Defn: Let $\underline{V}$ be a vector space over $\mathbb{F}$ $(\mathbb{R}, \mathbb{C})$. A function $f: V \times V \to \mathbb{F}$, denoted $f(u,v) \leftrightarrow \langle u,v \rangle$, and satisfies

i) $\langle u,v \rangle = \overline{\langle v,u \rangle}$

ii) $\langle \alpha u, v \rangle = \alpha \langle u,v \rangle$

iii) $\langle u+v, w \rangle = \langle u,w \rangle + \langle v,w \rangle$

iv) $\langle u,u \rangle \geq 0$ $\forall u \in V$ and $\langle u,u \rangle = 0 \Rightarrow u = 0$.

Implications: ① $\langle 0,0 \rangle = 0$

② $\langle u, \alpha v \rangle = \bar{\alpha} \langle u,v \rangle$

③ $\langle u,v \rangle = 0$ for all $v \in V \Rightarrow u = 0$.

Examples: ① $\mathbb{R}^3$ $\quad \langle x,y \rangle = y^T x = x_1y_1 + x_2y_2 + x_3y_3$

② $\mathbb{R}^n$ $\quad \langle x,y \rangle = y^T x = \sum_{i=1}^n x_i y_i$

③ $\mathbb{C}^n$ $\quad \langle x,y \rangle = y^* x = \sum_{i=1}^n x_i \overline{y_i}$.

④ $\mathbb{R}^2$ $\quad A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $\quad \langle x,y \rangle = y^T A x$

positive definite matrices

$$= [y_1 \; y_2] \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= 2x_1y_1 + y_1x_2 + y_2x_1 + 2x_2y_2$$

$\langle (i,0), (i,0) \rangle$

$(i,0) \cdot \overline{(i,0)}$

✓ $\langle x,y \rangle = \langle y,x \rangle$, $\langle \alpha x, y \rangle = y^T A(\alpha x) = \alpha y^T A x = \alpha \langle x,y \rangle$

$\langle x+y, z \rangle = z^T A(x+y) = z^T A x + z^T A y = \langle x,z \rangle + \langle y,z \rangle$

$\boxed{\langle x,x \rangle \geq 0 \;\; \forall x \;\; \text{and} \;\; \langle x,x \rangle = 0 \Rightarrow x = 0}$

$\langle x,x \rangle = 2x_1^2 + 2x_1x_2 + 2x_2^2$ = Try yourself

$\boxed{x^2 + x + 1}$

⑤ $\mathcal{C}([a,b],\mathbb{C})$ $\quad \langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx.$

$\underline{\underline{\mathbb{R}}} \longrightarrow \langle f,g\rangle = \int_a^b f(x)g(x)\,dx.$ $\quad$ LT

$\boxed{\langle f,f\rangle \geq 0}$ $\quad \langle f,f\rangle = \int_a^b |f(x)|^2\,dx$

$\int_a^b h(x)\,dx \geq 0$ and $|h(x)| \geq 0 \ \forall x$

⑥ $M_{n\times n}(\mathbb{C})$ $\quad \langle A,B\rangle = tr(B^*A)$

$tr(B^*A) = (B^*A)_{11} + (B^*A)_{22} + \cdots + (B^*A)_{nn}$

$= (B^*)_{11}a_{11} + (B^*)_{12}a_{21} + \cdots + (B^*)_{21}a_{12} + (B^*)_{22}a_{22} + \cdots$

$= \overline{b_{11}}a_{11} + \overline{b_{21}}a_{21} + \cdots + \overline{b_{n1}}a_{n1} + \overline{b_{12}}a_{12} + \cdots$

$\leftrightarrow$ Standard Inner Product / Dot Product.

Defn: Length / Norm of X $\quad V, \langle\ ,\ \rangle$ ← Inner Product

$\|X\| = \sqrt{\langle X,X\rangle}$ LT

Standard Inner Product / Dot product

$X \in \mathbb{R}^n \quad \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \|X\|$

$$\boxed{\cos\theta = \frac{X\cdot Y}{\|X\|\cdot\|Y\|}}$$

Thm: Cauchy-Schwartz Inequality $\quad$ LT

Let V be an Inner Product Space over $\mathbb{F}$. Then for every $x,y \in V$ $\quad \boxed{|\langle x,y\rangle| \leq \|x\|\cdot\|y\|.}$ The equality holds if and only if $y = \alpha x$ for some scalar $\alpha \in \mathbb{F}$.

