $u(x)$ such that $y_1(x) = u(x)y_0(x)$ satisfies $y'' + P(x)y' + Q(x)y = y_0(x)$

12. Find the Fourier transform of the function $f(x) = e^{-x^2}$.

13. Solve

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

   for $0 \le x \le L$ and $0 < t < \infty$ with the boundary condition $T(0, t) = T(L, t) = 0$ and initial condition $T(x, 0) = 1$. Use any method you wish. Sketch T as a function of x for several choices of t.

14. A string is stretched and fastened to two points l apart. Its transverse displacement is governed by the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}\,.$$

   The string is given an initial displacement of the form $y = a\sin(\pi x/l)$. Show that the displacement of any point at a distance x from the left end at time t is given by

$$y(x, t) = a\sin(\pi x/l)\cos(\pi ct/l)\,.$$

15. Find the deflection of a vibrating string of unit length having fixed ends with initial velocity as zero and initial deflection of (a) $2(x^2 - x^3)$, and (b) $2\sin(2\pi x)$. (hint: you may like to consider the D'Alembert's solution).

16. A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find its subsequent temperature distribution as a function of time (use symbols for any physical parameters needed).

17. A semi-infinite rectangular plate is bounded by two parallel edges which are π units apart. Its short edges (the finite-length end) is maintained at a uniform temperature of u_0 at all points. The semi-finite bounding edges are maintained at zero temperature. Determine the steady state temperature at an arbitrary point in the plate.

2 Problems posted earlier (Feb 12, 2017)

1. Parametric representation of curves. What curves are represented by the following? Can you sketch them?
   - $[t,\, t^3 - 2,\, 0]$
   - $[a + 2\cos 2t,\, b - 2\sin 2t,\, 1]$
   - $[\cosh t,\, \sinh t,\, 0]$
   - $[\cos t,\, \sin t,\, 0.1t]$
2. For the above curves, at the point represented by some t, find the unit tangent vector in each case.
3. Find the inverse(s) of the following matrices.
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 2 & 3 \\ 7 & 6 & 9 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 2 \\ 3 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}.$$
4. Which of the following matrices is/are positive definite?
$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 2 & 3 \\ 2 & 6 & 9 \\ 3 & 9 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 3 & 2 \\ 3 & 4 & 1 \\ 2 & 1 & 8 \end{bmatrix}, \quad \begin{bmatrix} 10 & 3 \\ 3 & 2 \end{bmatrix}.$$
5. For what values of a, if any, is the following matrix positive definite?
$$\begin{bmatrix} 10 & -2a \\ -2a & a \end{bmatrix}.$$
6. Consider an arbitrary three-dimensional solid region in space (something without internal holes; like a potato). Let the outward normal to the surface at any point be $\hat{n}$. Then the integral over the surface of the region,
$$\iint \hat{n} \cdot (x\,\hat{\imath})\, \mathrm{d}A,$$
has what units, and what interpretation?
7. Consider the boundary value problem
$$y'' + y' + \lambda y = 2, \quad y(0) = 1,\, y(2) = 7.$$
Solve the problem for $\lambda = 10$. Are there values of λ for which solutions do not exist? It may help you to think about this problem by setting the y' term to zero for a simpler version.
8. Solve the following boundary value problems. Power law solutions for complementary function; real roots *and* complex roots. (What about repeated roots?)
$$x^2 y'' + xy' - 4y = x, \quad x(1) = 7,\, x(7) = 1.$$
$$x^2 y'' + xy' + 4y = x, \quad x(1) = 0,\, x(2) = 1.$$
9. Use the Laplace transform to solve
$$\dot{y} + 3y = t, \quad y(0) = 1.$$
10. Find the Fourier series for the function with period $T = 2$, defined by
$$f(t) = t, \quad 0 < t \leq 2.$$
11. Transverse vibrations $u(x,t)$ of a string of unit length are governed by
$$u_{tt} = 4u_{xx} \quad u(0,t) = 0,\, u(1,t) = 0,\, u(x,0) = \sin 2\pi x.$$
Find the solution $u(x,t)$.

12. An infinitely long string governed by $u_{tt} = 4u_{xx}$ is stationary at $t = 0$ with an initial displacement of

$$u(x, 0) = \sin x,\ 0 \leq x \leq \pi, \qquad \text{and} \qquad u(x, 0) = 0 \text{ otherwise.}$$

Describe subsequent motions of the string. Sketch the displaced string at a later time of your choice.

