Multi-objective Optimization and Pareto optimality

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Abstract—Multi-objective optimization, an area of multiple criteria decision making finds applications in many fields like engineering, economics, finance where need for optimality is required. In this paper, we have focused on describing and solving the multi-objective optimization problem using classical methods like Global criterion method and Normal boundary intersection to find the solutions which we call as Pareto optimal points. The simulations have been carried out in MATLAB and the results have been produced. Also, we have discussed the use of the techniques of multi-objective optimization in Cognitive Radio, which basically addresses the imbalance between spectrum scarcity and spectrum under utilization. The formulation of multi-optimization problem in such cases into SDP is done in the paper.

Keywords—Multi-objective optimization, Global criterion method, Normal Boundary Intersection, Cognitive Radio, Pareto Solutions.

I. INTRODUCTION

A multi-objective optimization, as the name suggests, is concerned with optimization problems with more than one objective function to be optimized simultaneously. Often, a single solution optimizing all the objective functions simultaneously does not exist. Such cases involve a number of Pareto optimal solutions. Similar to the single-objective optimization problem, the problem usually has a number of constraints which any feasible solution, including the optimal solution must satisfy. Multi-objective optimization can be stated as follows in its general form:

$$\min_{x \in C} F(x) = [f_1(x), f_2(x), f_3(x), ..., f_n(x)]$$

where

$$C = x : h(x) = 0, g(x) \leq 0, a \leq x \leq b$$

$$F : R^N \to R^n, h : R^N \to R^{ne},$$

$$g : R^N \to R^{ni}$$

are twice continuously differentiable mappings, and $a \in (R \cup \{-\infty\})^N, b \in (R \cup \{\infty\})^N$, $N$ being the number of variables, $n$ the number of objectives and $ne$ and $ni$ the number of equality and inequality constraints respectively. Usually, a single minimizing or in some cases maximizing every function ($F_k$) simultaneously does not exist. Here, the concept of Pareto Optimality is introduced which is useful in multi-objective framework. It is explained below:

Pareto Solutions A solution is called Pareto optimal if none of the objective functions can be improved in value without degrading some of the other objective values. Without additional subjective preference information, all Pareto optimal solutions are considered equally good. In other words, $x^*$ is a Pareto optimal if there is no other $x$ such that $F(x) \leq F(x^*)$. $F(x^*)$ is called as non dominated point. Similarly, if $F(x_1) \leq F(x_2)$ then $x_1$ dominates $x_2$ and $F(x_1)$ dominates $F(x_2)$

II. GLOBAL CRITERION METHOD

The global criterion method (GCM) belongs to a category of solution methods with no articulation of preference information given. The advantage of this procedure is that the solution can be obtained from the decision making itself and not from the designer. The approach is better illustrated in the figure given above where two objective functions are minimized simultaneously. The preferred relative metric function $d_\alpha$ is stated as:

$$d_\alpha = \left[ \sum_{i=1}^{ne} \left| \frac{f_i(X) - f_i^{id}(X)}{f_i^{id}(X)} \right|^\alpha \right]^{\frac{1}{\alpha}}$$

If the choice of $\alpha$ is $\infty$, minimization of this metric function results in a commonly encountered min-max method, since for this metric the optimum $X^*$ is defined as

$$f(X^*) = \min_{x} \max_{i} \frac{f_i(X) - f_i^{id}(X)}{f_i^{id}(X)}$$

To solve this multi-objective optimization problem numerically, a scalar variable $p$ is introduced to facilitate the transformation of the min-max problem into an equivalent single objective optimization problem:
minimize $\beta$, subject to

\[
    g_i(X) \leq b_j, j = 1, 2, .., p \\
    h_k(X) = 0, k = 1, 2, ..., q \\
    X_i^l \leq X_i \leq X_i^u, i = 1, 2, ..., n \\
    X = [x_1, x_2, ..., x_n]^T
\]