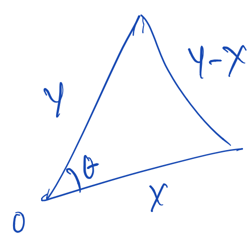
$$0 \le \langle x-\alpha y, x-\alpha y\rangle = \langle x,x\rangle - \alpha\langle y,x\rangle - \bar{\alpha}\langle x,y\rangle + \alpha\bar{\alpha}\langle y,y\rangle$$

Take $\alpha = \frac{\langle x,y\rangle}{\|y\|^2}$, $\|y\| \neq 0$, $y \neq 0$.

$$0 \le \|x\|^2 - \frac{\langle x,y\rangle\,\langle y,x\rangle}{\|y\|^2} - \cancel{\frac{\langle y,x\rangle\,\langle x,y\rangle}{\|y\|^2}} + \cancel{\frac{\langle x,y\rangle\,\langle y,x\rangle}{\|y\|^2\cdot\|y\|^2}\cdot\|y\|^2}$$

$$\Rightarrow \|x\|^2 \ge \frac{|\langle x,y\rangle|^2}{\|y\|^2} \Rightarrow |\langle x,y\rangle|^2 \le \|x\|^2\cdot\|y\|^2.$$

$$-1 \le \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|} \le 1$$



Over Real Inner Product

$$\cos\theta = \frac{\|x\|^2+\|y\|^2-\|y-x\|^2}{2\,\|x\|\cdot\|y\|}$$

$$= \frac{\|\ \|^2+\|\ \|^2-\left[\|x\|^2+\|y\|^2-2\langle x,y\rangle\right]}{2\,\|x\|\,\|y\|}$$

$$= \frac{\langle x,y\rangle}{\|x\|\cdot\|y\|}.$$

$$\|y-x\|^2 = \langle y-x, y-x\rangle = \langle y,y\rangle - \langle x,y\rangle - \langle y,x\rangle + \langle x,x\rangle$$

Defn: Let V be an Inner Product space. Two vectors x and y are said to be orthogonal/perpendicular if $\langle x,y\rangle = 0$.
$x \perp y$ (Notation)

IMP: $0 \perp x$ for all $x \in V$.

Defn: Let V be a vector space over $\mathbb{F}$. A fn. $f: V \to \mathbb{R}$ is called a 'Norm' if it satisfies the following. Notation: $\|\cdot\|$

i) $\|x\| \geq 0 \quad \forall x \in V$ and $\|x\| = 0 \Rightarrow x = 0$.

ii) $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{F},\ x \in V$.

iii) $\|x+y\| \leq \|x\| + \|y\|$ ← Triangle Inequality.

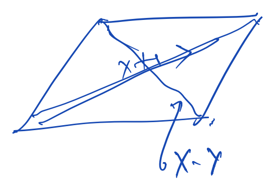
$d(x,y) \leftrightarrow$ distance between x & y

$$= \|x - y\|$$

Thm: Let V be a Normed space (There is a well-defined Norm) over $\mathbb{F}$. Then this norm comes from an Inner Product if and only if

Parallelogram Law holds

$$\|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right), \ \forall$$



Example: $\mathbb{R}^2$ $\qquad \|x\|_1 = \|(x_1, x_2)\|_1 := |x_1| + |x_2|$ ← Defn of norm.

$x = (1,0),\ y = (0,1)$

$x+y = (1,1),\ x-y = (1,-1)$ $\qquad$ LHS $= \|(1,1)\|^2 + \|(1,-1)\|^2$

$$= (1+1)^2 + (1+|-1|)^2 = 2^2 + 2^2 = 8$$

RHS $= 2\left(\|(1,0)\|^2 + \|(0,1)\|^2\right)$

$$= 2(1^2 + 1^2) = 4.$$

$$\begin{cases} x = (x_1, x_2, \cdots, x_n) & y_i, x_i \in \{0,1\} \\ y = (y_1, y_2, \cdots, y_n) & \text{mod } 2. \end{cases}$$

$d(x,y) = \left| \{ i \mid x_i \neq y_i \} \right|$ gives a distance in $\mathbb{Z}_2^n$

So, if $n = 4$ then for $x = (1,1,1,1)$
$x \cdot x = 0$. So, there are vectors which are self-orthogonal.

If we have real Inner product space

$$0 \leq \langle x - \alpha y, x - \alpha y \rangle = \langle x, x \rangle - \alpha \langle y, x \rangle - \alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle$$
$$= \langle x, x \rangle - 2\alpha \langle x, y \rangle + \alpha^2 \langle y, y \rangle.$$

A quadratic expression in α and is non-negative for each $\alpha \Rightarrow$ Discriminant ≤ 0

$\Rightarrow 4 \langle x, y \rangle^2 - 4 \langle x, x \rangle \cdot \langle y, y \rangle \leq 0 \Rightarrow$ Cauchy-Schwartz Inequality

Polar Identity.

