Linear Algebra through Matrices

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Contents

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Chapter 1

Introduction to Matrices

1.1 Motivation

Recall that at some stage, we have solved a linear system of 3 equations in 3 unknowns. But, for clarity, let us start with a few linear systems of 2 equations in 2 unknowns.

Example 1.1.1. 1. Consider the linear system

$$\left.\begin{aligned} 2x + 5y &= 7 \\ 2x + 4y &= 6. \end{aligned}\right\} \tag{1.1.1}$$

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The two linear systems represent a pair of non-parallel lines in $\mathbb{R}^2$. Note that $x = 1, y = 1$ is the unique solution of the given system as $(1,1)$ is the point of intersection of the two given lines $2x + 5y = 7$ and $2x + 4y = 6$. But, we also see that

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot 1 + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \cdot 1 = \begin{bmatrix} 7 \\ 6 \end{bmatrix},$$

which corresponds to the solution of

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} \cdot x + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \cdot y = \begin{bmatrix} 7 \\ 6 \end{bmatrix} \Leftrightarrow \left\{ \begin{aligned} 2x + 5y &= 7 \\ 2x + 4y &= 6. \end{aligned} \right\} \tag{1.1.2}$$

Equation (1.1.2) also implies that we can write the vector $\begin{bmatrix} 7 \\ 6 \end{bmatrix}$ as sum of the vectors $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$. So, even though we were looking at the point of intersection of two lines, an interpretation of the solution gives information about vectors in $\mathbb{R}^2$.

2. Consider the linear system

$$\left.\begin{aligned} x + 5y + 4z &= 11 \\ x + 6y - 7z &= 1 \\ 2x + 11y - 3z &= 12. \end{aligned}\right\} \tag{1.1.3}$$

Here, we have three planes in $\mathbb{R}^3$ and an easy observation implies that the third equation is the sum of the first two equations. Hence, the line of intersection of the first two planes is contained in the third plane. Hence, this system has infinite number of solutions given by

$$x = 61 - 59k, y = -10 + 11k, z = k \text{ with } k \text{ arbitrary real number.}$$

For example, verify that for $k = 1$, we get $x = 2, y = 1$ and $z = 1$ as a possible solution. Also,

$$\begin{bmatrix}1\\1\\2\end{bmatrix} \cdot 2 + \begin{bmatrix}5\\6\\11\end{bmatrix} \cdot 1 + \begin{bmatrix}4\\-7\\-3\end{bmatrix} \cdot 1 = \begin{bmatrix}11\\1\\12\end{bmatrix} = \begin{bmatrix}1\\1\\2\end{bmatrix} \cdot 61 + \begin{bmatrix}5\\6\\11\end{bmatrix} \cdot (-10) + \begin{bmatrix}4\\-7\\-3\end{bmatrix} \cdot 0,$$

where the second part corresponds to $k = 0$ as a possible solution. Thus, we again see that the vector $\begin{bmatrix}11\\1\\12\end{bmatrix}$ is a sum of the three vectors $\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\begin{bmatrix}5\\6\\11\end{bmatrix}$ and $\begin{bmatrix}4\\-7\\-3\end{bmatrix}$ (which are associated with the unknowns x, y and z, respectively) after multiplying by certain scalars which itself appear as solutions of the linear system.

Before going to the next example, also note that the numbers $-59, 11$ and 1, which appear as coefficients of k in the solution satisfies

$$\begin{bmatrix}1\\1\\2\end{bmatrix} \cdot (-59) + \begin{bmatrix}5\\6\\11\end{bmatrix} \cdot 11 + \begin{bmatrix}4\\-7\\-3\end{bmatrix} \cdot 1 = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

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3. As a last example, consider the linear system

$$\left.\begin{aligned} x + 5y + 4z &= 11 \\ x + 6y - 7z &= 1 \\ 2x + 11y - 3z &= 13. \end{aligned}\right\} \tag{1.1.4}$$

Here, we see that if we add the first two equations and subtract it with the third equation then we are left with $0x + 0y + 0z = 1$, which has no solution. That is, the above system has no solution. I leave it to the readers to verify that there does not exist any x, y and z such that

$$\begin{bmatrix}1\\1\\2\end{bmatrix} \cdot x + \begin{bmatrix}5\\6\\11\end{bmatrix} \cdot y + \begin{bmatrix}4\\-7\\-3\end{bmatrix} \cdot z = \begin{bmatrix}11\\1\\13\end{bmatrix}.$$

Remark 1.1.2. *So, what we see above is "each of the linear systems gives us certain 'relationships' between vectors which are 'associated' with the unknowns". These relationships will lead to the study of certain objects when we study "vector spaces". They are as follows:*

1. *The first idea of 'relationship' that helps us to write a vector in terms of other vectors will lead us to the study of 'linear combination' of vectors. So,* $\begin{bmatrix}7\\6\end{bmatrix}$ *is a 'linear combination' of* $\begin{bmatrix}2\\2\end{bmatrix}$ *and* $\begin{bmatrix}5\\4\end{bmatrix}$. *Similarly,* $\begin{bmatrix}11\\1\\12\end{bmatrix}$ *is a 'linear combination' of* $\begin{bmatrix}1\\1\\2\end{bmatrix}$, $\begin{bmatrix}5\\6\\11\end{bmatrix}$ *and* $\begin{bmatrix}4\\-7\\-3\end{bmatrix}$.

2. *Further, it also leads to the study of 'linear span' of a set. A positive answer leads to the vector being an element of the 'linear span; and a negative answer to 'NOT an element of the linear span'. For example, for* $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ -3 \end{bmatrix} \right\}$, *the vector* $\begin{bmatrix} 11 \\ 1 \\ 12 \end{bmatrix}$ *belongs to the 'linear span' of* S, *whereas,* $\begin{bmatrix} 11 \\ 1 \\ 13 \end{bmatrix}$ *does NOT belong to the 'linear span' of* S.
3. *The idea of a unique solution leads us to the statement that the corresponding vectors are 'linearly independent'. For example, the set* $\left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 4 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$ *is 'linearly independent'. Whereas, the set* $\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 6 \\ 11 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ -3 \end{bmatrix} \right\} \subseteq \mathbb{R}^3$ *is NOT 'linearly independent' as*
$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot (-59) + \begin{bmatrix} 5 \\ 6 \\ 11 \end{bmatrix} \cdot 11 + \begin{bmatrix} 4 \\ -7 \\ -3 \end{bmatrix} \cdot 1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

1.2 Definition of a Matrix

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Definition 1.2.1. A rectangular array of numbers is called a **matrix**.

The horizontal arrays of a matrix are called its **rows** and the vertical arrays are called its **columns**. A matrix A having m rows and n columns is said to be a matrix of **size/ order** $m \times n$ and can be represented in either of the following forms:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad \text{or} \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

where a_{ij} is the entry at the intersection of the i^{th} row and j^{th} column. One writes $A \in \mathbb{M}_{m,n}(\mathbb{F})$ to mean that A is an $m \times n$ matrix with entries from the set $\mathbb{F}$, or in short $A = [a_{ij}]$ or $A = (a_{ij})$. We write $A[i,:]$ to denote the i-th row of A, $A[:,j]$ to denote the j-th column of A and a_{ij} or $(A)_{ij}$ or $A[i,j]$, for the (i,j)-th entry of A.

For example, if $A = \begin{bmatrix} 1 & 3+\mathbf{i} & 7 \\ 4 & 5 & 6-5\mathbf{i} \end{bmatrix}$ then $A[1,:] = [1 \;\; 3+\mathbf{i} \;\; 7]$, $A[:,3] = \begin{bmatrix} 7 \\ 6-5\mathbf{i} \end{bmatrix}$ and $a_{22} = 5$. Sometimes commas are inserted to differentiate between entries of a row vector. Thus, $A[1,:]$ may also be written as $[1, \; 3+\mathbf{i}, \; 7]$. A matrix having only one column is called a **column vector** and a matrix with only one row is called a **row vector**. All our vectors will be **column vectors** and will be represented by **bold letters**. A matrix of size 1×1 is also called a **scalar** and is treated as such and hence we may or may not put it under brackets.

Definition 1.2.2. Two matrices $A = [a_{ij}], B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ are said to be **equal** if $a_{ij} = b_{ij}$, for each $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$.

In other words, two matrices are said to be equal if they have the same order and their corresponding entries are equal.

Example 1.2.3. 1. Consider a system of linear equations $2x + 5y = 7$ and $3x + 2y = 6$. Then, we identify it with the matrix $A = \left[\begin{array}{cc|c} 2 & 5 & 7 \\ 3 & 2 & 6 \end{array}\right]$. Here, $A[:,1] = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $A[:,2] = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ are associated with the variables/ unknowns x and y, respectively.

2. $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ Then, $A \neq B$ as $a_{12} \neq b_{12}$. Similarly, if $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ then $A \neq C$ as they are of different sizes.

1.2.1 Special Matrices

Definition 1.2.4. Let $A = [a_{ij}]$ be an $m \times n$ matrix with $a_{ij} \in \mathbb{F}$.

1. Then A is called a **zero-matrix**, denoted $\mathbf{0}$ (order is mostly clear from the context), if $a_{ij} = 0$ for all i and j. For example, $\mathbf{0}_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\mathbf{0}_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

2. Then A is called a **square matrix** if $m = n$ and is denoted by $A \in \mathbb{M}_n(\mathbb{F})$.

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3. Let $A \in \mathbb{M}_n(\mathbb{F})$.

   (a) Then, the entries $a_{11}, a_{22}, \ldots, a_{nn}$ are called the diagonal entries of A. They constitute the **principal diagonal** of A.

   (b) Then, A is said to be a **diagonal** matrix, , denoted $\text{diag}(a_{11}, \ldots, a_{nn})$, if $a_{ij} = 0$ for $i \neq j$. For example, the zero matrix $\mathbf{0}_n$ and $\begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$ are diagonal matrices.

   (c) Then, $A = \text{diag}(1, \ldots, 1)$ is called the **identity matrix**, denoted I_n, or in short I. For example, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

   (d) If $A = \alpha I$, for some $\alpha \in \mathbb{F}$, then A is called a **scalar matrix**.

   (e) Then, A is said to be an **upper triangular** matrix if $a_{ij} = 0$ for $i > j$.

   (f) Then, A is said to be a **lower triangular** matrix if $a_{ij} = 0$ for $i < j$.

   (g) Then, A is said to be **triangular** if it is an upper or a lower triangular matrix. For example, $\begin{bmatrix} 0 & 1 & 4 \\ 0 & 3 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ is upper triangular, $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ is lower triangular and the matrices $\mathbf{0}, I$ are upper as well as lower triangular matrices.

4. An $m \times n$ matrix $A = [a_{ij}]$ is said to have an **upper triangular form** if $a_{ij} = 0$ for all $i > j$. For example, the matrices $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}$ have upper triangular forms.

5. For $1 \le i \le n$, define $\mathbf{e}_i = I_n[:, i]$, a matrix of order $n \times 1$. Then the column matrices $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are called the **standard unit vectors** or the **standard basis** of $\mathbb{M}_{n,1}(\mathbb{C})$ or $\mathbb{C}^n$. The dependence of n is omitted as it is understood from the context. For example, if $\mathbf{e}_1 \in \mathbb{C}^2$ then, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and if $\mathbf{e}_1 \in \mathbb{C}^3$ then $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

1.3 Matrix Operations

As a first operation, we define 'transpose' and/or 'conjugate transpose' of a matrix. This allows us to interchange the ideas related with the rows of a matrix with the columns of a matrix and vice-versa. It's use also helps us in looking at geometrical ideas that are useful in applications.

1.3.1 Transpose and Conjugate Transpose of Matrices

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Definition 1.3.1. Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$. Then

1. the **transpose** of A, denoted A^T, is an $n \times m$ matrix with $(A^T)_{ij} = a_{ji}$, for all i, j.
2. the **conjugate transpose** of A, denoted A^*, is an $n \times m$ matrix with $(A^*)_{ij} = \overline{a_{ji}}$ (the complex-conjugate of a_{ji}), for all i, j.

If $A = \begin{bmatrix} 1 & 4+\mathbf{i} \\ 0 & 1-\mathbf{i} \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 4+\mathbf{i} & 1-\mathbf{i} \end{bmatrix}$ and $A^* = \begin{bmatrix} 1 & 0 \\ 4-\mathbf{i} & 1+\mathbf{i} \end{bmatrix}$. Note that $A^* \neq A^T$.

Note that if $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a column vector then $\mathbf{x}^T = \begin{bmatrix} 1 & 2 \end{bmatrix}$ and $\mathbf{x}^*$ are row vectors.

Theorem 1.3.2. *For any matrix A, $(A^*)^* = A$ and $(A^T)^T = A$.*

Proof. Let $A = [a_{ij}]$, $A^* = [b_{ij}]$ and $(A^*)^* = [c_{ij}]$. Clearly, the order of A and $(A^*)^*$ is the same. Also, by definition $c_{ij} = \overline{b_{ji}} = \overline{\overline{a_{ij}}} = a_{ij}$ for all i, j. ■

1.3.2 Sum and Scalar Multiplication of Matrices

Definition 1.3.3. Let $A = [a_{ij}], B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ and $k \in \mathbb{C}$.

1. . Then the **sum** of A and B, denoted $A + B$, is defined to be the matrix $C = [c_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ with $c_{ij} = a_{ij} + b_{ij}$ for all i, j.
2. Then, the **product** of $k \in \mathbb{C}$ with A, denoted kA, equals $kA = [ka_{ij}] = [a_{ij}k] = Ak$.

Example 1.3.4. If $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -4 & 6 \\ 1 & 1 & 7 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 2 & 0 & 11 \\ 1 & 2 & 9 \end{bmatrix} \text{ and } 5A = \begin{bmatrix} 5 & 20 & 25 \\ 0 & 5 & 10 \end{bmatrix}.$$

Theorem 1.3.5. *Let $A, B, C \in \mathbb{M}_{m,n}(\mathbb{C})$ and let $k, \ell \in \mathbb{C}$. Then*

1. $A + B = B + A$ *(commutativity).*
2. $(A + B) + C = A + (B + C)$ *(associativity).*
3. $k(\ell A) = (k\ell)A$.
4. $(k + \ell)A = kA + \ell A$.

Proof. (1). Let $A = [a_{ij}]$ and $B = [b_{ij}]$. Then by definition

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}] = [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}] = B + A$$

as complex numbers commute. The other parts are left for the reader. ∎

Definition 1.3.6. Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. Then

1. the matrix $\mathbf{0}_{m\times n}$ satisfying $A + \mathbf{0} = \mathbf{0} + A = A$ is called the **additive identity**.
2. the matrix B with $A + B = \mathbf{0}$ is called the **additive inverse** of A, denoted $-A = (-1)A$.

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EXERCISE **1.3.7.**

1. *Find non zero, non-identity matrices A satisfying*
   (a) $A^* = A$ *(such matrices are called Hermitian matrices).*
   (b) $A^* = -A$ *(such matrices are called skew-Hermitian matrices).*
2. *Suppose $A = [a_{ij}], B = [b_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$.*
   (a) *If $A + B = \mathbf{0}$ then show that $B = (-1)A = [-a_{ij}]$.*
   (b) *If $A + B = A$ then show that $B = \mathbf{0}$.*
3. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then there exists matrices B and C such that $A = B + C$, where $B^T = B$ (Symmetric matrix) and $C^T = -C$ (skew-symmetric matrix).*

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4. *Let $A = \begin{bmatrix} 1+\mathbf{i} & -1 \\ 2 & 3 \\ \mathbf{i} & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 1-\mathbf{i} & 2 \end{bmatrix}$. Compute $A + B^*$ and $B + A^*$.*
5. *Write the 3×3 matrices $A = [a_{ij}]$ satisfying*
   (a) $a_{ij} = 1$ *if* $i \neq j$ *and* 2 *otherwise.*
   (b) $a_{ij} = 1$ *if* $\mid i - j \mid \ \leq 1$ *and* 0 *otherwise.*
   (c) $a_{ij} = i + j$.
   (d) $a_{ij} = 2^{i+j}$.

1.3.3 Multiplication of Matrices

We now come to the most important operation between matrices, called the **matrix multiplication**. We define it as follows.

Definition 1.3.8. Let $A = [a_{ij}] \in \mathbb{M}_{m,n}(\mathbb{C})$ and $B = [b_{ij}] \in \mathbb{M}_{n,r}(\mathbb{C})$. Then, the **product** of A and B, denoted AB, is a matrix $C = [c_{ij}] \in \mathbb{M}_{m,r}(\mathbb{C})$ such that for $1 \le i \le m,\ 1 \le j \le r$

$$c_{ij} = A[i,:]B[:,j] = [a_{i1},\ a_{i2}, \ldots, a_{in}] \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

Thus, AB is defined if and only if **the number of columns of A = the number of rows of B.** The way matrix product is defined seems quite complicated. Most of you have already seen it. But, we will find other ways (3 more ways) to understand this matrix multiplication. These will be quite useful at different stages in our study. So, we need to spend enough time on it.

Example 1.3.9. Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix}$.

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1. **Entry-wise Method:** $(AB)_{11} = 1 \cdot 3 + (-1) \cdot (-1) = 3 + 1 = 4$. Similarly, compute the rest and verify that $AB = \begin{bmatrix} 4 & 4 & 4 \\ 6 & 8 & 10 \\ -1 & 0 & 1 \end{bmatrix}$.

2. **Row Method:** Note that $A[1,:]$ is a 1×2 matrix and B is a 2×3 matrix and hence $A[1,:]B$ is a 1×3 matrix. So, matrix multiplication is defined. Thus,

$$A[1,:]B = \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} + (-1) \cdot \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \end{bmatrix}$$

$$A[2,:]B = \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix} = 2 \cdot \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} + 0 \cdot \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 8 & 10 \end{bmatrix}$$

$$A[3,:]B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \\ -1 & 0 & 1 \end{bmatrix} = 0 \cdot \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 1 \end{bmatrix}.$$

Hence, if $A = \begin{bmatrix} A[1,:] \\ A[2,:] \\ A[3,:] \end{bmatrix}$ then $AB = \begin{bmatrix} A[1,:] \\ A[2,:] \\ A[3,:] \end{bmatrix} B = \begin{bmatrix} A[1,:]B \\ A[2,:]B \\ A[3,:]B \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 6 & 8 & 10 \\ -1 & 0 & 1 \end{bmatrix}$.

3. **Column Method:** Note that A is a 3×2 matrix and $B[:,1]$ is a 2×1 matrix and hence

$A(B[:,1])$ is a 3×1 matrix. So, matrix multiplication is defined. Thus,

$$A \cdot B[:,1] = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot 3 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot -1 = \begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix}$$

$$A \cdot B[:,2] = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot 4 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot 0 = \begin{bmatrix} 4 \\ 8 \\ 0 \end{bmatrix}$$

$$A \cdot B[:,1] = \begin{bmatrix} 1 & -1 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \cdot 5 + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \cdot 1 = \begin{bmatrix} 4 \\ 10 \\ 1 \end{bmatrix}$$

Thus, if $B = \begin{bmatrix} B[:,1] & B[:,2] & B[:,3] \end{bmatrix}$ then

$$AB = A\begin{bmatrix} B[:,1 & B[:,2] & B[:,3] \end{bmatrix} = \begin{bmatrix} A \cdot B[:,1 & A \cdot B[:,2] & A \cdot B[:,3] \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 6 & 8 & 10 \\ -1 & 0 & 1 \end{bmatrix}.$$

4. **Matrix Method:** We also have if $A = \begin{bmatrix} A[:,1] & A[:,2] \end{bmatrix}$ and $B = \begin{bmatrix} B[1,:] \\ B[2,:] \end{bmatrix}$ then $A[:,1]$ is a 3×1 matrix and $B[1,:]$ is a 1×3 matrix. Thus, the matrix product $A[:,1]\ B[1,:]$ is defined and is a 3×3 matrix. Hence,

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$$\begin{aligned} A[:,1]B[1,:] + A[:,2]B[2,:] &= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 3 & 4 & 5 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 4 & 4 \\ 6 & 8 & 10 \\ -1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

Remark 1.3.10. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ and $B \in \mathbb{M}_{n,p}(\mathbb{C})$. Then the product AB is defined and observe the following:*

1. *AB corresponds to operating (combining certain multiples of rows) on the* rows of B. *This is called the* **row method** *for calculating the matrix product. Here,*

$$(AB)[i,:] = A[i,:]B = a_{i1}B[1,:] + \cdots + a_{in}B[n,:], \text{ for } 1 \le i \le m.$$

2. *AB also corresponds to operating (combining certain multiples of columns) on the* columns of A. *This is called the* **column method** *for calculating the matrix product. Here,*

$$(AB)[:,j] = AB[:,j] = A[:,1]b_{1j} + \cdots + A[:,n]b_{nj}, \text{ for } 1 \le j \le p.$$

3. *Write* $A = \begin{bmatrix} A[1,:] \\ \vdots \\ A[m,:] \end{bmatrix}$ *and* $B = \begin{bmatrix} B[:,1] & \cdots & B[:,p] \end{bmatrix}$ *then*

$$AB = \begin{bmatrix} A[1,:]B[:,1] & A[1,:]B[:,2] & \cdots & A[1,:]B[:,p] \\ A[2,:]B[:,1] & A[2,:]B[:,2] & \cdots & A[2,:]B[:,p] \\ \vdots & \ddots & \cdots & \vdots \\ A[m,:]B[:,1] & A[m,:]B[:,2] & \cdots & A[m,:]B[:,p] \end{bmatrix}.$$

4. *Write* $A = \begin{bmatrix} A[:,1] & \cdots & A[:,n] \end{bmatrix}$ *and* $B = \begin{bmatrix} B[1,:] \\ \vdots \\ B[n,:] \end{bmatrix}$. *Then*

$$AB = A[:,1]B[1,:] + A[:,2]B[2,:] + \cdots + A[:,n]B[n,:].$$

5. *If* $m \neq p$ *then the product* BA *is NOT defined.*

6. *Let* $m = p$. *Here* BA *and* AB *can still be different. For example, if* $A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} -1 & 2 & 3 \end{bmatrix}$ *then* $AB = \begin{bmatrix} -1 & 2 & 3 \\ -2 & 4 & 6 \\ -3 & 6 & 9 \end{bmatrix}$ *whereas* $BA = -1 + 4 + 9 = 12$. *As matrices, they look quite different but it will be shown during the study of eigenvalues and eigenvectors that they have similar structure.*

7. *If* $m = n = p$, *then the orders of* AB *and* BA *are same. Even then* AB *may NOT equal* BA. *For example, if* $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ *then* $AB = \begin{bmatrix} 2 & 2 \\ -2 & -2 \end{bmatrix}$ *whereas* $BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. *Thus,* $AB \neq BA$ *and hence*

$$(A+B)^2 = A^2 + AB + BA + B^2 \neq A^2 + B^2 + 2AB.$$

Whereas if $C = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ *then* $BC = CB = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} = 3A \neq A = CA$. *Note that* **cancellation laws don't hold**.

Definition 1.3.11. Two square matrices A and B are said to **commute** if $AB = BA$.

Theorem 1.3.12. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{C}), B \in \mathbb{M}_{n,p}(\mathbb{C})$ *and* $C \in \mathbb{M}_{p,q}(\mathbb{C})$.

1. *Then* $(AB)C = A(BC)$, *i.e., the matrix multiplication is associative.*

2. *For any* $k \in \mathbb{C}$, $(kA)B = k(AB) = A(kB)$.

DRAFT

3. *Then $A(B+C) = AB + AC$, i.e., multiplication distributes over addition.*

4. *If $A \in \mathbb{M}_n(\mathbb{C})$ then $AI_n = I_n A = A$.*

Proof. (1). Verify that $(BC)_{kj} = \sum\limits_{\ell=1}^{p} b_{k\ell}c_{\ell j}$ and $(AB)_{i\ell} = \sum\limits_{k=1}^{n} a_{ik}b_{k\ell}$. Therefore,

$$\begin{aligned}\big(A(BC)\big)_{ij} &= \sum_{k=1}^{n} a_{ik}\big(BC\big)_{kj} = \sum_{k=1}^{n} a_{ik}\big(\sum_{\ell=1}^{p} b_{k\ell}c_{\ell j}\big) = \sum_{k=1}^{n}\sum_{\ell=1}^{p} a_{ik}\big(b_{k\ell}c_{\ell j}\big) \\ &= \sum_{k=1}^{n}\sum_{\ell=1}^{p}\big(a_{ik}b_{k\ell}\big)c_{\ell j} = \sum_{\ell=1}^{p}\big(\sum_{k=1}^{n} a_{ik}b_{k\ell}\big)c_{\ell j} = \sum_{\ell=1}^{T}\big(AB\big)_{i\ell}c_{\ell j} = \big((AB)C\big)_{ij}.\end{aligned}$$

Using a similar argument, the next part follows. The other parts are left for the reader. ■

EXERCISE **1.3.13.** 1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{M}_{n,1}(\mathbb{C})$ (see Definition 5). Then*

   (a) $A\mathbf{e}_1 = A[:,1], \ldots, A\mathbf{e}_n = A[:,n]$.

   (b) $\mathbf{e}_1^T A = \mathbf{e}_1^* A = A[1,:], \ldots, \mathbf{e}_n^T A = \mathbf{e}_n^* A = A[n,:]$.

2. *Let $L_1, L_2 \in \mathbb{M}_n(\mathbb{C})$ be lower triangular matrices. If $D \in \mathbb{M}_n(\mathbb{C})$ is a diagonal matrix then*

   (a) *L_1L_2 is a lower triangular matrix.*

   (b) *DL_1 and L_1D are lower triangular matrices.*

   The same holds for upper triangular matrices.

DRAFT

3. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ and $B \in \mathbb{M}_{n,p}(\mathbb{C})$.*

   (a) *Prove that $(AB)^* = B^*A^*$.*

   (b) *If $A[1,:] = \mathbf{0}^T$ then $(AB)[1,:] = \mathbf{0}^T$.*

   (c) *If $B[:,1] = \mathbf{0}$ then $(AB)[:,1] = \mathbf{0}$.*

   (d) *If $A[i,:] = A[j,:]$ for some i and j then $(AB)[i,:] = (AB)[j,:]$.*

   (e) *If $B[:,i] = B[:,j]$ for some i and j then $(AB)[:,i] = (AB)[:,j]$.*

4. *Construct matrices A and B that satisfy the following statements.*

   (a) *The product AB is defined but BA is not defined.*

   (b) *The products AB and BA are defined but they have different orders.*

   (c) *The products AB and BA are defined, they have the same order but $AB \neq BA$.*

   (d) *Construct a 2×2 matrix satisfying $A^2 = A$.*

   (e) *Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Guess a formula for A^n and B^n and prove it?*

   (f) *Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$. Is it true that $A^2 - 2A + I = \mathbf{0}$? What is $B^3 - 3B^2 + 3B - I$? Is $C^2 = 3C$?*

5. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. If $A\mathbf{x} = \mathbf{0}$ for all $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{C})$ then $A = \mathbf{0}$, the zero matrix.*

6. *Let $A, B \in \mathbb{M}_{m,n}(\mathbb{C})$. If $A\mathbf{x} = B\mathbf{x}$, for all $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{C})$ then prove that $A = B$.*

7. *Let $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{M}_{n,1}(\mathbb{C})$. Then $\mathbf{y}^*\mathbf{x} = \sum\limits_{i=1}^{n} \overline{y_i} x_i$, $\mathbf{x}^*\mathbf{x} = \sum\limits_{i=1}^{n} |x_i|^2$,*

$$\mathbf{x}\mathbf{y}^* = \begin{bmatrix} x_1\overline{y_1} & x_1\overline{y_2} & \cdots & x_1\overline{y_n} \\ \vdots & \ddots & \cdots & \vdots \\ x_n\overline{y_1} & x_n\overline{y_2} & \cdots & x_n\overline{y_n} \end{bmatrix} \text{ and } \mathbf{x}\mathbf{x}^* = \begin{bmatrix} |x_1|^2 & x_1\overline{x_2} & \cdots & x_1\overline{x_n} \\ x_2\overline{x_1} & |x_2|^2 & \cdots & x_2\overline{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ x_n\overline{x_1} & x_n\overline{x_2} & \cdots & |x_n|^2 \end{bmatrix}.$$

8. *Let A be an upper triangular matrix. If $A^*A = AA^*$ then prove that A is a diagonal matrix. The same holds for lower triangular matrix.*

9. *Let A be a 3×3 upper triangular matrix with diagonal entries a, b, c. Then*

$$(A - aI_3)(A - bI_3)(A - cI_3) = \mathbf{0}.$$

Note that $(A - aI_3)[:, 1] = \mathbf{0}$. So, if $A[:, 1] = \mathbf{0}$ then $B[1, :]$ doesn't play any role in AB.

DRAFT

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10. *Let A and B be two $m \times n$ matrices. Then, prove that $(A + B)^* = A^* + B^*$.*

11. *Find $A, B, C \in \mathbb{M}_2(\mathbb{C})$ such that $AB = AC$ but $B \neq C$ (**Cancellation laws don't hold**).*

12. *Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. Compute A^2 and A^3. Is $A^3 = I$? Determine $aA^3 + bA + cA^2$.*

1.3.4 Inverse of a Matrix

Definition 1.3.14. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then

1. $B \in \mathbb{M}_n(\mathbb{C})$ is said to be a **left inverse** of A if $BA = I_n$.

2. $C \in \mathbb{M}_n(\mathbb{C})$ is called a **right inverse** of A if $AC = I_n$.

3. A is **invertible** (has an **inverse**) if there exists $B \in \mathbb{M}_n(\mathbb{C})$ such that $AB = BA = I_n$.

Lemma 1.3.15. *Let $A \in \mathbb{M}_n(\mathbb{C})$. If there exist $B, C \in \mathbb{M}_n(\mathbb{C})$ such that $AB = I_n$ and $CA = I_n$ then $B = C$, i.e., If A has a left inverse and a right inverse then they are equal.*

Proof. Note that $C = CI_n = C(AB) = (CA)B = I_nB = B$. ∎

Remark 1.3.16. *Lemma 1.3.15 implies that whenever A is invertible, the inverse is unique. Thus, we denote the inverse of A by A^{-1}. That is, $AA^{-1} = A^{-1}A = I$.*

Theorem 1.3.17. *Let A and B be two invertible matrices. Then,*

1. $(A^{-1})^{-1} = A$.

2. $(AB)^{-1} = B^{-1}A^{-1}$.

3. $(A^*)^{-1} = (A^{-1})^*$.

Proof. (1). Let $B = A^{-1}$. Then $AB = BA = I$. Thus, by definition, B is invertible and $B^{-1} = A$. Or equivalently, $(A^{-1})^{-1} = A$.

(2). By associativity $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = I = (B^{-1}A^{-1})(AB)$.

(3). As $AA^{-1} = A^{-1}A = I$, we get $(AA^{-1})^* = (A^{-1}A)^* = I^*$. Or equivalently, $(A^{-1})^*A^* = A^*(A^{-1})^* = I$. Thus, by definition $(A^*)^{-1} = (A^{-1})^*$. ■

We will again come back to the study of invertible matrices in Sections 2.4 and 2.8.

Exercise **1.3.18.**

1. *If A is an invertible matrix then $(A^{-1})^r = A^{-r}$, for all $r \in \mathbb{N}$.*

2. *If $A_1, \ldots, A_r$ are invertible matrices then $B = A_1A_2 \cdots A_r$ is also invertible.*

3. *Find the inverse of* $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ \sin(\theta) & -\cos(\theta) \end{bmatrix}$ *and* $\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$.