and the following additional constraints

\[
    \omega_i \left| \frac{f_i(X) - f_i(x^*)}{f_i(x^*)} \right| - \beta \leq 0, i = 1, ..., m \\
    \sum_{i=1}^{m} \omega_i = 1
\]

where $\omega_i$ represents the weight of the $i$th objective function depending on the degree of importance. The results are unique if the above two constraints vanish in the formulation process. In this paper we’ll formulate global criterion as :

\[
    \min \sum_{i=1}^{n} \left( \frac{f_i(x^*) - f_i(x)}{f_i(x^*)} \right)^p
\]

such that $g_i(x) \leq 0, i=1,2,3,4,...,m$

The value of $p$ usually is between 1 and 2 (when p=1 the objective is linear, and when p=2 the objective is quadratic. Different problems might lead to the selection of a different value of p.

### A. MATLAB SIMULATIONS

We consider the problem given below:

\[
    \max f_1(x) = 0.5x_1 + 0.4x_2 \\
    f_2(x) = x_1 \\
    \text{subject to} \\
    g_1(x) = x_1 + x_2 \leq 600 \\
    g_2(x) = 2x_1 + x_2 \leq 800 \\
    x_1, x_2 \geq 0
\]

We consider 2 cases here p=1 and p=2 in the following steps:

1st: Find individual maxima of $f_1(x)$ and $f_2(x)$

$f_1(x^*) = 260$ where $x^*=(200,400)$;

$f_2(x^*) = 400$ where $x^*=(400,0)$;

2nd: Create the global criterion

Case 1: p=1;

\[
    \min F = \frac{260 - (0.5x_1 + 0.4x_2)}{260} + \frac{400 - x_1}{400}
\]

subject to

\[
    g_1(x) = x_1 + x_2 \leq 600 \\
    g_2(x) = 2x_1 + x_2 \leq 800 \\
    x_1, x_2 \geq 0
\]

Optimal solution is $f_1 = 200, f_2 = 400$ and $x^* = (400,0)$

Case 2: p=2;

\[
    \min F = \left( \frac{260 - (0.5x_1 + 0.4x_2)}{260} \right)^2 + \left( \frac{400 - x_1}{400} \right)^2
\]

subject to

\[
    g_1(x) = x_1 + x_2 \leq 600 \\
    g_2(x) = 2x_1 + x_2 \leq 800 \\
    x_1, x_2 \geq 0
\]

Optimal solution is $f_1 = 210.5367, f_2 = 364.8781$ and $x^* = 364.8781, 70.2438$

### III. NORMAL BOUNDARY INTERSECTION

NBI is a method for finding several Pareto optimal points for a general nonlinear multi criteria optimization problem. This method comes handy in solving the trade-off among the various conflicting objectives. Further, along with handling of more than two objectives, this method also retains the computational efficiency of continuation-type algorithms. In all we can say, this method is an improvement over continuation techniques since they cannot easily handle more than two objectives.

To understand the procedure in detail, we need to get acquainted with some terminology.

**Convex hull of individual minima (CHIM).** Let $x_i^*$ be the respective global minimizers of $f_i(x), i = 1, ..., n$ over $x \in C$. Let $F_i = F(x_i^*), i = 1, ..., n$. Let $\Phi$ be the $n \times n$ matrix whose $i$th column is $F_i - F^*$ also known as the pay-off matrix. The set of points which are convex combinations of $F_i^* - F^*$

\[
    \Phi \beta : \beta \in R^n, \sum_{i=1}^{n} \beta_i = 1, \beta_i \geq 0
\]

is the Convex Hull of Individual minima.

The set of objective vectors \{F(x) : x \in C\} attained is denoted by $\hat{F}$, so $F : C \rightarrow \hat{F}$. The space $R^n$ containing $\hat{F}$ is known as objective space. Boundary of $\hat{F}$ is denoted by $\partial \hat{F}$ while the set of all Pareto optimal by $P$. The surface of Pareto minima is referred as the trade-off function.