Complex Inner product.

$$4 \langle x, y \rangle = \|x+y\|^2 - \|x-y\|^2 + i \|x + iy\|^2 - i \|x - iy\|^2$$

$$\frac{\operatorname{Re} \langle x, y \rangle}{\|x\| \cdot \|y\|}$$ angle between x & y

Rank-Nullity Thm.

$\underline{AX=0}$

$$\begin{bmatrix} \boxed{1} & & \boxed{1} & & \boxed{1} & & \end{bmatrix}$$

non-pivots
Free variables

$$\left[\begin{array}{ccccccc|c} \boxed{1} & 2 & -3 & 0 & 0 & 3 & 4 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 & -4 & -3 & 0 \\ & 0 & & 0 & \boxed{1} & 0 & 1 & 0 \end{array}\right] \sim$$

$x_5 = -x_7, \quad x_4 = 4x_6 + 3x_7$

$x_1 = -2x_2 + 3x_3 - 3x_6 - 4x_7$

$$X = \begin{bmatrix} x_2 \\ x_3 \\ x_6 \\ x_7 \end{bmatrix} = \begin{bmatrix} \text{Free} \\ \text{var} \end{bmatrix}$$

$U_1 \rightarrow$ 1st Free variable -1
rest as 0

$$U_1 \quad \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \end{bmatrix}$$

$U_2 \rightarrow$ 2nd as 1
& rest as 0

$$U_2 \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$Y - y_2U_1 - y_3U_2 - y_6U_3 - y_7U_4$

$z =$

$Y \in \text{Null}(A)$

$Az = 0$

$Y - y_{i_1}U_1 - y_{i_2}U_2 - \cdots - y_{i_{n-k}}U_{n-k}$

$$Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

The i, th compo corresp to first free variable

$$\langle u, \alpha w \rangle = \overline{\langle \alpha w, u \rangle} = \overline{\alpha \langle w, u \rangle}$$

$$= \bar{\alpha} \; \overline{\langle w, u \rangle} = \bar{\alpha} \langle u, w \rangle$$

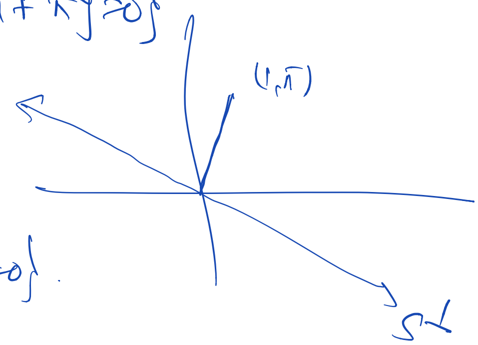
Defn: V as an IPS (Inner Product space) over $\mathbb{F}$. Then x is said to be orthogonal/perpendicular to y if $\langle x, y\rangle = 0$.

Defn: Let S be a subset of V. Then the orthogonal complement of S in V, denoted $S^\perp$, equals

$$S^\perp = \{ x \in V \mid \langle x, s\rangle = 0 \ \forall s \in S \}.$$

Example: $\mathbb{R}^2$, $\quad S = \{(1,\pi)\}$. $\quad S^\perp = ?$

$$S^\perp = \{(x,y) \in \mathbb{R}^2 \mid \langle (1,\pi), (x,y)\rangle = 0\}$$
$$= \{(x,y) \in \mathbb{R}^2 \mid x + \pi y = 0\}$$



$\mathbb{R}^3$, $S = \{(1,1,1)\}$

$$S^\perp = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}.$$

$S \subseteq V$, $\quad S = \{x, y\}$, $\qquad S^\perp \overset{?}{=} x^\perp \cap y^\perp$

$(S(S))^\perp$ and $S^\perp$?

Thm: Let S be a subset of a v.s. V (IPS) over $\mathbb{F}$. Then $S^\perp$ is a subspace of V over $\mathbb{F}$.

Defn: A set S is said to be orthogonal if any two vectors in S are orthogonal.