4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be an invertible matrix. Then*

   (a) $A[i,:] \neq \mathbf{0}^T$, *for any i.*

   (b) $A[:,j] \neq \mathbf{0}$, *for any j.*

   (c) $A[i,:] \neq A[j,:]$, *for any i and j.*

   (d) $A[:,i] \neq A[:,j]$, *for any i and j.*

   (e) $A[3,:] \neq \alpha A[1,:] + \beta A[2,:]$, *for any $\alpha, \beta \in \mathbb{C}$, whenever $n \geq 3$.*

   (f) $A[:,3] \neq \alpha A[:,1] + \beta A[:,2]$, *for any $\alpha, \beta \in \mathbb{C}$, whenever $n \geq 3$.*

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5. *Determine A that satisfies* $(I + 3A)^{-1} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

6. *Let A be an invertible matrix satisfying $A^3 + A - 2I = \mathbf{0}$. Then* $A^{-1} = \dfrac{1}{2}\left(A^2 + I\right)$.

7. *Let $A = [a_{ij}]$ be an invertible matrix and $B = [p^{i-j}a_{ij}]$, for some $p \in \mathbb{C}$, $p \neq 0$. Then $B^{-1} = [p^{i-j}(A^{-1})_{ij}]$.*

DRAFT

1.4 Some More Special Matrices

Definition 1.4.1. 1. For $1 \le k \le m$ and $1 \le \ell \le n$, define $\mathbf{e}_{k\ell} \in \mathbb{M}_{m,n}(\mathbb{C})$ by

$$(\mathbf{e}_{k\ell})_{ij} = \begin{cases} 1, & \text{if } (k,\ell) = (i,j) \\ 0, & \text{otherwise.} \end{cases}$$

Then, the matrices $\mathbf{e}_{k\ell}$ for $1 \le k \le m$ and $1 \le \ell \le n$ are called the **standard basis** elements for $\mathbb{M}_{m,n}(\mathbb{C})$.

So, if $\mathbf{e}_{k\ell} \in \mathbb{M}_{2,3}(\mathbb{C})$ then $\mathbf{e}_{11} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \mathbf{e}_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$ and $\mathbf{e}_{22} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.

In particular, if $\mathbf{e}_{ij} \in \mathbb{M}_n(\mathbb{C})$ then $\mathbf{e}_{ij} = \mathbf{e}_i \mathbf{e}_j^T = \mathbf{e}_i \mathbf{e}_j^*$, for $1 \le i, j \le n$.

2. Let $A \in \mathbb{M}_n(\mathbb{R})$. Then

   (a) A is called **symmetric** if $A^T = A$. For example, $A = \begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix}$.

   (b) A is called **skew-symmetric** if $A^T = -A$. For example, $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$.

   (c) A is called **orthogonal** if $AA^T = A^TA = I$. For example, $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

   (d) A is said to be a **permutation matrix** if A has exactly one non-zero entry, namely 1, in each row and column. For example, I_n for each positive integer n, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ are permutation matrices. Verify that permutation matrices are Orthogonal matrices.

3. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then

   (a) A is called **normal** if $A^*A = AA^*$. For example, $\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ is a normal matrix.

   (b) A is called **Hermitian** if $A^* = A$. For example, $A = \begin{bmatrix} 1 & 1+\mathbf{i} \\ 1-\mathbf{i} & 2 \end{bmatrix}$.

   (c) A is called **skew-Hermitian** if $A^* = -A$. For example, $A = \begin{bmatrix} 0 & 1+\mathbf{i} \\ -1+\mathbf{i} & 0 \end{bmatrix}$.

   (d) A is called **unitary** if $AA^* = A^*A = I$. For example, $A = \frac{1}{\sqrt{3}} \begin{bmatrix} 1+\mathbf{i} & 1 \\ -1 & 1-\mathbf{i} \end{bmatrix}$.

   Verify that Hermitian, skew-Hermitian and Unitary matrices are normal matrices.

4. A vector $\mathbf{u} \in \mathbb{M}_{n,1}(\mathbb{C})$ such that $\mathbf{u}^*\mathbf{u} = 1$ is called a **unit vector**.

DRAFT

5. A matrix A is called **idempotent** if $A^2 = A$. For example, $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ is idempotent.

6. An idempotent matrix which is also Hermitian is called a **projection** matrix. For example, if $\mathbf{u} \in \mathbb{M}_{n,1}(\mathbb{C})$ is a unit vector then $A = \mathbf{u}\mathbf{u}^*$ is a Hermitian, idempotent matrix. Thus A is a projection matrix.

   In particular, if $\mathbf{u} \in \mathbb{M}_{n,1}(\mathbb{R})$ is a unit vector then $A = \mathbf{u}\mathbf{u}^T$. Then verify that $\mathbf{u}^T(\mathbf{x}-A\mathbf{x}) = \mathbf{u}^T\mathbf{x} - \mathbf{u}^T A\mathbf{x} = \mathbf{u}^T\mathbf{x} - \mathbf{u}^T(\mathbf{u}\mathbf{u}^T)\mathbf{x} = 0$ (as $\mathbf{u}^T\mathbf{u} = 1$), for any $\mathbf{x} \in \mathbb{R}^3$. Thus, with respect to the dot product in $\mathbb{R}^3$, $A\mathbf{x}$ is the foot of the perpendicular from the point $\mathbf{x}$ on the vector $\mathbf{u}$. In particular, if $\mathbf{u} = \dfrac{1}{\sqrt{6}}[1, 2, -1]^T$ and $A = \mathbf{u}\mathbf{u}^T$. Then, for any vector $\mathbf{x} = [x_1, x_2, x_3]^T \in \mathbb{M}_{3,1}(\mathbb{R})$,

$$A\mathbf{x} = (\mathbf{u}\mathbf{u}^T)\mathbf{x} = \mathbf{u}(\mathbf{u}^T\mathbf{x}) = \frac{x_1 + 2x_2 - x_3}{\sqrt{6}}\mathbf{u} = \frac{x_1 + 2x_2 - x_3}{6}[1, 2, -1]^T.$$

7. Fix a unit vector $\mathbf{u} \in \mathbb{M}_{n,1}(\mathbb{R})$ and let $A = 2\mathbf{u}\mathbf{u}^T - I_n$. Then, verify that $A \in \mathbb{M}_n(\mathbb{R})$ and $A\mathbf{y} = 2(\mathbf{u}^T\mathbf{y})\mathbf{u} - \mathbf{y}$, for all $\mathbf{y} \in \mathbb{R}^n$. This matrix is called the **reflection** matrix about the line, say ℓ, containing the points $\mathbf{0}$ and $\mathbf{u}$. This matrix fixes each point on the line ℓ and send the vector $\mathbf{v}$, which is orthogonal to $\mathbf{u}$, to $-\mathbf{v}$.

8. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, A is said to be **nilpotent** if there exists a positive integer n such that $A^n = \mathbf{0}$. The least positive integer k for which $A^k = \mathbf{0}$ is called the **order of nilpotency**. For example, if $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ with a_{ij} equal to 1 if $i - j = 1$ and 0, otherwise then $A^n = \mathbf{0}$ and $A^\ell \neq \mathbf{0}$ for $1 \leq \ell \leq n - 1$.

DRAFT

Exercise **1.4.2.** *1. Consider the matrices $\mathbf{e}_{ij} \in \mathbb{M}_n(\mathbb{C})$ for $1 \leq i, j, \leq n$. Is $\mathbf{e}_{12}\mathbf{e}_{11} = \mathbf{e}_{11}\mathbf{e}_{12}$? What about $\mathbf{e}_{12}\mathbf{e}_{22}$ and $\mathbf{e}_{22}\mathbf{e}_{12}$?*

2. *Let $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ be three vectors in $\mathbb{R}^3$ such that $\mathbf{u}_i^*\mathbf{u}_i = 1$, for $1 \leq i \leq 3$, and $\mathbf{u}_i^*\mathbf{u}_j = 0$ whenever $i \neq j$. Prove the following.*

   (a) *If $U = [\mathbf{u}_1\ \mathbf{u}_2\ \mathbf{u}_3]$ then $U^*U = I$. What about $UU^* = \mathbf{u}_1\mathbf{u}_1^* + \mathbf{u}_2\mathbf{u}_2^* + \mathbf{u}_3\mathbf{u}_3^*$?*

   (b) *If $A = \mathbf{u}_i\mathbf{u}_i^*$, for $1 \leq i \leq 3$ then $A^2 = A$. Is A Hermitian? Is A a projection matrix?*

   (c) *If $A = \mathbf{u}_i\mathbf{u}_i^* + \mathbf{u}_j\mathbf{u}_j^*$, for $i \neq j$ then $A^2 = A$. Is A a projection matrix?*

3. *Let A be an $n \times n$ upper triangular matrix. If A is also an orthogonal matrix then A is a diagonal matrix with diagonal entries ± 1.*

4. *Prove that in $M_5(\mathbb{R})$, there are infinitely many orthogonal matrices of which only finitely many are diagonal (in fact, there number is just 32).*

5. *Prove that permutation matrices are real orthogonal.*

6. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be two unitary matrices. Then both AB and BA are unitary matrices.*

7. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix.*

   (a) *Then the diagonal entries of A are necessarily real numbers.*

   (b) *Then, for any* $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{C})$. $\mathbf{x}^*A\mathbf{x}$ *is a real number.*

   (c) *For each* $B \in \mathbb{M}_n(\mathbb{C})$ *the matrix* B^*AB *is Hermitian.*

   (d) *Further, if* $A^2 = \mathbf{0}$ *then* $A = \mathbf{0}$.

8. *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *If* $\mathbf{x}^*A\mathbf{x} \in \mathbb{R}$ *for every* $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{C})$ *then* A *is a Hermitian matrix. [Hint: Use* $\mathbf{e}_j, \mathbf{e}_j + \mathbf{e}_k$ *and* $\mathbf{e}_j + \mathbf{i}\mathbf{e}_k$ *of* $\mathbb{M}_{n,1}(\mathbb{C})$ *for* $\mathbf{x}$.*]*

9. *Let* A *and* B *be Hermitian matrices. Then* AB *is Hermitian if and only if* $AB = BA$.

10. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *be a skew-Hermitian matrix. Then prove that*

    (a) *the diagonal entries of* A *are either zero or purely imaginary.*

    (b) *for each* $B \in \mathbb{M}_n(\mathbb{C})$ *prove that* B^*AB *is a skew-Hermitian matrix.*

    (c) *Then, for any* $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{C})$, $\mathbf{x}^*A\mathbf{x}$ *is either* 0 *or purely imaginary.*

11. *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *Then* $A = S_1 + S_2$, *where* $S_1 = \frac{1}{2}(A + A^*)$ *is Hermitian and* $S_2 = \frac{1}{2}(A - A^*)$ *is skew-Hermitian.*

12. *Let* A, B *be skew-Hermitian matrices with* $AB = BA$. *Is the matrix* AB *Hermitian or skew-Hermitian?*

13. *Let* A *be a nilpotent matrix. Then prove that* $I + A$ *is invertible.*

14. *Let* $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & \sin\theta & -\cos\theta \end{bmatrix}$, *for* $\theta \in [-\pi, \ \pi)$. *Are they orthogonal?*

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1.5 Submatrix of a Matrix

Definition 1.5.1. For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. Also, let $A \in \mathbb{M}_{m \times n}(\mathbb{C})$.

1. Then, a matrix obtained by deleting some of the rows and/or columns of A is said to be a **submatrix** of A.
2. If $S \subseteq [m]$ and $T \subseteq [n]$ then by $\mathbf{A(S|T)}$, we denote the submatrix obtained from A by deleting the rows with indices in S and columns with indices in T. By $A[S, T]$, we mean $A(S^c|T^c)$, where $S^c = [m] \setminus S$ and $T^c = [n] \setminus T$. Whenever, S or T consist of a single element, then we just write the element. If $S = [m]$, then $A[S, T] = A[:, T]$ and if $T = [n]$ then $A[S, T] = A[S, :]$ which matches with our notation in Definition 1.2.1.
3. If $m = n$, the submatrix $A[S, S]$ is called a **principal submatrix** of A.

Example 1.5.2. 1. Let $A = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 2 \end{bmatrix}$. Then, $A[\{1, 2\}, \{1, 3\}] = A[:, \{1, 3\}] = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$, $A[1, 1] = [1], A[2, 3] = [2], A[\{1, 2\}, 1] = A[:, 1] = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $A[1, \{1, 3\}] = [1 \ 5]$ and A are a few sub-matrices of A. But the matrices $\begin{bmatrix} 1 & 4 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 4 \\ 0 & 2 \end{bmatrix}$ are not sub-matrices of A.

2. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 8 & 7 \end{bmatrix}$, $S = \{1,3\}$ and $T = \{2,3\}$. Then, $A[S,S] = \begin{bmatrix} 1 & 3 \\ 9 & 7 \end{bmatrix}$, $A(S \mid S) = \begin{bmatrix} 6 \end{bmatrix}$, $A[T,T] = \begin{bmatrix} 6 & 7 \\ 8 & 7 \end{bmatrix}$ and $A(T \mid T) = \begin{bmatrix} 1 \end{bmatrix}$ are principal sub-matrices of A.

Let $A \in \mathbb{M}_{n,m}(\mathbb{C})$ and $B \in \mathbb{M}_{m,p}(\mathbb{C})$. Then the product AB is defined. Suppose $r < m$. Then A and B can be decomposed as $A = [P\ Q]$ and $B = \begin{bmatrix} H \\ K \end{bmatrix}$, where $P \in \mathbb{M}_{n,r}(\mathbb{C})$ and $H \in \mathbb{M}_{r,p}(\mathbb{C})$ so that $AB = PH + QK$. This is proved next.

Theorem 1.5.3. *Let the matrices A, B, P, H, Q and K be defined as above. Then*

$$AB = PH + QK.$$

Proof. Verify that the matrix products PH and QK are valid. Further, their sum is defined as $PH, QK \in \mathbb{M}_{n,p}(\mathbb{C})$. Now, let $P = [P_{ij}]$, $Q = [Q_{ij}]$, $H = [H_{ij}]$, and $K = [K_{ij}]$. Then, for $1 \le i \le n$ and $1 \le j \le p$, we have

$$\begin{aligned} (AB)_{ij} &= \sum_{k=1}^{m} a_{ik}b_{kj} = \sum_{k=1}^{r} a_{ik}b_{kj} + \sum_{k=r+1}^{m} a_{ik}b_{kj} = \sum_{k=1}^{r} P_{ik}H_{kj} + \sum_{k=r+1}^{m} Q_{ik}K_{kj} \\ &= (PH)_{ij} + (QK)_{ij} = (PH+QK)_{ij}. \end{aligned}$$

Thus, the required result follows. ∎

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Remark 1.5.4. *Theorem 1.5.3 is very useful due to the following reasons:*

1. *The matrices P, Q, H and K can be further partitioned so as to form blocks that are either identity or zero or have certain nice properties. So, such partitions are useful during different matrix operations. Examples of such partitions appear throughout the notes. For example, let $A = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, $P = \begin{bmatrix} P_1 & P_2 \end{bmatrix}$ and $Q = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}$. Then, verify that $PAQ = P_1Q_1$. This is similar to the understanding that*

$$\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1a_{11}y_1 + x_1a_{12}y_2 + x_2a_{21}y_1 + x_2a_{22}y_2.$$

2. *Suppose one wants to prove a result for a square matrix A. If we want to prove it using induction then we can prove it for the 1×1 matrix (the initial step of induction). Then assume the result to hold for all $k \times k$ sub-matrices of A or just the first $k \times k$ principal sub-matrix of A. At the next step write $A = \begin{bmatrix} B & \mathbf{x} \\ \mathbf{x}^T & a \end{bmatrix}$, where B is a $k \times k$ matrix. Then the result holds for B and then one can proceed to prove it for A.*

Exercise **1.5.5.** 1. *Complete the proofs of Theorems 1.3.5 and 1.3.12.*

2. *Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and $B = \begin{bmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{bmatrix}$.*

(a) *Then* $\mathbf{y} = A\mathbf{x}$ *gives the counter-clockwise rotation through an angle* α.

(b) *Then* $\mathbf{y} = B\mathbf{x}$ *gives the reflection about the line* $y = \tan(\theta)x$.

(c) *Let* $\alpha = \theta$ *and compute* $\mathbf{y} = (AB)\mathbf{x}$ *and* $\mathbf{y} = (BA)\mathbf{x}$. *Do they correspond to reflection? If yes, then about which line(s)?*

(d) *Further, if* $\mathbf{y} = C\mathbf{x}$ *gives the counter-clockwise rotation through* β *and* $\mathbf{y} = D\mathbf{x}$ *gives the reflections about the line* $y = \tan(\delta)\, x$. *Then prove that*

i. $AC = CA$ *and* $\mathbf{y} = (AC)\mathbf{x}$ *gives the counter-clockwise rotation through* $\alpha + \beta$.

ii. $\mathbf{y} = (BD)\mathbf{x}$ *and* $\mathbf{y} = (DB)\mathbf{x}$ *give rotations. Which angles do they represent?*

3. *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *If* $AB = BA$ *for all* $B \in \mathbb{M}_n(\mathbb{C})$ *then* A *is a scalar matrix, i.e.,* $A = \alpha I$ *for some* $\alpha \in \mathbb{C}$ *(use the matrices* $\mathbf{e}_{ij}$ *in Definition 1.4.1.1).*

4. *For* $A_{n\times n} = [a_{ij}]$, *the* **trace** *of* A, *denoted* $\mathsf{tr(A)}$, *is defined by* $\mathsf{tr(A)} = \mathsf{a_{11}} + \mathsf{a_{22}} + \cdots + \mathsf{a_{nn}}$.

(a) *Compute* $\mathsf{tr(A)}$ *for* $A = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$ *and* $A = \begin{bmatrix} 4 & -3 \\ -5 & 1 \end{bmatrix}$.

(b) *Let* A *be a matrix with* $A\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ *and* $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. *Determine* $\mathsf{tr(A)}$?

(c) *Let* A *and* B *be two square matrices of the same order. Then*

i. $\mathsf{tr(A + B)} = \mathsf{tr(A)} + \mathsf{tr(B)}$.

ii. $\mathsf{tr(AB)} = \mathsf{tr(BA)}$.

DRAFT

(d) *Does there exist matrices* $A, B \in \mathbb{M}_n(\mathbb{C})$ *such that* $AB - BA = cI$, *for some* $c \neq 0$?

5. *Let* $J \in \mathbb{M}_n(\mathbb{R})$ *be a matrix having each entry* 1.

(a) *Verify that* $J = \mathbf{1}\mathbf{1}^T$, *where* $\mathbf{1}$ *is a column vector having all entries* 1.

(b) *Verify that* $J^2 = nJ$.

(c) *Also, for any* $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$, *verify that there exist* $\alpha_3, \beta_3 \in \mathbb{R}$ *such that*

$$(\alpha_1 I_n + \beta_1 J)\cdot(\alpha_2 I_n + \beta_2 J) = \alpha_3 I_n + \beta_3 J.$$

(d) *Let* $\alpha, \beta \in \mathbb{R}$ *such that* $\alpha \neq 0$ *and* $\alpha + n\beta \neq 0$. *Now, define* $A = \alpha I_n + \beta J$. *Then, use the above to prove that* A *is invertible.*

---

6. *Suppose the matrices* B *and* C *are invertible and the involved partitioned products are defined, then verify that that*

$$\begin{bmatrix} A & B \\ C & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{0} & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}.$$

7. *Let* $A = \begin{bmatrix} A_{11} & \mathbf{x} \\ \mathbf{y}^* & c \end{bmatrix}$, *where* $A_{11} \in \mathbb{M}_n(\mathbb{C})$ *is invertible and* $c \in \mathbb{C}$.

(a) *If* $p = c - \mathbf{y}^* A_{11}^{-1}\mathbf{x}$ *is non zero, then verify that*

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \frac{1}{p}\begin{bmatrix} A_{11}^{-1}\mathbf{x} \\ -1 \end{bmatrix}\begin{bmatrix} \mathbf{y}^* A_{11}^{-1} & -1 \end{bmatrix}.$$

(b) *Use the above to find the inverse of* $\left[\begin{array}{rr|r} 0 & -1 & 2 \\ 1 & 1 & 4 \\ \hline -2 & 1 & 1 \end{array}\right]$ *and* $\left[\begin{array}{rr|r} 0 & -1 & 2 \\ 3 & 1 & 4 \\ \hline -2 & 5 & -3 \end{array}\right]$.

8. *Let* $\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{R})$ *be a unit vector (recall the reflection matrix).*

   (a) *Define* $A = I_n - 2\mathbf{x}\mathbf{x}^T$. *Prove that A is symmetric and $A^2 = I$. The matrix A is commonly known as the* **Householder matrix**.

   (b) *Let $\alpha \neq 1$ be a real number and define $A = I_n - \alpha\mathbf{x}\mathbf{x}^T$. Prove that A is symmetric and invertible. [The inverse is also of the form $I_n + \beta\mathbf{x}\mathbf{x}^T$, for some β.]*

9. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be an invertible matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{M}_{n,1}(\mathbb{R})$. Also, let $\beta \in \mathbb{R}$ such that $\alpha = 1 + \beta\mathbf{y}^T A^{-1}\mathbf{x} \neq 0$. Then, verify the famous Shermon-Morrison formula*

$$(A + \beta\mathbf{x}\mathbf{y}^T)^{-1} = A^{-1} - \frac{\beta}{\alpha}A^{-1}\mathbf{x}\mathbf{y}^T A^{-1}.$$

   This formula gives the information about the inverse when an invertible matrix is modified by a rank (see Definition 2.5.1) one matrix.

DRAFT

10. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. Then, a matrix $G \in \mathbb{M}_{n,m}(\mathbb{C})$ is called a* **generalized inverse** *(for short, g-inverse) of A if $AGA = A$.*

    (a) *For example, a generalized inverse of the matrix $A = [1, 2]$ is a matrix $G = \begin{bmatrix} 1 - 2\alpha \\ \alpha \end{bmatrix}$, for all $\alpha \in \mathbb{R}$. So, for a fixed singular matrix A, there are infinitely many g-inverses.*

    (b) *A generalized inverse G is called a* **pseudo inverse** *or a* **Moore-Penrose inverse** *if $GAG = G$ and the matrices AG and GA are symmetric. Thus, verify that AG and GA are both idempotent matrices. It can also be shown that the pseudo inverse of a matrix is unique. Check that for $\alpha = \frac{2}{5}$ the matrix G is a pseudo inverse of A.*

    (c) *It turns out that among all the g-inverses, the inverse with the least euclidean norm is the pseudo inverse.*

1.6 Summary

In this chapter, we started with the definition of a matrix and came across lots of examples. We recall these examples as they will be used in later chapters to relate different ideas:

1. The zero matrix of size $m \times n$, denoted $\mathbf{0}_{m\times n}$ or $\mathbf{0}$.

2. The identity matrix of size $n \times n$, denoted I_n or I.

3. Triangular matrices.

4. Hermitian/Symmetric matrices.

5. Skew-Hermitian/skew-symmetric matrices.

6. Unitary/Orthogonal matrices.

7. Idempotent matrices.

8. Nilpotent matrices.

We also learnt product of two matrices. Even though it seemed complicated, it basically tells that multiplying by a matrix on the

1. left of A is same as operating on (playing with) the rows of A.

2. right of A is same as operating on (playing with) the columns of A.

The matrix multiplication is not commutative. We also defined the inverse of a matrix. Further, there were exercises that informs us that the rows and columns of invertible matrices cannot have certain properties.

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Chapter 2

System of Linear Equations

2.1 Introduction

We start this section with our understanding of the system of linear equations.

Example 2.1.1. Let us look at some examples of linear systems.

1. Suppose $a, b \in \mathbb{R}$. Consider the system $ax = b$ in the variable x. If

   (a) $a \neq 0$ then the system has a UNIQUE SOLUTION $x = \frac{b}{a}$.

   (b) $a = 0$ and

      i. $b \neq 0$ then the system has NO SOLUTION.

      ii. $b = 0$ then the system has INFINITE NUMBER OF SOLUTIONS, namely all $x \in \mathbb{R}$.

DRAFT

2. Recall that the linear system $ax + by = c$ for $(a, b) \neq (0, 0)$, in the variables x and y, represents a line in $\mathbb{R}^2$. So, let us consider the points of intersection of the two lines

$$a_1x + b_1y = c_1, \; a_2x + b_2y = c_2, \tag{2.1.1}$$

   where $a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$ with $(a_1, b_1), (a_2, b_2) \neq (0, 0)$ (see Figure 2.1 for illustration of different cases).

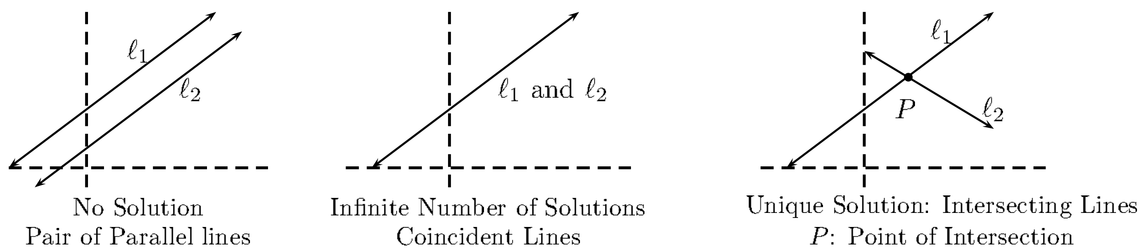

Figure 2.1: Examples in 2 dimension.

   (a) UNIQUE SOLUTION $(a_1b_2 - a_2b_1 \neq 0)$: The linear system $x - y = 3$ and $2x + 3y = 11$ has $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ as the unique solution.

(b) NO SOLUTION ($a_1b_2 - a_2b_1 = 0$ but $a_1c_2 - a_2c_1 \neq 0$): The linear system $x + 2y = 1$ and $2x + 4y = 3$ represent a pair of parallel lines which have no point of intersection.

(c) INFINITE NUMBER OF SOLUTIONS ($a_1b_2 - a_2b_1 = 0$ and $a_1c_2 - a_2c_1 = 0$): The linear system $x + 2y = 1$ and $2x + 4y = 2$ represent the same line. So, the solution set equals $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1-2y \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ with y arbitrary. Observe that the vector

   i. $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ corresponds to the solution $x = 1, y = 0$ of the given system.

   ii. $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ gives $x = -2, y = 1$ as the solution of $x + 2y = 0, 2x + 4y = 0$.

(d) If the linear system $ax + by = c$ has
   i. $(a, b) = (0, 0)$ and $c \neq 0$ then $ax + by = c$ has no solution.
   ii. $(a, b, c) = (0, 0, 0)$ then $ax + by = c$ has INFINITE NUMBER OF SOLUTIONS, namely whole of $\mathbb{R}^2$.

Let us now look at different interpretations of the solution concept.

Example 2.1.2. Observe the following of the linear system in Example 2.1.1.2a.

1. $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$ corresponds to the point of intersection of the corresponding two lines.

2. Using matrix multiplication, the given system equals $A\mathbf{x} = \mathbf{b}$, where $A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$. So, the solution is $\mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 11 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$.

3. Re-writing $A\mathbf{x} = \mathbf{b}$ as $\begin{bmatrix} 1 \\ 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ 3 \end{bmatrix} y = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$ gives us $4\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 11 \end{bmatrix}$. This corresponds to addition of vectors in the Euclidean plane.

Thus, there are three ways of looking at the linear system $A\mathbf{x} = \mathbf{b}$, where, as the name suggests, one of the ways is looking at the point of intersection of planes, the other is the vector sum approach and the third is the matrix multiplication approach. We will see that all the three approaches are fundamental to the understanding of linear algebra.

Definition 2.1.3. A system of m linear equations in n variables $x_1, x_2, \ldots, x_n$ is a set of equations of the form

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n & = & b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & = & b_2 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & = & b_m \end{array} \tag{2.1.2}$$

where for $1 \leq i \leq m$ and $1 \leq j \leq n$; $a_{ij}, b_i \in \mathbb{R}$. The linear system (2.1.2) is called **homogeneous** if $b_1 = 0 = b_2 = \cdots = b_m$ and **non-homogeneous**, otherwise.

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Definition 2.1.4. Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$. Then, Equation (2.1.2) can be re-written as $A\mathbf{x} = \mathbf{b}$, where A is called the **coefficient matrix** and the block matrix $[A \ \ \mathbf{b}]$ is called the **augmented matrix** .

In the above definition, note the following.

1. the i-th row of the augmented matrix, namely, $([A \ \ \mathbf{b}])[i,:]$, corresponds to the i-th linear equation.
2. the j-th column of the augmented matrix, namely, $([A \ \ \mathbf{b}])[:,j]$, corresponds to the j-th unknown/variable whenever $1 \le j \le n$ and
3. the $(n+1)$-th column, namely $([A \ \ \mathbf{b}])[:,n+1]$, corresponds to the vector $\mathbf{b}$.

Definition 2.1.5. Consider a linear system $A\mathbf{x} = \mathbf{b}$. Then

1. a **solution** of $A\mathbf{x} = \mathbf{b}$ is a vector $\mathbf{y}$ such that the matrix product $A\mathbf{y}$ indeed equals $\mathbf{b}$.
2. the set of all solutions is called the **solution set** of the system.
3. this linear system is called **consistent** if it admits a solution and is called **inconsistent** if it admits no solution.

For example, $A\mathbf{x} = \mathbf{b}$, with $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 4 & 2 \\ 4 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ has $\left\{ \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} \right\}$ as the solution set. Similarly, $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ has $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ as the solution set. Further, they are consistent systems. Whereas, the system $x + y = 2, 2x + 2y = 3$ is inconsistent (has no solution).

DRAFT

Definition 2.1.6. For the linear system $A\mathbf{x} = \mathbf{b}$ the corresponding linear homogeneous system $A\mathbf{x} = \mathbf{0}$ is called the **associated homogeneous system**.

The readers are advised to supply the proof of the next remark.

Remark 2.1.7. *Consider the linear system $A\mathbf{x} = \mathbf{b}$ with two distinct solutions, say $\mathbf{u}$ and $\mathbf{v}$.*

1. *Then $\mathbf{x}_h = \mathbf{u} - \mathbf{v}$ is a non-zero solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.*
2. *Thus, any two distinct solutions of $A\mathbf{x} = \mathbf{b}$ differs by a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$, i.e., $\{\mathbf{x}_0 + \mathbf{x}_h\}$ is the solution set of $A\mathbf{x} = \mathbf{b}$ with $\mathbf{x}_0$ as a particular solution and $\mathbf{x}_h$, a solution of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.*
3. *Equivalently, $A(\alpha\mathbf{u} + (1-\alpha)\mathbf{v}) = \alpha A\mathbf{u} + (1-\alpha)A\mathbf{v} = \alpha\mathbf{b} + (1-\alpha)\mathbf{b} = \mathbf{b}$. Thus, the line joining the two points $\mathbf{u}$ and $\mathbf{v}$ is also a solution of the system $A\mathbf{x} = \mathbf{b}$.*
4. *Now, consider the associated homogeneous linear system $A\mathbf{x} = \mathbf{0}$.*
   (a) *Then, $\mathbf{x} = \mathbf{0}$, the zero vector, is always a solution, called the* **trivial** *solution.*

(b) *Let $\mathbf{w} \neq \mathbf{0}$ be a solution of $A\mathbf{x} = \mathbf{0}$. Then $\mathbf{w}$ is called a* **non-trivial** *solution. Thus $\mathbf{y} = c\mathbf{w}$ is also a solution for all $c \in \mathbb{R}$. So, a non-trivial solution implies the solution set has infinite number of elements.*

(c) *Let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be solutions of $A\mathbf{x} = \mathbf{0}$. Then, $\sum\limits_{i=1}^{k} a_i\mathbf{w}_i$ is also a solution of $A\mathbf{x} = \mathbf{0}$, for each choice of $a_i \in \mathbb{R}, 1 \leq i \leq k$.*

Example 2.1.8. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then $\mathbf{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is a non-trivial solution of $A\mathbf{x} = \mathbf{0}$.

EXERCISE **2.1.9.** *1. Consider a system of 2 equations in 3 variables. If this system is consistent then how many solutions does it have?*

2. Give a linear system of 3 equations in 2 variables such that the system is inconsistent whereas it has 2 equations which form a consistent system.