**CHIM**: Consider $CHIM_{\infty}$ as the affine subspace of lowest dimension that contains the $CHIM$. $CHIM_+$ then will be the smallest simply connected set formed by intersection of $\partial \hat{F}$ and $CHIM_{\infty}$. In other words, consider extending the boundary of $CHIM$ simplex to touch $\partial \hat{F}$, the extension of $CHIM$ thus obtained is defined as $CHIM_+$. From now on, $F(x)$ is redefined as:

\[
    F(x) \leftarrow F(x) - F^* + \gamma
\]

Where $\Phi \gamma$ is the least convex combination of the $F_i^* - F^*$ that is a point on the surface of $\hat{F}$.
In the figure above, the set $\hat{F}$ is in the objective space, point A is $F^*_1$, B is $F^*_2$, O is the shadow minima, the broken line segment is the $CHIM$, and the arc ACB is the set of all ePa for etro minima in the objective. In this problem $n$ is 2 or bi=objective, $CHIM = CHIM_+$ and the matrix $\Phi$ is anti-diagonal.

The main idea involved in this method is that the intersection point between the normal drawn from any point in the $CHIM$ and the boundary is probably a Pareto optimal point while the point closest to origin is a Pareto minimal point. 'Probably' because this may not always be the case, for example when the boundary closer to origin is folded. Though it is true when the trade-off surface in the objective space is convex. For a given convex weighing vector $\beta$, $\Phi \beta$ represents a point in the $CHIM$. Let $n$ denote the unit normal drawn to the $CHIM$ pointing towards the origin; then $\Phi \beta + tn, t \in \mathbb{R}$, denotes the sets of points towards the origin. Then the point of intersection between the normal and the boundary of $\hat{F}$ closest to the origin is the global solution of problem formulated as below:

$$\max_{x,t} t$$

s.t. $\Phi \beta + tn = F(x)$

$h(x) = 0$

$g(x) \leq 0$

$a \leq x \leq b$

The above subproblem is referred to as the $NBI$ subproblem, written as $NBI_{\beta}$. , and the solution of these sub problems is referred to as $NBI$ points. The crux is to solve $NBI_{\beta}$ for various $\beta$ and find several points on the boundary of $\hat{F}$

**Structure of $\Phi$**

The $i^{th}$ column of $\Phi$ is described by

$$\Phi(\cdot,i) = F(x^*_i) - F^*$$

Since

$$f_i(x^*_i) = f^*_i,$$

clearly,

$$\Phi(i,i) = 0$$

Also, if $x^*_i$ is the global maximizer of $f_i(x)$, then

$$\Phi(j,i) \geq 0, j \neq i$$

A negative element in position $(j, k)$ of $\Phi$ tells us that $x^*_j$ is not the point which minimizes $f_k(x)$ globally, and $f_k(x^*_j) < f_k(x^*_k)$. i.e., $x^*_j$ is better minimizer than $x^*_k$ of $f_k(x)$. This occurrence can help refine the local minimum of an objective by a simple examination of $\Phi$.

In the NBI method, sufficient condition for points to be globally Pareto optimal is that the components of the shadow minimum is the global minima of the objective and the Pareto surface is convex, although it is not a necessary condition. NBI guarantees mostly locally Pareto points. In situations like in the figure above where part of $\partial \hat{F}$ is folded, the point obtained may not be the one furthest out on the boundary along that normal. Thus the point is not globally optimal.

**A. MATLAB SIMULATION**

The problem taken here is different from the paper. Objective functions are different whereas the constraints are same:

$$\min f_1(x) = x_1 + x_3;$$

$$f_2(x) = x_4 + x_2$$

such that

$$x_1 + 2x_2 - x_3 - 0.5x_4 + x_5 = 2$$

$$4x_1 - 2x_2 + 0.8x_3 + 0.6x_4 + 0.5x_5^2 \leq 0$$
\[ x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 \leq 10 \]

For the start, 2nd inequality constraint is converted into convex SOCP form.

\[
\| \sqrt{2}x_5 - 4x_1 + 2x_2 - 0.8x_3 - 0.6x_4 - 1 \| \leq
-4x_1 + 2x_2 - 0.8x_3 - 0.6x_4 + 1
\]

