ME-752 Project

Stationary Point Analysis

Determination of Saddle Points, Points of Maxima and Minima

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Objective

- Saddle point test for a given point
- Determination of stationary points using gradient vanishing
- Maxima/Minima/Saddle point test at detected stationary points

Definitions

- Stationary Point : A point in the domain of a function where all its partial derivatives are zero.
- Saddle Point : a point in the domian of a function that is a stationary point but not a local extremum.

Saddle Point Algorithm

- Taylor Series:

$$f(\vec{x}) = f(\vec{p}) + \vec{\nabla} f|_{\vec{p}}^{T} (\vec{x} - \vec{p}) + \frac{1}{2} (\vec{x} - \vec{p})^{T} [\nabla^2 f] (\vec{x} - \vec{p}) + \ldots$$

- First Derivative: Gradient must be zero for a stationary point
- 2^{nd} Derivative: Hessian
  - Positive Definite => Minima
  - Negative Definite => Maxima
  - Indefinite => Saddle point
  - Semi-Definite => Can be maxima, minima or saddle (Inconclusive)

Analysis for Semi-Definite Hessian

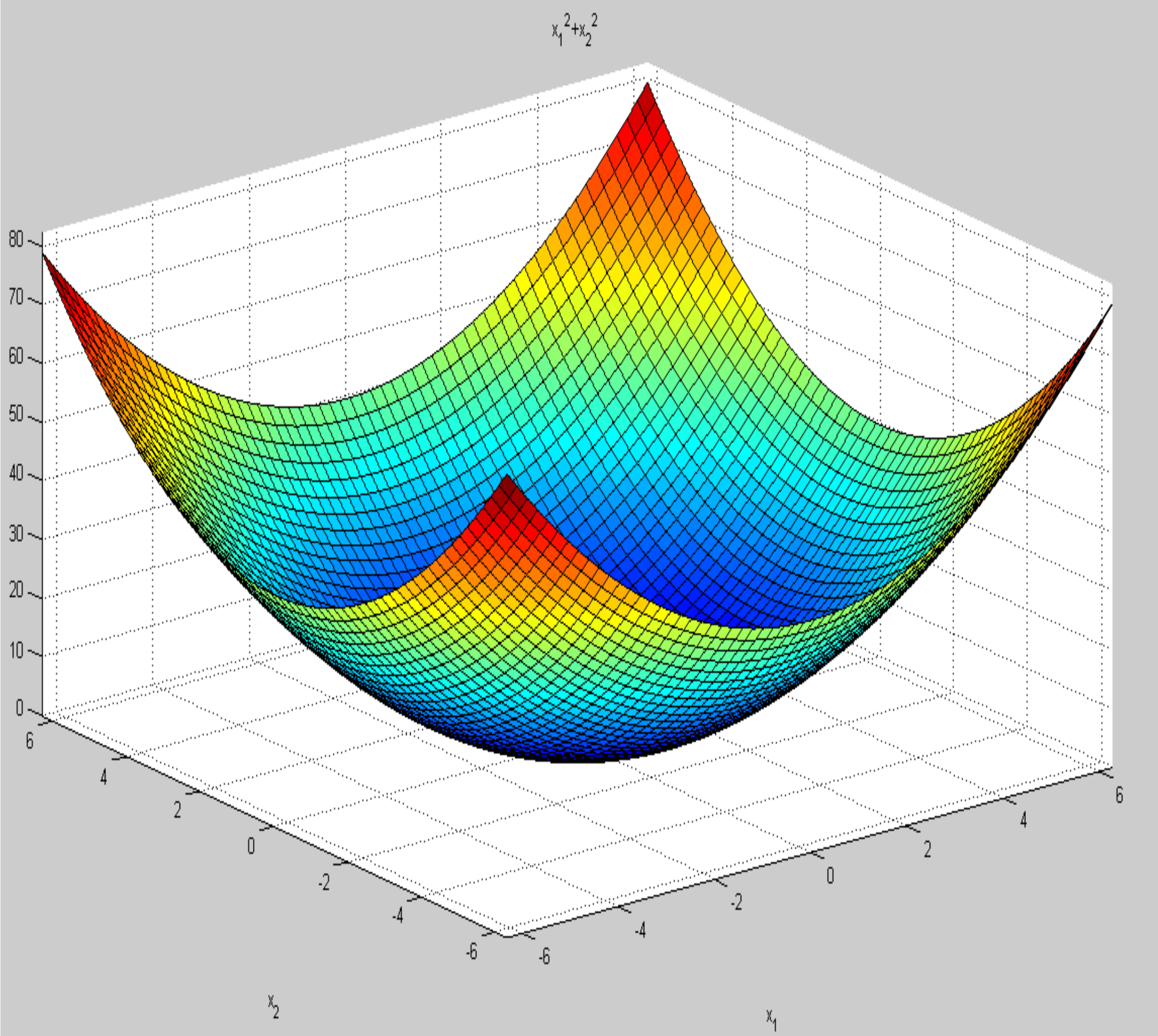
- Positive Semi-Definite -> Minima/Saddle, Negative Semi-Definite->Maxima/Saddle
- N-variable problem -> n eigenvectors of Hessian exist
- Let any m eigenvalues be zero
- Check for each of the m eigenvector corresponding to zero eigenvalue:
  - Parameterize the curve in terms of the eigenvector
  - Calculate the third derivative
    - Third Derivative = 0
      - Check for the first non-zero derivative.
        - Odd -> Saddle Point
        - Even -> Result depends on the higher order derivatives for other eigenvectors with zero eigenvalue
    - Third Derivative is not zero
      - Saddle Point
- In case all first non zero derivatives along eigenvectors with zero eigenvalues are even, the point is a maxima/minima, otherwise saddle point.