13. A unit square domain conducts heat, and the temperature therein satisfies the equation

$$\nabla^2 T = 0$$

at steady state. The lower edge of the square is maintained at 0° C, while all other edges are at 10° C. What is the temperature at the center of the square? (Compare with figure 268 in Kreyszig, 8th ed.)

14. A semi-infinite rod obeys the transient 1D heat equation. It is initially at a temperature of zero degrees everywhere. At $t = 0$, its free end is suddenly heated to a temperature $T \neq 0$, and that end temperature is maintained thereafter. Find the subsequent temperature distribution in the rod as a function of position and time, in terms of system parameters.

15. Same problem as above, but for a finite rod of length L that is initially at zero degrees; and whose left end is suddenly raised to $T \neq 0$ while the right end is maintained at zero degrees.

16. We have the data

x	0	1	2	2.2	3
y	0	2	3	3	2

Fit, in a least squares sense, the curve

$$y = a_0 + a_1 x + a_2 x^2.$$

17. We have the boundary value problem

$$y'' + xy' + y = 4 \sin x, \quad y(0) = 0,\ y(1) = 1.$$

We wish to solve this numerically using a finite-difference approximation. Using 3 uniformly spaced internal points between 0 and 1 (i.e., at $x = 0.25$, 0.5 and 0.75), set up simultaneous equations and solve them. Clearly state the approximations being made for the derivatives.

18. A random variable x has the probability density function

$$f(x) = 1 \text{ for } 0 \leq x \leq 1, \qquad \text{and} \qquad 0 \text{ otherwise.}$$

Find its mean, median, and standard deviation.

19. Same problem as above, except

$$f(x) = 2 - x \text{ for } 0 \leq x \leq 1, \qquad \text{and} \qquad 0 \text{ otherwise.}$$

20. Same problem as above, except

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} \text{ for all } x.$$

21. In the distribution of problem 19, what is the probability that x lies between 0.3 and 0.5?

22. If the probability of success in a random game is 1/3, what is the probability of getting exactly 6 successes in 10 trials? (See the binomial distribution.)

23. If the expected number of emails arriving for you in one hour (9-10 am) in the morning is 7, what is the probability of exactly 5 such emails arriving on a given morning (same time interval: 9-10 am)? (See the Poisson distribution.)

3 Problems posted still earlier (Feb 1, 2017)

1. Sketch the curve (by hand) of the following functions
$$y = x - x^3, \quad y = x - x^4,$$
and find their maximum values for $x > 0$.

2. In three dimensions, with coordinates (x, y, z), the surface given by $x^2 + y^2 + 3z^2 = 5$ is a (or an) (a) circle, (b) paraboloid, (c) sphere, (d) cylinder, (e) ellipsoid, (f) cuboid, (g) parallelogram. Find the unit outward normals to this surface at the points $(1, 1, 1)$ and $(1, 1, -1)$.

3. Find the divergence and the curl of the vector fields
$$\cos(xy)\,\hat{\imath} - \sin(z)\,\hat{\jmath} + xyz\,\hat{k},$$
$$\cos(x + y)\,\hat{\imath} - \sin(y)\,\hat{\jmath} + (xy + z)\,\hat{k}.$$

4. Find the derivative of the function $7xy + 4\cos(z) - yz^2$ at (1,2,3) along a unit vector parallel to $7\hat{\imath} - 4\hat{\jmath}$.

5. If $\dfrac{\partial f(x, y)}{\partial x} = \cos(xy)$, what is f?

6. Derive the formula for
$$\underline{A} \times (\underline{B} \times \underline{C}) = (?)\underline{B} + (?)\underline{C}.$$

7. Given that the arc length of a curve is $\displaystyle\int_{x_1}^{x_2} \sqrt{1 + y'^2}\, dx$, find the arc length of the thick portion of the curve shown in figure 1 by integration (not from prior knowledge).

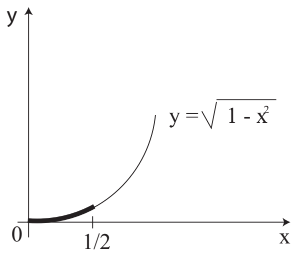

Figure 1: For question 7.

8. Integrate $x^2 + y$ over the shaded region shown in figure 2.

9. Given
$$\mathbf{F} = 3x\,\hat{\imath} + 4xy\,\hat{\jmath} + xyz\,\hat{k}$$
and the domain of figure 3, find the area integral $\displaystyle\iint (\operatorname{curl}\mathbf{F}) \cdot \hat{\mathbf{n}}\, dA$ and separately find the contour integral $\oint \mathbf{F} \cdot \mathbf{r}'(s)\, ds$, and thus verify that Stokes's theorem holds. (Is there any relation to Green's theorem?)