3. Give a linear system of 4 equations in 3 variables such that the system is inconsistent whereas it has three equations which form a consistent system.

4. Let $A\mathbf{x} = \mathbf{b}$ be a system of m equations in n variables, where $A \in \mathbb{M}_{m,n}(\mathbb{R})$.

(a) *Can the system, $A\mathbf{x} = \mathbf{b}$ have exactly two distinct solutions for any choice of m and n? Give reasons for your answer.*

(b) *Can the system $A\mathbf{x} = \mathbf{b}$ have only a finitely many (greater than 1) solutions for any choice of m and n? Give reasons for your answer.*

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2.2 Row-Reduced Echelon Form (RREF)

A system of linear equations can be solved by people differently. But, the final solution set remains the same. In this section, we use a systematic way to solve any linear system which is popularly known as the Guass - Jordan. We start with Gauss Elimination Method.

2.2.1 Gauss Elimination Method

To proceed with the understanding of the solution set of a system of linear equations, we start with the definition of a pivot.

Definition 2.2.1. Let A be a non-zero matrix. Then, in each non-zero row of A, the left most non-zero entry is called a **pivot/leading entry**. The column containing the pivot is called a **pivotal column**.

If a_{ij} is a pivot then we denote it by $\boxed{a_{ij}}$. For example, the entries a_{12} and a_{23} are pivots in $A = \begin{bmatrix} 0 & \boxed{3} & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{2} & 1 \end{bmatrix}$. Thus, columns 2 and 3 are pivotal columns.

Definition 2.2.2. A matrix is in **row echelon form (REF)** (staircase/ ladder like)

1. if the zero rows are at the bottom;

2. if the pivot of the $(i+1)$-th row, if it exists, comes to the right of the pivot of the i-th row.

3. if the entries below the pivot in a pivotal column are 0.

Example 2.2.3. 1. The following matrices are in echelon form.

$$\begin{bmatrix} 0 & \boxed{2} & 4 & 2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \boxed{1} & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & \boxed{3} & 4 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} \boxed{1} & 2 & 0 & 5 \\ 0 & \boxed{2} & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}.$$

2. The following matrices are not in echelon form (determine the rule(s) that fail).

$$\begin{bmatrix} 0 & \boxed{1} & 4 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{bmatrix} \text{ and } \begin{bmatrix} \boxed{1} & 1 & 0 & 2 & 3 \\ 0 & 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & \boxed{1} & 4 \end{bmatrix}.$$

We now start with solving two systems of linear equations. The idea is to manipulate the rows of the augmented matrix in place of the linear equations themselves. Since, multiplying a matrix on the left corresponds to row operations, we left multiply by certain matrices to the augmented matrix so that the final matrix is in row echelon form (REF). The process of obtaining the REF of a matrix is called the **Gauss Elimination method**. The readers should carefully look at the matrices being multiplied on the left in the examples given below.

Example 2.2.4. 1. Solve the linear system $y+z=2,\ 2x+3z=5,\ x+y+z=3$.

Solution: Let $B_0 = [A \ \ \mathbf{b}]$, the augmented matrix. Then, $B_0 = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 2 & 0 & 3 & 5 \\ 1 & 1 & 1 & 3 \end{bmatrix}$. We now systematically proceed to get the solution.

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(a) Interchange 1-st and 2-nd equations (interchange $B_0[1,:]$ and $B_0[2,:]$ to get B_1).

$$\begin{aligned} 2x+3z &= 5 \\ y+z &= 2 \\ x+y+z &= 3 \end{aligned} \qquad B_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} B_0 = \begin{bmatrix} \boxed{2} & 0 & 3 & 5 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 3 \end{bmatrix}.$$

(b) In the new system, replace 3-rd equation by 3-rd equation minus $\frac{1}{2}$ times the 1-st equation (replace $B_1[3,:]$ by $B_1[3,:] - \frac{1}{2}B_1[1,:]$ to get B_2).

$$\begin{aligned} 2x+3z &= 5 \\ y+z &= 2 \\ y-\tfrac{1}{2}z &= \tfrac{1}{2} \end{aligned} \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1/2 & 0 & 1 \end{bmatrix} B_1 = \begin{bmatrix} \boxed{2} & 0 & 3 & 5 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 1 & -1/2 & 1/2 \end{bmatrix}.$$

(c) In the new system, replace 3-rd equation by 3-rd equation minus 2-nd equation (replace $B_2[3,:]$ by $B_2[3,:] - B_2[2,:]$ to get B_3).

$$\begin{aligned} 2x+3z &= 5 \\ y+z &= 2 \\ -\tfrac{3}{2}z &= -\tfrac{3}{2} \end{aligned} \qquad B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} B_2 = \begin{bmatrix} \boxed{2} & 0 & 3 & 5 \\ 0 & \boxed{1} & 1 & 2 \\ 0 & 0 & \boxed{\text{-3/2}} & -3/2 \end{bmatrix}.$$

Observe that the matrix B_3 is in REF. Using the last row of B_3, we get $z = 1$. Using this and the second row of B_3 gives $y = 1$. Finally, the first row gives $x = 1$. Hence, the solution set of $A\mathbf{x} = \mathbf{b}$ is $\{[x, y, z]^T \mid [x, y, z] = [1, 1, 1]\}$, A UNIQUE SOLUTION. The method of finding the values of the unknowns y and x, using the 2-nd and 1-st row of B_3 and the value of z is called **back substitution**.

2. Solve the linear system $x + y + z = 4,\ 2x + 3z = 5,\ y + z = 3$.

Solution: Let $B_0 = [A\ \ \mathbf{b}] = \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}$ be the augmented matrix. Then

(a) The given system looks like (correspond to the augment matrix B_0).

$$\begin{aligned} x + y + z &= 4 \\ 2x + 3z &= 5 \\ y + z &= 3 \end{aligned} \qquad B_0 = \begin{bmatrix} \boxed{1} & 1 & 1 & 4 \\ 2 & 0 & 3 & 5 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

(b) In the new system, replace 2-nd equation by 2-nd equation minus 2 times the 1-st equation (replace $B_0[2, :]$ by $B_0[2, :] - 2 \cdot B_0[1, :]$ to get B_1).

$$\begin{aligned} x + y + z = &\ \ 4 \\ -2y + z = &\ -3 \\ y + z = &\ \ 3 \end{aligned} \qquad B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} B_0 = \begin{bmatrix} \boxed{1} & 1 & 1 & 4 \\ 0 & \boxed{-2} & 1 & -3 \\ 0 & 1 & 1 & 3 \end{bmatrix}.$$

(c) In the new system, replace 3-rd equation by 3-rd equation plus 1/2 times the 2-nd equation (replace $B_1[3, :]$ by $B_1[3, :] + 1/2 \cdot B_1[2, :]$ to get B_2).

$$\begin{aligned} x + y + z = &\ \ 4 \\ -2y + z = &\ -3 \\ \frac{3}{2}z = &\ \ \frac{3}{2} \end{aligned} \qquad B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1/2 & 1 \end{bmatrix} B_1 = \begin{bmatrix} \boxed{1} & 1 & 1 & 4 \\ 0 & \boxed{-2} & 1 & -3 \\ 0 & 0 & \boxed{3/2} & 3/2 \end{bmatrix}.$$

Observe that the matrix B_2 is in REF. Verify that the solution set is $\{[x, y, z]^T \mid [x, y, z] = [1, 2, 1]\}$, again A UNIQUE SOLUTION.

In both the Examples, observe the following.

1. Each operation on the linear system corresponds to a similar operation on the rows of the augmented matrix.

2. At each stage, the new augmented matrix was obtained by left multiplication by a matrix, say E. Note that E is obtained by changing exactly one row of the identity matrix. The readers should find the relationship between the matrix E and the row operations that have been made on the augmented matrix.

3. Thus, the task of understanding the solution set of a linear system reduces to understanding the form of the matrix E.

4. Note that E corresponds to a row operation made on the identity matrix I_3.

DRAFT

5. Also, for each matrix E note that we have a matrix F, again a variant of I_3 such that $EF = I_3 = FE$.

We use the above ideas to define elementary row operations and the corresponding elementary matrices in the next subsection.

2.2.2 Elementary Row Operations

Definition 2.2.5. Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. Then, the **elementary row operations** are

1. E_{ij}: Interchange the i-th and j-th rows, namely, interchange $A[i,:]$ and $A[j,:]$.
2. $E_k(c)$ for $c \neq 0$: Multiply the k-th row by c, namely, multiply $A[k,:]$ by c.
3. $E_{ij}(c)$ for $c \neq 0$: Replace the i-th row by i-th row plus c-times the j-th row, namely, replace $A[i,:]$ by $A[i,:] + cA[j,:]$.

Definition 2.2.6. A matrix $E \in \mathbb{M}_m(\mathbb{R})$ is called an **elementary matrix** if it is obtained by applying exactly one elementary row operation to the identity matrix I_m.

For better understanding we give the elementary matrices for $m = 3$.

Notation	**Operations on Equations**	**Elementary Row Operations: A is $3 \times n$ matrix**	**Elementary Matrix**
$E_2(c)$, $c \neq 0$; $E_k(c)$, $c \neq 0$	Multiply the 2-th row by c	$A[2,:] \leftarrow cA[2,:]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & 1 \end{bmatrix}$
$E_{21}(c)$, $c \neq 0$; $E_{ij}(c)$, $c \neq 0$	Replace 2-th row by 2-nd row plus c-times 1-st row	$A[2,:] \leftarrow A[2,:] + cA[1,:]$	$\begin{bmatrix} 1 & 0 & 0 \\ c & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
E_{23}; E_{ij}	Interchange 2-nd and 3-rd rows	Interchange $A[2,:]$ and $A[3,:]$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

DRAFT

Example 2.2.7. Verify that $E_2(5) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{12}(-5) = \begin{bmatrix} 1 & -5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $E_{13} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ are elementary matrices.

EXERCISE **2.2.8.** *1. Which of the following matrices are elementary?*

$$\begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

2. Find some elementary matrices $E_1, \ldots, E_k$ such that $E_k \cdots E_1 \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = I_2$.

We now give the elementary matrices for general n.

Example 2.2.9. Let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be the standard unit vectors of $\mathbb{M}_{n,1}(\mathbb{R})$. Then, using $\mathbf{e}_i^T\mathbf{e}_j = 0 = \mathbf{e}_j^T\mathbf{e}_i$ and $\mathbf{e}_i^T\mathbf{e}_i = 1 = \mathbf{e}_j^T\mathbf{e}_j$, verify that each elementary matrix is invertible.

1. $E_k(c) = I_n + (c-1)\mathbf{e}_k\mathbf{e}_k^T$ for $c \neq 0$. Verify that
$$E_k(c)E_k(1/c) = \left(I_n + (c-1)\mathbf{e}_k\mathbf{e}_k^T\right)\left(I_n + (1/c-1)\mathbf{e}_k\mathbf{e}_k^T\right) = I_n = E_k(1/c)E_k(c).$$

2. $E_{ij}(c) = I_n + c\ \mathbf{e}_i\mathbf{e}_j^T$ for $c \neq 0$. Verify that
$$E_{ij}(c)E_{ij}(-c) = \left(I_n + c\ \mathbf{e}_i\mathbf{e}_j^T\right)\left(I_n - c\ \mathbf{e}_i\mathbf{e}_j^T\right) = I_n = E_{ij}(-c)E_{ij}(c).$$

3. $E_{ij} = I_n - \mathbf{e}_i\mathbf{e}_i^T - \mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T$. Verify that
$$E_{ij}E_{ij} = \left(I_n - \mathbf{e}_i\mathbf{e}_i^T - \mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T\right)\left(I_n - \mathbf{e}_i\mathbf{e}_i^T - \mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T\right) = I_n.$$

We now show that the above elementary matrices correspond to respective row operations.

Remark 2.2.10. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$.*

1. *For $c \neq 0$, $E_k(c)A$ corresponds to the replacement of $A[k,:]$ by $cA[k,:]$.*
Using $\mathbf{e}_k^T\mathbf{e}_k = 1$ and $A[k,:] = \mathbf{e}_k^TA$, we get
$$\begin{aligned}(E_k(c)A)[k,:] &= \mathbf{e}_k^T\left(E_k(c)A\right) = \mathbf{e}_k^T\left(I_m + (c-1)\mathbf{e}_k\mathbf{e}_k^T\right)A = \left(\mathbf{e}_k^T + (c-1)\mathbf{e}_k^T(\mathbf{e}_k\mathbf{e}_k^T)\right)A \\ &= \left(\mathbf{e}_k^T + (c-1)\mathbf{e}_k^T\right)A = c\mathbf{e}_k^TA = cA[k,:].\end{aligned}$$
A similar argument with $\mathbf{e}_i^T\mathbf{e}_k = 0$, for $i \neq k$, gives $(E_k(c)A)[i,:] = A[i,:]$, for $i \neq k$.

2. *For $c \neq 0$, $E_{ij}(c)A$ corresponds to the replacement of $A[i,:]$ by $A[i,:] + cA[j,:]$.*
Using $\mathbf{e}_i^T\mathbf{e}_i = 1$ and $A[i,:] = \mathbf{e}_i^TA$, we get
$$\begin{aligned}(E_{ij}(c)A)[i,:] &= \mathbf{e}_i^T\left(E_{ij}(c)A\right) = \mathbf{e}_i^T\left(I_m + c\ \mathbf{e}_i\mathbf{e}_j^T\right)A = \left(\mathbf{e}_i^T + c\ \mathbf{e}_i^T(\mathbf{e}_i\mathbf{e}_j^T)\right)A \\ &= \left(\mathbf{e}_i^T + c\ \mathbf{e}_j^T\right)A = A[i,:] + cA[j,:].\end{aligned}$$
A similar argument with $\mathbf{e}_k^T\mathbf{e}_i = 0$, for $k \neq i$, gives $(E_{ij}(c)A)[k,:] = A[k,:]$, for $k \neq i$.

3. *$E_{ij}A$ corresponds to interchange of $A[i,:]$ and $A[j,:]$.*
Using $\mathbf{e}_i^T\mathbf{e}_i = 1, \mathbf{e}_i^T\mathbf{e}_j = 0$ and $A[i,:] = \mathbf{e}_i^TA$, we get
$$\begin{aligned}(E_{ij}A)[i,:] &= \mathbf{e}_i^T\left(E_{ij}A\right) = \mathbf{e}_i^T\left(I_m - \mathbf{e}_i\mathbf{e}_i^T - \mathbf{e}_j\mathbf{e}_j^T + \mathbf{e}_i\mathbf{e}_j^T + \mathbf{e}_j\mathbf{e}_i^T\right)A \\ &= \left(\mathbf{e}_i^T - \mathbf{e}_i^T - \mathbf{0}^T + \mathbf{e}_j^T + \mathbf{0}^T\right)A = \mathbf{e}_j^TA = A[j,:]\end{aligned}$$
Similarly, using $\mathbf{e}_j^T\mathbf{e}_j = 1, \mathbf{e}_j^T\mathbf{e}_i = 0$ and $A[j,:] = \mathbf{e}_j^TA$ show that $(E_{ij}A)[j,:] = A[i,:]$. Further, using $\mathbf{e}_k^T\mathbf{e}_i = 0 = \mathbf{e}_k^T\mathbf{e}_j$, for $k \neq i,j$ show that $(E_{ij}A)[k,:] = A[k,:]$.

Definition 2.2.11. Two matrices A and B are said to be **row equivalent** if one can be obtained from the other by a finite number of elementary row operations. Or equivalently, there exists elementary matrices $E_1, \ldots, E_k$ such that $B = E_1 \cdots E_kA$.

DRAFT

Definition 2.2.12. The linear systems $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ are said to be **row equivalent** if their respective augmented matrices, $[A \ \ \mathbf{b}]$ and $[C \ \ \mathbf{d}]$, are row equivalent.

Thus, note that the linear systems at each step in Example 2.2.4 are row equivalent to each other. We now prove that the solution set of two row equivalent linear systems are same.

Theorem 2.2.13. *Let $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ be two row equivalent linear systems. Then they have the same solution set.*

Proof. Let $E_1, \ldots, E_k$ be the elementary matrices such that $E_1 \cdots E_k[A \ \ \mathbf{b}] = [C \ \ \mathbf{d}]$. Put $E = E_1 \cdots E_k$. Then

$$EA = C, \ \ E\,\mathbf{b} = \mathbf{d}, \ \ A = E^{-1}C \text{ and } \mathbf{b} = E^{-1}\,\mathbf{d}. \tag{2.2.3}$$

Now assume that $A\mathbf{y} = \mathbf{b}$ holds. Then, by Equation (2.2.3)

$$C\,\mathbf{y} = EA\,\mathbf{y} = E\,\mathbf{b} = \mathbf{d}. \tag{2.2.4}$$

On the other hand if $C\,\mathbf{z} = \mathbf{d}$ holds then using Equation (2.2.3), we have

$$A\,\mathbf{z} = E^{-1}C\,\mathbf{z} = E^{-1}\mathbf{d} = \mathbf{b}. \tag{2.2.5}$$

Therefore, using Equations (2.2.4) and (2.2.5) the required result follows. ∎

The following result is a particular case of Theorem 2.2.13.

DRAFT

Corollary 2.2.14. *Let A and B be two row equivalent matrices. Then, the systems $A\mathbf{x} = \mathbf{0}$ and $B\mathbf{x} = \mathbf{0}$ have the same solution set.*

Example 2.2.15. Are the matrices $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 0 \end{bmatrix}$ row equivalent?

Solution: No, as $\begin{bmatrix} a \\ b \\ -1 \end{bmatrix}$ is a solution of $B\mathbf{x} = \mathbf{0}$ but it isn't a solution of $A\mathbf{x} = \mathbf{0}$.

The following exercise shows that every square matrix is row equivalent to an upper triangular matrix. We will come back to this idea again in the chapter titled "Advanced Topics".

EXERCISE **2.2.16.** *1. Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R})$. Then there exists an orthogonal matrix U such that UA is upper triangular. The proof uses the following ideas.*

(a) If $A[1,:] = \mathbf{0}$ then proceed to the next column. Else, $A[:,1] \neq \mathbf{0}$.

(b) If $A[:,1] = \alpha\mathbf{e}_1$, for some $\alpha \in \mathbb{R}, \alpha \neq 0$, proceed to the next column. Else, either $a_{11} = 0$ or $a_{11} \neq 0$.

(c) If $a_{11} = 0$ then left multiply A with E_{1i} (an orthogonal matrix) so that the $(1,1)$ entry of $B = E_{1i}A$ is non-zero. Hence, without loss of generality, let $a_{11} \neq 0$.

(d) Let $[w_1, \ldots, w_n]^T = \mathbf{w} \in \mathbb{R}^n$ with $w_1 \neq 0$. Then use the Householder matrix H such that $H\mathbf{w} = w_1\mathbf{e}_1$, i.e., find $\mathbf{x} \in \mathbb{R}^n$ such that $(I_n - 2\mathbf{x}\mathbf{x}^T)\mathbf{w} = w_1\mathbf{e}_1$.

(e) *So, Part 1d gives an orthogonal matrix H_1 with* $H_1A = \begin{bmatrix} w_1 & * \\ \mathbf{0} & A_1 \end{bmatrix}$.

(f) *Use induction to get $H_2 \in \mathbb{M}_{n-1}(\mathbb{R})$ satisfying $H_2A_1 = T_1$, an upper triangular matrix.*

(g) *Define* $H = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & H_2 \end{bmatrix} H_1$. *Then H is an orthogonal matrix and* $HA = \begin{bmatrix} w_1 & * \\ \mathbf{0} & T_1 \end{bmatrix}$, *an upper triangular matrix.*

2. *Let $A \in \mathbb{M}_n(\mathbb{R})$ such that $\mathsf{tr}(\mathsf{A}) = \mathsf{0}$. Then prove that there exists a non-singular matrix S such that $SAS^{-1} = B$ with $B = [b_{ij}]$ and $b_{ii} = 0$, for $1 \le i \le n$.*

2.3 Initial Results on LU Decomposition

Consider the linear system $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{M}_n(\mathbb{R})$ and $\mathbf{b} \in \mathbb{M}_n(\mathbb{R})$ are known matrices and $\mathbf{x} \in \mathbb{M}_n(\mathbb{R})$ is the unknown matrix. Recall that in Example 2.2.4 we used the Gauss Elimination method to get the REF of the augmented matrix. If the REF was $[C\ \mathbf{d}]$ then C was an upper triangular matrix. The upper triangular form of C was used during back substitution.

The decomposition of a square matrix A as LU, where L is a lower triangular matrix and U is an upper triangular matrix plays an important role in numerical linear algebra. To get a better understanding of the LU**-decomposition** (LU**-factorization**), we recall a few observations and give a few examples.

DRAFT

1. Observe that solving the system $A\mathbf{x} = \mathbf{b}$ is quite easy whenever A is a triangular matrix.
2. As L is a triangular matrix, the linear system $L\mathbf{y} = \mathbf{b}$ can be easily solved. Let $\mathbf{y}_0$ be a solution of $L\mathbf{y} = \mathbf{b}$.
3. Now consider the linear system $U\mathbf{z} = \mathbf{y}_0$. As U is again triangular, this system can again be easily solved. Let $\mathbf{u}_0$ be a solution of $U\mathbf{z} = \mathbf{y}_0$. Then, $\mathbf{z}_0$ is a solution of the system $A\mathbf{x} = \mathbf{b}$ as
$$A\mathbf{z}_0 = (LU)\mathbf{z}_0 = L(U\mathbf{z}_0) = L\mathbf{y}_0 = \mathbf{b}.$$

Hence, we observe that solving the system $A\mathbf{x} = \mathbf{b}$ reduces to solving two easier linear systems, namely $L\mathbf{y} = \mathbf{b}$ and $U\mathbf{z} = \mathbf{y}$, where $\mathbf{y}$ is obtained as a solution of $L\mathbf{y} = \mathbf{b}$.

To give the LU-decomposition for a square matrix A, we need to know the determinant of A, namely $\det(A)$, and its properties. Since, we haven't yet studied it, we just give the idea of the LU-decomposition. For the general case, the readers should see the chapter titled "Advanced Topics". Let us start with a few examples.

Example 2.3.1. 1. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Then A cannot be decomposed into LU.

For if, $A = LU = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix} \begin{bmatrix} e & f \\ 0 & g \end{bmatrix}$ then the numbers $a, b, c, e, f, g \in \mathbb{R}$ satisfy
$$ae = 0, af = 1, be = 1 \text{ and } bf + cg = 0.$$

But, $ae = 0$ implies either $a = 0$ or $e = 0$, contradicting either $af = 1$ or $be = 1$.

2. Let $\epsilon > 0$ and $A = \begin{bmatrix} \epsilon & 1 \\ 1 & 0 \end{bmatrix}$. Then, $A = LU$, where $L = \begin{bmatrix} 1 & 0 \\ \frac{1}{\epsilon} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} \epsilon & 1 \\ 0 & -\frac{1}{\epsilon} \end{bmatrix}$. Thus, comparing it with the previous example, we see that the LU-decomposition is highly unstable.

3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$. Then, for any choice of $\alpha \neq 0$, $L = \begin{bmatrix} 1 & 0 \\ 2 & \alpha \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 2 \\ 0 & -\frac{2}{\alpha} \end{bmatrix}$ gives $A = LU$. Check that if we restrict ourselves with the condition that the diagonal entries of L are all 1 then the decomposition is unique.

4. Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$. Then, using ideas in Example 2.2.4.2 verify that $A = LU$, where $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -1/2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 3/2 \end{bmatrix}$.

5. Recall that in Example 2.2.4.2, we had pivots pivots at each stage. Whereas, in Example 2.2.4.1, we had to interchange the first and second row to get a pivot. So, it is not possible to write $A = LU$.

6. Finally, using $A = LU$, the system $A\mathbf{x} = \mathbf{b}$ reduces to $LU\mathbf{x} = \mathbf{b}$. Here, the solution of $L\mathbf{y} = \mathbf{b}$, for $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 3 \end{bmatrix}$ equals $\mathbf{y} = \begin{bmatrix} 4 \\ -3 \\ 3/2 \end{bmatrix}$. This, in turn implies $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ as the solution of both $U\mathbf{x} = \mathbf{y}$ or $A\mathbf{x} = \mathbf{b}$.

DRAFT

So, to proceed further, let $A \in \mathbb{M}_n(\mathbb{R})$. Then, recall that for any $S \subseteq \{1, 2, \ldots, n\}$, $A[S, S]$ denotes the principal submatrix of A corresponding to the elements of S (see Page 21). Then, we assume that $\det(A[S, S]) \neq 0$, for every $S = \{1, 2, \ldots, i\}$, $1 \leq i \leq n$.

We need to show that there exists an invertible lower triangular matrix L such that LA is an invertible upper triangular matrix. The proof uses the following ideas.

1. By assumption $A[1, 1] = a_{11} \neq 0$. Write $A = \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$, where A_{22} is a $(n-1) \times (n-1)$.

2. Let $L_1 = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{x} & I_{n-1} \end{bmatrix}$, where $\mathbf{x} = \dfrac{-1}{a_{11}} A_{21}$. Then L_1 is a lower triangular matrix and

$$L_1 A = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{x} & I_{n-1} \end{bmatrix} \begin{bmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & A_{12} \\ a_{11}\mathbf{x} + A_{21} & \mathbf{x}A_{12} + A_{22} \end{bmatrix} = \begin{bmatrix} a_{11} & A_{12} \\ \mathbf{0} & \mathbf{x}A_{12} + A_{22} \end{bmatrix}.$$

3. Note that $(2, 2)$-th entry of $L_1 A$ equals the $(1, 1)$-th entry of $\mathbf{x}A_{12} + A_{22}$. This equals

$$\left(\frac{-1}{a_{11}} \begin{bmatrix} a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} \begin{bmatrix} a_{12} & \cdots & a_{1n} \end{bmatrix} \right)_{11} + (A_{22})_{11} = \frac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}} = \frac{A[\{1, 2\}, \{1, 2\}]}{a_{11}} \neq 0.$$

4. Thus, L_1 is an invertible lower triangular matrix with $L_1A = \begin{bmatrix} a_{11} & * \\ \mathbf{0} & A_1 \end{bmatrix}$ and $(A_1)_{11} \neq 0$. Hence, $\det(A) = a_{11}\det(A_1)$ and $\det(A_1[S,S]) \neq 0$, for all $S \subseteq \{1, 2, \ldots, n-1\}$ as
   (a) the determinant of a lower triangular matrix equals product of diagonal entries and
   (b) if A and B are two $n \times n$ matrices then $\det(AB) = \det(A) \cdot \det(B)$.
5. Now, using induction, we get $L_2 \in \mathbb{M}_{n-1}(\mathbb{R})$, an invertible lower triangular matrix, with 1's on the diagonal such that $L_2A_1 = T_1$, an invertible upper triangular matrix.
6. Define $\widetilde{L} = \begin{bmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & L_2 \end{bmatrix} L_1$. Then, verify that $\widetilde{L}A = \begin{bmatrix} a_{11} & * \\ \mathbf{0} & T_1 \end{bmatrix}$, is an upper triangular matrix with $\widetilde{L}$ as an invertible lower triangular matrix.
7. Defining $L = \left(\widetilde{L}\right)^{-1}$, we see that L is a lower triangular matrix (inverse of a lower triangular matrix is lower triangular) with $A = LU$ and $U = \begin{bmatrix} a_{11} & * \\ \mathbf{0} & T_1 \end{bmatrix}$, an upper triangular invertible matrix.

As mentioned above, we will again come back to this at a later stage.

2.4 Row-Reduced Echelon Form (RREF)

We now proceed to understand the row-reduced echelon form (RREF) of a matrix. This understanding will be used to define the row-rank of a matrix in the next section. In subsequent sections and chapters, RREF is used to obtain very important results.

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Definition 2.4.1. A matrix C is said to be in **row-reduced echelon form (RREF)**

1. if C is already in REF,
2. if the pivot of each non-zero row is 1,
3. if every other entry in each pivotal column is zero.

A matrix in RREF is also called a row-reduced echelon matrix.

Example 2.4.2. 1. The following matrices are in RREF.

$$\begin{bmatrix} 0 & \boxed{1} & 0 & -2 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & 6 \\ 0 & 0 & \boxed{1} & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} \boxed{1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{bmatrix}.$$

Note that if we look at the pivotal rows and columns then I_2 is present in the first two matrices and I_3 is there in the next two. Also, the subscripts 2 and 3, respectively, in I_2 and I_3 correspond to the number of pivots.

2. The following matrices are not in RREF (determine the rule(s) that fail).

$$\begin{bmatrix} 0 & \boxed{3} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & \boxed{1} & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} \end{bmatrix}, \begin{bmatrix} 0 & \boxed{1} & 3 & 1 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Even though, we have two pivots in examples 1 and 3, the matrix I_2 doesn't appear as a submatrix in pivotal rows and columns. In the first one, we have $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$ as a submatrix and in the third the corresponding submatrix is $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

We now give another examples to indicate its application to the theory of the system of linear equations.

Example 2.4.3. Consider a linear system $A\mathbf{x} = \mathbf{b}$, where $A \in \mathbb{M}_3(\mathbb{C})$ and $A[:,1] \neq \mathbf{0}$. If $[C \quad \mathbf{d}] = \mathrm{RREF}([A \quad \mathbf{b}])$ then what are the possible choices for $[C \quad \mathbf{d}]$ and what are its implication?

Solution: Since there are 3 rows, the number of pivots can be at most 3. So, let us verify that there are 7 different choices for $[C \;\; \mathbf{d}] = \mathrm{RREF}([A \;\; \mathbf{b}])$.

1. There are exactly 3 pivots. These pivots can be in either the columns $\{1, 2, 3\}$, $\{1, 2, 4\}$ and $\{1, 3, 4\}$ as we have assumed $A[:,1] \neq \mathbf{0}$. The corresponding cases are given below.
   (a) Pivots in the columns $1, 2, 3 \Rightarrow [C \;\mathbf{d}] = \begin{bmatrix} 1 & 0 & 0 & d_1 \\ 0 & 1 & 0 & d_2 \\ 0 & 0 & 1 & d_3 \end{bmatrix}$. Here, $A\mathbf{x} = \mathbf{b}$ is consistent.

   The UNIQUE SOLUTION equals $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$.

   (b) Pivots in the columns $1, 2, 4$ or $1, 3, 4 \Rightarrow [C \;\mathbf{d}]$ equals $\begin{bmatrix} 1 & 0 & \alpha & 0 \\ 0 & 1 & \beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & \alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   Here, $A\mathbf{x} = \mathbf{b}$ is inconsistent for any choice of α, β as there is a row of $[0\;0\;0\;1]$. This corresponds to solving $0 \cdot x + 0 \cdot y + 0 \cdot z = 1$, an equation which has no solution.

2. There are exactly 2 pivots. These pivots can be in either the columns $\{1, 2\}$, $\{1, 3\}$ or $\{1, 4\}$ as we have assumed $A[:,1] \neq \mathbf{0}$. The corresponding cases are given below.
   (a) Pivots in the columns $1, 2$ or $1, 3 \Rightarrow [C \;\mathbf{d}]$ equals $\begin{bmatrix} 1 & 0 & \alpha & d_1 \\ 0 & 1 & \beta & d_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 1 & \alpha & 0 & d_1 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Here, for the first matrix, the solution set equals

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 - \alpha z \\ d_2 - \beta z \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + z \begin{bmatrix} -\alpha \\ -\beta \\ 1 \end{bmatrix},$$

   where z is arbitrary. Here, z is called the "Free variable" as z can be assigned any value and x and y are called "Basic Variables" and they can be written in terms of the free variable z and constant.

DRAFT

(b) Pivots in the columns $1, 4 \Rightarrow [C\ \mathbf{d}] = \begin{bmatrix} 1 & \alpha & \beta & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which has a row of $[0\ 0\ 0\ 1]$.