Presentation Scheme

Problem 1	Problem 5
Positive Definite Hessian (1 Minimum)	Positive Semi-Definite Hessian (with $\frac{d^3}{dx^3}\boldsymbol{\phi} \neq 0$)
Problem 2	Problem 6
Negative Definite Hessian (1 Maximum)	Positive Semi-Definite Hessian (with $\frac{d^3}{dx^3}\boldsymbol{\phi} = 0$)
Problem 3	Problem 7
Indefinite Hessian (1 Saddle Point)	Null Hessian
Problem 4	Problem 8
Multiple Stationary Points (Non – Semi Definite Hessian)	8 Variable Problem

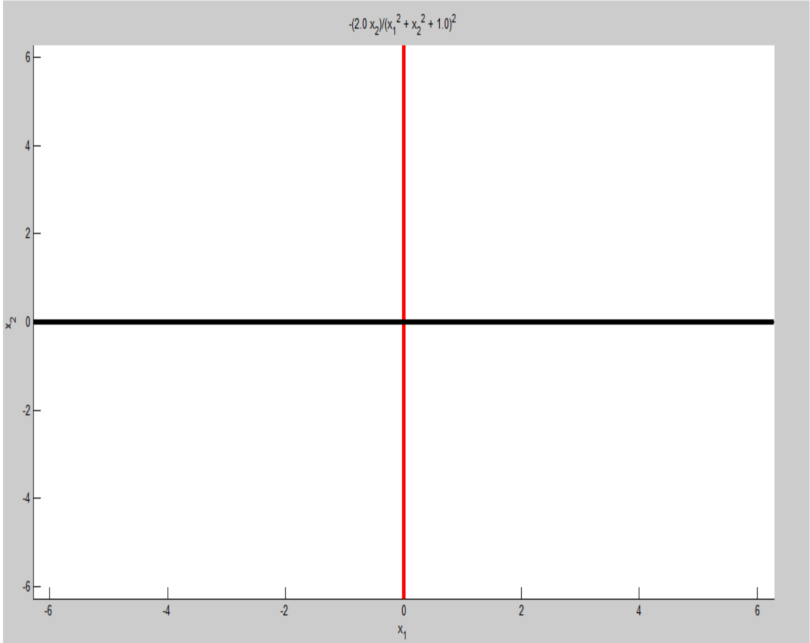
Problem 1: Single Minimum, Positive Definite Hessian

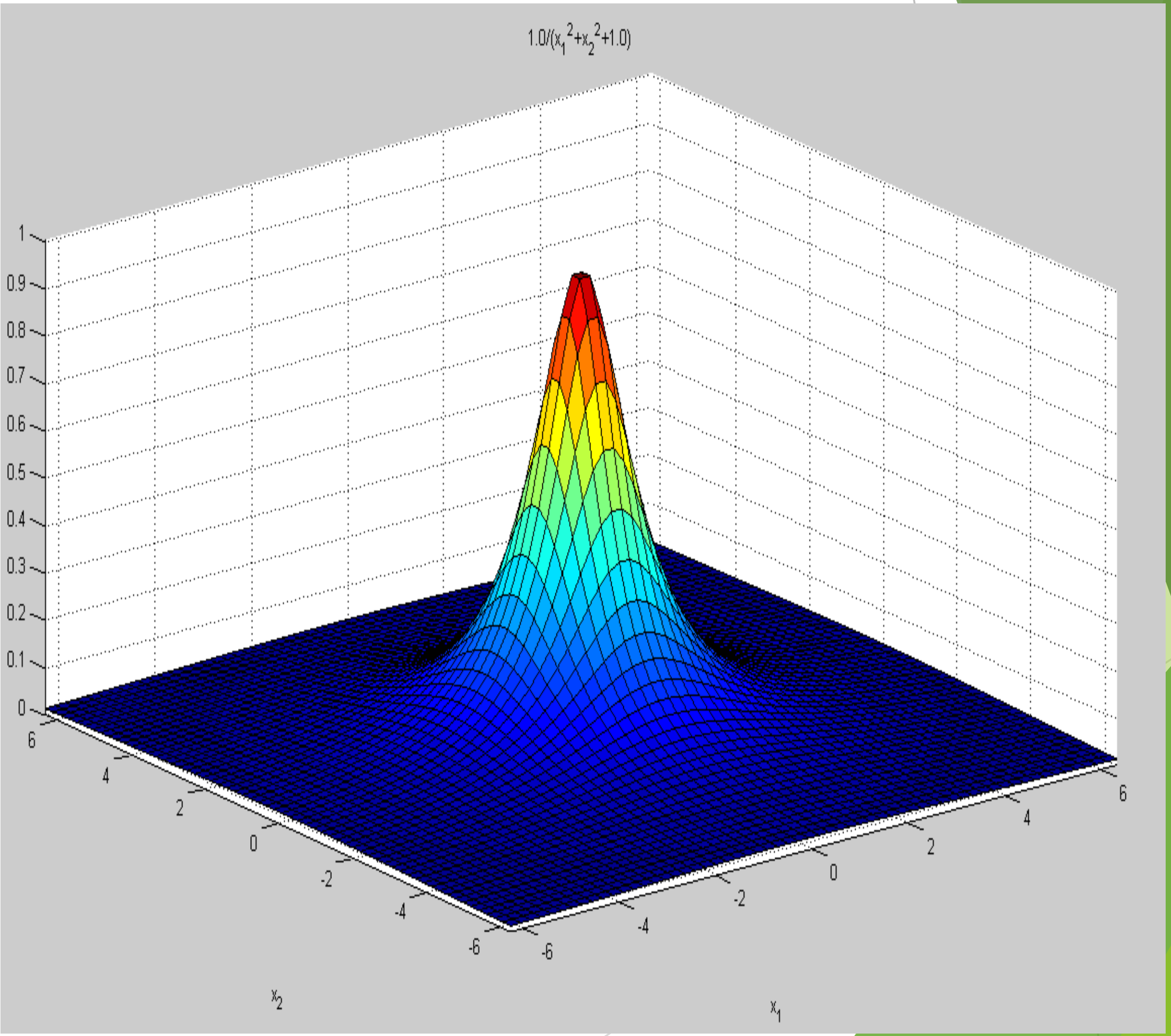
- $f(x_1, x_2) = x_1^2 + x_2^2$
- Stationary Point: (0,0)
- Hessian: $\begin{matrix} 2 & 0 \\ 0 & 2 \end{matrix}$
- Eigen Values: (2,2)