$u_i \neq 0$ $S = \{u_1, u_2, \dots, u_k\} \subseteq V \to (IPS)$

S is orthogonal $\iff$ $\langle u_i, u_j \rangle = 0$ $\forall i \neq j$, we

orthonormal $\iff$ orthogonal & $\boxed{\|u_i\| = 1 \ \forall i}$

$\mathbb{R}^3 \to$ $\{e_1, e_2, e_3\}$

$$A = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

$$\boxed{A^T A = I = A A^T.}$$

Thm: Let $S = \{u_1, u_2, \dots, u_k\}$ be an ~~orthogonal~~ orthonormal set in an IPS V over $\mathbb{F}$. Then S is l.l. subset of S over $\mathbb{F}$.

Pf: $\alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k = 0$ unknowns are $\alpha_1, \dots, \alpha_k$.

$$0 = \langle 0, u_1 \rangle = \langle \alpha_1 u_1 + \alpha_2 u_2 + \dots + \alpha_k u_k, u_1 \rangle$$

$$= \alpha_1 \langle u_1, u_1 \rangle + \alpha_2 \underbrace{\langle u_2, u_1 \rangle}_{=0} + \dots + \alpha_k \underbrace{\langle u_k, u_1 \rangle}_{=0}$$

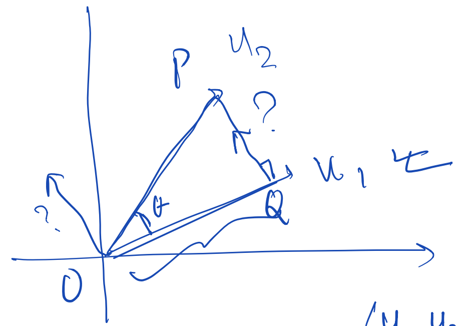
$$= \alpha_1 \|u_1\|^2$$

$$\Rightarrow \underline{\alpha_1 = 0}.$$

Ques: Given a l.l. set, say $S = \{u_1, u_2, \dots, u_k\}$, we can we get an orthonormal set, say $T = \{v_1, v_2, \dots\}$ such that $\boxed{LS(u_1, \dots, u_\ell) = LS(v_1, \dots, v_\ell), \ 1 \le \ell \le k?}$

$v_1 = \frac{u_1}{\|u_1\|} \Rightarrow \|v_1\| = 1.$

$\leftarrow$ unit vector in the direction of u_1.



$\vec{OP} = u_2,$

$\cos\theta = \frac{\|\vec{OQ}\|}{\|\vec{OP}\|} \Rightarrow \|\vec{OQ}\| = \|\vec{OP}\| \cdot \cos\theta = \|u_2\| \cdot \frac{\langle u_1, u_2 \rangle}{\|u_1\| \cdot \|u_2\|}$

$= \frac{\langle u_1, u_2 \rangle}{\|u_1\|}$

$\Rightarrow \vec{OQ} = \|\vec{OQ}\| \cdot$ unit vector in the direction of $\vec{OQ}$

$= \frac{\langle u_1, u_2 \rangle}{\|u_1\|} \cdot \frac{u_1}{\|u_1\|}$.

Projection of $\vec{u_2}$ on $\vec{u_1}$ $= \left\langle \frac{u_1}{\|u_1\|}, u_2 \right\rangle \frac{u_1}{\|u_1\|}$.

$w_2 = u_2 - \vec{OQ} = u_2 - \underbrace{\left\langle \frac{u_1}{\|u_1\|}, u_2 \right\rangle}_{\alpha} \frac{u_1}{\|u_1\|}$

$\langle u_1, w_2 \rangle = \langle u_1, u_2 - \vec{OQ} \rangle = \left\langle u_1, u_2 - \alpha \frac{u_1}{\|u_1\|} \right\rangle$

$= \langle u_1, u_2 \rangle - \left\langle u_1, \alpha \frac{u_1}{\|u_1\|} \right\rangle.$

$= \langle u_1, u_2 \rangle - \bar{\alpha} \left\langle u_1, \frac{u_1}{\|u_1\|} \right\rangle = \langle u_1, u_2 \rangle - \bar{\alpha} \|u_1\|$

$\alpha = \frac{1}{\|u_1\|} \cdot \langle u_1, u_2 \rangle$

$\boxed{w_2 = u_2 - \alpha u_1}$

(1) $w_2 \perp v_1 = \frac{u_1}{\|u_1\|}, \quad w_2 \neq 0$?