This corresponds to solving $0 \cdot x + 0 \cdot y + 0 \cdot z = 1$, an equation which has no solution.

3. There is exactly one pivot. In this case $[C\ \mathbf{d}] = \begin{bmatrix} 1 & \alpha & \beta & d_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Here, $A\mathbf{x} = \mathbf{b}$ is consistent and has INFINITE NUMBER OF SOLUTIONS for every choice of α, β as RREF($[A\ \mathbf{b}]$) has no row of the form $[0\ 0\ 0\ 1]$.

2.4.1 The Gauss-Jordan Elimination Method

So, having seen the application of the RREF to the augmented matrix, let us proceed with the algorithm, commonly known as the Gauss-Jordan Elimination (GJE), which helps us compute the RREF.

1. Input: $A \in \mathbb{M}_{m,n}(\mathbb{R})$.
2. Output: a matrix B in RREF such that A is row equivalent to B.
3. **Step 1:** Put 'Region' $= A$.
4. **Step 2:** If all entries in the Region are 0, STOP. Else, in the Region, find the leftmost nonzero column and find its topmost nonzero entry. Suppose this nonzero entry is $a_{ij} = c$ (say). Box it. This is a pivot.
5. **Step 3:** Interchange the row containing the pivot with the top row of the region. Also, make the pivot entry 1 by dividing this top row by c. Use this pivot to make other entries in the pivotal column as 0.
6. **Step 4:** Put Region $=$ the submatrix below and to the right of the current pivot. Now, go to step 2.

   Important: The process will stop, as we can get at most $\min\{m, n\}$ pivots.

DRAFT

Example 2.4.4. Apply GJE to $\begin{bmatrix} 0 & 2 & 3 & 7 \\ 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

1. Region $= A$ as $A \neq \mathbf{0}$.
2. Then, $E_{12}A = \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 1 & 3 & 4 & 8 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Also, $E_{31}(-1)E_{12}A = \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & 2 & 3 & 7 \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = B$ (say).
3. Now, Region $= \begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 0 & 0 & 1 \end{bmatrix} \neq \mathbf{0}$. Then, $E_2(\frac{1}{2})B = \begin{bmatrix} \boxed{1} & 1 & 1 & 1 \\ 0 & \boxed{1} & \frac{3}{2} & \frac{7}{2} \\ 0 & 2 & 3 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix} = C$(say). Then,

$$E_{12}(-1)E_{32}(-2)C = \begin{bmatrix} \boxed{1} & 0 & \frac{-1}{2} & \frac{-5}{2} \\ 0 & \boxed{1} & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = D(\text{say}).$$

4. Now, Region $= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then, $E_{34}D = \begin{bmatrix} \boxed{1} & 0 & \frac{-1}{2} & \frac{-5}{2} \\ 0 & \boxed{1} & \frac{3}{2} & \frac{7}{2} \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Now, multiply on the left by $E_{13}(\frac{5}{2})$ and $E_{23}(\frac{-7}{2})$ to get $\begin{bmatrix} \boxed{1} & 0 & -\frac{1}{2} & 0 \\ 0 & \boxed{1} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$, a matrix in RREF. Thus, A is row equivalent to F, where $F = \text{RREF}(A) = \begin{bmatrix} \boxed{1} & 0 & -\frac{1}{2} & 0 \\ 0 & \boxed{1} & \frac{3}{2} & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

5. Note that we have multiplied A on the **left** by the elementary matrices, $E_{12}, E_{31}(-1)$, $E_2(1/2)$, $E_{32}{-2}, E_{12}(-1), E_{34}, E_{23}(-7/2), E_{13}(5/2)$, *i.e.*,
$$E_{13}(5/2)E_{23}(-7/2)E_{34}E_{12}(-1)E_{32}(-2)E_2(1/2)E_{31}(-1)E_{12}A = F = \text{RREF}(A).$$

6. Or equivalently, we have an invertible matrix P such that $PA = F = \text{RREF}(A)$, where
$$P = E_{13}(5/2)E_{23}(-7/2)E_{34}E_{12}(-1)E_{32}(-2)E_2(1/2)E_{31}(-1)E_{12}.$$

DRAFT

EXERCISE **2.4.5.** *Let* $A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 3 \\ 3 & 0 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & 0 \end{bmatrix}$ *and* $C = \begin{bmatrix} 0 & -1 & 1 \\ -2 & 0 & 3 \\ -5 & 1 & 0 \end{bmatrix}$. *Determine their RREF.*

2.4.2 Results on RREF

The proof of the next result is beyond the scope of this book and hence is omitted.

Theorem 2.4.6. *Let A and B be two row equivalent matrices in RREF. Then $A = B$.*

As an immediate corollary, we obtain the following important result.

Corollary 2.4.7. *The RREF of a matrix A is unique.*

Proof. Suppose there exists a matrix A having B and C as RREFs. As the RREFs are obtained by left multiplication of elementary matrices, there exist elementary matrices $E_1, \ldots, E_k$ and $F_1, \ldots, F_\ell$ such that $B = E_1 \cdots E_k A$ and $C = F_1 \cdots F_\ell A$. Thus,
$$B = E_1 \cdots E_k A = E_1 \cdots E_k (F_1 \cdots F_\ell)^{-1} C = E_1 \cdots E_k F_\ell^{-1} \cdots F_2^{-1} F_1^{-1} C.$$

As inverse of an elementary matrix is an elementary matrix, B and C are are row equivalent. As B and C are in RREF, using Theorem 2.4.6, $B = C$. ∎

Remark 2.4.8. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$.*

1. *Then, the uniqueness of RREF implies that RREF(A) is independent of the choice of the row operations used to get the final matrix which is in RREF.*
2. *Let $B = EA$, for some elementary matrix E. Then, RREF(A) = RREF(B).*
3. *Then RREF$(A) = PA$, for some invertible matrix P.*
4. *Let $F =$ RREF(A) and $B = [A[:,1], \ldots, A[:,s]]$, for some $s \le n$. Then,*

$$\text{RREF}(B) = [F[:,1], \ldots, F[:,s]].$$

Proof. By Remark 2.4.8.3, there exist an invertible matrix P, such that

$$F = PA = [PA[:,1], \ldots, PA[:,n]] = [F[:,1], \ldots, F[:,n]].$$

But, $PB = [PA[:,1], \ldots, PA[:,s]] = [F[:,1], \ldots, F[:,s]]$. As F is in RREF, its first s columns are also in RREF. Thus, by Corollary 2.4.7, RREF$(PB) = [F[:,1], \ldots, F[:,s]]$. Now, a repeated application of Remark 2.4.8.2 implies RREF$(B) = [F[:,1], \ldots, F[:,s]]$. Thus, the required result follows. ■

Proposition 2.4.9. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then, A is invertible if and only if RREF$(A) = I_n$, i.e., every invertible matrix is a product of elementary matrices.*

Proof. If RREF$(A) = I_n$ then there exist elementary matrices $E_1, \ldots, E_k$ such that $I_n =$ RREF$(A) = E_1 \cdots E_k A$. As elementary matrices are invertible and product of invertible matrices are invertible, we get $A = (E_1 \cdots E_k)^{-1} \Leftrightarrow A^{-1} = E_1 \cdots E_k$.

Now, let A be invertible with $B =$ RREF$(A) = E_1 \cdots E_k A$, for some elementary matrices $E_1, \ldots, E_k$. As A and E_i's are invertible, the matrix B is invertible. Hence, B doesn't have any zero row. Thus, all the n rows of B have pivots. Therefore, B has n pivotal columns. As B has exactly n columns, each column is a pivotal column and hence $B = I_n$. Thus, the required result follows. ■

EXERCISE **2.4.10.**
1. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then RREF$(SA)$ = RREF(A), for any invertible matrix $S \in \mathbb{M}_n(\mathbb{R})$.*
2. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be an invertible matrix. Then, for any matrix B, define $C = \begin{bmatrix} A & B \end{bmatrix}$ and $D = \begin{bmatrix} A \\ B \end{bmatrix}$. Then, RREF$(C) = \begin{bmatrix} I_n & A^{-1}B \end{bmatrix}$ and RREF$(D) = \begin{bmatrix} I_n \\ \mathbf{0} \end{bmatrix}$.*

2.4.3 Computing the Inverse using GJE

Recall that if $A \in \mathbb{M}_n(\mathbb{C})$ is invertible then there exists a matrix B such that $AB = I_n = BA$. So, we want to find a B such that

$$\begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix} = I_n = AB = A\begin{bmatrix} B[:,1] & \cdots & B[:,n] \end{bmatrix} = \begin{bmatrix} AB[:,1] & \cdots & AB[:,n] \end{bmatrix}.$$

So, if $B = \begin{bmatrix} B[:,1] & \cdots & B[:,n] \end{bmatrix}$ is the matrix of unknowns then we need to solve n-system of linear equations $AB[:,1] = \mathbf{e}_1, \ldots,\ AB[:,n] = \mathbf{e}_n$. Thus, we have n-augmented matrices

DRAFT

$[A \mid \mathbf{e}_1], \ldots, [A \mid \mathbf{e}_n]$. So, in place of solving the n-augmented matrices separately, the idea of GJE is to consider the augmented matrix

$$\left[A \,\middle|\, \mathbf{e}_1 \quad \mathbf{e}_2 \quad \cdots \quad \mathbf{e}_n\right] = \left[A \,\middle|\, I_n\right].$$

Thus, if E is an invertible matrix such that $E\left[A \,\middle|\, I_n\right] = \left[I_n \,\middle|\, B\right]$ then $EA = I_n$ and $E = B$. Hence, invertibility of E implies $AE = I_n$ and hence, $B = E = A^{-1}$. This idea together with Remark 2.4.8.4 is used to compute A^{-1} whenever it exists.
Let $A \in \mathbb{M}_n(\mathbb{R})$. Compute RREF$(C)$, where $C = [A\ I\ I_n]$. Then RREF$(C) = [\text{RREF}(A)\ \ B]$. Now, either RREF$(A) = I_n$ or RREF$(A) \neq I_n$. Thus, if RREF$(A) = I_n$ then we must have $B = A^{-1}$. Else, A is not invertible. We show this with an example.

Example 2.4.11. Use GJE to find the inverse of $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

Solution: Applying GJE to $[A \mid I_3] = \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$ gives

$$[A \mid I_3] \xrightarrow{E_{13}} \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right] \xrightarrow{E_{13}(-1), E_{23}(-2)} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{E_{12}(-1)} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{array}\right].$$

DRAFT

Thus, $A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

EXERCISE **2.4.12.** *Use GJE to compute the inverse* $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$ *and* $B = \begin{bmatrix} 1 & 3 & 3 \\ 2 & 3 & 2 \\ 3 & 5 & 4 \end{bmatrix}$.

2.5 Rank of a Matrix

Recall that the RREF of a matrix is unique. So, we use RREF(A) to define the rank of a A.

Definition 2.5.1. Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. Then, the **rank** of A, denoted $\mathsf{Rank}(A)$, is the number of pivots in the RREF(A).

Note that Rank(A) is defined using the number of pivots in $RREF(A)$. These pivots were obtained using the row operations. The question arises, what if we had applied column operations? That is, what happens when we multiply by invertible matrices on the right?

Will the pivots using column operations remain the same or change? This question cannot be answered at this stage. Using the ideas in vector spaces, we can show that the number of pivots do not change and hence, we just use the word $\mathsf{Rank}(A)$.

We now illustrate the calculation of the rank by giving a few examples.

Example 2.5.2. Determine the rank of the following matrices.

1. $\mathsf{Rank}(I_n) = n$ and $\mathsf{Rank}(\mathbf{0}) = 0$.
2. Let $A = \text{diag}(d_1, \ldots, d_n)$. Then, $\mathsf{Rank}(A)$ equals the number of non-zero d_i's.
3. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$. Then $\text{RREF}(A) = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}(A) = 1$.
4. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$. Then $\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}(A) = 1$.
5. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix}$. Then $\text{RREF}(A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{Rank}(A) = 2$.
6. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}$. Then, $\mathsf{Rank}(A) = \mathsf{Rank}(B) = 1$. Also, verify that $AB = \mathbf{0}$ and $BA = \begin{bmatrix} -6 & -12 \\ 3 & 6 \end{bmatrix}$. So, $\mathsf{Rank}(AB) = 0 \neq 1 = \mathsf{Rank}(BA)$. Observe that A and B are not invertible. So, the rank can either remain the same or reduce.
7. Let $A = \begin{bmatrix} 1 & 2 & 1 & 1 & 1 \\ 2 & 3 & 1 & 2 & 2 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix}$. Then, $\mathsf{Rank}(A) = 2$ as it's REF has two pivots.

DRAFT

Remark 2.5.3. *Before proceeding further, for $A, B \in \mathbb{M}_{m,n}(\mathbb{C})$, we observe the following.*

1. *If A and B are row-equivalent then $\mathsf{Rank}(A) = \mathsf{Rank}(B)$.*
2. *The number of pivots in the RREF(A) equals the number of pivots in REF of A. Hence, one needs to compute only the REF to determine the rank.*

EXERCISE **2.5.4.** *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$.*

1. *Then $\mathsf{Rank}(A) \leq \min\{m, n\}$.*
2. *If $B = \begin{bmatrix} A & C \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ then $\mathsf{Rank}(B) = \mathsf{Rank}([A\ C])$.*
3. *If $B = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ then $\mathsf{Rank}(B) = \mathsf{Rank}(A)$*
4. *If $A = PB$, for some invertible matrix P then $\mathsf{Rank}(A) = \mathsf{Rank}(B)$.*
5. *If $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ then $\mathsf{Rank}(A) \leq \mathsf{Rank}\left(\begin{bmatrix} A_{11} & A_{12} \end{bmatrix}\right) + \mathsf{Rank}\left(\begin{bmatrix} A_{21} & A_{22} \end{bmatrix}\right)$.*

We now have the following result.

Corollary 2.5.5. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $B \in \mathbb{M}_{n,q}(\mathbb{R})$. Then,* $\mathsf{Rank}(AB) \leq \mathsf{Rank}(A)$.

In particular, if $B \in \mathbb{M}_n(\mathbb{R})$ is invertible then $\mathsf{Rank}(AB) = \mathsf{Rank}(A)$.

Proof. Let $\mathsf{Rank}(A) = r$. Then, there exists an invertible matrix P and $A_1 \in \mathbb{M}_{r,n}(\mathbb{R})$ such that $PA = \text{RREF}(A) = \begin{bmatrix} A_1 \\ \mathbf{0} \end{bmatrix}$. Then, $PAB = \begin{bmatrix} A_1 \\ \mathbf{0} \end{bmatrix} B = \begin{bmatrix} A_1 B \\ \mathbf{0} \end{bmatrix}$. So, using Remark 2.5.3 and Exercise 2.5.4.2

$$\mathsf{Rank}(AB) = \mathsf{Rank}(PAB) = \mathsf{Rank}\left(\begin{bmatrix} A_1 B \\ \mathbf{0} \end{bmatrix}\right) = \mathsf{Rank}(A_1 B) \leq r = \mathsf{Rank}(A). \tag{2.5.1}$$

In particular, if B is invertible then, using Equation (2.5.1), we get

$$\mathsf{Rank}(A) = \mathsf{Rank}(ABB^{-1}) \leq \mathsf{Rank}(AB)$$

and hence the required result follows. ∎

Proposition 2.5.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be an invertible matrix and let S be any subset of $\{1, 2, \ldots, n\}$. Then* $\mathsf{Rank}(A[S, :]) = |S|$ *and* $\mathsf{Rank}(A[:, S]) = |S|$.

Proof. Without loss of generality, let $S = \{1, \ldots, r\}$ and $S^c = \{r+1, \ldots, n\}$. Write $A_1 = A[:, S]$ and $A_2 = A[:, S^c]$. Since A is invertible, $\text{RREF}(A) = I_n$. Hence, by Remark 2.4.8.3, there exists an invertible matrix P such that $PA = I_n$. Thus,

$$\begin{bmatrix} PA_1 & PA_2 \end{bmatrix} = P\begin{bmatrix} A_1 & A_2 \end{bmatrix} = PA = I_n = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & I_{n-r} \end{bmatrix}.$$

Thus, $PA_1 = \begin{bmatrix} I_r \\ \mathbf{0} \end{bmatrix}$ and $PA_2 = \begin{bmatrix} \mathbf{0} \\ I_{n-r} \end{bmatrix}$. So, using Corollary 2.5.5, $\mathsf{Rank}(A_1) = r$.

For the second part, let $B_1 = A[S, :]$, $B_2 = A[S^c, :]$ and let $\mathsf{Rank}(B_1) = t < s$. Then, by Exercise 2, there exists an $s \times s$ invertible matrix Q and a matrix C in RREF, of size $t \times n$ and having exactly t pivots, such that

$$QB_1 = \text{RREF}(B_1) = \begin{bmatrix} C \\ \mathbf{0} \end{bmatrix}. \tag{2.5.2}$$

As $t < s$, QB_1 has at least one zero row. Then

$$\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & I_{n-r} \end{bmatrix} A = \begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & I_{n-r} \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} QB_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} C \\ \mathbf{0} \\ B_2 \end{bmatrix}.$$

As $\begin{bmatrix} Q & \mathbf{0} \\ \mathbf{0} & I_{n-r} \end{bmatrix}$ and A are invertible, their product is invertible. But, their product has a zero row, a contradiction. Thus, $\mathsf{Rank}(B_1) = s$. ∎

Let us also look at the following example to understand the prove of the next theorem.

DRAFT

Example 2.5.7. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 5 & 1 \\ 1 & 2 & 4 & 7 \end{bmatrix}$. Then $\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. Thus, we have an invertible matrix P such that $PA = \text{RREF}(A)$. Note that $\mathsf{Rank}(A) = 3$ and hence I_3 is a submatrix of $\text{RREF}(A)$. So, we need to find $Q \in \mathbb{M}_4(\mathbb{R})$ such that $PAQ = \begin{bmatrix} I_3 & \mathbf{0} \end{bmatrix}$. Now, consider the following column operations.

1. Let $B = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Then $B = \text{RREF}(A)\ E_{23}E_{34}$.

2. $B(E_{41}(-2))^T = BE_{14}(-2) = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} I_3 & \mathbf{0} \end{bmatrix}$.

3. Thus, define $Q = E_{23}E_{34}E_{14}(-2)$ to get $PAQ = \begin{bmatrix} I_3 & \mathbf{0} \end{bmatrix}$.

Theorem 2.5.8. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. If $\mathsf{Rank}(A) = r$ then, there exist invertible matrices P and Q such that*

$$P\ A\ Q = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

DRAFT

Proof. Let $C = \text{RREF}(A)$. Then, by Remark 2.4.8.3 there exists as invertible matrix P such that $C = PA$. Note that C has r pivots and they appear in columns, say $i_1 < i_2 < \cdots < i_r$.

Now, let $Q_1 = E_{1i_1}E_{2i_2} \cdots E_{ri_r}$. As E_{ji_j}'s are elementary matrices that interchange the columns of C, one has $D = CQ_1 = \begin{bmatrix} I_r & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$, where $B \in \mathbb{M}_{r,n-r}(\mathbb{R})$.

Now, let $Q_2 = \begin{bmatrix} I_r & -B \\ \mathbf{0} & I_{n-r} \end{bmatrix}$ and $Q = Q_1Q_2$. Then Q is invertible and

$$PAQ = CQ = CQ_1Q_2 = DQ_2 = \begin{bmatrix} I_r & B \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} I_r & -B \\ 0 & I_{n-r} \end{bmatrix} = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Thus, the required result follows. ∎

Since we are multiplying by invertible matrices on the right, the idea of the above theorem cannot be used for solving the system of linear equations. But, this idea can be used to understand the properties of the given matrix, such as ideas related to rank-factorization, row-space, column space and so on which have not yet been defined.

As a corollary of Theorem 2.5.8, we now give the rank-factorization of a matrix A.

Corollary 2.5.9. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. If $\mathsf{Rank}(A) = r$ then there exist matrices $B \in \mathbb{M}_{m,r}(\mathbb{R})$ and $C \in \mathbb{M}_{r,n}(\mathbb{R})$ such that $\mathsf{Rank}(B) = \mathsf{Rank}(C) = r$ and $A = BC$. Furthermore, $A = \sum\limits_{i=1}^{r} \mathbf{x}_i\mathbf{y}_i^T$, for some $\mathbf{x}_i \in \mathbb{R}^m$ and $\mathbf{y}_i \in \mathbb{R}^n$.*

Proof. By Theorem 2.5.8, there exist invertible matrices P and Q such that $P\,A\,Q = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. Or equivalently, $A = P^{-1}\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} Q^{-1}$. Decompose $P^{-1} = [B\ D]$ and $Q^{-1} = \begin{bmatrix} C \\ F \end{bmatrix}$ such that $B \in \mathbb{M}_{m,r}(\mathbb{R})$ and $C \in \mathbb{M}_{r,n}(\mathbb{R})$. Then $\mathsf{Rank}(B) = \mathsf{Rank}(C) = r$ (see Proposition 2.5.6) and

$$A = [B\ D]\begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} C \\ F \end{bmatrix} = \begin{bmatrix} B & \mathbf{0} \end{bmatrix}\begin{bmatrix} C \\ F \end{bmatrix} = BC.$$

Furthermore, assume that $B = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_r \end{bmatrix}$ and $C = \begin{bmatrix} \mathbf{y}_1^T \\ \vdots \\ \mathbf{y}_r^T \end{bmatrix}$. Then $A = BC = \sum\limits_{i=1}^{r} \mathbf{x}_i \mathbf{y}_i^T$. ∎

Proposition 2.5.10. *Let $A, B \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, prove that* $\mathsf{Rank}(A+B) \leq \mathsf{Rank}(A) + \mathsf{Rank}(B)$. *In particular, if $A = \sum\limits_{i=1}^{k} \mathbf{x}_i \mathbf{y}_i^T$, for some $\mathbf{x}_i, \mathbf{y}_i \in \mathbb{R}$, for $1 \leq i \leq k$, then* $\mathsf{Rank}(A) \leq k$.

Proof. Let $\mathsf{Rank}(A) = r$. Then, there exists an invertible matrix P and a matrix $A_1 \in \mathbb{M}_{r,n}(\mathbb{R})$ such that $PA = \mathrm{RREF}(A) = \begin{bmatrix} A_1 \\ \mathbf{0} \end{bmatrix}$. Then,

$$P(A+B) = PA + PB = \begin{bmatrix} A_1 \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} A_1 + B_1 \\ B_2 \end{bmatrix}.$$

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Now using Corollary 2.5.5, we have

$$\mathsf{Rank}(A+B) = \mathsf{Rank}(P(A+B)) \leq r + \mathsf{Rank}(B_2) \leq r + \mathsf{Rank}(B) = \mathsf{Rank}(A) + \mathsf{Rank}(B).$$

Thus, the required result follows. The other part follows, as $\mathsf{Rank}(\mathbf{x}_i\mathbf{y}_i^T) = 1$, for $1 \leq i \leq k$. ∎

EXERCISE **2.5.11.**

1. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ be a matrix of rank 1. Then prove that $A = \mathbf{x}\mathbf{y}^T$, for non-zero vectors $\mathbf{x} \in \mathbb{R}^m$ and $\mathbf{y} \in \mathbb{R}^n$.*
2. *Let $A \in \mathbb{M}_m(\mathbb{R})$. If* $\mathsf{Rank}(A) = 1$ *then prove that $A^2 = \alpha A$, for some scalar α.*
3. *Let $A = \begin{bmatrix} 2 & 4 & 8 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.*
   (a) *Find P and Q such that $B = PAQ$. Thus, $A = P^{-1}\begin{bmatrix} I_2 & \mathbf{0} \end{bmatrix} Q^{-1}$.*
   (b) *Define $G = Q\begin{bmatrix} I_2 \\ \mathbf{x}^T \end{bmatrix} P$. Then, verify that $AGA = A$. Hence, G is a g-inverse of A.*
   (c) *In particular, if $\mathbf{b} = \mathbf{0}$ then $G = Q\begin{bmatrix} I_2 \\ \mathbf{0}^T \end{bmatrix} P$. In this case, verify that $GAG = G$, $(AG)^T = AG$ and $(GA)^T = GA$. Hence, this G is the pseudo-inverse of A.*
4. *Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \end{bmatrix}$.*

(a) *Find a matrix* G *such that* $AG = I_2$. *Hint: Let* $G = \begin{bmatrix} a & \alpha \\ b & \beta \\ c & \gamma \end{bmatrix}$. *Now, use* $AG = I_2$ *to get the solution space and proceed.*

(b) *What can you say about the number of such matrices? Give reasons for your answer.*

(c) *Does the choice of* G *in part (a) also satisfies* $(AG)^T = AG$ *and* $(GA)^T = GA$*? Give reasons for your answer.*

(d) *Does there exist a matrix* C *such that* $CA = I_3$*? Give reasons for your answer.*

(e) *Could you have used the ideas from Exercise 2.5.11.3 to get your answers?*

5. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$ *with* $\mathsf{Rank}(A) = r$. *Then, using Theorem 2.5.8, we can find invertible matrices* P *and* Q *such that* $P\ A\ Q = \begin{bmatrix} I_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$. *Choose arbitrary matrices* U, V *and* W *such that the matrix* $\begin{bmatrix} I_r & U \\ V & W \end{bmatrix}$ *is an* $n \times m$ *matrix. Define* $G = Q \begin{bmatrix} I_r & U \\ V & W \end{bmatrix} P$. *Then, prove that* G *is a g-inverse of* A.

6. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$ *with* $\mathsf{Rank}(A) = r$. *Then, using Corollary 2.5.9 there exist matrices* $B \in \mathbb{M}_{m,r}(\mathbb{R})$ *and* $C \in \mathbb{M}_{r,n}(\mathbb{R})$ *such that* $\mathsf{Rank}(B) = \mathsf{Rank}(C) = r$ *and* $A = BC$. *Thus,* $B^T B$ *and* CC^T *are invertible matrices. Now, define* $G_1 = (B^T B)^{-1} B^T$ *and* $G_2 = C^T (CC^T)^{-1}$. *Then, prove that* $G = G_2 G_1$ *is a pseudo-inverse of* A.

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2.6 Solution set of a Linear System

Having learnt RREF of a matrix and the properties of the rank function, let us go back to the system of linear equations and apply those ideas to have a better understanding of the system of linear equations. So, let us consider the linear system $A\mathbf{x} = \mathbf{b}$. Then, using Remark 2.4.8.4 $\mathsf{Rank}([A\ \ \mathbf{b}]) \geq \mathsf{Rank}(A)$. Further, the augmented matrix has exactly one extra column. Hence, either

$$\mathsf{Rank}([A\ \ \mathbf{b}]) = \mathsf{Rank}(\text{Augmented Matrix}) = \mathsf{Rank}(\text{Coefficient matrix}) + 1 = \mathsf{Rank}(A) + 1$$
$$\text{or}$$
$$\mathsf{Rank}([A\ \ \mathbf{b}]) = \mathsf{Rank}(\text{Augmented Matrix}) = \mathsf{Rank}(\text{Coefficient matrix}) = \mathsf{Rank}(A).$$

In the first case, there is a pivot in the $(n+1)$-th column of the augmented matrix $[A\ \ \mathbf{b}]$. Thus, the column corresponding to $\mathbf{b}$ has a pivot. This implies $\mathbf{b} \neq \mathbf{0}$. This implies that the row corresponding to this pivot in RREF$([A\ \ \mathbf{b}])$ has all entries before this pivot as 0. Thus, in RREF$([A\ \ \mathbf{b}])$ this pivotal row equals $[0\ 0\ \cdots\ 0\ 1]$. But, this corresponds to the equation $0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 1$. This implies that the $A\mathbf{x} = \mathbf{b}$ has no solution whenever

$$\mathbf{b} \neq \mathbf{0} \text{ and } \mathsf{Rank}(\text{Augmented Matrix}) > \mathsf{Rank}(\text{Coefficient matrix}).$$

We now define the words "basic variables" and "free variables".

Definition 2.6.1. Consider the linear system $A\mathbf{x} = \mathbf{b}$. If $\text{RREF}([A \ \ \mathbf{b}]) = [C \ \ \mathbf{d}]$. Then, the variables corresponding to the pivotal columns of C are called the **basic** variables and the variables that correspond to non-pivotal columns are called **free** variables.

To understand the second case, we look at the homogeneous system $A\mathbf{x} = \mathbf{0}$.

Example 2.6.2. Consider a linear system $A\mathbf{x} = \mathbf{0}$. Suppose $\text{RREF}(A) = C$, where

$$C = \begin{bmatrix} \boxed{1} & 0 & 2 & -1 & 0 & 0 & 2 \\ 0 & \boxed{1} & 1 & 3 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then to get the solution set, observe that C has 4 pivotal columns, namely, the columns $1, 2, 5$ and 6. Thus, x_1, x_2, x_5 and x_6 are basic variables. Therefore, the remaining variables x_3, x_4 and x_7 are free variables. Hence, the solution set is given by

$$\begin{bmatrix} x_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \\ \mathbf{x}_4 \\ \mathbf{x}_5 \\ \mathbf{x}_6 \\ \mathbf{x}_7 \end{bmatrix} = \begin{bmatrix} -2x_3 + x_4 - 2x_7 \\ -x_3 - 3x_4 - 5x_7 \\ x_3 \\ x_4 \\ 4x_7 \\ 4 - x_7 \\ x_7 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_7 \begin{bmatrix} -2 \\ -5 \\ 0 \\ 0 \\ 4 \\ -1 \\ 1 \end{bmatrix},$$

DRAFT

where x_3, x_4 and x_7 are arbitrary. Let

$$\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix} = \begin{bmatrix} -2 & 1 & -2 \\ -1 & -3 & -5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, for $1 \le i \le 3$, $C\mathbf{u}_i = \mathbf{0} \Rightarrow A\mathbf{u}_i = \mathbf{0}$. Further, as x_3, x_4 and x_7 are the free variables, observe that the submatrix of $\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \mathbf{u}_3 \end{bmatrix}$ corresponding to the 3-rd, 4-th and 7-th rows equals I_3.

Let us now summarize the above ideas and examples.

Theorem 2.6.3. *Let $A\mathbf{x} = \mathbf{b}$ be a linear system in n variables with RREF$([A \ \ \mathbf{b}]) = [C \ \ \mathbf{d}]$.*

1. *Then the system $A\mathbf{x} = \mathbf{b}$ is inconsistent if* $\mathsf{Rank}([A\ \mathbf{b}]) > \mathsf{Rank}(A)$.

2. *Then the system $A\mathbf{x} = \mathbf{b}$ is consistent if* $\mathsf{Rank}([A\ \mathbf{b}]) = \mathsf{Rank}(A)$.

(a) *Further,* $A\mathbf{x} = \mathbf{b}$ *has* A UNIQUE SOLUTION *if* $\mathsf{Rank}(A) = n$.

(b) *Further,* $A\mathbf{x} = \mathbf{b}$ *has* INFINITE NUMBER OF SOLUTIONS *if* $\mathsf{Rank}(A) < n$. *In this case, there exist vectors* $\mathbf{x}_0, \mathbf{u}_1, \ldots, \mathbf{u}_{n-r} \in \mathbb{R}^n$ *with* $A\mathbf{x}_0 = \mathbf{b}$ *and* $A\mathbf{u}_i = \mathbf{0}$*, for* $1 \le i \le n-r$. *Furthermore, the solution set is given by*

$$\{\mathbf{x}_0 + k_1\mathbf{u}_1 + k_2\mathbf{u}_2 + \cdots + k_{n-r}\mathbf{u}_{n-r} \mid k_i \in \mathbb{R},\ 1 \le i \le n-r\}.$$

Proof. PART 1: As $\mathsf{Rank}([A\ \mathbf{b}]) > \mathsf{Rank}(A)$, by Remark 2.4.8.4 $([C\ \ \mathbf{d}])[r+1,:] = [\mathbf{0}^T\ 1]$. Note that this row corresponds to the linear equation

$$0 \cdot x_1 + 0 \cdot x_2 + \cdots + 0 \cdot x_n = 1$$

which clearly has no solution. Thus, by Theorem 2.2.13, $A\mathbf{x} = \mathbf{b}$ is inconsistent.