Individual minima and minimizers are found out to create the payoff matrix. Next, we find the Pareto optimal points for equally spaced \( \beta \) starting from 0 to 1 with a 0.05 gap between successive \( \beta \). Given below is the table showing values of Objective for different \( \beta \):

<table>
<thead>
<tr>
<th>( \beta )</th>
<th>Objective Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.00, 1.00</td>
<td>-0.6345, -3.5056</td>
</tr>
<tr>
<td>0.05, 0.95</td>
<td>-0.9780, -3.4763</td>
</tr>
<tr>
<td>0.10, 0.90</td>
<td>-1.2802, -3.4033</td>
</tr>
<tr>
<td>0.15, 0.85</td>
<td>-1.5646, -3.3117</td>
</tr>
<tr>
<td>0.20, 0.80</td>
<td>-1.8347, -3.2049</td>
</tr>
<tr>
<td>0.25, 0.75</td>
<td>-2.0914, -3.0840</td>
</tr>
<tr>
<td>0.30, 0.70</td>
<td>-2.3362, 2.9498</td>
</tr>
<tr>
<td>0.35, 0.65</td>
<td>-2.5675, -2.8027</td>
</tr>
<tr>
<td>0.40, 0.60</td>
<td>-2.7876, -2.6431</td>
</tr>
<tr>
<td>0.45, 0.55</td>
<td>-2.9959, -2.4712</td>
</tr>
<tr>
<td>0.50, 0.50</td>
<td>-3.1926, -2.2868</td>
</tr>
<tr>
<td>0.55, 0.45</td>
<td>-3.3772, -2.0898</td>
</tr>
<tr>
<td>0.60, 0.40</td>
<td>-3.5497, -1.8799</td>
</tr>
<tr>
<td>0.65, 0.35</td>
<td>-3.7094, -1.6566</td>
</tr>
<tr>
<td>0.70, 0.30</td>
<td>-3.8555, -1.4189</td>
</tr>
<tr>
<td>0.75, 0.25</td>
<td>-3.9870, -1.1658</td>
</tr>
<tr>
<td>0.80, 0.20</td>
<td>-4.1025, -0.8958</td>
</tr>
<tr>
<td>0.85, 0.15</td>
<td>-4.1999, -0.6067</td>
</tr>
<tr>
<td>0.90, 0.10</td>
<td>-4.2764, -0.2955</td>
</tr>
<tr>
<td>0.95, 0.05</td>
<td>-4.3277, 0.0422</td>
</tr>
<tr>
<td>1.00, 0.00</td>
<td>-4.3470, 0.4138</td>
</tr>
</tbody>
</table>

### IV. DIFFERENCES BETWEEN NBI AND GLOBAL CRITERION METHOD

Global Criterion Method comes under the category of No-Preference methods that is no preference information is available to us. For example when we need to carry out some online optimization and we do not take into account which problem is solved. The main idea in the Global criterion method is to minimize the distance to the ideal objective vector. The norm taken into account is an \( L_p \) norm with common choices including \( L_1, L_2 \) and \( L_\infty \).

Normal Boundary Intersection is a Posteriori method which computes different Pareto solutions and then selects the most preferred one. The main advantage of this method over the above one is that it is well suited for two objectives since the solution set can be easily visualized and gives a better understanding of the whole set. Also, the NBI method finds a uniform spread of Pareto points. Though, approximating the set often is a time consuming process as it has to choose the most preferred solution among large number of solutions.

### V. APPLICATION IN COGNITIVE RADIO

Cognitive radio (CR) technology is envisioned to be a promising solution to address the imbalance between spectrum scarcity and spectrum under utilisation. Two types of interference originate with the introduction of CR networks. One is the interference from CR to primary networks (CR-primary interference) and the other one is the interference among spectrum-sharing CR nodes (CR-CR interference). The interference should be well controlled and managed in order not to damage the operation of the primary network and as well improve the performance of CR systems.