$v_2 = \frac{w_2}{\|w_2\|}$

$$\boxed{w_3 = u_3 - \langle u_3, v_1\rangle v_1 - \langle u_3, v_2\rangle v_2}$$

$$(1,2,3) - (1,0,0) - (0,2,0) = (0,0,3)$$

$w_3 \neq 0$, $w_3 = 0 \Rightarrow u_3 - \alpha v_1 - \beta v_2 = 0$

$\Rightarrow u_3 \in LS(v_1, v_2) = LS(u_1, u_2)$.

$$v_3 = \frac{w_3}{\|w_3\|}$$

$\{v_1, v_2, v_3\}$ ~~are~~ is orthonormal set

$$S = \left\{ \underset{u_1}{\begin{pmatrix}1\\1\\1\end{pmatrix}}, \underset{u_2}{\begin{pmatrix}1\\2\\3\end{pmatrix}}, \underset{u_3}{\begin{pmatrix}2\\1\\3\end{pmatrix}}, \underset{u_4}{\begin{pmatrix}1\\2\\1\end{pmatrix}} \right\}$$

$$\boxed{u_1 \to v_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix}, \quad \|u_1\| = \sqrt{3}}$$

$$w_2 = \begin{pmatrix}1\\2\\3\end{pmatrix} - \left\langle \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix} \right\rangle \cdot \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\\3\end{pmatrix} - 6/3 \begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1\\2\\3\end{pmatrix} - \begin{pmatrix}2\\2\\2\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$w_3 = (2,1,3) - \left\langle (2,1,3), \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)\right\rangle \left(\tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}, \tfrac{1}{\sqrt{3}}\right)$$
$$- \left\langle (2,1,3), \left(\tfrac{-1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right)\right\rangle \left(\tfrac{-1}{\sqrt{2}}, 0, \tfrac{1}{\sqrt{2}}\right)$$

$$= (2,1,3) - \frac{6}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}(1,1,1) - \frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$$

$$= (2,1,3) - (2,2,2) + \left(\frac{1}{2}, 0, -\frac{1}{2}\right) = \left(\frac{1}{2}, -1, \frac{1}{2}\right)$$

$$\rightarrow \frac{1}{2}(1,-2,1)$$

$$v_3 = \frac{w_3}{\|w_3\|} = \frac{1}{\sqrt{6}}(1,-2,1)$$

$$\boxed{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right), \left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \frac{1}{\sqrt{6}}(1,-2,1)}$$

$$\left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)$$

$$(1,2,1) = (1,2,1) - \underbrace{\langle (1,2,1), v_1\rangle}_{\alpha} v_1 - \underbrace{\langle (1,2,1), v_2\rangle}_{\beta} v_2 - \underbrace{\langle (1,2,1), v_3\rangle}_{\gamma} v_3$$

$$= 0.$$

$$(1,2,1) = \alpha v_1 + \beta v_2 + \gamma v_3 \qquad \text{Find } \alpha, \beta, \gamma$$

$$\langle (1,2,1), v_1\rangle = \langle \alpha v_1 + \beta v_2 + \gamma v_3, v_1\rangle$$

$$= \alpha \langle v_1, v_1\rangle + \beta\langle v_2, v_1\rangle + \gamma\langle v_3, v_1\rangle$$

$$= \alpha \|v_1\|^2 = \alpha.$$

QR-algorithm.

$$[u_1\ u_2\ u_3\ u_4] = [\underbrace{v_1\ v_2\ v_3}_{\text{Orthonormal}}] \begin{bmatrix} a_{11} & a_{12} & a_{13} & ? \\ 0 & a_{22} & a_{23} & ? \\ 0 & & a_{33} & ? \end{bmatrix}$$

Know: $LS(u_1) = LS(v_1)$

$u_2 = a_{12} v_1 + a_{22} v_2$

$\underline{\underline{u_2}} \in LS(\underline{\underline{v_1}}, v_2) = LS(u_1, u_2)$

$x \longmapsto Ax$

$\|x\| = \langle x, x \rangle$
$= x^T x$

$\|Ax\| \longrightarrow (Ax)^T (Ax)$
$= x^T \underbrace{(A^T A)}_{= I} x$

$\Downarrow$

$a_{11} v_1 + 0 \cdot v_2 + 0 \cdot v_3$
$= a_{11} v_1 = u_1$
for some choice of a_{11}.

$\{u_1, u_2, \cdots, u_n\}$ ↔ $\{u_2, u_3, u_1, u_4, \cdots\}$

$\text{Proj}_{\underline{\underline{v}}}(u) = \left\langle u, \frac{v}{\|v\|} \right\rangle \frac{v}{\|v\|}$, $f = \frac{v}{\|v\|}$

$= \underline{\underline{\langle u, f\rangle f}}$ f is a unit $\underline{\text{vector}}$.