Problem 2 : Single Maxima, Negative Definite Hessian

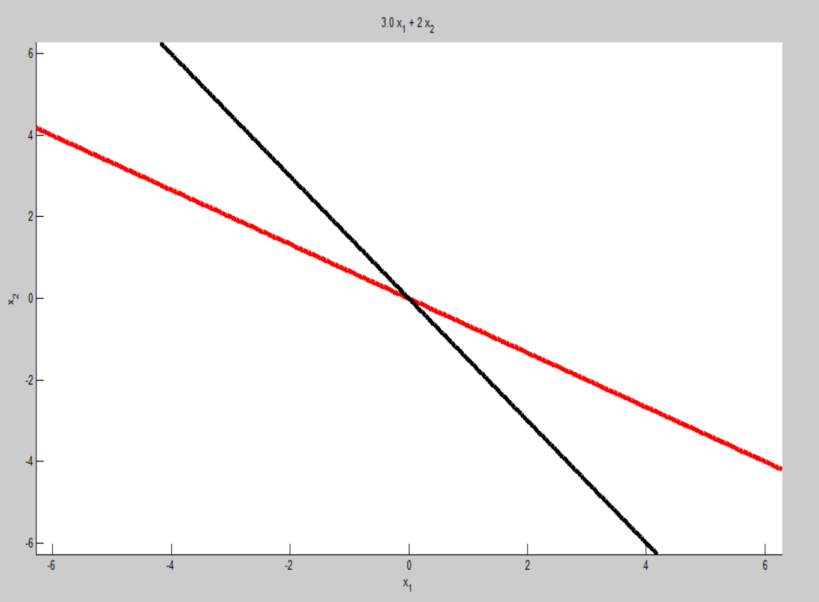
- $f(x_1, x_2) = (\frac{1}{x_1^2 + x_2^2 + 1})$
- Stationary Point: (0,0)
- Hessian: $\begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$
- Eigen Values: (-2,-2)