PART 2: As $\mathsf{Rank}([A\ \mathbf{b}]) = \mathsf{Rank}(A)$, by Remark 2.4.8.4, $[C\ \ \mathbf{d}]$ doesn't have a row of the form $[\mathbf{0}^T\ 1]$. Further, the number of pivots in $[C\ \ \mathbf{d}]$ and that in C is same, namely, r pivots.

PART 2A: As $\mathsf{Rank}(A) = r = n$, there are no free variables. Hence, $x_i = d_i$, for $1 \le i \le n$, is the unique solution.

PART 2B: As $\mathsf{Rank}(A) = r < n$. Suppose the pivots appear in columns $i_1, \ldots, i_r$ with $1 \le i_1 < \cdots < i_r \le n$. Thus, the variables x_{i_j}, for $1 \le j \le r$, are basic variables and the remaining $n-r$ variables, say $x_{t_1}, \ldots, x_{t_{n-r}}$, are free variables with $t_1 < \cdots < t_{n-r}$. Since C is in RREF, in terms of the free variables and basic variables, the ℓ-th row of $[C\ \ \mathbf{d}]$, for $1 \le \ell \le r$, corresponds to the equation (writing basic variables in terms of a constant and free variables)

$$x_{i_\ell} + \sum_{k=1}^{n-r} c_{\ell t_k} x_{t_k} = d_\ell \Leftrightarrow x_{i_\ell} = d_\ell - \sum_{k=1}^{n-r} c_{\ell t_k} x_{t_k}.$$

DRAFT

Thus, the system $C\mathbf{x} = \mathbf{d}$ is consistent. Hence, by Theorem 2.2.13 the system $A\mathbf{x} = \mathbf{b}$ is consistent and the solution set of the system $A\mathbf{x} = \mathbf{b}$ and $C\mathbf{x} = \mathbf{d}$ are the same. Therefore, the solution set of the system $C\mathbf{x} = \mathbf{d}$ (or equivalently $A\mathbf{x} = \mathbf{b}$) is given by

$$\begin{bmatrix} x_{i_1} \\ \vdots \\ x_{i_r} \\ x_{t_1} \\ x_{t_2} \\ \vdots \\ x_{t_{n-r}} \end{bmatrix} = \begin{bmatrix} d_1 - \sum\limits_{k=1}^{n-r} c_{1t_k} x_{t_k} \\ \vdots \\ d_r - \sum\limits_{k=1}^{n-r} c_{rt_k} x_{t_k} \\ x_{t_1} \\ x_{t_2} \\ \vdots \\ x_{t_{n-r}} \end{bmatrix} = \begin{bmatrix} d_1 \\ \vdots \\ d_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{t_1} \begin{bmatrix} c_{1t_1} \\ \vdots \\ c_{rt_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{t_2} \begin{bmatrix} c_{1t_2} \\ \vdots \\ c_{rt_2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_{t_{n-r}} \begin{bmatrix} c_{1t_{n-r}} \\ \vdots \\ c_{rt_{n-r}} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.6.3}$$

Define $\mathbf{x}_0 = \begin{bmatrix} d_1 \\ \vdots \\ d_r \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ and $\mathbf{u}_1 = \begin{bmatrix} c_{1t_1} \\ \vdots \\ c_{rt_1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{u}_{n-r} = \begin{bmatrix} c_{1t_{n-r}} \\ \vdots \\ c_{rt_{n-r}} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$. Then, it can be easily verified that $A\mathbf{x}_0 = \mathbf{b}$ and, for $1 \le i \le n-r$, $A\mathbf{u}_i = \mathbf{0}$. Also, by Equation (2.6.3) the solution set has indeed the required form, where k_i corresponds to the free variable x_{t_i}. As there is at least one free variable the system has infinite number of solutions. ∎

Thus, note that the solution set of $A\mathbf{x} = \mathbf{b}$ depends on the rank of the coefficient matrix, the rank of the augmented matrix and the number of unknowns. In some sense, it is independent of the choice of m.

EXERCISE **2.6.4.** *Consider the linear system given below. Use GJE to find the RREF of its augmented matrix and use it to find the solution.*

$$\begin{array}{cccccccc} x & +y & & -2\,u & +v & & = & 2 \\ & & z & +u & +2\,v & & = & 3 \\ & & & & v & +w & = & 3 \\ & & & & v & +2\,w & = & 5 \end{array}$$

Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, $\mathsf{Rank}(A) \le m$. Thus, using Theorem 2.6.3 the next result follows.

Corollary 2.6.5. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. If $\mathsf{Rank}(A) = r < n$ then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has at least one non-trivial solution.*

Remark 2.6.6. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, Theorem 2.6.3 implies that $A\mathbf{x} = \mathbf{b}$ is consistent if and only if $\mathsf{Rank}(A) = \mathsf{Rank}([A\ \mathbf{b}])$. Further, the the vectors $\mathbf{u}_i$'s associated with the free variables in Equation (2.6.3) are solutions of the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.*

DRAFT

We end this subsection with some applications.

Example 2.6.7. 1. Determine the equation of the circle passing through the points $(-1,4)$, $(0,1)$ and $(1,4)$.

Solution: The equation $a(x^2+y^2)+bx+cy+d = 0$, for $a,b,c,d \in \mathbb{R}$, represents a circle. Since this curve passes through the given points, we get a homogeneous system having 3 equations in4 unknowns, namely

$$\begin{bmatrix} (-1)^2+4^2 & -1 & 4 & 1 \\ (0)^2+1^2 & 0 & 1 & 1 \\ 1^2+4^2 & 1 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \mathbf{0}.$$

Solving this system, we get $[a,b,c,d] = [\frac{3}{13}d, 0, -\frac{16}{13}d, d]$. Hence, choosing $d = 13$, the required circle is given by $3(x^2+y^2) - 16y + 13 = 0$.

2. Determine the equation of the plane that contains the points $(1,1,1)$, $(1,3,2)$ and $(2,-1,2)$.

Solution: The general equation of a plane in space is given by $ax+by+cz+d = 0$, where a,b,c and d are unknowns. Since this plane passes through the 3 given points, we get a homogeneous system in 3 equations and 4 variables. So, it has a non-trivial solution, namely $[a,b,c,d] = [-\frac{4}{3}d, -\frac{d}{3}, -\frac{2}{3}d, d]$. Hence, choosing $d = 3$, the required plane is given by $-4x - y + 2z + 3 = 0$.

3. Let $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & -1 & 0 \\ 0 & -3 & 4 \end{bmatrix}$. Then, find a non-trivial solution of $A\mathbf{x} = 2\mathbf{x}$. Does there exist a nonzero vector $\mathbf{y} \in \mathbb{R}^3$ such that $A\mathbf{y} = 4\mathbf{y}$?

   Solution: Solving for $A\mathbf{x} = 2\mathbf{x}$ is equivalent to solving $(A - 2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system equals $\begin{bmatrix} 0 & 3 & 4 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & -3 & 2 & 0 \end{bmatrix}$. Verify that $\mathbf{x}^T = [1, 0, 0]$ is a nonzero solution. For the other part, the augmented matrix for solving $(A - 4I)\mathbf{y} = \mathbf{0}$ equals $\begin{bmatrix} -2 & 3 & 4 & 0 \\ 0 & -5 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{bmatrix}$. Thus, verify that $\mathbf{y}^T = [2, 0, 1]$ is a nonzero solution.

EXERCISE **2.6.8.** *1. Let $A \in \mathbb{M}_n(\mathbb{R})$. If $A^2\mathbf{x} = \mathbf{0}$ has a non trivial solution then show that $A\mathbf{x} = \mathbf{0}$ also has a non trivial solution.*

2. *Let $\mathbf{u} = (1, 1, -2)^T$ and $\mathbf{v} = (-1, 2, 3)^T$. Find condition on x, y and z such that the system $c\mathbf{u} + d\mathbf{v} = (x, y, z)^T$ in the unknowns c and d is consistent.*

3. *Find condition(s) on x, y, z so that the systems given below (in the unknowns a, b and c) is consistent?*

   (a) $a + 2b - 3c = x,\ 2a + 6b - 11c = y,\ a - 2b + 7c = z.$

   (b) $a + b + 5c = x,\ a + 3c = y,\ 2a - b + 4c = z.$

4. *For what values of c and k, the following systems have $i)$ no solution, $ii)$ a unique solution and $iii)$ infinite number of solutions.*

   (a) $x + y + z = 3,\ \ x + 2y + cz = 4,\ \ 2x + 3y + 2cz = k.$

   (b) $x + y + z = 3,\ x + y + 2cz = 7,\ x + 2y + 3cz = k.$

   (c) $x + y + 2z = 3,\ x + 2y + cz = 5,\ x + 2y + 4z = k.$

   (d) $x + 2y + 3z = 4,\ \ 2x + 5y + 5z = 6,\ \ 2x + (c^2 - 6)z = c + 20.$

   (e) $x + y + z = 3,\ 2x + 5y + 4z = c,\ 3x + (c^2 - 8)z = 12.$

5. *Consider the linear system $A\mathbf{x} = \mathbf{b}$ in m equations and 3 unknowns. Then, for each of the given solution set, determine the possible choices of m? Further, for each choice of m, determine a choice of A and $\mathbf{b}$.*

   (a) $(1, 1, 1)^T$ *is the only solution.*

   (b) $\{c(1, 2, 1)^T | c \in \mathbb{R}\}$ *as the solution set.*

   (c) $\{(1, 1, 1)^T + c(1, 2, 1)^T | c \in \mathbb{R}\}$ *as the solution set.*

   (d) $\{c(1, 2, 1)^T + d(2, 2, -1)^T | c, d \in \mathbb{R}\}$ *as the solution set.*

   (e) $\{(1, 1, 1)^T + c(1, 2, 1)^T + d(2, 2, -1)^T | c, d \in \mathbb{R}\}$ *as the solution set.*

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2.7 Square Matrices and Linear Systems

In this section, we apply our ideas to the linear system $A\mathbf{x} = \mathbf{b}$, where the coefficient matrix is square. We start with proving a few equivalent conditions that relate different ideas.

Theorem 2.7.1. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then, the following statements are equivalent.*

1. *A is invertible.*
2. *RREF$(A) = I_n$.*
3. *A is a product of elementary matrices.*
4. $\mathsf{Rank}(A) = n$.
5. *The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
6. *The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.*
7. *The system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b}$.*

Proof. $1 \Leftrightarrow 2$ and $2 \Leftrightarrow 3$ See Proposition 2.4.9.

$2 \Leftrightarrow 4$ By definition. For the converse, $\mathsf{Rank}(A) = n \Rightarrow A$ has n pivots and A has n columns. So, all columns are pivots. Thus, RREF$(A) = I_n$.

$1 \Longrightarrow 5$ As A is invertible $A^{-1}A = I_n$. So, if $\mathbf{x}_0$ is any solution of the homogeneous system $A\mathbf{x} = \mathbf{0}$ then

$$\mathbf{x}_0 = I_n \cdot \mathbf{x}_0 = (A^{-1}A)\mathbf{x}_0 = A^{-1}(A\mathbf{x}_0) = A^{-1}\,\mathbf{0} = \mathbf{0}.$$

DRAFT

$5 \Longrightarrow 1$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution implies that there are no free variables. So, all the unknowns are basic variables. So, each column is a pivotal column. Thus, RREF$(A) = I_n$.

$1 \Rightarrow 6$ Note that $\mathbf{x}_0 = A^{-1}\mathbf{b}$ is the unique solution of $A\mathbf{x} = \mathbf{b}$.

$6 \Rightarrow 7$ A unique solution implies that is at least one solution. So, nothing to show.

$7 \Longrightarrow 1$ Given assumption implies that for $1 \le i \le n$, the linear system $A\mathbf{x} = \mathbf{e}_i$ has a solution, say $\mathbf{u}_i$. Define $B = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$. Then

$$AB = A[\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] = [A\mathbf{u}_1, A\mathbf{u}_2, \ldots, A\mathbf{u}_n] = [\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n] = I_n.$$

Now, consider the linear homogeneous system $B\mathbf{x} = \mathbf{0}$. Then $AB = I_n$ implies that

$$\mathbf{x}_0 = I_n\mathbf{x}_0 = (AB)\mathbf{x}_0 = A(B\mathbf{x}_0) = A\mathbf{0} = \mathbf{0}.$$

Thus, the homogeneous system $B\mathbf{x} = \mathbf{0}$ has a only the trivial solution. Hence, using Part 5, B is invertible. As $AB = I_n$ and B is invertible, we get $BA = I_n$. Thus $AB = I_n = BA$. Thus, A is invertible as well. ∎

We now give an immediate application of Theorem 2.7.1 without proof.

Theorem 2.7.2. *The following two statements cannot hold together for $A \in \mathbb{M}_n(\mathbb{R})$.*

1. *The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.*

2. *The system* $A\mathbf{x} = \mathbf{0}$ *has a non-trivial solution.*

As an immediate consequence of Theorem 2.7.1, the readers should prove that one needs to compute either the left or the right inverse to prove invertibility of $A \in \mathbb{M}_n(\mathbb{R})$.

Corollary 2.7.3. *Let* $A \in \mathbb{M}_n(\mathbb{R})$. *Then the following holds.*

1. *If there exists* C *such that* $CA = I_n$ *then* A^{-1} *exists.*
2. *If there exists* B *such that* $AB = I_n$ *then* A^{-1} *exists.*

Corollary 2.7.4. *(***Theorem of the Alternative***) The following two statements cannot hold together for* $A \in \mathbb{M}_n(\mathbb{C})$ *and* $\mathbf{b} \in \mathbb{R}^n$.

1. *The system* $A\mathbf{x} = \mathbf{b}$ *has a solution.*
2. *The system* $\mathbf{y}^T A = \mathbf{0}^T, \mathbf{y}^T\mathbf{b} \neq 0$ *has a solution.*

Proof. Observe that if $\mathbf{x}_0$ is a solution of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}_0$ is a solution of $\mathbf{y}^T A = \mathbf{0}^T$ then $\mathbf{y}_0^T\mathbf{b} = \mathbf{y}_0^T(A\mathbf{x}_0) = (\mathbf{y}_0^T A)\mathbf{x}_0 = \mathbf{0}^T\mathbf{x}_0 = 0$. ■

Note that one of the requirement in the last corollary is $\mathbf{y}^T\mathbf{b} \neq 0$. Thus, we want non-zero vectors $\mathbf{x}_0$ and $\mathbf{y}_0$ in $\mathbb{R}^n$ such that they are solutions of $A\mathbf{x} = \mathbf{b}$ and $\mathbf{y}^T A = \mathbf{0}^T$, respectively, with the added condition that $\mathbf{y}_0$ and $\mathbf{b}$ are not orthogonal or perpendicular (their dot product is not zero).

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EXERCISE **2.7.5.**

1. *Give the proof of Theorem 2.7.2 and Corollary 2.7.3.*
2. *Let* $A \in \mathbb{M}_{n,m}(\mathbb{R})$ *and* $B \in \mathbb{M}_{m,n}(\mathbb{R})$. *Either use Theorem 2.7.1.5 or multiply the matrices to verify the following statementes.*
   (a) *Then, prove that* $I - BA$ *is invertible if and only if* $I - AB$ *is invertible.*
   (b) *If* $I - AB$ *is invertible then, prove that* $(I - BA)^{-1} = I + B(I - AB)^{-1}A$.
   (c) *If* $I - AB$ *is invertible then, prove that* $(I - BA)^{-1}B = B(I - AB)^{-1}$.
   (d) *If* A, B *and* $A + B$ *are invertible then, prove that* $(A^{-1} + B^{-1})^{-1} = A(A + B)^{-1}B$.
3. *Let* $\mathbf{b}^T = [1, 2, -1, -2]$. *Suppose* A *is a* 4×4 *matrix such that the linear system* $A\mathbf{x} = \mathbf{b}$ *has no solution. Mark each of the statements given below as* TRUE *or* FALSE*?*
   (a) *The homogeneous system* $A\mathbf{x} = \mathbf{0}$ *has only the trivial solution.*
   (b) *The matrix* A *is invertible.*
   (c) *Let* $\mathbf{c}^T = [-1, -2, 1, 2]$. *Then, the system* $A\mathbf{x} = \mathbf{c}$ *has no solution.*
   (d) *Let* $B = RREF(A)$. *Then,*
      i. $B[4, :] = [0, 0, 0, 0]$.
      ii. $B[4, :] = [0, 0, 0, 1]$.

2.8 Determinant

Recall the notations used in Section 1.5 on Page 21 . If $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 4 & 7 \end{bmatrix}$ then $A(1 \mid 2) = \begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$ and $A(\{1,2\} \mid \{1,3\}) = [4]$. The actual definition of the determinant requires an understanding of group theory. So, we will just give an inductive definition which will help us to compute the determinant and a few results. The advanced students can find the main definition of the determinant in Appendix 9.2.22, where it is proved that the definition given below corresponds to the expansion of determinant along the first row.

Definition 2.8.1. Let A be a square matrix of order n. Then, the determinant of A, denoted $\det(A)$ (or $\mid A \mid$) is defined by

$$\det(A) = \begin{cases} a, & \text{if } A = [a] \text{ (corresponds to } n = 1), \\ \sum\limits_{j=1}^{n} (-1)^{1+j} a_{1j} \det\big(A(1 \mid j)\big), & \text{otherwise.} \end{cases}$$

Example 2.8.2. 1. Let $A = [-2]$. Then, $\det(A) = \mid A \mid = -2$.

2. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, $\det(A) = a \ \det(A(1 \mid 1)) - b \ \det(A(1 \mid 2)) = ad - bc$.

   (a) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$ then $\det(A) = 1 \cdot 5 - 2 \cdot 0 = 5$.

   (b) If $B = \begin{bmatrix} 2 & 1 \\ 5 & 0 \end{bmatrix}$ then $\det(B) = 2 \cdot 0 - 1 \cdot 5 = -5$.

   Observe that B is obtained from A by interchanging the columns. This has resulted in the value of the determinant getting multiplied by -1. So, if we think of the columns of the matrix as vectors in $\mathbb{R}^2$ then, the sign of determinant gets related with the orientation of the parallelogram formed using the two column vectors.

3. Let $A = [a_{ij}]$ be a 3×3 matrix. Then,

$$\begin{aligned} \det(A) = \mid A \mid &= a_{11} \det(A(1 \mid 1)) - a_{12} \det(A(1 \mid 2)) + a_{13} \det(A(1 \mid 3)) \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ &= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22}). \end{aligned}$$

   For $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$, $\det(A) = 1 \cdot \begin{vmatrix} 3 & 1 \\ 2 & 2 \end{vmatrix} - 2 \cdot \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 4 - 2(3) + 3(1) = 1$.

EXERCISE **2.8.3.** *Find the determinant of the following matrices.*

i) $\begin{bmatrix} 1 & 2 & 7 & 8 \\ 0 & 4 & 3 & 2 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}$ *ii)* $\begin{bmatrix} 3 & 0 & 0 & 1 \\ 0 & 2 & 0 & 5 \\ 6 & -7 & 1 & 0 \\ 3 & 2 & 0 & 6 \end{bmatrix}$ *iii)* $\begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$.

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Definition 2.8.4. A matrix A is said to be a SINGULAR if $\det(A) = 0$ and is called NON-SINGULAR if $\det(A) \neq 0$.

It turns out that the determinant of a matrix equals the volume of the parallelepiped formed using the columns of the matrix. With this understanding, the singularity of A gets related with the dimension in which we are looking at the parallelepiped. For, example, the length makes sense in one-dimension but it doesn't make sense to talk of area (which is a two-dimensional idea) of a line segment. Similarly, it makes sense to talk of volume of a cube but it doesn't make sense to talk of the volume of a square or rectangle or parallelogram which are two-dimensional objects.

We now state a few properties of the determinant function. For proof, see Appendix 9.3.

Theorem 2.8.5. *Let A be an $n \times n$ matrix.*

1. $\det(I_n) = 1$.
2. *If A is a triangular matrix with $d_1, \ldots, d_n$ on the diagonal then* $\det(A) = d_1 \cdots d_n$.
3. *If $B = E_{ij}A$, for $1 \le i \neq j \le n$, then* $\det(B) = -\det(A)$. *In particular,* $\det(E_{ij}) = -1$
4. *If $B = E_i(c)A$, for $c \neq 0, 1 \le i \le n$, then* $\det(B) = c\det(A)$. *In particular,* $\det(E_i(c)) = c$.
5. *If $B = E_{ij}(c)A$, for $c \neq 0$ and $1 \le i \neq j \le n$, then* $\det(B) = \det(A)$. *In particular,* $\det(E_{ij}(c)) = 1$.
6. *If $A[i,:] = \mathbf{0}^T$, for $1 \le i, j \le n$ then* $\det(A) = 0$.
7. *If $A[i,:] = A[j,:]$ for $1 \le i \neq j \le n$ then* $\det(A) = 0$.

DRAFT

Example 2.8.6. Let $A = \begin{bmatrix} 2 & 2 & 6 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{bmatrix}$. Then $A \overset{E_1(\frac{1}{2})}{\to} \begin{vmatrix} 1 & 1 & 3 \\ 1 & 3 & 2 \\ 1 & 1 & 2 \end{vmatrix} \overset{E_{21}(-1)E_{31}(-1)}{\to} \begin{vmatrix} 1 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & -1 \end{vmatrix}$. Thus, using Theorem 2.8.5, $\det(A) = 2 \cdot (1 \cdot 2 \cdot (-1)) = -4$, where the first 2 appears from the elementary matrix $E_1(\frac{1}{2})$.

EXERCISE **2.8.7.** *Prove the following without computing the determinant (use Theorem 2.8.5).*

1. *Let* $A = \begin{bmatrix} \mathbf{u} & \mathbf{v} & 2\mathbf{u} + 3\mathbf{v} \end{bmatrix}$, *where* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$. *Then,* $\det(A) = 0$.
2. *Let* $A = \begin{bmatrix} a & b & c \\ e & f & g \\ h & j & \ell \end{bmatrix}$. *If $x \neq 0$ and* $B = \begin{bmatrix} a & e & x^2a + xe + h \\ b & f & x^2b + xf + j \\ c & g & x^2c + xg + \ell \end{bmatrix}$ *then* $\det(A) = \det(B)$.

   Hence, conclude that 3 divides $\begin{vmatrix} 3 & 1 & 2 \\ 4 & 7 & 1 \\ 1 & 4 & -2 \end{vmatrix}$.

Remark 2.8.8. *Theorem 2.8.5.3 implies that the determinant can be calculated by expanding along any row. Hence, the readers are advised to verify that*

$$\det(A) = \sum_{j=1}^{n} (-1)^{k+j} a_{kj} \det(A(k \mid j)), \quad \textit{for } 1 \le k \le n.$$

Example 2.8.9. Using Remark 2.8.8, one has

$$\begin{vmatrix} 2 & 2 & 6 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 2 & 1 & 1 \end{vmatrix} = (-1)^{2+3} \cdot 2 \cdot \begin{vmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{vmatrix} + (-1)^{2+4} \cdot \begin{vmatrix} 2 & 2 & 6 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{vmatrix} = -2 \cdot 1 + (-8) = -10.$$

2.8.1 Inverse of a Matrix

Definition 2.8.10. Let $A \in \mathbb{M}_n(\mathbb{R})$. Then, the **cofactor** matrix, denoted $\mathsf{Cof}(A)$, is an $\mathbb{M}_n(\mathbb{R})$ matrix with $\mathsf{Cof}(A) = [C_{ij}]$, where

$$C_{ij} = (-1)^{i+j} \det\left(A(i \mid j)\right), \text{ for } 1 \le i, j \le n.$$

And, the **Adjugate** (classical Adjoint) of A, denoted $Adj(A)$, equals $\mathsf{Cof}^T(A)$.

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Example 2.8.11. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$. Then,

$$\begin{aligned} Adj(A) &= \mathsf{Cof}^T(A) = \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \\ &= \begin{bmatrix} (-1)^{1+1}\det(A(1|1)) & (-1)^{2+1}\det(A(2|1)) & (-1)^{3+1}\det(A(3|1)) \\ (-1)^{1+2}\det(A(1|2)) & (-1)^{2+2}\det(A(2|2)) & (-1)^{3+2}\det(A(3|2)) \\ (-1)^{1+3}\det(A(1|3)) & (-1)^{2+3}\det(A(2|3)) & (-1)^{3+3}\det(A(3|3)) \end{bmatrix} \\ &= \begin{bmatrix} 10 & -2 & -7 \\ -7 & 1 & 5 \\ 1 & 0 & -1 \end{bmatrix}. \end{aligned}$$

Now, verify that $AAdj(A) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix} = Adj(A)A.$

Consider $xI_3 - A = \begin{bmatrix} x-1 & -2 & -3 \\ -2 & x-3 & -1 \\ -1 & -2 & x-4 \end{bmatrix}$. Then,

$$\begin{aligned} Adj(xI-A) &= \begin{bmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} = \begin{bmatrix} x^2-7x+10 & 2x-2 & 3x-7 \\ 2x-7 & x^2-5x+1 & x+5 \\ x+1 & 2x & x^2-4x-1 \end{bmatrix} \\ &= x^2 I + x \begin{bmatrix} -7 & 2 & 3 \\ 2 & -5 & 1 \\ 1 & 2 & -4 \end{bmatrix} + Adj(A) = x^2 I + Bx + C(\text{say}). \end{aligned}$$

Hence, we observe that $Adj(xI - A) = x^2I + Bx + C$ is a polynomial in x with coefficients as matrices. Also, note that $(xI - A)Adj(xI - A) = (x^3 - 8x^2 + 10x - \det(A))I_3$. Thus

$$(xI - A)(x^2I + Bx + C) = (x^3 - 8x^2 + 10x - \det(A))I_3.$$

That is, we have obtained a matrix identity. Hence, replacing x by A makes sense. But, then the LHS is $\mathbf{0}$. So, for the RHS to be zero, we must have $A^3 - 8A^2 + 10A - \det(A)I = \mathbf{0}$ (this equality is famously known as the Cayley-Hamilton Theorem).

The next result relates adjugate matrix with the inverse, in case $\det(A) \neq 0$.

DRAFT

Theorem 2.8.12. *Let $A \in \mathbb{M}_n(\mathbb{R})$.*

1. *Then* $\sum\limits_{j=1}^{n} a_{ij}\, C_{ij} = \sum\limits_{j=1}^{n} a_{ij}(-1)^{i+j}\ \det(A(i|j)) = \det(A)$, *for* $1 \leq i \leq n$.
2. *Then* $\sum\limits_{j=1}^{n} a_{ij}\, C_{\ell j} = \sum\limits_{j=1}^{n} a_{ij}(-1)^{i+j}\ \det(A(\ell|j)) = 0$, *for* $i \neq \ell$.
3. *Thus* $A(Adj(A)) = \det(A)I_n$. *Hence,*

$$\text{whenever } \det(A) \neq 0 \ \text{ one has } \ A^{-1} = \frac{1}{\det(A)} Adj(A). \tag{2.8.1}$$

Proof. Part 1: It follows directly from Remark 2.8.8 and the definition of the cofactor.

Part 2: Fix positive integers i, ℓ with $1 \leq i \neq \ell \leq n$. Suppose that the i-th and ℓ-th rows of B are equal to the i-th row of A and $B[t,:] = A[t,:]$, for $t \neq i, \ell$. Since two rows of B are equal, $\det(B) = 0$. Now, let us expand the determinant of B along the ℓ-th row. We see that

$$\begin{aligned} 0 = \det(B) &= \sum_{j=1}^{n} (-1)^{\ell+j} b_{\ell j} \det\big(B(\ell \mid j)\big) && (2.8.2) \\ &= \sum_{j=1}^{n} (-1)^{\ell+j} a_{ij} \det\big(B(\ell \mid j)\big) && (b_{ij} = b_{\ell j} = a_{ij} \text{ for all } j) \\ &= \sum_{j=1}^{n} (-1)^{\ell+j} a_{ij} \det\big(A(\ell \mid j)\big) = \sum_{j=1}^{n} a_{ij} C_{\ell j}. && (2.8.3) \end{aligned}$$

This completes the proof of Part 2.

Part 3: Using the first two parts, observe that

$$\Big[A\big(\mathrm{Adj}(A)\big)\Big]_{ij} = \sum_{k=1}^{n} a_{ik}\big(\mathrm{Adj}(A)\big)_{kj} = \sum_{k=1}^{n} a_{ik}C_{jk} = \begin{cases} 0, & \text{if } i \neq j, \\ \det(A), & \text{if } i = j. \end{cases}$$

Thus, $A(Adj(A)) = \det(A)I_n$. Therefore, if $\det(A) \neq 0$ then $A\left(\frac{1}{\det(A)}Adj(A)\right) = I_n$. Hence, by Proposition 2.4.9, $A^{-1} = \frac{1}{\det(A)}Adj(A)$. ∎

Example 2.8.13. For $A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$, $Adj(A) = \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -3 & 1 \end{bmatrix}$ and $\det(A) = -2$. Thus, by Theorem 2.8.12.3, $A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 \\ 1/2 & 3/2 & -1/2 \end{bmatrix}$.

Let A be a non-singular matrix. Then, by Theorem 2.8.12.3, $A^{-1} = \frac{1}{\det(A)}Adj(A)$. Thus $A\big(Adj(A)\big) = \big(Adj(A)\big)A = \det(A)\ I_n$ and this completes the proof of the next result

Corollary 2.8.14. *Let A be a non-singular matrix. Then,*

$$\sum_{i=1}^{n} C_{ik}\ a_{ij} = \begin{cases} \det(A), & \text{if } j = k, \\ 0, & \text{if } j \neq k. \end{cases}$$

DRAFT

The next result gives another equivalent condition for a square matrix to be invertible.

Theorem 2.8.15. *A square matrix A is non-singular if and only if A is invertible.*

Proof. Let A be non-singular. Then, $\det(A) \neq 0$ and hence $A^{-1} = \frac{1}{\det(A)}Adj(A)$.

Now, let us assume that A is invertible. Then, using Theorem 2.7.1, $A = E_1 \cdots E_k$, a product of elementary matrices. Thus, a repeated application of Parts 3, 4 and 5 of Theorem 2.8.5 gives $\det(A) \neq 0$. ∎

2.8.2 Results on the Determinant

The next result relates the determinant of a matrix with the determinant of its transpose. Thus, the determinant can be computed by expanding along any column as well.

Theorem 2.8.16. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then* $\det(A) = \det(A^T)$. *Further,* $\det(A^*) = \overline{\det(A)}$.

Proof. If A is singular then, by Theorem 2.8.15, A is not invertible. So, A^T is also not invertible and hence by Theorem 2.8.15, $\det(A^T) = 0 = \det(A)$.

Now, let A be a non-singular and let $A^T = B$. Then, by definition,

$$\begin{aligned} \det(A^T) &= \det(B) = \sum_{j=1}^{n}(-1)^{1+j}b_{1j}\det\big(B(1 \mid j)\big) = \sum_{j=1}^{n}(-1)^{1+j}a_{j1}\det\big(A(j \mid 1)\big) \\ &= \sum_{j=1}^{n} a_{j1}C_{j1} = \det(A) \end{aligned}$$

using Corollary 2.8.14. Further, using induction and the first part, one has

$$\begin{aligned}\det(A^*) &= \det((\overline{A})^T) = \det(\overline{A}) = \sum_{j=1}^{n}(-1)^{1+j}\overline{a_{1j}}\det\big(\overline{A(1\mid j)}\big)\\ &= \overline{\sum_{j=1}^{n}(-1)^{1+j}a_{1j}\det\big(A(1\mid j)\big)} = \overline{\det(A)}\end{aligned}$$

Hence, the required result follows. ∎

The next result relates the determinant of product of two matrices with their determinants.