$\mathbb{R}^n$, f is a unit vector in $\mathbb{R}^n$

The projection of u on f ↔ $\text{Proj}_f(u) = \langle u, f\rangle f$

$= (f^T u) f$

$= f (f^T u)$

$= (f f^T) u$

The matrix of the projection onto the line $\underline{\underline{f}}$ ↔ $LS(f)$ is given by $A = f\underline{\underline{f^T}}$.

$A^2 = (ff^T)(ff^T) = f(f^T f)f^T = f \cdot (1) \cdot f^T = ff^T = A$.

$\underline{\underline{A^T = A}}$.

$\underline{\underline{f^T = (1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})}}$

$\underset{3\times 3}{A} \to \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$\text{Proj}_f(u) = \underline{\underline{Au}}$.

Proved so that what we did during Gram–Schmidt process made sense.

the

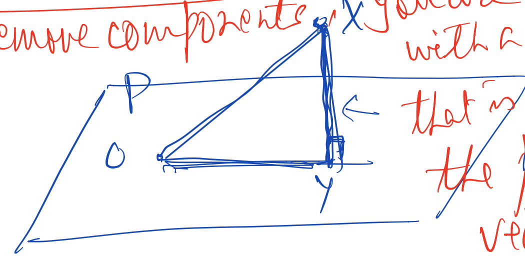
Thm: Let H be a finite dimensional vector subspace of an IPS V over $\mathbb{F}$. If $\{u_1, u_2, \dots, u_k\}$ is an orthonormal basis of H then for any $x \in V$

① $y = \boxed{\langle x, u_1\rangle u_1 + \langle x, u_2\rangle u_2 + \dots + \langle x, u_k\rangle u_k} \in \underline{H}.$
is the ONLY closest point to x in H

② $\boxed{x-y \text{ is orthogonal to } H \leftrightarrow x-y \in H^{\perp}}$

whenever you remove components x you are left out with a new vector

$Proj_H(x) \leftrightarrow$ projection of x on the plane H.



that is $\perp$ to the previous vectors.

Pf: Take $\alpha_i = \langle x, u_i\rangle$ for $1 \le i \le k$.

Then $\langle x-y, u_1\rangle = \langle x, u_1\rangle - \langle y, u_1\rangle$

$= \langle x, u_1\rangle - \langle \sum_{i=1}^{k} \alpha_i u_i, u_1\rangle$

$= \langle x, u_1\rangle - \sum_{i=1}^{n} \alpha_i \langle u_i, u_1\rangle$

$= \langle x, u_1\rangle - \alpha_1$

$= \langle x, u_1\rangle - \langle x, u_1\rangle = \underline{0}.$

$$\|x\|^2 = \|y + x - y\|^2 = \|y\|^2 + \|x-y\|^2$$

as $y \in H$

$$\Rightarrow \|x\|^2 \geq \boxed{\underline{\|y\|^2} = |\langle x,u_1\rangle|^2 + |\langle x,u_2\rangle|^2 + \cdots + |\langle x,u_k\rangle|^2}$$

<u>Bessel's Inequality.</u>

$$\|\alpha u_1 + \beta u_2\|^2 \stackrel{?}{=} \langle \alpha u_1 + \beta u_2, \alpha u_1 + \beta u_2\rangle$$

$$= \alpha\langle u_1, \alpha u_1\rangle + \alpha\langle u_1, \beta u_2\rangle + \beta\langle u_2, \alpha u_1\rangle + \beta\langle u_2, \beta u_2\rangle$$

$$= |\alpha|^2\langle u_1,u_1\rangle + \alpha\bar\beta\langle u_1,u_2\rangle + \beta\bar\alpha\langle u_2,u_1\rangle + |\beta|^2\langle u_2,u_2\rangle$$

$$= |\alpha|^2 + |\beta|^2.$$

<u>Bessel's Inequality:</u> H a f.d. subspace of an IPS V with orthonormal basis $\{u_1, u_2, \ldots, u_k\}$. Let $x \in V$.

Then $\|x\|^2 \geq \sum_{i=1}^{k} |\langle x,u_i\rangle|^2$.

Equality holds if and only if $\underline{\underline{x \in H}}$.

$H \leftrightarrow \{u_1, u_2, \ldots, u_k\}$ orthonormal.

If $x \in H$ then $\underbrace{\|x\|^2 = \sum_{i=1}^{k} |\langle x,u_i\rangle|^2}_{\text{Pythagoras Thm}}$

<u>Thm:</u> Let $A_{m\times n}$ with real entries and $b \in \mathbb{R}^m$. Then every solution of the system $A^TAX = A^Tb$ is a <u>least square solution</u> of the system $AX = b$.