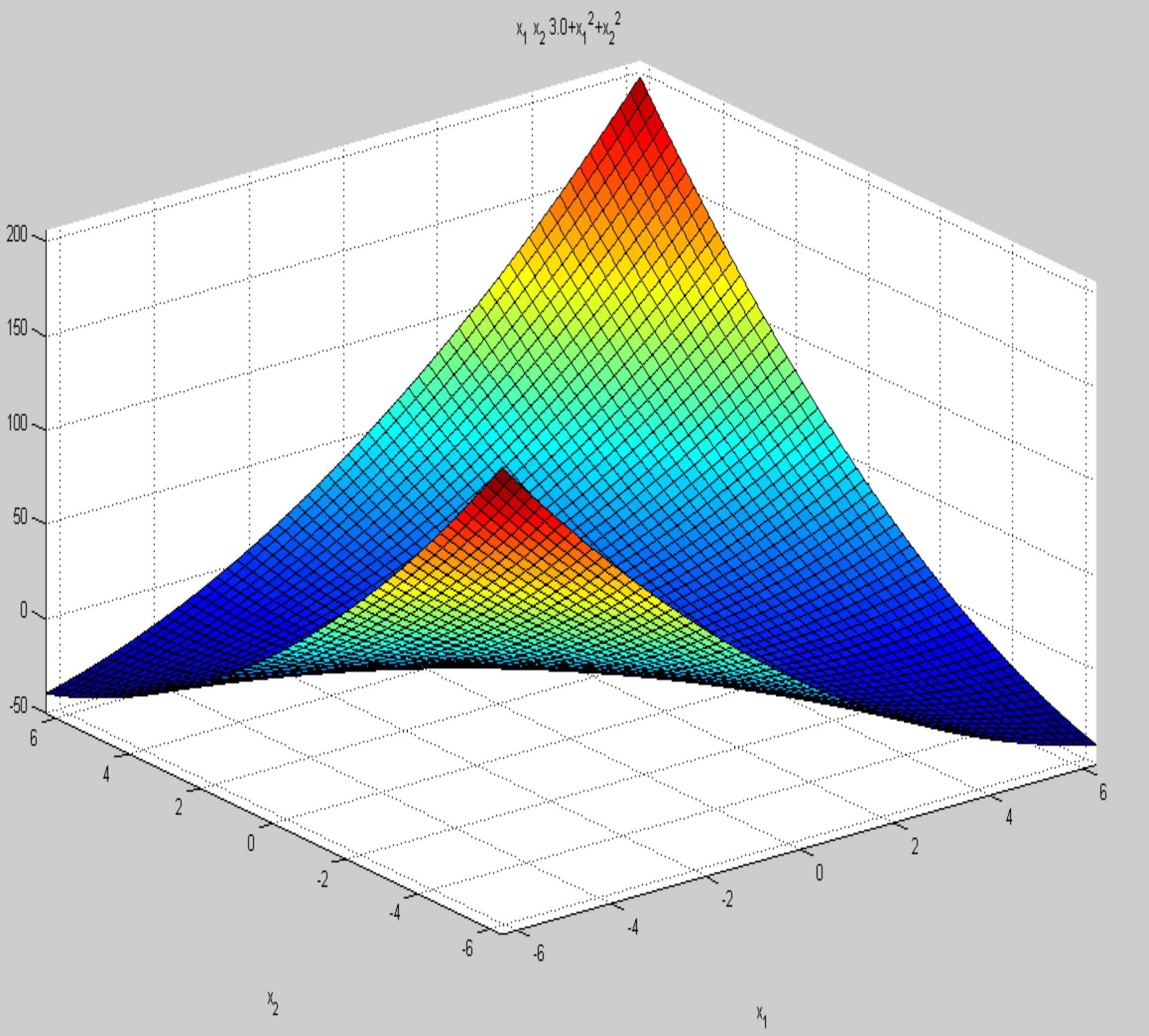




Problem 3 : Single Saddle Point, Indefinite Hessian

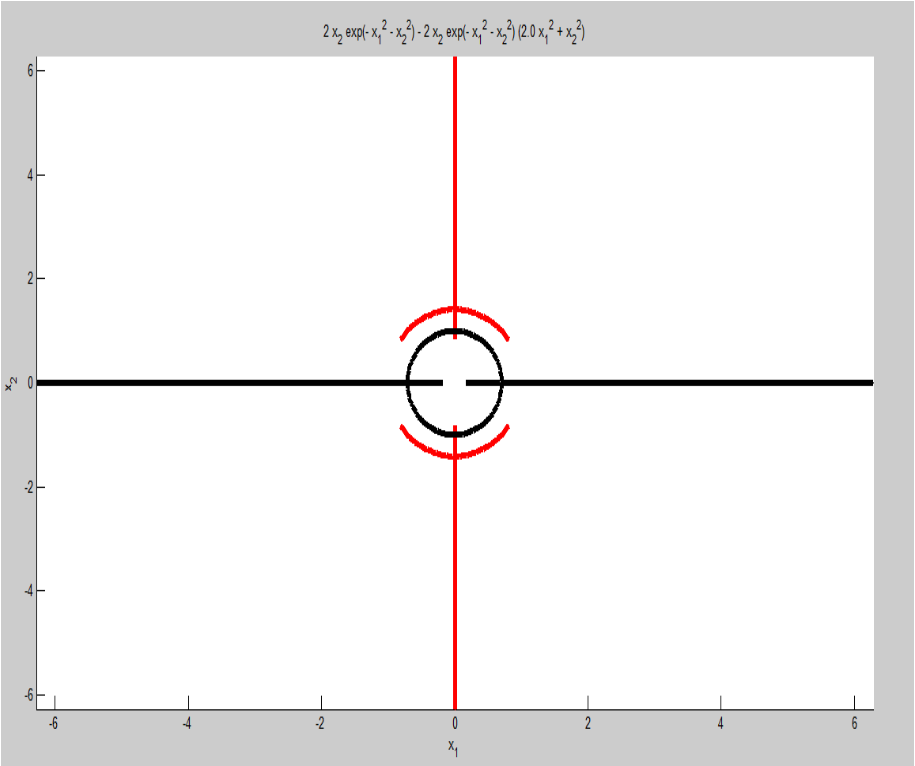
- $f(x_1, x_2) = x_1^2 + 3x_1x_2 + x_2^2$
- Stationary Point: (0,0)
- Hessian: $\begin{matrix} 2 & 3 \\ 3 & 2 \end{matrix}$
- Eigen Values: (5,-1)