Theorem 2.8.17. *Let A and B be square matrices of order n. Then,*

$$\det(AB) = \det(A)\cdot\det(B) = \det(BA).$$

Proof. Case 1: Let A be non-singular. Then, by Theorem 2.8.12.3, A is invertible and by Theorem 2.7.1, $A = E_1\cdots E_k$, a product of elementary matrices. Thus, a repeated application of Parts 3, 4 and 5 of Theorem 2.8.5 and an inductive argument gives the desired result as

$$\begin{aligned}\det(AB) &= \det(E_1\cdots E_kB) = \det(E_1)\det(E_2\cdots E_kB)\\ &= \det(E_1)\det(E_2\cdots E_k)\det(B) = \det(E_1E_2\cdots E_k)\det(B)\\ &= \det(A)\det(B).\end{aligned}$$

DRAFT

Case 2: Let A be singular. Then, by Theorem 2.8.15 A is not invertible. So, by Proposition 2.4.9 there exists an invertible matrix P such that $PA = \begin{bmatrix}C_1\\ \mathbf{0}\end{bmatrix}$. So $A = P^{-1}\begin{bmatrix}C_1\\ \mathbf{0}\end{bmatrix}$. As P is invertible, using Part 1, we have

$$\begin{aligned}\det(AB) &= \det\left(\left(P^{-1}\begin{bmatrix}C_1\\ \mathbf{0}\end{bmatrix}\right)B\right) = \det\left(P^{-1}\begin{bmatrix}C_1B\\ \mathbf{0}\end{bmatrix}\right) = \det(P^{-1})\cdot\det\left(\begin{bmatrix}C_1B\\ \mathbf{0}\end{bmatrix}\right)\\ &= \det(P)\cdot 0 = 0 = 0\cdot\det(B) = \det(A)\det(B).\end{aligned}$$

Thus, the proof of the theorem is complete. ∎

We now give an application of Theorem 2.8.17.

Example 2.8.18. Let $A \in \mathbb{M}_n(\mathbb{R})$ be an orthogonal matrix then, by definition, $AA^T = I$. Thus, by Theorems 2.8.17 and 2.8.16

$$1 = \det(I) = \det(AA^T) = \det(A)\det(A^T) = \det(A)\det(A) = (\det(A))^2.$$

Hence $\det A = \pm 1$. In particular, if $A = \begin{bmatrix}a & b\\ c & d\end{bmatrix} \in \mathbb{M}_2(\mathbb{R})$ then the following holds.

1. $I = AA^T = \begin{bmatrix}a^2+b^2 & ac+bd\\ ac+bd & c^2+d^2\end{bmatrix}$.
2. Thus $a^2+b^2 = 1$ and hence there exists $\theta \in [-\pi,\ \pi)$ such that $a = \cos\theta$ and $b = \sin\theta$.
3. Further, $ac+bd = 0$ and $c^2+d^2 = 1 \Rightarrow c = \sin\theta, d = -\cos\theta$ or $c = -\sin\theta, d = \cos\theta$.

4. Thus $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$.

5. For $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, $\det(A) = -1$. Then A represents a reflection about the line $y = m\mathbf{x}$. Determine m? (see Exercise 2.2b).

6. For $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, $\det(A) = 1$. Then A represents a rotation through the angle θ. Is the rotation clockwise or counter-clockwise (see Exercise 2.2a)?

EXERCISE **2.8.19.** *1. Let A be a square matrix. Then, prove that A is invertible $\Leftrightarrow$ A^T is invertible $\Leftrightarrow$ A^TA is invertible $\Leftrightarrow$ AA^T is invertible.*

2. Let A and B be two matrices having positive entries and of orders $1 \times n$ and $n \times 1$, respectively. Which of BA or AB is invertible? Give reasons.

2.8.3 Cramer's Rule

Consider the linear system $A\mathbf{x} = \mathbf{b}$. Then, using Theorems 2.7.1 and 2.8.15, we conclude that $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$ if and only if $\det(A) \neq 0$. The next theorem, commonly known as the **Cramer's rule** gives a direct method of finding the solution of the linear system $A\mathbf{x} = \mathbf{b}$ when $\det(A) \neq 0$.

DRAFT

Theorem 2.8.20. *Let A be an $n \times n$ non-singular matrix. Then, the unique solution of the linear system $A\mathbf{x} = \mathbf{b}$ with the unknown vector $\mathbf{x}^T = [x_1, \ldots, x_n]$ is given by*

$$x_j = \frac{\det(A_j)}{\det(A)}, \quad \textit{for } j = 1, 2, \ldots, n,$$

where A_j is the matrix obtained from A by replacing the j-th column of A, namely $A[:, j]$, by $\mathbf{b}$.

Proof. Since $\det(A) \neq 0$, A is invertible. Thus $A^{-1}[A \mid \mathbf{b}] = [I \mid A^{-1}\mathbf{b}]$. Let $\mathbf{d} = A^{-1}\mathbf{b}$. Then $A\mathbf{x} = \mathbf{b}$ has the unique solution $x_j = \mathbf{d}_j$, for $1 \leq j \leq n$. Thus,

$$\begin{aligned} A^{-1}A_j &= A^{-1}\Big[A[:,1], \ldots, A[:,j-1], \mathbf{b}, A[:,j+1], \ldots, A[:,n]\Big] \\ &= \Big[A^{-1}A[:,1], \ldots, A^{-1}A[:,j-1], A^{-1}\mathbf{b}, A^{-1}A[:,j+1], \ldots, A^{-1}A[:,n]\Big] \\ &= [\mathbf{e}_1, \ldots, \mathbf{e}_{j-1}, \mathbf{d}, \mathbf{e}_{j+1}, \ldots, \mathbf{e}_n]. \end{aligned}$$

Thus, $\det(A^{-1}A_j) = \mathbf{d}_j$, for $1 \leq j \leq n$. Also,

$$\mathbf{d}_j = \frac{\mathbf{d}_j}{1} = \frac{\det(A^{-1}A_j)}{\det(I)} = \frac{\det(A^{-1}A_j)}{\det(A^{-1}A)} = \frac{\det(A^{-1})\det(A_j)}{\det(A^{-1})\det(A)} = \frac{\det(A_j)}{\det(A)}.$$

Hence, $x_j = \dfrac{\det(A_j)}{\det(A)}$ and the required result follows. ∎

Example 2.8.21. Solve $A\mathbf{x} = \mathbf{b}$ using Cramer's rule, where $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: Check that $\det(A) = 1$ and $\mathbf{x}^T = [-1, 1, 0]$ as

$$x_1 = \begin{vmatrix} 1 & 2 & 3 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{vmatrix} = -1, \; x_2 = \begin{vmatrix} 1 & 1 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 2 \end{vmatrix} = 1, \quad \text{and} \quad x_3 = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0.$$

2.9 Miscellaneous Exercises

EXERCISE **2.9.1.**

1. *Determine the determinant of an orthogonal matrix.*

2. *Let A be a unitary matrix then what can you say about $\mid \det(A) \mid$?*

3. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Prove that the following statements are equivalent:*

   (a) A is not invertible.

   (b) $\mathsf{Rank}(A) \neq n$.

   (c) $\det(A) = 0$.

   (d) A is not row-equivalent to I_n.

   (e) The homogeneous system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

   (f) The system $A\mathbf{x} = \mathbf{b}$ is either inconsistent or it has an infinite number of solutions.

   (g) A is not a product of elementary matrices.

4. *Let $A \in \mathbb{M}_{2n+1}(\mathbb{R})$ be a skew-symmetric matrix. Then $\det(A) = 0$.*

5. *If A is a Hermitian matrix then $\det A$ is a real number.*

6. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then A is invertible if and only if $Adj(A)$ is invertible.*

7. *Let A and B be invertible matrices. Prove that $Adj(AB) = Adj(B)Adj(A)$.*

8. *Let A be an $n \times n$ invertible matrix and let $P = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then, show that $\mathsf{Rank}(P) = n$ if and only if $D = CA^{-1}B$.*

9. *Let A be a 2×2 matrix with $\mathsf{tr(A)} = 0$ and $\det(A) = 0$. Then, A is a nilpotent matrix.*

10. *Determine necessary and sufficient condition for a triangular matrix to be invertible.*

11. *Let A and B be two non-singular matrices. Are the matrices $A + B$ and $A - B$ non-singular? Justify your answer.*

12. *For what value(s) of λ does the following systems have non-trivial solutions? Also, for each value of λ, determine a non-trivial solution.*

DRAFT

(a) $(\lambda - 2)x + y = 0,\ x + (\lambda + 2)y = 0.$

(b) $\lambda x + 3y = 0,\ (\lambda + 6)y = 0.$

13. *Let* $a_1, \ldots, a_n \in \mathbb{R}$ *and define* $A = [a_{ij}]_{n \times n}$ *with* $a_{ij} = a_i^{j-1}$. *Prove that* $\det(A) = \prod\limits_{1 \le i < j \le n} (a_j - a_i)$. *This matrix is usually called the van der monde matrix.*

14. *Let* $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R})$ *with* $a_{ij} = \max\{i, j\}$. *Prove that* $\det A = (-1)^{n-1} n$.

15. *Let* $p \in \mathbb{R}, p \neq 0$. *Let* $A = [a_{ij}], B = [b_{ij}] \in \mathbb{M}_n(\mathbb{R})$ *with* $b_{ij} = p^{i-j} a_{ij}$, *for* $1 \le i, j \le n$. *Then, compute* $\det(B)$ *in terms of* $\det(A)$.

16. *The position of an element* a_{ij} *of a determinant is called even or odd according as* $i + j$ *is even or odd. Prove that if all the entries in*

    (a) *odd positions are multiplied with* -1 *then the value of determinant doesn't change.*

    (b) *even positions are multiplied with* -1 *then the value of determinant*

        i. *does not change if the matrix is of even order.*

        ii. *is multiplied by* -1 *if the matrix is of odd order.*

2.10 Summary

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In this chapter, we started with a system of m linear equations in n variables and formally wrote it as $A\mathbf{x} = \mathbf{b}$ and in turn to the augmented matrix $[A \mid \mathbf{b}]$. Then, the basic operations on equations led to multiplication by elementary matrices on the right of $[A \mid \mathbf{b}]$. These elementary matrices are invertible and applying the GJE on a matrix A, resulted in getting the RREF of A. We used the pivots in RREF matrix to define the rank of a matrix. So, if $\mathsf{Rank}(A) = r$ and $\mathsf{Rank}([A \mid \mathbf{b}]) = r_a$

1. then, $r < r_a$ implied the linear system $A\mathbf{x} = \mathbf{b}$ is inconsistent.

2. then, $r = r_a$ implied the linear system $A\mathbf{x} = \mathbf{b}$ is consistent. Further,

    (a) if $r = n$ then the system $A\mathbf{x} = \mathbf{b}$ has a unique solution.

    (b) if $r < n$ then the system $A\mathbf{x} = \mathbf{b}$ has an infinite number of solutions.

We have also seen that the following conditions are equivalent for $A \in \mathbb{M}_n(\mathbb{R})$.

1. A is invertible.

2. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

3. The row reduced echelon form of A is I.

4. A is a product of elementary matrices.

5. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.

6. The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.

7. $\mathsf{Rank}(A) = n$.

8. $\det(A) \neq 0$.

So, overall we have learnt to solve the following type of problems:

1. Solving the linear system $A\mathbf{x} = \mathbf{b}$. This idea will lead to the question "is the vector $\mathbf{b}$ a linear combination of the columns of A"?

2. Solving the linear system $A\mathbf{x} = \mathbf{0}$. This will lead to the question "are the columns of A linearly independent/dependent"? In particular, we will see that

   (a) if $A\mathbf{x} = \mathbf{0}$ has a unique solution then the columns of A are linear independent.

   (b) if $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution then the columns of A are linearly dependent.

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Chapter 3

Vector Spaces

In this chapter, we will mainly be concerned with finite dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$. Please note that the real and complex numbers have the property that any pair of elements can be added, subtracted or multiplied. Also, division is allowed by a non-zero element. Such sets in mathematics are called **field**. So, $\mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ are examples of field and they have infinite number of elements. But, in mathematics, we do have fields that have only finitely many elements. For example, consider the set $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$. In $\mathbb{Z}_5$, we define addition and multiplication, respectively, as

$+$	0	1	2	3	4
0	0	1	2	3	4
1	1	2	3	4	0
2	2	3	4	0	1
3	3	4	0	1	2
4	4	0	1	2	3

DRAFT

and

$\cdot$	0	1	2	3	4
0	0	0	0	0	0
1	0	1	2	3	4
2	0	2	4	1	3
3	0	3	1	4	2
4	0	4	3	2	1

.

Then, we see that the elements of $\mathbb{Z}_5$ can be added, subtracted and multiplied. Note that 4 behaves as -1 and 3 behaves as -2. Thus, 1 behaves as -4 and 2 behaves as -3. Also, we see that in this multiplication $2 \cdot 3 = 1$ and $4 \cdot 4 = 1$. Hence,

1. the division by 2 is similar to multiplying by 3,
2. the division by 3 is similar to multiplying by 2, and
3. the division by 4 is similar to multiplying by 4.

Thus, $\mathbb{Z}_5$ indeed behaves like a field. So, in this chapter, $\mathbb{F}$ will represent a field.

3.1 Vector Spaces: Definition and Examples

Let us recall that the vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ satisfy the following properties:

1. **Vector Addition:** To every pair $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ there corresponds a unique element $\mathbf{u}+\mathbf{v} \in \mathbb{R}^3$ (called the **addition of vectors**) such that

   (a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (Commutative law).

(b) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w})$ (Associative law).

(c) $\mathbb{R}^3$ has a unique element, denoted $\mathbf{0}$, called **the zero vector** that satisfies $\mathbf{u}+\mathbf{0} = \mathbf{u}$, for every $\mathbf{u} \in \mathbb{R}^3$ (called **the additive identity**).

(d) for every $\mathbf{u} \in \mathbb{R}^3$ there is an element $\mathbf{w} \in \mathbb{R}^3$ that satisfies $\mathbf{u}+\mathbf{w} = \mathbf{0}$.

2. **Scalar Multiplication:** For each $\mathbf{u} \in \mathbb{R}^3$ and $\alpha \in \mathbb{R}$, there corresponds a unique element $\alpha \cdot \mathbf{u} \in \mathbb{R}^3$ (called the **scalar multiplication**) such that

   (a) $\alpha \cdot (\beta \cdot \mathbf{u}) = (\alpha \cdot \beta) \cdot \mathbf{u}$ for every $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u} \in \mathbb{R}^3$.

   (b) $1 \cdot \mathbf{u} = \mathbf{u}$ for every $\mathbf{u} \in \mathbb{R}^3$, where $1 \in \mathbb{R}$.

3. **Distributive Laws: relating vector addition with scalar multiplication**
   For any $\alpha, \beta \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$, the following **distributive laws** hold:

   (a) $\alpha \cdot (\mathbf{u}+\mathbf{v}) = (\alpha \cdot \mathbf{u}) \; + \; (\alpha \cdot \mathbf{v})$.

   (b) $(\alpha+\beta) \cdot \mathbf{u} = (\alpha \cdot \mathbf{u}) \; + \; (\beta \cdot \mathbf{u})$.

So, we want the above properties to hold for any collection of vectors. Thus, formally, we have the following definition.

Definition 3.1.1. A **vector space** $\mathbb{V}$ over $\mathbb{F}$, denoted $\mathbb{V}(\mathbb{F})$ or in short $\mathbb{V}$ (if the field $\mathbb{F}$ is clear from the context), is a non-empty set, in which one can define vector addition, scalar multiplication. Further, with these definitions, the properties of vector addition, scalar multiplication and distributive laws (see items 1, 2 and 3 above) are satisfied.

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Remark 3.1.2. *1. The elements of* $\mathbb{F}$ *are called* **scalars**.

2. The elements of $\mathbb{V}$ *are called* **vectors**.

3. We denote the zero element of $\mathbb{F}$ *by* 0*, whereas the zero element of* $\mathbb{V}$ *will be denoted by* $\mathbf{0}$.

4. Observe that Condition 1d implies that for every $\mathbf{u} \in \mathbb{V}$*, the vector* $\mathbf{w} \in \mathbb{V}$ *such that* $\mathbf{u}+\mathbf{w} = \mathbf{0}$ *holds, is unique. For if,* $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{V}$ *with* $\mathbf{u}+\mathbf{w}_i = \mathbf{0}$*, for* $i = 1, 2$ *then by commutativity of vector addition, we see that*

$$\mathbf{w}_1 = \mathbf{w}_1 + \mathbf{0} = \mathbf{w}_1 + (\mathbf{u}+\mathbf{w}_2) = (\mathbf{w}_1+\mathbf{u}) + \mathbf{w}_2 = \mathbf{0} + \mathbf{w}_2 = \mathbf{w}_2.$$

Hence, we represent this unique vector by $-\mathbf{u}$ *and call it* **the additive inverse**.

5. If $\mathbb{V}$ *is a vector space over* $\mathbb{R}$ *then* $\mathbb{V}$ *is called a* **real vector space**.

6. If $\mathbb{V}$ *is a vector space over* $\mathbb{C}$ *then* $\mathbb{V}$ *is called a* **complex vector space**.

7. In general, a vector space over $\mathbb{R}$ *or* $\mathbb{C}$ *is called a* **linear space**.

Some interesting consequences of Definition 3.1.1 is stated next. Intuitively, they seem obvious. The proof are given for better understanding of the given conditions.

Theorem 3.1.3. *Let* $\mathbb{V}$ *be a vector space over* $\mathbb{F}$*. Then,*

1. $\mathbf{u}+\mathbf{v} = \mathbf{u}$ *implies* $\mathbf{v} = \mathbf{0}$.

2. $\alpha \cdot \mathbf{u} = \mathbf{0}$ *if and only if either* $\mathbf{u} = \mathbf{0}$ *or* $\alpha = 0$.

3. $(-1) \cdot \mathbf{u} = -\mathbf{u}$, *for every* $\mathbf{u} \in \mathbb{V}$.

Proof. Part 1: By Condition 1d and Remark 3.1.2.4, for each $\mathbf{u} \in \mathbb{V}$ there exists $-\mathbf{u} \in \mathbb{V}$ such that $-\mathbf{u} + \mathbf{u} = \mathbf{0}$. Hence $\mathbf{u} + \mathbf{v} = \mathbf{u}$ implies

$$\mathbf{0} = -\mathbf{u} + \mathbf{u} = -\mathbf{u} + (\mathbf{u} + \mathbf{v}) = (-\mathbf{u} + \mathbf{u}) + \mathbf{v} = \mathbf{0} + \mathbf{v} = \mathbf{v}.$$

Part 2: As $\mathbf{0} = \mathbf{0} + \mathbf{0}$, using Condition 3, $\alpha \cdot \mathbf{0} = \alpha \cdot (\mathbf{0} + \mathbf{0}) = (\alpha \cdot \mathbf{0}) + (\alpha \cdot \mathbf{0})$. Thus, using Part 1, $\alpha \cdot \mathbf{0} = \mathbf{0}$ for any $\alpha \in \mathbb{F}$. Similarly, $0 \cdot \mathbf{u} = (0 + 0) \cdot \mathbf{u} = (0 \cdot \mathbf{u}) + (0 \cdot \mathbf{u})$ implies $0 \cdot \mathbf{u} = \mathbf{0}$, for any $\mathbf{u} \in \mathbb{V}$.

Now suppose $\alpha \cdot \mathbf{u} = \mathbf{0}$. If $\alpha = 0$ then the proof is over. So, assume that $\alpha \neq 0, \alpha \in \mathbb{F}$. Then $(\alpha)^{-1} \in \mathbb{F}$ and using, $1 \cdot \mathbf{u} = \mathbf{u}$ for every vector $\mathbf{u} \in \mathbb{V}$ (see Condition 2.2b), we have

$$\mathbf{0} = (\alpha)^{-1} \cdot \mathbf{0} = (\alpha)^{-1} \cdot (\alpha \cdot \mathbf{u}) = ((\alpha)^{-1} \cdot \alpha) \cdot \mathbf{u} = 1 \cdot \mathbf{u} = \mathbf{u}.$$

Thus, if $\alpha \neq 0$ and $\alpha \cdot \mathbf{u} = \mathbf{0}$ then $\mathbf{u} = \mathbf{0}$.

Part 3: As $\mathbf{0} = 0 \cdot \mathbf{u} = (1 + (-1))\mathbf{u} = \mathbf{u} + (-1) \cdot \mathbf{u}$, one has $(-1) \cdot \mathbf{u} = -\mathbf{u}$. ■

Example 3.1.4. The readers are advised to justify the statements given below.

1. Let $\mathbb{V} = \{\mathbf{0}\}$. Then, $\mathbb{V}$ is a real as well as a complex vector space.

2. Let $A \in \mathbb{M}_{m,n}(\mathbb{F})$ and define $\mathbb{V} = \{\mathbf{x} \in \mathbb{M}_{n,1}(\mathbb{F}) : A\mathbf{x} = \mathbf{0}\}$. Then, by Theorem 2.1.7, $\mathbb{V}$ satisfies:

   (a) $\mathbf{0} \in \mathbb{V}$ as $A\mathbf{0} = \mathbf{0}$.

   (b) if $\mathbf{x} \in \mathbb{V}$ then $\alpha\mathbf{x} \in \mathbb{V}$, for all $\alpha \in \mathbb{F}$. In particular, for $\alpha = -1$, $-\mathbf{x} \in \mathbb{V}$.

   (c) if $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ then, for any $\alpha, \beta \in \mathbb{F}$, $\alpha\mathbf{x} + \beta\mathbf{y} \in \mathbb{V}$.

   Thus, $\mathbb{V}$ is a vector space over $\mathbb{F}$.

3. Consider $\mathbb{R}$ with the usual addition and multiplication. Then $\mathbb{R}$ forms a real vector space.

4. Let $\mathbb{R}^n = \{(a_1, \ldots, a_n)^T \mid a_i \in \mathbb{R}, 1 \leq i \leq n\}$. For $\mathbf{u} = (a_1, \ldots, a_n)^T$, $\mathbf{v} = (b_1, \ldots, b_n)^T \in \mathbb{V}$ and $\alpha \in \mathbb{R}$, define

   $$\mathbf{u} + \mathbf{v} = (a_1 + b_1, \ldots, a_n + b_n)^T \text{ and } \alpha \cdot \mathbf{u} = (\alpha a_1, \ldots, \alpha a_n)^T$$

   (CALLED COMPONENT-WISE OPERATIONS). Then, $\mathbb{V}$ is a real vector space. The vector space $\mathbb{R}^n$ is called **the real vector space of n-tuples**.

   Recall that the symbol i represents the complex number $\sqrt{-1}$.

5. Let $\mathbb{C}^n = \{(z_1, \ldots, z_n)^T \mid z_i \in \mathbb{C}, 1 \leq i \leq n\}$. For $\mathbf{z} = (z_1, \ldots, z_n), \mathbf{w} = (w_1, \ldots, w_n)^T \in \mathbb{C}^n$ and $\alpha \in \mathbb{F}$, define component-wise vector sum and scalar multiplication. Then, verify that $\mathbb{C}^n$ forms a vector space over $\mathbb{C}$ (called the **complex** vector space) as well as over $\mathbb{R}$ (called the **real** vector space). Unless specified otherwise, $\mathbb{C}^n$ will be considered a complex vector space.

DRAFT

Remark 3.1.5. *If $\mathbb{F} = \mathbb{C}$ then $i(1, 0, \ldots, 0)^T = (i, 0, \ldots, 0)^T$ is allowed. Whereas, if $\mathbb{F} = \mathbb{R}$ then $i(1, 0, \ldots, 0)^T$ doesn't make sense as $i \notin \mathbb{R}$.*

6. Fix $m, n \in \mathbb{N}$ and let $\mathbb{M}_{m,n}(\mathbb{C}) = \{A_{m\times n} = [a_{ij}] \mid a_{ij} \in \mathbb{C}\}$. Then, with usual addition and scalar multiplication of matrices, $\mathbb{M}_{m,n}(\mathbb{C})$ is a complex vector space. If $m = n$, the vector space $\mathbb{M}_{m,n}(\mathbb{C})$ is denoted by $\mathbb{M}_n(\mathbb{C})$.

7. Let S be a non-empty set and let $\mathbb{R}^S = \{f \mid f$ is a function from S to $\mathbb{R}\}$. For $f, g \in \mathbb{R}^S$ and $\alpha \in \mathbb{R}$, define $(f + \alpha g)(x) = f(x) + \alpha g(x)$, for all $x \in S$. Then, $\mathbb{R}^S$ is a real vector space. In particular, for $S = \mathbb{N}$, observe that $\mathbb{R}^{\mathbb{N}}$ consists of all real sequences and forms a real vector space.

8. Fix $a, b \in \mathbb{R}$ with $a < b$ and let $\mathcal{C}([a, b], \mathbb{R}) = \{f : [a, b] \to \mathbb{R} \mid f$ is continuous$\}$. Then, $\mathcal{C}([a, b], \mathbb{R})$ with $(f + \alpha g)(x) = f(x) + \alpha g(x)$, for all $x \in [a, b]$, is a real vector space.

9. Let $\mathcal{C}(\mathbb{R}, \mathbb{R}) = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is continuous$\}$. Then, $\mathcal{C}(\mathbb{R}, \mathbb{R})$ is a real vector space, where $(f + \alpha g)(x) = f(x) + \alpha g(x)$, for all $x \in \mathbb{R}$.

10. Fix $a < b \in \mathbb{R}$ and let $\mathcal{C}^2((a, b), \mathbb{R}) = \{f : (a, b) \to \mathbb{R} \mid f''$ is continuous$\}$. Then, $\mathcal{C}^2((a, b), \mathbb{R})$ with $(f + \alpha g)(x) = f(x) + \alpha g(x)$, for all $x \in (a, b)$, is a real vector space.

11. Let $\mathbb{R}[x] = \{a_0 + a_1x + \cdots + a_nx^n \mid a_i \in \mathbb{R}$, for $0 \le i \le n\}$. Now, let $p(x), q(x) \in \mathbb{R}[x]$. Then, we can choose m such that $p(x) = a_0 + a_1x + \cdots + a_mx^m$ and $q(x) = b_0 + b_1x + \cdots + b_mx^m$, where some of the a_i's or b_j's may be zero. Then, we define

DRAFT

$$p(x) + q(x) = (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_m + b_m)x^m$$

and $\alpha p(x) = (\alpha a_0) + (\alpha a_1)x + \cdots + (\alpha a_m)x^m$, for $\alpha \in \mathbb{R}$. With these operations "component-wise addition and multiplication", it can be easily verified that $\mathbb{R}[x]$ forms a real vector space.

12. Fix $n \in \mathbb{N}$ and let $\mathbb{R}[x; n] = \{p(x) \in \mathbb{R}[x] \mid p(x)$ has degree $\le n\}$. Then, with component-wise addition and multiplication, the set $\mathbb{R}[x; n]$ forms a real vector space.

13. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$, with operations $(+, \bullet)$ and $(\oplus, \odot)$, respectively. Let $\mathbb{V} \times \mathbb{W} = \{(\mathbf{v}, \mathbf{w}) \mid \mathbf{v} \in \mathbb{V}, \mathbf{w} \in \mathbb{W}\}$. Then, $\mathbb{V} \times \mathbb{W}$ forms a vector space over $\mathbb{F}$, if for every $(\mathbf{v}_1, \mathbf{w}_1), (\mathbf{v}_2, \mathbf{w}_2) \in \mathbb{V} \times \mathbb{W}$ and $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} (\mathbf{v}_1, \mathbf{w}_1) \oplus' (\mathbf{v}_2, \mathbf{w}_2) &= (\mathbf{v}_1 + \mathbf{v}_2, \mathbf{w}_1 \oplus \mathbf{w}_2), \text{ and} \\ \alpha \circ (\mathbf{v}_1, \mathbf{w}_1) &= (\alpha \bullet \mathbf{v}_1, \alpha \odot \mathbf{w}_1). \end{aligned}$$

$\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{w}_1 \oplus \mathbf{w}_2$ on the right hand side mean vector addition in $\mathbb{V}$ and $\mathbb{W}$, respectively. Similarly, $\alpha \bullet \mathbf{v}_1$ and $\alpha \odot \mathbf{w}_1$ correspond to scalar multiplication in $\mathbb{V}$ and $\mathbb{W}$, respectively. Note that $\mathbb{R}^2$ is similar to $\mathbb{R} \times \mathbb{R}$, where the operations are the same in both spaces.

14. Let $\mathbb{Q}$ be the set of scalars. Then,

(a) $\mathbb{R}$ is a vector space over $\mathbb{Q}$. In this space, all the irrational numbers are vectors but not scalars.

(b) $\mathbb{V} = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a vector space.

(c) $\mathbb{V} = \{a + b\sqrt{2} + c\sqrt{3} + d\sqrt{6} : a, b, c, d \in \mathbb{Q}\}$ is a vector space.

(d) $\mathbb{V} = \{a + b\sqrt{-3} : a, b \in \mathbb{Q}\}$ is a vector space.

15. Let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Then,

(a) $\mathbb{R}^+$ is not a vector space under usual operations of addition and scalar multiplication.

(b) $\mathbb{R}^+$ is a real vector space with 1 as the additive identity if we define

$$\mathbf{u} \oplus \mathbf{v} = \mathbf{u} \cdot \mathbf{v} \text{ and } \alpha \odot \mathbf{u} = \mathbf{u}^{\alpha}, \text{ for all } \mathbf{u}, \mathbf{v} \in \mathbb{R}^+ \text{ and } \alpha \in \mathbb{R}.$$

16. For any $\alpha \in \mathbb{R}$ and $\mathbf{x} = (x_1, x_2)^T, \mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$, define

$$\mathbf{x} \oplus \mathbf{y} = (x_1 + y_1 + 1, x_2 + y_2 - 3)^T \text{ and } \alpha \odot \mathbf{x} = (\alpha x_1 + \alpha - 1, \alpha x_2 - 3\alpha + 3)^T.$$

Then, $\mathbb{R}^2$ is a real vector space with $(-1, 3)^T$ as the additive identity.

17. Recall the field $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ given on the first page of this chapter. Then, $\mathbb{V} = \{(a, b) \mid a, b \in \mathbb{Z}_5\}$ is a vector space over $\mathbb{Z}_5$ having 25 elements/vectors.

From now on, we will use '$\mathbf{u} + \mathbf{v}$' for '$\mathbf{u} \oplus \mathbf{v}$' and '$\alpha\mathbf{u}$ or $\alpha \cdot \mathbf{u}$' for '$\alpha \odot \mathbf{u}$'.

DRAFT

Exercise **3.1.6.** *1. Verify that the vectors spaces mentioned in Example 3.1.4 do satisfy all the conditions for vector spaces.*

2. Does $\mathbb{R}$ with $x \oplus y = x - y$ and $\alpha \odot x = -\alpha x$, for all $x, y, \alpha \in \mathbb{R}$ form a vector space?

3. Let $\mathbb{V} = \mathbb{R}^2$. For $\mathbf{x} = (x_1, x_2)^T, \mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$ and $\alpha \in \mathbb{R}$, define

(a) $(x_1, y_1)^T \oplus (x_2, y_2)^T = (x_1 + x_2, 0)^T$ and $\alpha \odot (x_1, y_1)^T = (\alpha x_1, 0)^T$.

(b) $\mathbf{x} + \mathbf{y} = (x_1 + y_1, x_2 + y_2)^T$ and $\alpha\mathbf{x} = (\alpha x_1, 0)^T$.

Then, does $\mathbb{V}$ form a vector space under any of the two operations?