The implementation of an interference-tolerant CR requires the primary receiver to provide Cognitive Radio systems the information of how much interference it can tolerate across the spectrum. This threshold limit is known as interference temperature limit. Thus, a real-time feedback mechanism from the primary to the CR networks is essential to inform the CR network of the interference temperature limit.

### Cognitive Radio Network in Action

In a cognitive radio network(CRN) the interference tolerance of the primary system effects the achievable throughput of the network. The criteria used in investigating performance of such systems is often based on considering a fixed maximum tolerable interference at the primary receivers. A good example in this respect is, when a 3G interference limited primary network is experiencing a lower traffic load, implying that for a given receiver performance, it is able(theoretically) to handle a higher level of interference from the secondary system. A case where the primary receiver has an higher interference threshold, greater level of the secondary system interference can be also tolerated.

Investigating the fundamental trade-off between the interference tolerance in the primary system and the achievable throughput of the network is one of the major tasks in CR. It is obvious that for a given interference tolerance level,
if a lower actual interference is imposed on the Primary Users (PUs), then more number of Secondary Users (SUs) can be handled. Here comes the application of the multi-objective optimization (MO) technique which helps us in developing analytical framework for the case that the cognitive system is capable of beamforming.

Beamforming employs an array of antennas to transmit radio frequency signals to multiple users over a shared channel. The phases and transmit power of the transmission across those antenna elements are controlled such that useful signals are constructively added up at a desired receiver while interfering signals are eliminated at unintended user terminals. Phases and power allocations across antenna elements corresponding to each user terminal are then represented by a complex vector which is referred to as the beamforming vector. In such systems, the design problem is to obtain the optimal beamforming vectors.

A multi-objective optimization problem (MOP) in such cases includes two types of objectives contradicting each other. 

First Minimizing the interference arising due to the transmission of the cognitive Base Station (BS) on each PU. Second The second objective is to maximize the signal which is being received at each of the SUs. This in turn consists of maximizing the signal-to-interference-plus-noise-ratio (SINR) at each secondary user.

As the SUs are sharing the same resource, conflicts arises between the secondary users. Therefore, a set consisting of SINR constraints guaranteeing that each SU is served with its required level at least are introduced. Interference levels at PUs are in general kept below their thresholds in order to protect the primary system. The proposed MOP is then mapped to an equivalent single-objective optimization problem (SOP) and then the problem is transformed into a semi-definite programming (SDP). The SDP being a convex problem can be solved using interior-point methods.

A. System Model

Consider a cognitive cellular system consisting of a cognitive BS, U active Secondary Users s and N Primary Users. A secondary (cognitive) BS is supporting a set of U secondary users while it is not interfering with a set of N primary users. Let $S_s = \{1, ..., U\}$ and $S_p = \{1, ..., K\}$ be the sets of indices of SUs and PUs, respectively. We assume that the cognitive BS is equipped with M antenna elements and each SU or PU has a single antenna. The beamforming technique is adopted at the cognitive BS which is equipped with M antennas. We assume single antenna setting at the SUs and PUs. The received signal at the secondary user is,

$$y_i = h_{s,i}^H w_i s_i + \sum_{j=1, j\neq i}^{U} h_{s,i}^H w_j s_j + n_i$$

where $h_{s,i}^H \in \mathbb{C}^{1 \times M}$ is the channel between the cognitive BS and SU i. Furthermore, the average energy in transmitting a symbol to the SU i is assumed to be unity, i.e., $E|s_i|^2 = 1$. Let $R_{s,i} = E(h_{s,i}^H, h_{s,i}^{H*})$ for the statistical channel state information (CSI) $R_{s,i} = E(h_{s,i}^H, h_{s,i}^{H*})$ for the instantaneous CSI, we then express the SINR at any SU i as :

$$\text{SINR}_i = \frac{w_i^H R_{s,i} w_i}{\sum_{j \in S_s, j \neq i} W_j^H R_{s,j} w_j + \sigma_i^2}$$

B. Multi-Objective Optimization

We’ll consider here p objective functions $f_1(x), f_2(x), ..., f_p(x)$. An MOP can be defined as

$$\min_{x \in X} f(x) = \min_{x \in X} (f_1(x), f_2(x), ..., f_p(x))$$

where X is the decision space whereas $R^p$ is the objective space.