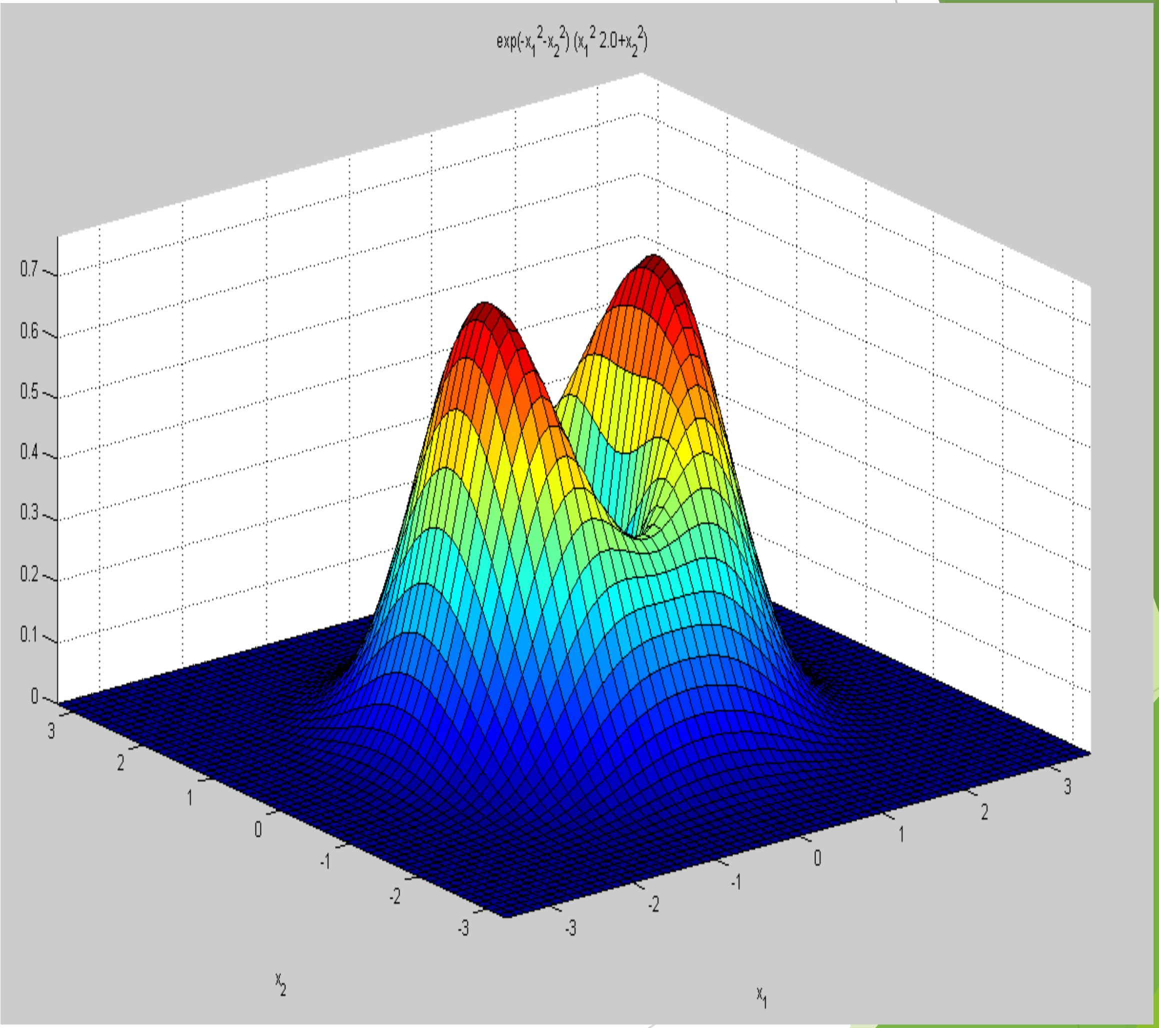




Problem 4 : Multiple minima/maxima/saddle points

- $f(x_1, x_2) = (2x_1^2 + x_2^2)e^{-(x_1^2+x_2^2)}$



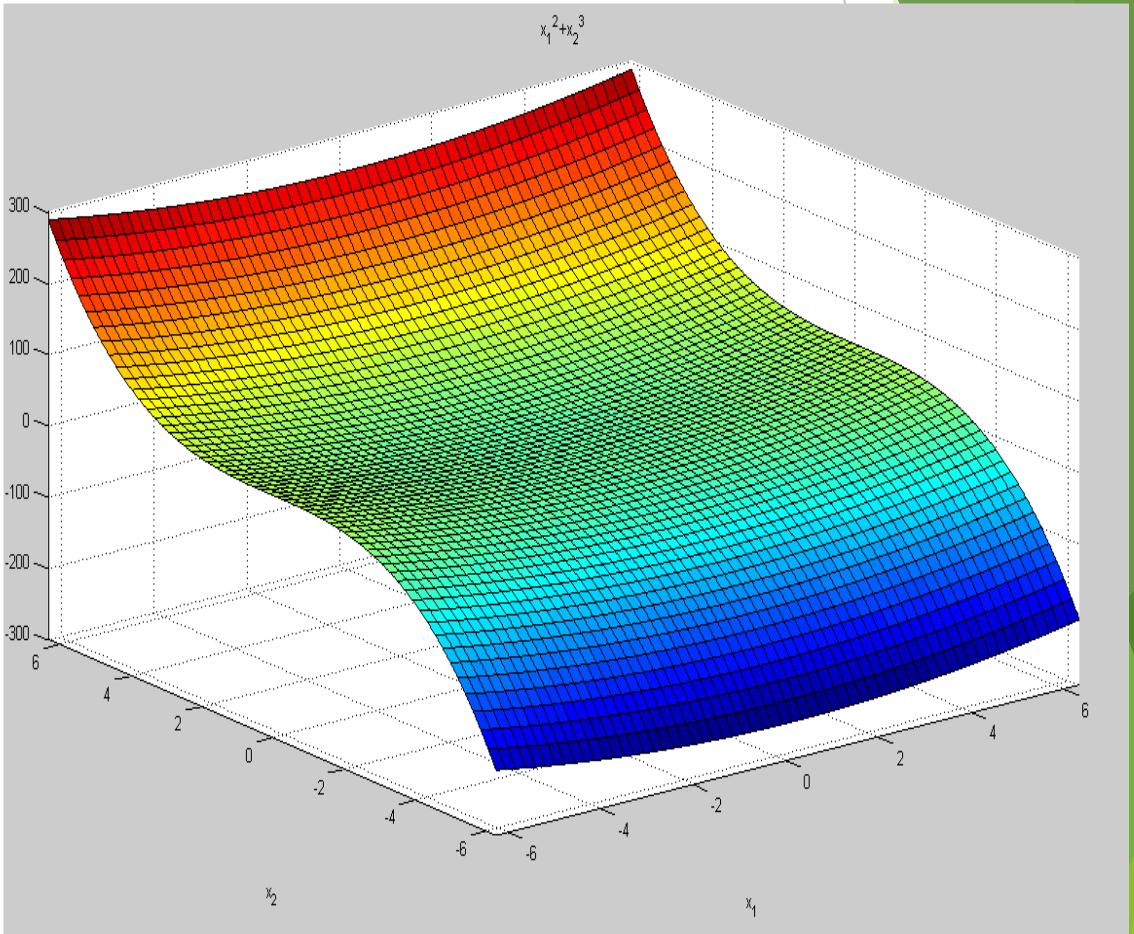


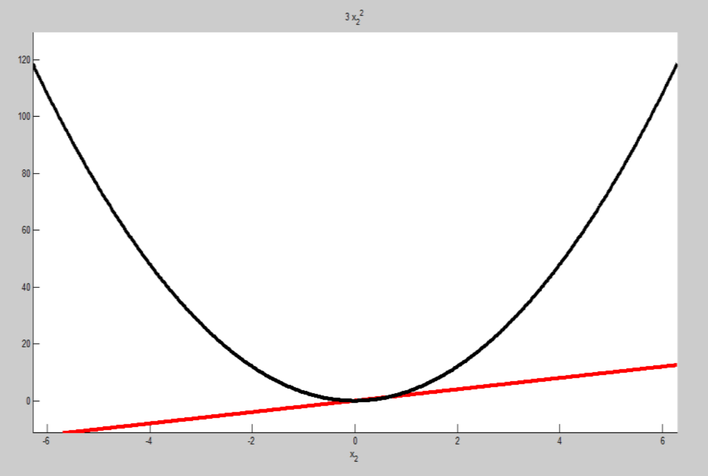
- Stationary Points

Stationary Points	Hessian	Eigen Values	Type of Hessian	Point
(0,0)	$\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$	(4,2)	Positive Definite	Minima