3.1.1 Vector Subspace

Definition 3.1.7. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Then, a non-empty subset $\mathbb{W}$ of $\mathbb{V}$ is called a **subspace** of $\mathbb{V}$ if $\mathbb{W}$ is also a vector space with vector addition and scalar multiplication in $\mathbb{W}$ coming from that in $\mathbb{V}$ (compute the vector addition and scalar multiplication in $\mathbb{V}$ and then the computed vector should be an element of $\mathbb{W}$).

Example 3.1.8.

1. The vector space $\mathbb{R}[x; n]$ is a subspace of $\mathbb{R}[x]$.

2. Is $\mathbb{V} = \{xp(x) \mid p(x) \in \mathbb{R}[x]\}$ a subspace of $\mathbb{R}[x]$?

3. Let $\mathbb{V}$ be a vector space. Then $\mathbb{V}$ and $\{\mathbf{0}\}$ are subspaces, called **trivial subspaces**.

4. The real vector space $\mathbb{R}$ has no non-trivial subspace. To check this, let $\mathbb{V} \neq \{\mathbf{0}\}$ be a vector subspace of $\mathbb{R}$. Then, there exists $x \in \mathbb{R}, x \neq \mathbf{0}$ such that $x \in \mathbb{V}$. Now, using scalar multiplication, we see that $\{\alpha x \mid \alpha \in \mathbb{R}\} \subseteq \mathbb{V}$. As, $x \neq \mathbf{0}$, the set $\{\alpha x \mid \alpha \in \mathbb{R}\} = \mathbb{R}$. This in turn implies that $\mathbb{V} = \mathbb{R}$.

5. $\mathbb{W} = \{\mathbf{x} \in \mathbb{R}^3 \mid \ [1, 2, -1]\mathbf{x} = 0\}$ is a subspace. It represents a plane in $\mathbb{R}^3$ containing $\mathbf{0}$.

6. $\mathbb{W} = \left\{\mathbf{x} \in \mathbb{R}^3 \mid \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{bmatrix}\mathbf{x} = \mathbf{0}\right\}$ is a subspace. What does it represent?

7. Verify that $\mathbb{W} = \{(x, 0)^T \in \mathbb{R}^2 \ \mid \ x \in \mathbb{R}\}$ is a subspace of $\mathbb{R}^2$.

8. Is the set of sequences converging to 0 a subspace of the set of all bounded sequences?

9. Let $\mathbb{V}$ be the vector space of Example 3.1.4.16. Then,
   (a) $S = \{(x, 0)^T \mid \ x \in \mathbb{R}\}$ is not a subspace of $\mathbb{V}$ as $(x, 0)^T \oplus (y, 0)^T = (x+y+1, -3)^T \notin S$.
   (b) Verify that $\mathbb{W} = \{(x, 3)^T \ \mid \ x \in \mathbb{R}\}$ is a subspace of $\mathbb{V}$.

10. The vector space $\mathbb{R}^+$ defined in Example 3.1.4.15 is not a subspace of $\mathbb{R}$.

Let $\mathbb{V}(\mathbb{F})$ be a vector space and $\mathbb{W} \subseteq \mathbb{V}, \mathbb{W} \neq \emptyset$. We now prove a result which implies that to check $\mathbb{W}$ to be a subspace, we need to verify only one condition.

DRAFT

Theorem 3.1.9. *Let $\mathbb{V}(\mathbb{F})$ be a vector space and $\mathbb{W} \subseteq \mathbb{V}, \mathbb{W} \neq \emptyset$. Then, $\mathbb{W}$ is a subspace of $\mathbb{V}$ if and only if $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathbb{W}$ whenever $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{W}$.*

Proof. Let $\mathbb{W}$ be a subspace of $\mathbb{V}$ and let $\mathbf{u}, \mathbf{v} \in \mathbb{W}$. As $\mathbb{W}$ is a subspace, the scalar multiplication and vector addition gives elements of $\mathbb{W}$ itself. Hence, for every $\alpha, \beta \in \mathbb{F}$, $\alpha\mathbf{u}, \beta\mathbf{v} \in \mathbb{W}$ and $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathbb{W}$.

Now, we assume that $\alpha\mathbf{u} + \beta\mathbf{v} \in \mathbb{W}$, whenever $\alpha, \beta \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{W}$. To show, $\mathbb{W}$ is a subspace of $\mathbb{V}$:

1. Taking $\alpha = 0$ and $\beta = 0 \Rightarrow \mathbf{0} \in \mathbb{W}$. So, $\mathbb{W}$ is non-empty.

2. Taking $\alpha = 1$ and $\beta = 1$, we see that $\mathbf{u} + \mathbf{v} \in \mathbb{W}$, for every $\mathbf{u}, \mathbf{v} \in \mathbb{W}$.

3. Taking $\beta = 0$, we see that $\alpha\mathbf{u} \in \mathbb{W}$, for every $\alpha \in \mathbb{F}$ and $\mathbf{u} \in \mathbb{W}$. Hence, using Theorem 3.1.3.3, $-\mathbf{u} = (-1)\mathbf{u} \in \mathbb{W}$ as well.

4. The commutative and associative laws of vector addition hold as they hold in $\mathbb{V}$.

5. The conditions related with scalar multiplication and the distributive laws also hold as they hold in $\mathbb{V}$. ∎

EXERCISE **3.1.10.** *1. Prove that a line in $\mathbb{R}^2$ is a subspace if and only if it passes through origin.*

2. Prove that $\{(x, y, z)^T \in \mathbb{R}^3 \mid \ ax + by + cz = d\}$ is a subspace of $\mathbb{R}^3$ if and only if $d = 0$.

3. *Does the set* $\mathbb{V}$ *given below form a subspace? Give reasons for your answer.*
   (a) *Let* $\mathbb{V} = \{(x, y, z)^T \mid x + y + z = 1\}$.
   (b) *Let* $\mathbb{V} = \{(x, y)^T \in \mathbb{R}^2 \mid x \cdot y = 0\}$.
   (c) *Let* $\mathbb{V} = \{(x, y)^T \in \mathbb{R}^2 \mid x = y^2\}$.
   (d) *Let* $\mathbb{V} = \{(x, y)^T \in \mathbb{R}^2 \mid x, y \geq 0\}$.
4. *Determine all the subspaces of* $\mathbb{R}$ *and* $\mathbb{R}^2$.
5. *Fix* $n \in \mathbb{N}$. *In the examples given below, is* $\mathbb{W}$ *a subspace of* $M_n(\mathbb{R})$, *where*
   (a) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A$ *is upper triangular*$\}$?
   (b) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A$ *is symmetric*$\}$?
   (c) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A$ *is skew-symmetric*$\}$?
   (d) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A$ *is a diagonal matrix*$\}$?
   (e) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid \textit{trace}(A) = 0\}$?
   (f) $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A^T = 2A\}$?
6. *Fix* $n \in \mathbb{N}$. *Then, is* $\mathbb{W} = \{A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R} \mid a_{11} + \overline{a_{22}} = 0\}$ *a subspace of the complex vector space* $M_n(\mathbb{C})$? *What if* $M_n(\mathbb{C})$ *is a real vector space?*
7. *Is* $\mathbb{W} = \{f \in C([-1, 1]) \mid f(-1/2) = 0, f(1/2) = 0\}$ *a subspace of* $\mathcal{C}([-1, 1])$?
8. *Are all the sets given below subspaces of* $\mathbb{R}[x]$?
   (a) $\mathbb{W} = \{f(x) \in \mathbb{R}[x] \mid \deg(f(x)) = 3\}$.
   (b) $\mathbb{W} = \{f(x) \in \mathbb{R}[x] \mid xg(x)$ *for some* $g(x) \in \mathbb{R}[x]\}$.
9. *Among the following, determine the subspaces of the complex vector space* $\mathbb{C}^n$?
   (a) $\{(z_1, z_2, \ldots, z_n)^T \mid z_1$ *is real* $\}$.
   (b) $\{(z_1, z_2, \ldots, z_n)^T \mid z_1 + z_2 = \overline{z_3}\}$.
   (c) $\{(z_1, z_2, \ldots, z_n)^T \mid \mid z_1 \mid = \mid z_2 \mid\}$.
10. *Prove that* $G = \{A \in M_n(\mathbb{R}) \mid \det(A) = 0\}$ *is not subspaces of* $M_n(\mathbb{R})$.

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3.2 Linear Combination and Linear Span

Let us recollect that system $A\mathbf{x} = \mathbf{b}$ was either consistent (has a solution) or inconsistent (no solution). It turns out that the system $A\mathbf{x} = \mathbf{b}$ is consistent leads to the idea that the vector $\mathbf{b}$ is a linear combination of the columns of A. Let us try to understand them using examples.

Example 3.2.1.

1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$. Then, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$. Thus, $\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$ is a linear combination of the vectors in $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$. Similarly, the vector $\begin{bmatrix} 10 \\ 16 \\ 22 \end{bmatrix}$ is a linear combination of the vectors in S as $\begin{bmatrix} 10 \\ 16 \\ 22 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = A \begin{bmatrix} 4 \\ 6 \end{bmatrix}$.

2. Let $\mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}$. Then, the system $A\mathbf{x} = \mathbf{b}$ has no solution as $REF([A\ \mathbf{b}]) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

Formally, we have the following definition.

Definition 3.2.2. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subseteq \mathbb{V}$. Then, a vector $\mathbf{u} \in \mathbb{V}$ is called a **linear combination** of elements of S if we can find $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$ such that

$$\mathbf{u} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n = \sum_{i=1}^{n} \alpha_i \mathbf{u}_i.$$

Or equivalently, any vector of the form $\sum\limits_{i=1}^{n} \alpha_i \mathbf{u}_i$, where $\alpha_1, \ldots, \alpha_n \in \mathbb{F}$, is said to be a linear combination of the elements of S.

DRAFT

Thus, the system $A\mathbf{x} = \mathbf{b}$ has a solution $\Rightarrow$ $\mathbf{b}$ is a linear combination of the columns of A. Or equivalently, $\mathbf{b}$ is a linear combination means the system $A\mathbf{x} = \mathbf{b}$ has a solution. So, recall that when we were solving a system of linear equations, we looked at the point of intersections of lines or plane etc. But, here it leads us to the study of whether a given vector is a linear combination of a given set S or not? Or in the language of matrices, is $\mathbf{b}$ a linear combination of columns of the matrix A or not?

Example 3.2.3.

1. $(3, 4, 5)$ is not a linear combination of $(1, 1, 1)$ and $(1, 2, 1)$ as the linear system $(3, 4, 5) = a(1, 1, 1) + b(1, 2, 1)$, in the unknowns a and b has no solution.

2. Is $(4, 5, 5)$ a linear combination of $(1, 0, 0)$, $(2, 1, 0)$ and $(3, 3, 1)$?

   Solution: Define $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix}$. Then, does the system $A\mathbf{x} = \mathbf{b}$ has a solution? Verify that $\mathbf{x} = [9,\ -10,\ 5]^T$ is a solution.

EXERCISE **3.2.4.** *1. Let $\mathbf{x} \in \mathbb{R}^3$. Prove that $\mathbf{x}^T$ is a linear combination of* $(1, 0, 0)$, $(2, 1, 0)$ *and* $(3, 3, 1)$.

2. *Find condition(s) on $x, y, z \in \mathbb{R}$ such that*
   (a) *(x, y, z) is a linear combination of $(1,2,3), (-1,1,4)$ and $(3,3,2)$.*
   (b) *(x, y, z) is a linear combination of $(1,2,1), (1,0,-1)$ and $(1,1,0)$.*
   (c) *(x, y, z) is a linear combination of $(1,1,1), (1,1,0)$ and $(1,-1,0)$.*

3.2.1 Linear Span

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and S a subset of $\mathbb{V}$. We now look at 'linear span' of a collection of vectors. So, here we ask "what is the largest collection of vectors that can be obtained as linear combination of vectors from S"? Or equivalently, what is the smallest subspace of $\mathbb{V}$ that contains S? We first look at an example for clarity.

Example 3.2.5. Let $S = \{(1,0,0), (1,2,0)\} \subseteq \mathbb{R}^3$. We want the largest possible subspace of $\mathbb{R}^3$ which contains vectors of the form $\alpha(1,0,0), \beta(1,2,0)$ and $\alpha(1,0,0) + \beta(1,2,0)$ for all possible choices of $\alpha, \beta \in \mathbb{R}$. Note that

1. $\ell_1 = \{\alpha(1,0,0) : \alpha \in \mathbb{R}\}$ gives the X-axis.
2. $\ell_2 = \{\beta(1,2,0) : \beta \in \mathbb{R}\}$ gives the line passing through $(0,0,0)$ and $(1,2,0)$.

So, we want the largest subspace of $\mathbb{R}^3$ that contains vectors which are formed as sum of any two points on the two lines ℓ_1 and ℓ_2. Or the smallest subspace of $\mathbb{R}^3$ that contains S? We give the definition next.

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Definition 3.2.6. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $S \subseteq \mathbb{V}$.

1. Then, the **linear span** of S, denoted $LS(S)$, is defined as
$$LS(S) \quad = \quad \{\alpha_1 \mathbf{u}_1 + \cdots + \alpha_n \mathbf{u}_n \mid \alpha_i \in \mathbb{F}, \mathbf{u}_i \in S, \text{ for } 1 \le i \le n\}.$$
   That is, $LS(S)$ is the set of all possible linear combinations of finitely many vectors of S. If S is an empty set, we define $LS(S) = \{\mathbf{0}\}$.
2. $\mathbb{V}$ is said to be **finite dimensional** if there exists a finite set S such that $\mathbb{V} = LS(S)$.
3. If there does not exist any finite subset S of $\mathbb{V}$ such that $\mathbb{V} = LS(S)$ then $\mathbb{V}$ is called **infinite dimensional**.

Example 3.2.7. For the set S given below, determine $LS(S)$.

1. $S = \{(1,0)^T, (0,1)^T\} \subseteq \mathbb{R}^2$.
   Solution: $LS(S) = \{a(1,0)^T + b(0,1)^T \mid a, b \in \mathbb{R}\} = \{(a,b)^T \mid a, b \in \mathbb{R}\} = \mathbb{R}^2$. Thus, $\mathbb{R}^2$ is finite dimensional.

2. $S = \{(1,1,1)^T, (2,1,3)^T\}$. What does $LS(S)$ represent in $\mathbb{R}^3$?
   Solution: $LS(S) = \{a(1,1,1)^T + b(2,1,3)^T \mid a, b \in \mathbb{R}\} = \{(a+2b, a+b, a+3b)^T \mid a, b \in \mathbb{R}\}$. Note that $LS(S)$ represents a plane passing through the points $(0,0,0)^T, (1,1,1)^T$ and $(2,1,3)^T$. To get he equation of the plane, we proceed as follows:
   Find conditions on x, y and z such that $(a+2b, a+b, a+3b) = (x, y, z)$. Or equivalently,

find conditions on x, y and z such that $a+2b = x, a+b = y$ and $a+3b = z$ has a solution for all $a, b \in \mathbb{R}$. The RREF of the augmented matrix equals $\begin{bmatrix} 1 & 0 & 2y-x \\ 0 & 1 & x-y \\ 0 & 0 & z+y-2x \end{bmatrix}$. Thus, the required condition on x, y and z is given by $z+y-2x = 0$. Hence,

$$LS(S) = \{a(1,1,1)^T + b(2,1,3)^T \mid a, b \in \mathbb{R}\} = \{(x,y,z)^T \in \mathbb{R}^3 \mid 2x-y-z = 0\}.$$

Verify that if $T = S \cup \{(1,1,0)^T\}$ then $LS(T) = \mathbb{R}^3$. Hence, $\mathbb{R}^3$ is finite dimensional. In general, for every fixed $n \in \mathbb{N}$, $\mathbb{R}^n$ is finite dimensional as $\mathbb{R}^n = LS(\{\mathbf{e}_1, \ldots, \mathbf{e}_n\})$.

3. $S = \{1+2x+3x^2, 1+x+2x^2, 1+2x+x^3\}$.
   Solution: To understand $LS(S)$, we need to find condition(s) on $\alpha, \beta, \gamma, \delta$ such that the linear system

$$a(1+2x+3x^2) + b(1+x+2x^2) + c(1+2x+x^3) = \alpha + \beta x + \gamma x^2 + \delta x^3$$

   in the unknowns a, b, c is always consistent. An application of GJE method gives $\alpha + \beta - \gamma - 3\delta = 0$ as the required condition. Thus,

$$LS(S) = \{\alpha + \beta x + \gamma x^2 + \delta x^3 \in \mathbb{R}[x] \mid \alpha + \beta - \gamma - 3\delta = 0\}.$$

   Note that, for every fixed $n \in \mathbb{N}$, $\mathbb{R}[x; n]$ is finite dimensional as $\mathbb{R}[x; n] = LS(\{1, x, \ldots, x^n\})$.

DRAFT

4. $S = \left\{ I_3, \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 4 \end{bmatrix} \right\} \subseteq \mathbb{M}_3(\mathbb{R})$.
   Solution: To get the equation, we need to find conditions on a_{ij}'s such that the system

$$\begin{bmatrix} \alpha & \beta+\gamma & \beta+2\gamma \\ \beta+\gamma & \alpha+\beta & 2\beta+2\gamma \\ \beta+2\gamma & 2\beta+2\gamma & \alpha+2\gamma \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

   in the unknowns α, β, γ is always consistent. Now, verify that the required condition equals

$$LS(S) = \left\{A = [a_{ij}] \in \mathbb{M}_3(\mathbb{R}) \mid A = A^T, a_{11} = \frac{a_{22}+a_{33}-a_{13}}{2}, \right. \\ \left. a_{12} = \frac{a_{22}-a_{33}+3a_{13}}{4}, a_{23} = \frac{a_{22}-a_{33}+3a_{13}}{2} \right\}.$$

   In general, for each fixed $m, n \in \mathbb{N}$, the vector space $\mathbb{M}_{m,n}(\mathbb{R})$ is finite dimensional as $\mathbb{M}_{m,n}(\mathbb{R}) = LS(\{\mathbf{e}_{ij} : 1 \le i \le m, 1 \le j \le n\})$.

5. $\mathbb{C}[x]$ is not finite dimensional as the degree of a polynomial can be any large positive integer. Indeed, verify that $\mathbb{C}[x] = LS(\{1, x, x^2, \ldots, x^n, \ldots\})$.

6. The vector space $\mathbb{R}$ over $\mathbb{Q}$ is infinite dimensional.

EXERCISE **3.2.8.** *Determine the equation of the geometrical object represented by $LS(S)$.*

1. *$S = \{\pi\} \subseteq \mathbb{R}$.*
2. *$S = \{(x, y)^T : x, y < 0\} \subseteq \mathbb{R}^2$.*
3. *$S = \{(x, y)^T :$ either $x \neq 0$ or $y \neq 0\} \subseteq \mathbb{R}^2$.*
4. *$S = \{(1, 0, 1)^T, (0, 1, 0)^T, (2, 0, 2)^T\} \subseteq \mathbb{R}^3$. Give two examples of vectors $\mathbf{u}, \mathbf{v}$ different from the given set such that $LS(S) = LS(\mathbf{u}, \mathbf{v})$.*
5. *$S = \{(x, y, z)^T : x, y, z > 0\} \subseteq \mathbb{R}^3$.*
6. *$S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \right\} \subseteq \mathbb{M}_3(\mathbb{R})$.*
7. *$S = \{(1, 2, 3, 4)^T, (-1, 1, 4, 5)^T, (3, 3, 2, 3)^T\} \subseteq \mathbb{R}^4$.*
8. *$S = \{1 + 2x + 3x^2, -1 + x + 4x^2, 3 + 3x + 2x^2\} \subseteq \mathbb{C}[x; 2]$.*
9. *$S = \{1, x, x^2, \ldots\} \subseteq \mathbb{C}[x]$.*

Lemma 3.2.9. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $S \subseteq \mathbb{V}$. Then $LS(S)$ is a subspace of $\mathbb{V}$.*

Proof. By definition, $\mathbf{0} \in LS(S)$. So, $LS(S)$ is non-empty. Let $\mathbf{u}, \mathbf{v} \in LS(S)$. To show, $a\mathbf{u} + b\mathbf{v} \in LS(S)$ for all $a, b \in \mathbb{F}$. As $\mathbf{u}, \mathbf{v} \in LS(S)$, there exist $n \in \mathbb{N}$, vectors $\mathbf{w}_i \in S$ and scalars $\alpha_i, \beta_i \in \mathbb{F}$ such that $\mathbf{u} = \alpha_1\mathbf{w}_1 + \cdots + \alpha_n\mathbf{w}_n$ and $\mathbf{v} = \beta_1\mathbf{w}_1 + \cdots + \beta_n\mathbf{w}_n$. Hence,

$$a\mathbf{u} + b\mathbf{v} = (a\alpha_1 + b\beta_1)\mathbf{w}_1 + \cdots + (a\alpha_n + b\beta_n)\mathbf{w}_n \in LS(S)$$

as $a\alpha_i + b\beta_i \in \mathbb{F}$ for $1 \leq i \leq n$. Thus, by Theorem 3.1.9, $LS(S)$ is a vector subspace. ∎

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Exercise **3.2.10.** *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $W \subseteq \mathbb{V}$.*

1. *Then $LS(W) = W$ if and only if W is a subspace of $\mathbb{V}$.*
2. *If W is a subspace of $\mathbb{V}$ and $S \subseteq W$ then $LS(S)$ is a subspace of W as well.*

Theorem 3.2.11. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $S \subseteq \mathbb{V}$. Then $LS(S)$ is the smallest subspace of $\mathbb{V}$ containing S.*

Proof. For every $\mathbf{u} \in S$, $\mathbf{u} = 1 \cdot \mathbf{u} \in LS(S)$. Thus, $S \subseteq LS(S)$. Need to show that $LS(S)$ is the smallest subspace of $\mathbb{V}$ containing S. So, let $\mathbb{W}$ be any subspace of $\mathbb{V}$ containing S. Then, by Exercise 3.2.10, $LS(S) \subseteq \mathbb{W}$ and hence the result follows. ∎

Definition 3.2.12. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and S, T be two subsets of $\mathbb{V}$. Then, the **sum** of S and T, denoted $S + T$ equals $\{\mathbf{s} + \mathbf{t} | \ \mathbf{s} \in S, \mathbf{t} \in T\}$.

Example 3.2.13.

1. If $\mathbb{V} = \mathbb{R}$, $S = \{0, 1, 2, 3, 4, 5, 6\}$ and $T = \{5, 10, 15\}$ then $S + T = \{5, 6, \ldots, 21\}$.
2. If $\mathbb{V} = \mathbb{R}^2$, $S = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $T = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ then $S + T = \left\{ \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}$.

3. If $\mathbb{V} = \mathbb{R}^2$, $S = \left\{\begin{bmatrix}1\\1\end{bmatrix}\right\}$ and $T = LS\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)$ then $S + T = \left\{\begin{bmatrix}1\\1\end{bmatrix} + c\begin{bmatrix}-1\\1\end{bmatrix} \mid c \in \mathbb{R}\right\}$.

EXERCISE **3.2.14.** *Let P and Q be two non-trivial, distinct subspaces of $\mathbb{R}^2$. Then $P + Q = \mathbb{R}^2$.*

We leave the proof of the next result for readers.

Lemma 3.2.15. *Let P and Q be two subspaces of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then $P + Q$ is a subspace of $\mathbb{V}$. Furthermore, $P + Q$ is the smallest subspace of $\mathbb{V}$ containing both P and Q.*

EXERCISE **3.2.16.** *1. Let $\mathbf{a} \in \mathbb{R}^2, \mathbf{a} \neq \mathbf{0}$. Then $\{\mathbf{x} \in \mathbb{R}^2 \mid \mathbf{a}^T\mathbf{x} = 0\}$ is a non-trivial subspace of $\mathbb{R}^2$. Geometrically, what does this set represent in $\mathbb{R}^2$?*

2. Find all subspaces of $\mathbb{R}^3$.

3. Let $\mathbb{U} = \left\{\begin{bmatrix}a & b\\-b & 0\end{bmatrix} \mid a, b \in \mathbb{R}\right\}$ and $\mathbb{W} = \left\{\begin{bmatrix}a & 0\\0 & d\end{bmatrix} \mid a, d \in \mathbb{R}\right\}$ be subspaces of $\mathbb{M}_2(\mathbb{R})$. Determine $\mathbb{U} \cap \mathbb{W}$. Is $\mathbb{M}_2(\mathbb{R}) = \mathbb{U} + \mathbb{W}$?

4. Let $\mathbb{W}$ and $\mathbb{U}$ be two subspaces of a vector space $\mathbb{V}$ over $\mathbb{F}$.

(a) Prove that $\mathbb{W} \cap \mathbb{U}$ is a subspace of $\mathbb{V}$.

(b) Give examples of $\mathbb{W}$ and $\mathbb{U}$ such that $\mathbb{W} \cup \mathbb{U}$ is not a subspace of $\mathbb{V}$.

(c) Determine conditions on $\mathbb{W}$ and $\mathbb{U}$ such that $\mathbb{W} \cup \mathbb{U}$ a subspace of $\mathbb{V}$?

(d) Prove that $LS(\mathbb{W} \cup \mathbb{U}) = \mathbb{W} + \mathbb{U}$.

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5. Let $S = \{\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \mathbf{x}_4\}$, where $\mathbf{x}_1 = (1, 0, 0)^T, \mathbf{x}_2 = (1, 1, 0)^T, \mathbf{x}_3 = (1, 2, 0)^T$ and $\mathbf{x}_4 = (1, 1, 1)^T$. Then, determine all $\mathbf{x}_i$ such that $LS(S) = LS(S \setminus \{\mathbf{x}_i\})$.

6. Let $\mathbb{W} = LS((1, 0, 0)^T, (1, 1, 0)^T)$ and $\mathbb{U} = LS((1, 1, 1)^T)$. Prove that $\mathbb{W} + \mathbb{U} = \mathbb{R}^3$ and $\mathbb{W} \cap \mathbb{U} = \{\mathbf{0}\}$. If $\mathbf{v} \in \mathbb{R}^3$, determine $\mathbf{w} \in \mathbb{W}$ and $\mathbf{u} \in \mathbb{U}$ such that $\mathbf{v} = \mathbf{w} + \mathbf{u}$. Is it necessary that $\mathbf{w}$ and $\mathbf{u}$ are unique?

7. Let $\mathbb{W} = LS((1, -1, 0), (1, 1, 0))$ and $\mathbb{U} = LS((1, 1, 1), (1, 2, 1))$. Prove that $\mathbb{W} + \mathbb{U} = \mathbb{R}^3$ and $\mathbb{W} \cap \mathbb{U} \neq \{\mathbf{0}\}$. Find $\mathbf{v} \in \mathbb{R}^3$ such that $\mathbf{v} = \mathbf{w} + \mathbf{u}$, for 2 different choices of $\mathbf{w} \in \mathbb{W}$ and $\mathbf{u} \in \mathbb{U}$. Thus, the choice of vectors $\mathbf{w}$ and $\mathbf{u}$ is not unique.

8. Let $S = \{(1, 1, 1, 1)^T, (1, -1, 1, 2)^T, (1, 1, -1, 1)^T\} \subseteq \mathbb{R}^4$. Does $(1, 1, 2, 1)^T \in LS(S)$? Furthermore, determine conditions on x, y, z and u such that $(x, y, z, u)^T \in LS(S)$.

3.3 Linear Independence

Let us now go back to homogeneous system $A\mathbf{x} = \mathbf{0}$. Here, we saw that this system has either a non-trivial solution or only the trivial solution. The idea of a non-trivial solution leads to linear dependence of vectors and the idea of only the trivial solution leads to linear independence. We look at a few examples for better understanding.

Example 3.3.1.

1. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. So, we say that the columns of A are linearly independent. Thus, the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \right\}$, consisting of columns of A, is linearly independent.

2. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 5 \end{bmatrix}$. As $REF(A) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A\mathbf{x} = \mathbf{0}$ has only the trivial solution. Hence, the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \right\}$, consisting of columns of A, is linearly independent.

3. Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$. As $REF(A) = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution. Hence, the set $S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \right\}$, consisting of columns of A, is linearly dependent.

DRAFT

Formally, we have the following definition.

Definition 3.3.2. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$ be a non-empty subset of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then, S is said to be **linearly independent** if the linear system

$$\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_m\mathbf{u}_m = \mathbf{0}, \tag{3.3.1}$$

in the unknowns α_i's, $1 \leq i \leq m$, has only the trivial solution. If Equation (3.3.1) has a non-trivial solution then S is said to be **linearly dependent**. If S has infinitely many vectors then S is said to be **linearly independent** if for every finite subset T of S, T is linearly independent.

Observe that we are solving a linear system over $\mathbb{F}$. Hence, whether a set is linearly independent or linearly dependent depends on the set of scalars.

Example 3.3.3.

1. Consider $\mathbb{C}^2$ as a vector space over $\mathbb{R}$. Let $S = \{(1,2)^T, (i, 2i)^T\}$. Then, the linear system $a \cdot (1,2)^T + b \cdot (i, 2i)^T = (0,0)^T$, in the unknowns $a, b \in \mathbb{R}$ has only the trivial solution, namely $a = b = 0$. So, S is a **linear independent** subset of the vector space $\mathbb{C}^2$ over $\mathbb{R}$.

2. Consider $\mathbb{C}^2$ as a vector space over $\mathbb{C}$. Then $S = \{(1,2)^T, (i, 2i)^T\}$ is a **linear dependent** subset of the vector space $\mathbb{C}^2$ over $\mathbb{C}$ as $a = -i$ and $b = 1$ is a non-trivial solution.

3. Let $\mathbb{V}$ be the vector space of all real valued continuous functions with domain $[-\pi,\ \pi]$. Then $\mathbb{V}$ is a vector space over $\mathbb{R}$. Question: What can you say about the linear independence or dependence of the set $S = \{1, \sin(x), \cos(x)\}$?

   Solution: For all $x \in [-\pi,\ \pi]$, consider the system

$$\begin{bmatrix} 1 & \sin(x) & \cos(x) \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \Leftrightarrow a \cdot 1 + b \cdot \sin(x) + c \cdot \cos(x) = 0, \tag{3.3.2}$$

   in the unknowns a, b and c. Even though we seem to have only one linear system, we we can obtain the following two linear systems (the first using differentiation and the second using evaluation at $0, \dfrac{\pi}{2}$ and π of the domain).

$$\left.\begin{array}{ll} a + b\sin x + c\cos x & = 0 \\ 0 \cdot a + b\cos x - c\sin x & = 0 \\ 0 \cdot a - b\sin x - c\cos x & = 0 \end{array}\right\} \text{ or } \left\{\begin{array}{ll} a + c & = 0 \\ a + b & = 0 \\ a - c & = 0 \end{array}\right.$$

   Clearly, the above systems has only the trivial solution. Hence, S is linearly independent.

4. Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. If Rank$(A) < m$ then, the rows of A are linearly dependent.

   Solution: As $\mathsf{Rank}(A) < m$, there exists an invertible matrix P such that $PA = \begin{bmatrix} C \\ \mathbf{0} \end{bmatrix}$. Thus, $\mathbf{0}^T = (PA)[m,:] = \sum\limits_{i=1}^{m} p_{mi} A[i,:]$. As P is invertible, at least one $p_{mi} \neq 0$. Thus, the required result follows.

5. Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. If Rank$(A) < n$ then, the columns of A are linearly dependent.

   Solution: As $\mathsf{Rank}(A) < n$ the system $A\mathbf{x} = \mathbf{0}$ has a non-trivial solution.

6. Let $S = \{\mathbf{0}\}$. Is S linearly independent?

   Solution: Let $\mathbf{u} = \mathbf{0}$. So, consider the system $\alpha\mathbf{u} = \mathbf{0}$. This has a non-trivial solution $\alpha = 1$ as $1 \cdot \mathbf{0} = \mathbf{0}$.

7. Let $S = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. Then $A\mathbf{x} = \mathbf{0}$ corresponds to $A = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$. This has a non-trivial solution $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hence, S is linearly dependent.