C. Multi-Objective Problem Formulation

We want to obtain the optimum beamforming vector $w_i$ for each secondary user i in the cognitive base station keeping in mind their required SINR. The objective remains is to maximize the signal power received at each secondary user i, that is 

$$f_{s,t}(W) = w_i^H R_{s,i} w_i$$

while minimizing the interference which is inflicted at each PU t, that is

$$f_{p,t}(W) = \sum_{i=1}^{U} w_i^H R_{p,t} w_i$$

The SINR constraints confirm that each of the secondary user is being served with its threshold(required) level at least. The optimization problem at hand is to tune the beam to further improve each SUs received signal strength and thus to raise the achievable throughput above the required level as far as possible.

We’ll use the weighted mean approach for this problem as follows: Let

$$\lambda_{p,t} > 0 \ \forall \ t, \lambda_{s,i} > 0 \ \forall \ i$$
and
\[ \sum_{i=1}^{N} \lambda_{p,t} + \sum_{i=1}^{U} \lambda_{s,i} = 1. \]

If \( W^* \) is the optimal solution to the following SOP:
\[
\min_{W \in \mathcal{D}} \left( \sum_{t=1}^{N} \lambda_{p,t} \sum_{i=1}^{U} w_{i}^{H} R_{p,t} w_{i} - \sum_{i=1}^{U} \lambda_{s,i} w_{i}^{H} R_{s,i} w_{i} \right),
\]
such that
\[
\frac{w_{i}^{H} R_{s,i} w_{i}}{\sum_{j=1, j \neq i}^{U} w_{j}^{H} R_{s,i} w_{j} + \sigma_i^2} \geq \gamma_i \quad \forall i
\]
\[
\sum_{i=1}^{U} w_{i}^{H} R_{p,t} w_{i} \leq I_t, \quad \forall t
\]
\[
\sum_{i=1}^{U} w_{i}^{H} w_{i} \leq P_m,
\]
Now we’ll consider a rank one matrix \( W_i \) such that \( W_i = w_i w_i^H \). Also by rearranging the constraints using \( x^H Y x = \text{Tr}(Yxx^H) \), the problem is converted to the following Semi Definite Programming form:
\[
\min_{\{W_i\}} \text{Tr}(\sum_{t=1}^{N} \lambda_{p,t} \sum_{i=1}^{U} W_i - \sum_{i=1}^{U} \lambda_{s,i} R_{s,i} W_i)
\]
s.t.
\[
\text{Tr}(R_{s,i} W_i) - \gamma_i \sum_{j=1, j \neq i}^{U} \text{Tr}(R_{s,i} W_j) - \gamma_i \sigma_i^2 \geq 0, W_i \succeq 0
\]
\[
I_t \geq \text{Tr}(R_{p,t} \sum_{i=1}^{U} W_i), \forall t,
\]
\[
P_m \geq \sum_{i=1}^{U} \text{Tr}(W_i)
\]
where \( \{W_i\} = \{W_1, ..., W_U\} \) is the set of beamforming matrices.

**D. SIMULATION RESULTS**

We consider the problem as given in the paper and thus simulated for two SUs and two PUs.

S.U. is at 0.5 km and P.U. is at 1km from Cognitive Base station. The channel is taken as flat fading Rayleigh with the values taken from the reference paper. PUs are located at -50 and 50 degrees and the SUs are located at -10 and 10.

**Case 1:-** We assigned weights corresponding to both primary user in objective function as .10 and weights corresponding to both secondary user as .40. We then obtained the total throughput as function of Interference temperature.

**Case 2:** We assigned weights corresponding to both primary user in objective function as .25 and weights corresponding to both secondary user as .25. We then obtained the total throughput as function of Interference temperature.

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