(1,0)	$\begin{bmatrix} -8/e & 0 \\ 0 & -2/e \end{bmatrix}$	(-8/e,-2/e)	Negative Definite	Maxima
(-1,0)	$\begin{bmatrix} -8/e & 0 \\ 0 & -2/e \end{bmatrix}$	(-2/e,-8/e)	Negative Definite	Maxima
(0,-1)	$\begin{bmatrix} 2/e & 0 \\ 0 & -4/e \end{bmatrix}$	(2/e,-4/e)	Indefinite	Saddle
(0,1)	$\begin{bmatrix} 2/e & 0 \\ 0 & -4/e \end{bmatrix}$	(2/e,-4/e)	Indefinite	Saddle

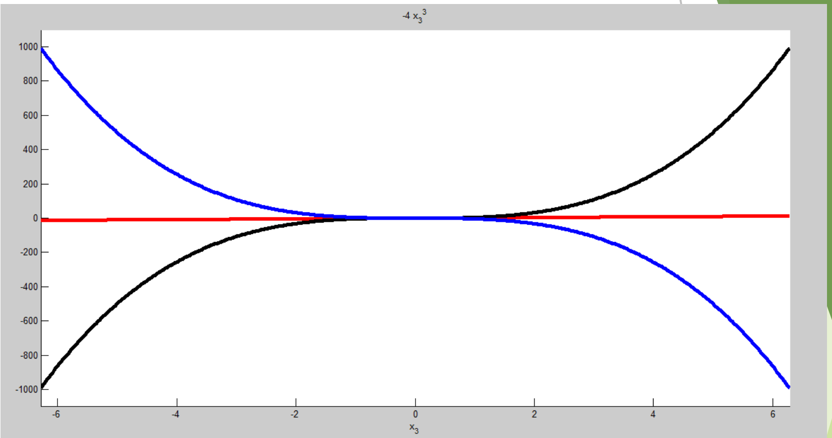
Problem 5 : Positive Semi-Definite Hessian (2-variables)