Conversely, every <u>least square</u> solution of the system $AX = b$ is a solution of the system $\boxed{A^TAX = A^Tb.}$

$$\boxed{AX = b}$$

$$\|err\| = \|AX - b\|$$

least $X_0 \in \mathbb{R}^n$ such that $\|AX_0 - b\| = \min_{X\in\mathbb{R}^n} \|AX - b\|$

<u>Pf:</u>

Let X_0 be a least square solution of the system $AX = b$. $\Rightarrow \|AX_0 - b\| = \min_{X\in\mathbb{R}^n} \|AX - b\|$

<u>Need to show:</u> $A^TAX_0 = A^Tb$.

<u>To show:</u> $A^T(AX_0 - b) = 0$.

$$\|AX - b\|^2 = \|AX - AX_0 + AX_0 - b\|^2$$

$$\stackrel{?}{=} \|AX - AX_0\|^2 + \|AX_0 - b\|^2 + 2\langle AX - AX_0, AX_0 - b\rangle$$

$$\geq \|AX_0 - b\|^2 \text{ for all } X \in \mathbb{R}^n.$$

$$\underset{X\in\mathbb{R}^n}{\text{Minimize}} \|AX - b\|^2 = \min_{X\in\mathbb{R}^n} (AX - b)^T(AX - b)$$

$$= \min_{X\in\mathbb{R}^n} (X^TA^T - b^T)(AX - b)$$

$$= \min_{X \in \mathbb{R}^n} \left(\boxed{X^T A^T A X} - X^T A^T b - b^T A X + b^T b \right)$$

$$\frac{d}{dx} \to \boxed{2A^TAX - 2A^Tb} = 0 \Rightarrow A^TAX = A^Tb.$$

$$f : \mathbb{R}^n \to \mathbb{R} \to \nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n} \right)$$

$$x_1 (A^TAX)_1 + x_2 (A^TAX)_2 + \cdots + x_n (A^TAX)_n$$

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = a_{11} x^2 + 2a_{12} xy + a_{22} y^2$$

$$\frac{\partial}{\partial x}, \frac{\partial}{\partial y} \qquad \frac{\partial}{\partial x} \to 2a_{11} x + 2a_{12} y$$

$$\frac{\partial}{\partial y} \to 2a_{12} x + 2a_{22} y$$

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} = 2 \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

$AX = b \Leftrightarrow$ If No soln then look for the Best possible solution.

$\Updownarrow$

soln of $A^TAX = A^Tb$.

Pseudo-Inverse / Moore-Penrose Inverse $\Leftrightarrow (G)$

$AGA = A$, $GAG = G$, $(AG)^T = AG$, $(GA)^T = GA$.

Thm: Let V be a f.d. IPS and W a proper subspace of V. Then $V = \boxed{W \oplus W^{\perp}}$ direct sum

($\Leftrightarrow$ each element of V can be uniquely expressed as the sum of an element from W and an element from $W^{\perp}$)

$\mathbb{R}^3$, $W_1 = \{(x,y,z) \in \mathbb{R}^3 \mid x+y=z\}$

$W_2 = \{(x,y,z) \in \mathbb{R}^3 \mid x+y=0\}$

$W_1 \to \{(1,0,1), (0,1,1)\}$ $\quad W_2 \leftrightarrow \{(1,-1,0), (0,0,1)\}$

$\{\underbrace{(1,0,1),(0,1,1)}, (1,\underbrace{-1,0),(0,0,1)}\} \longleftrightarrow \underline{\underline{S}}$.

W_1 is a plane, W_2 is also a plane in $\mathbb{R}^3$

$W_1 \cap W_2$ is a line in $\mathbb{R}^3$

① $\Rightarrow \{0\} \subsetneq W_1 \cap W_2$. $\quad W_1 \cap W_2$ contains a non-zero vector.

② $\mathbb{R}^3 = LS\{(1,0,1), (0,1,1), (1,-1,0), (0,0,1)\}$.

$= W_1 + W_2$.

$V = W_1 + W_2$.
Any $x \in V$ can be written as
$x = w_1 + w_2$, $w_1 \in W_1$
& $w_2 \in W_2$.

$W \cap W^\perp = \{0\}$.

$W \subseteq V = W \oplus W^\perp$?

Let $\{u_1, u_2, \cdots, u_k\}$ be a basis of W.
Apply Gram-Schmidt to get $\underbrace{\{v_1, v_2, \cdots, v_k\}}_{\text{orthonormal set}}$
as a basis of W.