10. Repeat problem 9, only changing the domain to the unit circle centered at the origin.

11. For the same $\mathbf{F}$ as in problem 9, and using the unit cube centered at the origin as the domain (that is, the length of every edge of the cube is unity, and the center of the cube is at the origin), evaluate the integral of $\nabla \cdot \mathbf{F}$; and also evaluate a suitable integral on the boundary of the domain to verify that the divergence theorem holds.

12. Find the rank of each of the following matrices
$$\begin{bmatrix} 4 & 2 & 3 \\ 7 & 6 & 9 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 4 & 3 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 0 \\ -1 & 0 & -7 \end{bmatrix}.$$

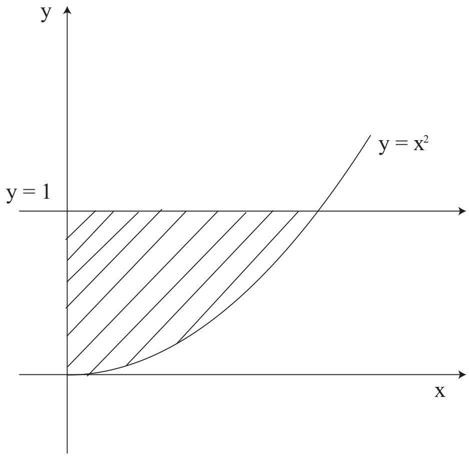

Figure 2: For question 8.

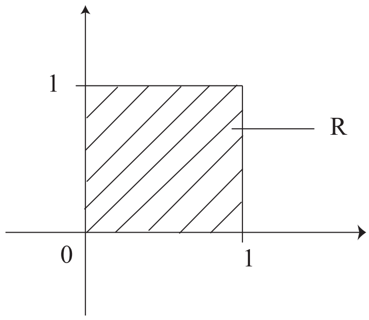

Figure 3: For question 9.

13. For
$$A = \begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 0 \\ -1 & 0 & -7 \end{bmatrix}$$
what should be the restriction on b such that $Ax = b$ has solutions? For such a b, what is the general solution?

14. Find the eigenvalues of
$$\begin{bmatrix} 4 & 3 & 2 \\ 4 & 3 & 2 \\ 4 & 3 & 2 \end{bmatrix}$$
and find its corresponding eigenvectors (are there 3 independent ones?)

15. Same as the previous question, only use the matrix
$$\begin{bmatrix} 1 & 2 & 7 \\ 0 & 2 & 4 \\ 3 & 9 & 2 \end{bmatrix}$$
and use a calculator if needed.

16. Find the eigenvalues and eigenvectors of
$$\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}, \quad \begin{bmatrix} 7 & 4 \\ 4 & 7 \end{bmatrix}.$$

17. Solve
$$\frac{dy}{dx} = \frac{\cos x}{\sin y}, \quad y(1) = 2.$$

$$\frac{dy}{dx} = \cos x \, \sin y, \quad y(1) = 2.$$

$$\frac{dy}{dx} + xy = x, \quad y(1) = 3.$$

$$y'' + 7y' + 22y = 6x, \quad y(0) = 1, \, y'(0) = 6.$$

$$y'' - 2y' + 14y = 0, \quad y(0) = 1, \, y'(0) = 0.$$

$$y'' + 4y' + 4y = 0, \quad y(0) = 1, \, y'(0) = 0.$$

18. Find the Laplace transforms of the following

    (a) $\sin 2t$, (b) $\cos^2 t$, (c) $t \sin 2t$, (d) 1 (unit step function), (e) t, (f) e^{4t}

19. We wish to numerically solve the equations

    $$x^2 + 3x + \cos(xy) - \tan(y) = 4, \quad \text{and} \quad x + x^2 + \sin(y) = 0.5.$$

    Starting from the initial guess of $(x, y) = (1{,}1.4)$, compute *one* Newton-Raphson iteration to find the next guess. (This example has 2 equations; if you had just one equation in one variable it would be easier.)

20. Numerically find, using both the trapezoidal and Simpson's rules, and using *five* (5) points in each case,

    $$\int_0^1 e^x \, dx,$$

    and compare with the exact answer. (How does accuracy relate to order of the method?)