- $f(x_1, x_2) = x_1^2 + x_2^3$
- Stationary Point : (0,0)
- Hessian : $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$
- Eigen Values: (2,0)
- Consider uni-variable problem along $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
- Third order derivative =6 ($\neq$ 0)
- Saddle point





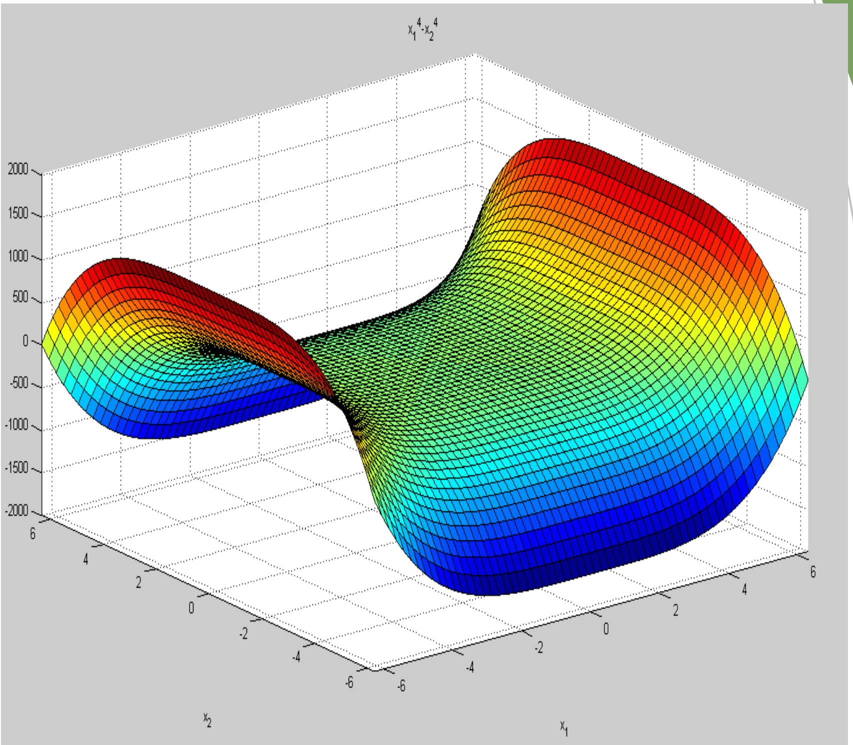
Problem 6 : Positive Semi Definite Hessian

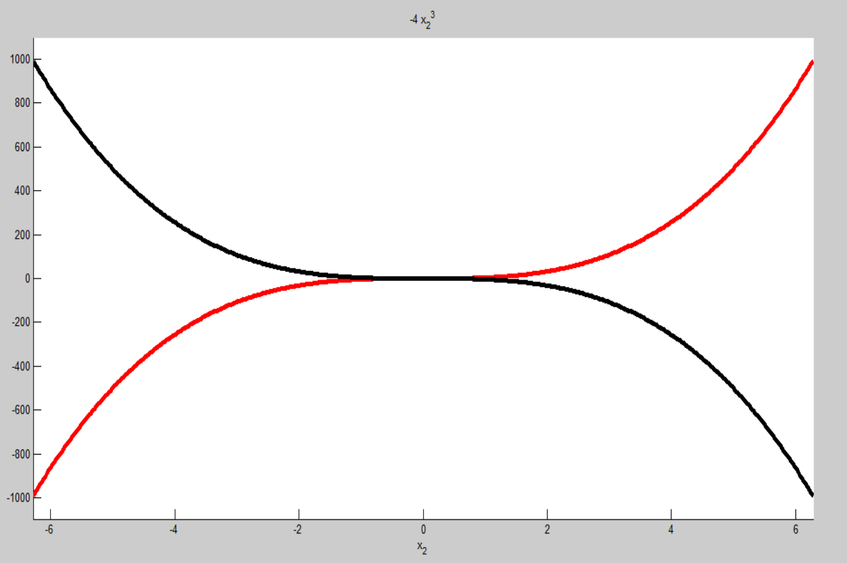


- $f(x_1, x_2, x_3) = (x_1^2 + \ x_2^4 - x_3^4)$
- Stationary Point: (0,0,0)
- Hessian: $\begin{matrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}$
- Eigen Values: (2,0,0)
- Consider uni-variable problem along eigenvector $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ (eigenvalue = 0).
- Third order derivative = 0, Fourth order derivative= 24 (≠0).
- Consider uni-variable problem along $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ (eigenvalue = 0).
- Third order derivative=0, Fourth order derivative= -24 (≠0).
- Along one eigenvector 4^{th} order derivative is positive and negative along another.
- Hence, Saddle Point

Problem 7: Zero Hessian

- $f(x_1, x_2) = x_1^4 - x_2^4$
- Stationary Point: (0,0)
- Hessian: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- Eigen Values: (0,0)
- Consider uni-variable problem along $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Third order derivative = 0.
- Fourth order derivative= 24 (≠0).
- Consider uni-variable problem along $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
- Third order derivative = 0.
- Fourth order derivative= -24 (≠0).
- Maximum along $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, Minimum along $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$
- Hence, Saddle Point