Extend $\{v_1, v_2, \cdots, v_k\}$ to get a basis of V.
Apply Gram-Schmidt again:

$$\{\underbrace{v_1, v_2, \cdots, v_k}_{W}, \underbrace{v_{k+1}, v_{k+2}, \cdots, v_n}\}$$

is an orthonormal basis of V.

$W = LS(v_1, v_2, \cdots, v_k)$. ? $W^\perp = LS(v_{k+1}, \cdots, v_n)$

$W \cap W^{\perp} = \{0\}$.

$v \in V \qquad v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$

$= (\underbrace{\alpha_1 v_1 + \cdots + \alpha_k v_k}_{W}) + (\underbrace{\alpha_{k+1} v_{k+1} + \cdots + \alpha_n v_n}_{W^{\perp}})$.

$\underline{\underline{Ax = b}}$ Know: $b \notin Col(A)$. $\underline{\underline{A_{m \times n}}}$.

$err = Ax - b, \ x \in \mathbb{R}^n$

$b \in V, \quad b \notin Col(A) \Rightarrow \exists$ a unique $y \in Col(A)$

s.t. $\boxed{b = y + v,}$ $v \in col(A)^{\perp}$
$\qquad\qquad\qquad\qquad\quad \|$

$\underline{\underline{Null(A)}} = Row(A)^{\perp} \quad (Row(A^T))^{\perp}$

$Ax = 0$

Claim: the vector $y \in Col(A)$ is the one that we are looking for.

$\Downarrow$

$\exists x_0 \in \mathbb{R}^n$ s.t. $y = \underline{Ax_0}$.

$A^T A x_0 = A^T y = A^T(b - v) = A^T b - A^T v$

$= A^T b - 0 = A^T b$.

Need to show $\|AX_0 - b\| = \min_X \{\|AX - b\|\}$.

$$\|AX - b\|^2 = \|AX - y + y - b\|^2 = \|AX - y\|^2 + \|y - b\|^2 + 2\underbrace{\langle AX - y, y - b\rangle}_{=0}.$$

$$= \|AX - y\|^2 + \|y - b\|^2$$

$$\geq \underline{\|y - b\|^2}.$$

Conversely, if X_0 satisfies $A^T A X = A^T b$ then

$\|AX_0 - b\| = \min_X \{\|AX - b\|\}$. ✓ yourself.

Q. Plot the curve ~~of~~ set

$$\left\{(x, y, z) \in \mathbb{R}^3 \,\middle|\, (x, y, z)\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 10\right\}.$$

$$\boxed{2x^2 + 2y^2 + 2z^2 + 2xy + 2yz + 2xz = 10}$$

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \qquad A\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} = 4\begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}, \quad A\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$A\begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ -2/\sqrt{6} \end{bmatrix} \qquad U = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

$$UU^T = I \Rightarrow \underline{U^{-1} = U^T}.$$

$$A\breve{U} = [A\,U[:,1],\ A\,U[:,2],\ A\,U[:,3]]$$

$$= [4\,U[:,1],\ U[:,2],\ U[:,3]] = U \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ?$$

$$\underline{\underline{A}} = U\,\mathrm{Diag}(4,1,1)\,U^{-1} = U\,\mathrm{Diag}(4,1,1)\,U^T.$$

$$X^T A X = 10 \iff X^T (U\,\mathrm{Diag}(4,1,1)\,U^T) X = 10$$

Define $\underline{\underline{Y = U^T X}} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \iff [y_1\ y_2\ y_3] \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 10$

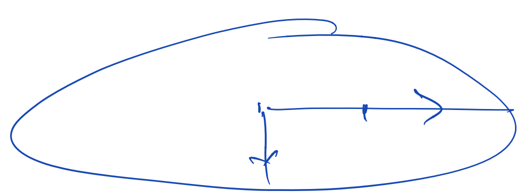
$$\iff 4y_1^2 + y_2^2 + y_3^2 = 10 \iff \frac{y_1^2}{10/4} + \frac{y_2^2}{10} + \frac{y_3^2}{10} = 1$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = U^T X = \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{pmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{pmatrix} \frac{x+y+z}{\sqrt{3}} \\ \frac{x-y}{\sqrt{2}} \\ \frac{x+y-2z}{\sqrt{6}} \end{pmatrix}$$

<u>Principal Axes</u> $x+y+z=0$

$x-y=0, \quad x+y-2z=0$

$A_{n \times n}$ $\{AX : X \in \mathbb{R}^n\} \iff$



$A_{n\times n} \in \mathbb{M}_{n\times n}(\mathbb{C})$ Eigenvalues & Eigenvectors

$\mathbb{C}^n$ over $\mathbb{C}$.