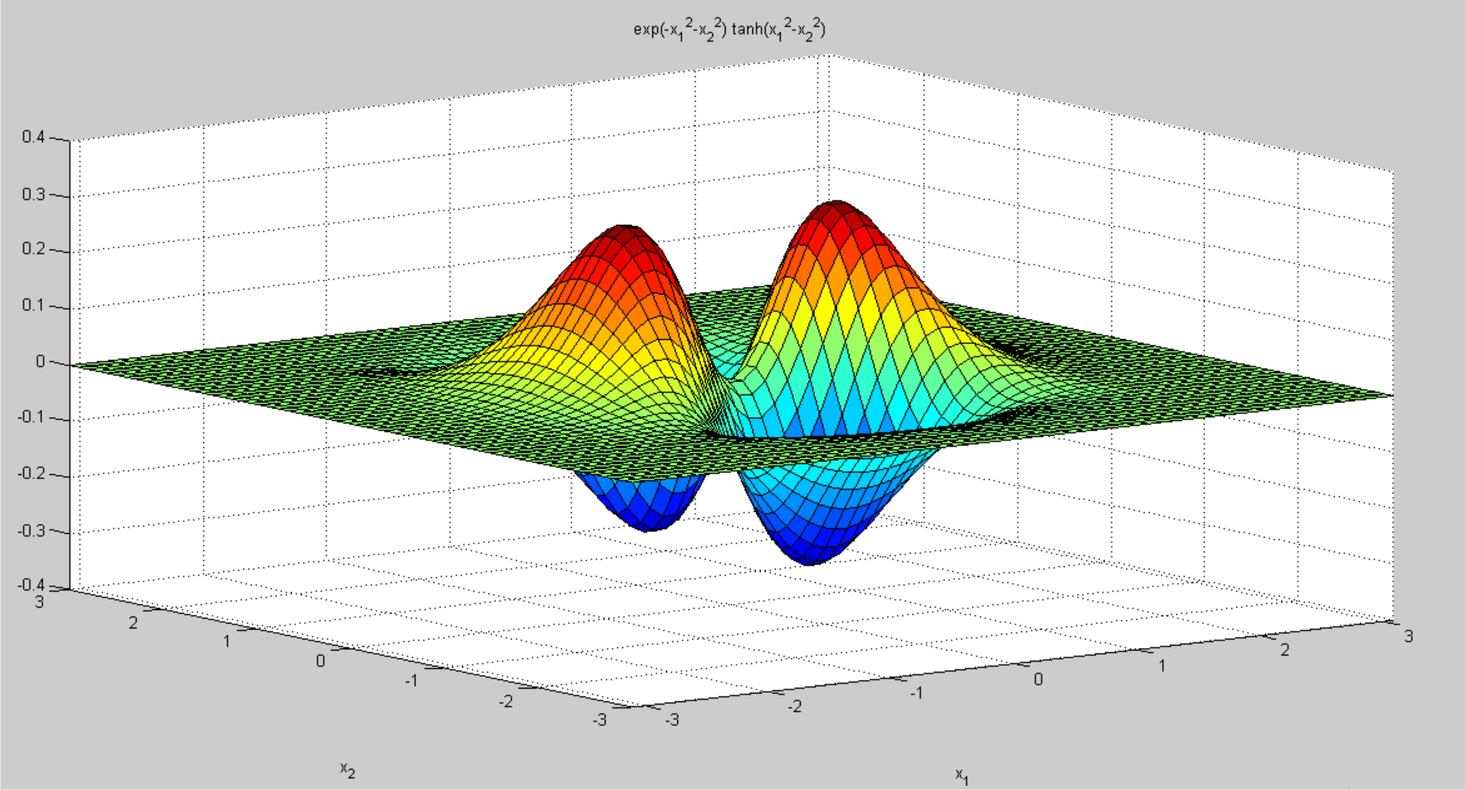


Problem 8 : 7 variables (Semi – Definite Hessian)

- $f(x_1, x_2) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7$
- Stationary Point: (10,12,0,11,0,1.13,1.46)
- Eigen Values of Hessian: (26.9,12.7,10,6,2,0,0)
- Consider uni-variable problem along [0 0 1 0 0 0 0]'
- Third Derivative =0, Fourth Derivative= 24
- Consider uni-variable problem along [0 0 0 0 1 0 0]'
- Third Derivative =0, Fourth Derivative= 0, Fifth Derivative=0, Sixth Derivative=7200
- As the 4^{th} and 6^{th} Derivative are positive for direction along the two eigenvectors the point is a MINIMUM point.

Special Case in Complex Nonlinear Problems

- $f(x_1, x_2) = \ tanh(x_1^2 - x_2^2) \,.\, e^{-(x_1^2 + x_2^2)}$
- Stationary Point using Matlab : (0,0)
- Number of Stationary points from plot : 5
- Limitation of solving non-linear simultaneous equations in Matlab.



- Plotting the two components of the Jacobian
- Incomplete plotting of points using ezplot() in Matlab
- 5 intersections visible in graph
- 1 intersection -> returned by Matlab