8. Let $S = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. Is S linearly independent?

   Solution: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Then the system $\alpha\mathbf{u} = \mathbf{0}$ has only the trivial solution. Hence S is linearly independent.

So, we observe that $\mathbf{0}$, the zero-vector cannot belong to any linearly independent set. Further, a set consisting of a single non-zero vector is linearly independent.

EXERCISE **3.3.4.** *1. Show that $S = \{(1,2,3)^T, (-2,1,1)^T, (8,6,10)^T\} \subseteq \mathbb{R}^3$ is linearly dependent.*

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2. *Let $A \in M_n(\mathbb{R})$. Suppose $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $A\mathbf{x} = 3\mathbf{x}$ and $A\mathbf{y} = 2\mathbf{y}$. Then, prove that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent.*

3. *Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 3 \\ 3 & -2 & 5 \end{bmatrix}$. Determine $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $A\mathbf{x} = 6\mathbf{x}$, $A\mathbf{y} = 2\mathbf{y}$ and $A\mathbf{z} = -2\mathbf{z}$. Use the vectors $\mathbf{x}, \mathbf{y}$ and $\mathbf{z}$ obtained above to prove the following.*

   (a) *$A^2\mathbf{v} = 4\mathbf{v}$, where $\mathbf{v} = c\mathbf{y} + d\mathbf{z}$ for any $c, d \in \mathbb{R}$.*

   (b) *The set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ is linearly independent.*

   (c) *Let $P = [\mathbf{x},\ \mathbf{y},\ \mathbf{z}]$ be a 3×3 matrix. Then, P is invertible.*

   (d) *Let $D = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$. Then $AP = PD$.*

3.3.1 Basic Results on Linear Independence

The reader is expected to supply the proof of the next proposition.

Proposition 3.3.5. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$.*

1. *Then, $\mathbf{0}$, the zero-vector, cannot belong to a linearly independent set.*

2. *Then, every subset of a linearly independent set in $\mathbb{V}$ is also linearly independent.*

3. *Then, a set containing a linearly dependent set of $\mathbb{V}$ is also linearly dependent.*

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We now prove a couple of results which will be very useful in the next section.

Proposition 3.3.6. *Let S be a linearly independent subset of a vector space $\mathbb{V}$ over $\mathbb{F}$. If T_1, T_2 are two subsets of S such that $T_1 \cap T_2 = \emptyset$ then, $LS(T_1) \cap LS(T_2) = \{\mathbf{0}\}$. That is, if $\mathbf{v} \in LS(T_1) \cap LS(T_2)$ then $\mathbf{v} = \mathbf{0}$.*

Proof. Let $\mathbf{v} \in LS(T_1) \cap LS(T_2)$. Then, there exist vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k \in T_1$, $\mathbf{w}_1, \ldots, \mathbf{w}_\ell \in T_2$ and scalars α_i's and β_j's (not all zero) such that $\mathbf{v} = \sum\limits_{i=1}^{k} \alpha_i \mathbf{u}_i$ and $\mathbf{v} = \sum\limits_{j=1}^{\ell} \beta_j \mathbf{w}_j$. Thus, we see that $\sum\limits_{i=1}^{k} \alpha_i \mathbf{u}_i + \sum\limits_{j=1}^{\ell} (-\beta_j) \mathbf{w}_j = \mathbf{0}$. As the scalars α_i's and β_j's are not all zero, we see that a non-trivial linear combination of some vectors in $T_1 \cup T_2 \subseteq S$ is $\mathbf{0}$. This contradicts the assumption that S is a linearly independent subset of $\mathbb{V}$. Hence, each of α's and β_j's is zero. That is $\mathbf{v} = \mathbf{0}$. ∎

Lemma 3.3.7. *Let S be a linearly independent subset of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then, each $\mathbf{v} \in LS(S)$ is a unique linear combination of vectors from S.*

Proof. Suppose there exists $\mathbf{v} \in LS(S)$ with $\mathbf{v} \in LS(T_1), LS(T_2)$ with $T_1, T_2 \subseteq S$. Let $T_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ and $T_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_\ell\}$, for some $\mathbf{v}_i$'s and $\mathbf{w}_j$'s in S. Define $T = T_1 \cup T_2$. Then, T is a subset of S. Hence, using Proposition 3.3.5, the set T is linearly independent. Let $T =$

$\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$. Then, there exist α_i's and β_j's in $\mathbb{F}$, not all zero, such that $\mathbf{v} = \alpha_1 \mathbf{u}_1 + \cdots + \alpha_p \mathbf{u}_p$ as well as $\mathbf{v} = \beta_1 \mathbf{u}_1 + \cdots + \beta_p \mathbf{u}_p$. Equating the two expressions for $\mathbf{v}$ gives

$$(\alpha_1 - \beta_1)\mathbf{u}_1 + \cdots + (\alpha_p - \beta_p)\mathbf{u}_p = \mathbf{0}. \tag{3.3.3}$$

As T is a linearly independent subset of $\mathbb{V}$, the system $c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0}$, in the variables $c_1, \ldots, c_p$, has only the trivial solution. Thus, in Equation (3.3.3), $\alpha_i - \beta_i = 0$, for $1 \le i \le p$. Thus, for $1 \le i \le p$, $\alpha_i = \beta_i$ and the required result follows. ∎

Theorem 3.3.8. *Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ be a non-empty subset of a vector space $\mathbb{V}$ over $\mathbb{F}$. If $Z \subseteq LS(S)$ having more than k vectors then, Z is a linearly dependent subset in $\mathbb{V}$.*

Proof. Let $Z = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$. As $\mathbf{w}_i \in LS(S)$, there exist $a_{ij} \in \mathbb{F}$ such that

$$\mathbf{w}_i = a_{i1}\mathbf{u}_1 + \cdots + a_{ik}\mathbf{u}_k, \text{ for } 1 \le i \le m.$$

So,

$$\begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m \end{bmatrix} = \begin{bmatrix} a_{11}\mathbf{u}_1 + \cdots + a_{1k}\mathbf{u}_k \\ \vdots \\ a_{m1}\mathbf{u}_1 + \cdots + a_{mk}\mathbf{u}_k \end{bmatrix} = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mk} \end{bmatrix} \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix}.$$

As $m > k$, the homogeneous system $A^T\mathbf{x} = \mathbf{0}$ has a non-trivial solution, say $\mathbf{y} \neq \mathbf{0}$, *i.e.*, $A^T\mathbf{y} = \mathbf{0} \Leftrightarrow \mathbf{y}^T A = \mathbf{0}^T$. Thus,

DRAFT

$$\mathbf{y}^T \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_m \end{bmatrix} = \mathbf{y}^T \left(A \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix} \right) = (\mathbf{y}^T A) \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix} = \mathbf{0}^T \begin{bmatrix} \mathbf{u}_1 \\ \vdots \\ \mathbf{u}_k \end{bmatrix} = \mathbf{0}^T.$$

As $\mathbf{y} \neq \mathbf{0}$, a non-trivial linear combination of vectors in Z is $\mathbf{0}$. Thus, the set Z is linearly dependent subset of $\mathbb{V}$. ∎

Corollary 3.3.9. *Fix $n \in \mathbb{N}$. Then, any subset S of $\mathbb{R}^n$ with $\mid S \mid \ge n+1$ is linearly dependent.*

Proof. Observe that $\mathbb{R}^n = LS(\{\mathbf{e}_1, \ldots, \mathbf{e}_n\})$, where $\mathbf{e}_i = I_n[:, i]$, is the i-th column of I_n. Hence, using Theorem 3.3.8, the required result follows. ∎

Theorem 3.3.10. *Let S be a linearly independent subset of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then, for any $\mathbf{v} \in \mathbb{V}$ the set $S \cup \{\mathbf{v}\}$ is linearly dependent if and only if $\mathbf{v} \in LS(S)$.*

Proof. Let us assume that $S \cup \{\mathbf{v}\}$ is linearly dependent. Then, there exist $\mathbf{v}_i$'s in S such that the linear system

$$\alpha_1\mathbf{v}_1 + \cdots + \alpha_p\mathbf{v}_p + \alpha_{p+1}\mathbf{v} = \mathbf{0} \tag{3.3.4}$$

in the variables α_i's has a non-trivial solution, say $\alpha_i = c_i$, for $1 \le i \le p+1$. We claim that $c_{p+1} \neq 0$.

For, if $c_{p+1} = 0$ then, Equation (3.3.4) has a non-trivial solution corresponds to having a non-trivial solution of the linear system $\alpha_1\mathbf{v}_1 + \cdots + \alpha_p\mathbf{v}_p = \mathbf{0}$ in the variables $\alpha_1, \ldots, \alpha_p$. This contradicts Proposition 3.3.5.2 as $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\} \subseteq S$, a linearly independent set. Thus, $c_{p+1} \neq 0$ and we get

$$\mathbf{v} = -\frac{1}{c_{p+1}}(c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p) \in LS(\mathbf{v}_1, \ldots, \mathbf{v}_p) \text{ as } -\frac{c_i}{c_{p+1}} \in \mathbb{F}, \text{ for } 1 \leq i \leq p.$$

Now, assume that $\mathbf{v} \in LS(S)$. Then, there exists $\mathbf{v}_i \in S$ and $c_i \in \mathbb{F}$, not all zero, such that $\mathbf{v} = \sum\limits_{i=1}^{p} c_i\mathbf{v}_i$. Thus, the linear system $\alpha_1\mathbf{v}_1 + \cdots + \alpha_p\mathbf{v}_p + \alpha_{p+1}\mathbf{v} = \mathbf{0}$ in the variables α_i's has a non-trivial solution $[c_1, \ldots, c_p, -1]$. Hence, $S \cup \{\mathbf{v}\}$ is linearly dependent. ∎

We now state a very important corollary of Theorem 3.3.10 without proof. This result can also be used as an alternative definition of linear independence and dependence.

Corollary 3.3.11. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let S be a subset of $\mathbb{V}$ containing a non-zero vector $\mathbf{u}_1$.*

1. *If S is linearly dependent then, there exists k such that $LS(\mathbf{u}_1, \ldots, \mathbf{u}_k) = LS(\mathbf{u}_1, \ldots, \mathbf{u}_{k-1})$. Or equivalently, if S is a linearly dependent set then there exists a vector $\mathbf{u}_k$, for $k \geq 2$, which is a linear combination of the previous vectors.*

2. *If S linearly independent then, $\mathbf{v} \in \mathbb{V} \setminus LS(S)$ if and only if $S \cup \{\mathbf{v}\}$ is also a linearly independent subset of $\mathbb{V}$.*

3. *If S is linearly independent then, $LS(S) = \mathbb{V}$ if and only if each proper superset of S is linearly dependent.*

DRAFT

As an application, we have the following result about finite dimensional vector spaces. We leave the proof for the reader as it directly follows from Corollary 3.3.11 and the idea that an algorithm has to finally stop if it has finite number of steps to implement.

Theorem 3.3.12. *Let $\mathbb{V}$ is a finite dimensional vector space over $\mathbb{F}$.*

1. *If S is a finite subset of $\mathbb{V}$ such that $LS(S) = \mathbb{V}$ then we can find a subset T of S such that T is linearly independent and $LS(T) = \mathbb{V}$.*

2. *Let T be a linearly independent subset of $\mathbb{V}$. Then, we can find a superset S of T such that S is linearly independent and $LS(S) = \mathbb{V}$.*

Exercise **3.3.13.**

1. *Prove Corollary 3.3.11.*

2. *Let $\mathbb{V}$ and $\mathbb{W}$ be subspaces of $\mathbb{R}^n$ such that $\mathbb{V} + \mathbb{W} = \mathbb{R}^n$ and $\mathbb{V} \cap \mathbb{W} = \{\mathbf{0}\}$. Prove that each $\mathbf{u} \in \mathbb{R}^n$ is uniquely expressible as $\mathbf{u} = \mathbf{v} + \mathbf{w}$, where $\mathbf{v} \in \mathbb{V}$ and $\mathbf{w} \in \mathbb{W}$.*

3. *Let $\mathbb{W}$ be a subspace of a vector space $\mathbb{V}$ over $\mathbb{F}$. For $\mathbf{u}, \mathbf{v} \in \mathbb{V} \setminus \mathbb{W}$, define $K = LS(\mathbb{W}, \mathbf{u})$ and $M = LS(\mathbb{W}, \mathbf{v})$. Then, prove that $\mathbf{v} \in K$ if and only if $\mathbf{u} \in M$.*

4. *Suppose $\mathbb{V}$ is a vector space over $\mathbb{R}$ as well as over $\mathbb{C}$. Then, prove that $\{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is a linearly independent subset of $\mathbb{V}$ over $\mathbb{C}$ if and only if $\{\mathbf{u}_1, \ldots, \mathbf{u}_k, i\mathbf{u}_1, \ldots, i\mathbf{u}_k\}$ is a linearly independent subset of $\mathbb{V}$ over $\mathbb{R}$.*

5. *Is the set $\{1, x, x^2, \ldots\}$ a linearly independent subset of the vector space $\mathbb{C}[x]$ over $\mathbb{C}$?*

6. *Is the set $\{\mathbf{e}_{ij} | \ 1 \leq i \leq m, 1 \leq j \leq n\}$ a linearly independent subset of the vector space $\mathbb{M}_{m,n}(\mathbb{C})$ over $\mathbb{C}$ (see Definition 1.4.1.1)?*

3.3.2 Application to Matrices

In this subsection, we use the understanding of vector spaces to relate the rank of a matrix with linear independence and dependence of rows and columns of a matrix. We start with our understanding of the RREF.

Theorem 3.3.14. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ with $\mathsf{Rank}(A) = r$. Then,*

1. *there exist r rows of A that are linearly independent.*
2. *every collection of $(r+1)$ rows of A are linearly dependent.*
3. *there exist r columns of A that are linearly independent.*
4. *every collection of $(r+1)$ columns of A are linearly dependent.*

Proof. As $\mathsf{Rank}(A) = r$, there exist an invertible matrix P and an $r \times n$ matrix B having r pivots such that $PA = \text{RREF}(A) = R = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}$. As B is in RREF, the matrix I_r is a submatrix of B. Hence, the rows of B are linearly independent. Thus, we have shown that the pivotal rows of R are linearly independent. These pivotal rows would have come from certain initial rows, say $i_1, \ldots, i_r$, of A. Thus, the rows $\{A[i_1,:], \ldots, A[i_r,:\}]$ is a linearly independent set.

Further, $PA = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}$ implies, $A = P^{-1}\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} = [P_1 \;\; P_2]\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix} = P_1 B$, for some matrix P_1 Thus, every row of A is a linear combination of the r-rows of B. Hence, using Theorem 3.3.8 any collection of $(r+1)$ rows of A are linearly dependent.

DRAFT

Let $B[:,i_1], \ldots, B[:,i_r]$ be the pivotal columns of B. Then, they are linearly independent due to pivotal 1's. As $B = \text{RREF}(A)$, there exists an invertible matrix P such that $B = PA$. Then, the corresponding columns of A satisfy

$$[A[:,i_1], \ldots, A[:,i_r]] = [P^{-1}B[:,i_1], \ldots, P^{-1}B[:,i_r]] = P^{-1}[B[:,i_1], \ldots, B[:,i_r]].$$

As P is invertible, the systems $[A[:,i_1], \ldots, A[:,i_r]]\begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = \mathbf{0}$ and $[B[:,i_1], \ldots, B[:,i_r]]\begin{bmatrix} x_1 \\ \vdots \\ x_r \end{bmatrix} = \mathbf{0}$ are row-equivalent. Thus, they have the same solution set. Hence, $\{A[:,i_1], \ldots, A[:,i_r]\}$ is linearly independent if and only if $\{B[:,i_1], \ldots, B[:,i_r]\}$ is linear independent. Thus, the required result follows. ∎

We consider an example for clarity of the above result.

Example 3.3.15. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & -1 & 1 \\ 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ with $\text{RREF}(A) = B = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

1. Then, $B[:,3] = -B[:,1] + 2B[:,2]$. Thus, $A[:,3] = -A[:,1] + 2A[:,2]$.
2. As the 1-st, 2-nd and 4-th columns of B are linearly independent, the set consisting of corresponding columns $\{A[:,1], A[:,2], A[:,4]\}$ is linearly independent.

3. Also, note that during the application of GJE, the 3-rd and 4-th rows were interchanged. Hence, the rows $A[1,:], A[2,:]$ and $A[4,:]$ are linearly independent.

As an immediate corollary of Theorem 3.3.14 one has the following result.

Corollary 3.3.16. *The following statements are equivalent for $A \in \mathbb{M}_n(\mathbb{C})$.*

1. *A is invertible.*
2. *The columns of A are linearly independent.*
3. *The rows of A are linearly independent.*

EXERCISE **3.3.17.** *1. Let $S_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ and $S_2 = \{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ be subsets of a complex vector space $\mathbb{V}$. Also, let $\begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_n\end{bmatrix} = \begin{bmatrix}\mathbf{u}_1 & \cdots & \mathbf{u}_n\end{bmatrix} A$ for some matrix $A \in \mathbb{M}_n(\mathbb{C})$.*

   (a) If $A = [a_{ij}]$ is invertible then S_1 is a linearly independent if and only if S_2 is linearly independent.

   (b) If S_2 is linearly independent then prove that A is invertible. Further, in this case, the set S_1 is necessarily linearly independent.

2. Let $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\} \subseteq \mathbb{C}^n$ and $T = \{A\mathbf{u}_1, \ldots, A\mathbf{u}_n\}$, for some matrix $A \in \mathbb{M}_n(\mathbf{C})$.

   (a) If S is linearly dependent then prove that T is linear dependent.

   (b) If S is linearly independent then prove that T is linearly independent for every invertible matrix A.

   (c) If T is linearly independent then S is linearly independent. Further, in this case, the matrix A is necessarily invertible.

DRAFT

3.4 Basis of a Vector Space

Definition 3.4.1. Let S be a subset of a set T. Then, S is said to be a **maximal subset** of T having property P if

1. S has property P and
2. no proper superset of S in T has property P.

Example 3.4.2. Let $T = \{2, 3, 4, 7, 8, 10, 12, 13, 14, 15\}$. Then, a maximal subset of T of consecutive integers is $S = \{2, 3, 4\}$. Other maximal subsets are $\{7, 8\}, \{10\}$ and $\{12, 13, 14, 15\}$. Note that $\{12, 13\}$ is not maximal. Why?

Definition 3.4.3. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Then, S is called a **maximal linearly independent** subset of $\mathbb{V}$ if

1. S is linearly independent and
2. no proper superset of S in $\mathbb{V}$ is linearly independent.

Example 3.4.4.

1. In $\mathbb{R}^3$, the set $S = \{\mathbf{e}_1, \mathbf{e}_2\}$ is linearly independent but not maximal as $S \cup \{(1,1,1)^T\}$ is a linearly independent set containing S.
2. In $\mathbb{R}^3$, $S = \{(1,0,0)^T, (1,1,0)^T, (1,1,-1)^T\}$ is a maximal linearly independent set as S is linearly independent and any collection of 4 or more vectors from $\mathbb{R}^3$ is linearly dependent (see Corollary 3.3.9).
3. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Now, form the matrix $A = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$ and let $B = \text{RREF}(A)$. Then, using Theorem 3.3.14, we see that if $B[:, i_1], \ldots, B[:, i_r]$ are the pivotal columns of B then $\{\mathbf{v}_{i_1}, \ldots, \mathbf{v}_{i_r}\}$ is a maximal linearly independent subset of S.
4. Is the set $\{1, x, x^2, \ldots\}$ a maximal linearly independent subset of $\mathbb{C}[x]$ over $\mathbb{C}$?
5. Is the set $\{\mathbf{e}_{ij} | \ 1 \le i \le m, 1 \le j \le n\}$ a maximal linearly independent subset of $\mathbb{M}_{m,n}(\mathbb{C})$ over $\mathbb{C}$?

Theorem 3.4.5. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and S a linearly independent set in $\mathbb{V}$. Then, S is maximal linearly independent if and only if $LS(S) = \mathbb{V}$.*

Proof. Let $\mathbf{v} \in \mathbb{V}$. As S is linearly independent, using Corollary 3.3.11.2, the set $S \cup \{\mathbf{v}\}$ is linearly independent if and only if $\mathbf{v} \in \mathbb{V} \setminus LS(S)$. Thus, the required result follows. ∎

Let $\mathbb{V} = LS(S)$ for some set S with $\mid S \mid \ = k$. Then, using Theorem 3.3.8, we see that if $T \subseteq \mathbb{V}$ is linearly independent then $\mid T \mid \ \le k$. Hence, a maximal linearly independent subset of $\mathbb{V}$ can have at most k vectors. Thus, we arrive at the following important result.

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Theorem 3.4.6. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and let S and T be two finite maximal linearly independent subsets of $\mathbb{V}$. Then $\mid S \mid \ = \ \mid T \mid$.*

Proof. By Theorem 3.4.5, S and T are maximal linearly independent if and only if $LS(S) = \mathbb{V} = LS(T)$. Now, use the previous paragraph to get the required result. ∎

Let $\mathbb{V}$ be a finite dimensional vector space. Then, by Theorem 3.4.6, the number of vectors in any two maximal linearly independent set is the same. We use this number to now define the dimension of a vector space.

Definition 3.4.7. Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$. Then, the number of vectors in any maximal linearly independent set is called the **dimension** of $\mathbb{V}$, denoted $\dim(\mathbb{V})$. By convention, $\dim(\{\mathbf{0}\}) = 0$.

Example 3.4.8.

1. As $\{1\}$ is a maximal linearly independent subset of $\mathbb{R}$, $\dim(\mathbb{R}) = 1$.
2. As $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a maximal linearly independent subset in $\mathbb{R}^n$, $\dim(\mathbb{R}^n) = n$.
3. As $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a maximal linearly independent subset in $\mathbb{C}^n$ over $\mathbb{C}$, $\dim(\mathbb{C}^n) = n$.
4. Using Exercise 3.3.13.4, $\{\mathbf{e}_1, \ldots, \mathbf{e}_n, i\mathbf{e}_1, \ldots, i\mathbf{e}_n\}$ is a maximal linearly independent subset in $\mathbb{C}^n$ over $\mathbb{R}$. Thus, as a real vector space, $\dim(\mathbb{C}^n) = 2n$.
5. As $\{\mathbf{e}_{ij} | \ 1 \le i \le m, 1 \le j \le n\}$ is a maximal linearly independent subset of $\mathbb{M}_{m,n}(\mathbb{C})$ over $\mathbb{C}$, $\dim(\mathbb{M}_{m,n}(\mathbb{C})) = mn$.

Definition 3.4.9. Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$. Then, a maximal linearly independent subset of $\mathbb{V}$ is called a **basis** of $\mathbb{V}$. The vectors in a basis are called **basis** vectors. By convention, a basis of $\{\mathbf{0}\}$ is the empty set.

Thus, using Theorem 3.3.12 we see that every finite dimensional vector space has a basis.

Remark 3.4.10 (Standard Basis)**.** *The readers should verify the statements given below.*

1. *All the maximal linearly independent set given in Example 3.4.8 form the standard basis of the respective vector space.*
2. $\{1, x, x^2, \ldots\}$ *is the standard basis of* $\mathbb{R}[x]$ *over* $\mathbb{R}$.
3. *Fix a positive integer* n. *Then* $\{1, x, x^2, \ldots, x^n\}$ *is the standard basis of* $\mathbb{R}[x; n]$ *over* $\mathbb{R}$.
4. *Let* $\mathbb{V} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A = A^T\}$. *Then,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$ *with standard basis* $\{\mathbf{e}_{ii}, \mathbf{e}_{ij} + \mathbf{e}_{ji} \mid 1 \le i < j \le n\}$.
5. *Let* $\mathbb{V} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A^T = -A\}$. *Then,* $\mathbb{V}$ *is a vector space over* $\mathbb{R}$ *with standard basis* $\{\mathbf{e}_{ij} - \mathbf{e}_{ji} \mid 1 \le i < j \le n\}$.

Definition 3.4.11. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Then, a subset S of $\mathbb{V}$ is called **minimal spanning** if $LS(S) = \mathbb{V}$ and no proper subset of S spans $\mathbb{V}$.

Example 3.4.12.

1. Note that $\{-2\}$ is a basis and a minimal spanning subset in $\mathbb{R}$.
2. Let $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3 \in \mathbb{R}^2$. Then $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ can neither be a basis nor a minimal spanning subset of $\mathbb{R}^2$.
3. Let $\mathbb{V} = \{(x, y, 0)^T \mid x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$. Then, $\mathcal{B} = \{(1, 0, 0)^T, (1, 3, 0)^T\}$ is a basis of $\mathbb{V}$.
4. Let $\mathbb{V} = \{(x, y, z)^T \in \mathbb{R}^3 \mid x + y - z = 0\} \subseteq \mathbb{R}^3$. As each element $(x, y, z)^T \in \mathbb{V}$ satisfies $x + y - z = 0$. Or equivalently $z = x + y$, we see that
$$(x, y, z) = (x, y, x + y) = (x, 0, x) + (0, y, y) = x(1, 0, 1) + y(0, 1, 1).$$
Hence, $\{(1, 0, 1)^T, (0, 1, 1)^T\}$ forms a basis of $\mathbb{V}$.
5. Let $S = \{a_1, \ldots, a_n\}$. Then, $\mathbb{R}^S$ is a real vector space (see Example 3.1.4.7). For $1 \le i \le n$, define the functions
$$\mathbf{e}_i(a_j) = \begin{cases} 1 & \text{if } j = i \\ 0 & \text{otherwise} \end{cases}.$$
Then, prove that $\mathcal{B} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ is a linearly independent subset of $\mathbb{R}^S$ over $\mathbb{R}$. Is it a basis of $\mathbb{R}^S$ over $\mathbb{R}$? What can you say if S is a countable set?
6. Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$. Define $A = [\mathbf{v}_1, \ldots, \mathbf{v}_k]$. Then, using Example 3.4.4.3, we see that $\dim(LS(S)) = \text{Rank}(A)$. Further, using Theorem 3.3.14, the columns of A corresponding to the pivotal columns in $\text{RREF}(A)$ form a basis of $LS(S)$.

3.4.1 Main Results associated with Bases

Theorem 3.4.13. *Let $\mathbb{V}$ be a non-zero vector space over $\mathbb{F}$. Then, the following statements are equivalent.*

1. *$\mathcal{B}$ is a basis (maximal linearly independent subset) of $\mathbb{V}$.*
2. *$\mathcal{B}$ is linearly independent and spans $\mathbb{V}$.*
3. *$\mathcal{B}$ is a minimal spanning set in $\mathbb{V}$.*

Proof. $1 \Rightarrow 2$ By definition, every basis is a maximal linearly independent subset of $\mathbb{V}$. Thus, using Corollary 3.3.11.2, we see that $\mathcal{B}$ spans $\mathbb{V}$.

$2 \Rightarrow 3$ Let S be a linearly independent set that spans $\mathbb{V}$. As S is linearly independent, for any $\mathbf{x} \in S$, $\mathbf{x} \notin LS\left(S - \{\mathbf{x}\}\right)$. Hence $LS\left(S - \{\mathbf{x}\}\right) \subsetneq LS(S) = \mathbb{V}$.

$3 \Rightarrow 1$ If $\mathcal{B}$ is linearly dependent then using Corollary 3.3.11.1, $\mathcal{B}$ is not minimal spanning. A contradiction. Hence, $\mathcal{B}$ is linearly independent.

We now need to show that $\mathcal{B}$ is a maximal linearly independent set. Since $LS(\mathcal{B}) = \mathbb{V}$, for any $\mathbf{x} \in \mathbb{V} \setminus \mathcal{B}$, using Corollary 3.3.11.2, the set $\mathcal{B} \cup \{\mathbf{x}\}$ is linearly dependent. That is, every proper superset of $\mathcal{B}$ is linearly dependent. Hence, the required result follows. ∎

Now, using Lemma 3.3.7, we get the following result.

Remark 3.4.14. *Let $\mathcal{B}$ be a basis of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then, for each $\mathbf{v} \in \mathbb{V}$, there exist unique $\mathbf{u}_i \in \mathcal{B}$ and unique $\alpha_i \in \mathbb{F}$, for $1 \le i \le n$, such that $\mathbf{v} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{u}_i$.*

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The next result is generally known as "every linearly independent set can be extended to form a basis of a finite dimensional vector space". Also, recall Theorem 3.3.12.

Theorem 3.4.15. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $\dim(\mathbb{V}) = n$. If S is a linearly independent subset of $\mathbb{V}$ then there exists a basis T of $\mathbb{V}$ such that $S \subseteq T$.*

Proof. If $LS(S) = \mathbb{V}$, done. Else, choose $\mathbf{u}_1 \in \mathbb{V} \setminus LS(S)$. Thus, by Corollary 3.3.11.2, the set $S \cup \{\mathbf{u}_1\}$ is linearly independent. We repeat this process till we get n vectors in T as $\dim(\mathbb{V}) = n$. By Theorem 3.4.13, this T is indeed a required basis. ∎

3.4.2 Constructing a Basis of a Finite Dimensional Vector Space

We end this section with an algorithm which is based on the proof of the previous theorem.

Step 1: Let $\mathbf{v}_1 \in \mathbb{V}$ with $\mathbf{v}_1 \neq \mathbf{0}$. Then, $\{\mathbf{v}_1\}$ is linearly independent.

Step 2: If $\mathbb{V} = LS(\mathbf{v}_1)$, we have got a basis of $\mathbb{V}$. Else, pick $\mathbf{v}_2 \in \mathbb{V} \setminus LS(\mathbf{v}_1)$. Then, by Corollary 3.3.11.2, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent.

Step i: Either $\mathbb{V} = LS(\mathbf{v}_1, \ldots, \mathbf{v}_i)$ or $LS(\mathbf{v}_1, \ldots, \mathbf{v}_i) \subsetneq \mathbb{V}$. In the first case, $\{\mathbf{v}_1, \ldots, \mathbf{v}_i\}$ is a basis of $\mathbb{V}$. Else, pick $\mathbf{v}_{i+1} \in \mathbb{V} \setminus LS(\mathbf{v}_1, \ldots, \mathbf{v}_i)$. Then, by Corollary 3.3.11.2, the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{i+1}\}$ is linearly independent.

This process will finally end as $\mathbb{V}$ is a finite dimensional vector space.

EXERCISE **3.4.16.**

1. *Let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be a basis of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then, does the condition $\sum\limits_{i=1}^{n} \alpha_i \mathbf{u}_i = \mathbf{0}$ in α_i's imply that $\alpha_i = 0$, for $1 \le i \le n$?*

2. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ be a subset of a vector space $\mathbb{V}$ over $\mathbb{F}$. Suppose $LS(S) = \mathbb{V}$ but S is not a linearly independent set. Then, does this imply that each $\mathbf{v} \in \mathbb{V}$ is expressible in more than one way as a linear combination of vectors from S? Is it possible to get a subset T of S such that T is a basis of $\mathbb{V}$ over $\mathbb{F}$? Give reasons for your answer.*

3. *Let $\mathbb{V}$ be a vector space of dimension n and let S be a subset of $\mathbb{V}$ having n vectors .*

   (a) *If S is linearly independent then prove that S forms a basis of $\mathbb{V}$.*

   (b) *If $LS(S) = \mathbb{V}$ then prove that S forms a basis of $\mathbb{V}$.*

4. *Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{C}^n$. Then, prove that the two matrices $B = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and $C = \begin{bmatrix} \mathbf{v}_1^T \\ \vdots \\ \mathbf{v}_n^T \end{bmatrix}$ are invertible.*

5. *Let $\mathbb{W}_1$ and $\mathbb{W}_2$ be two subspaces of a finite dimensional vector space $\mathbb{V}$ such that $\mathbb{W}_1 \subseteq \mathbb{W}_2$. Then, prove that $\mathbb{W}_1 = \mathbb{W}_2$ if and only if $\dim(\mathbb{W}_1) = \dim(\mathbb{W}_2)$.*

6. *Let $\mathbb{W}_1$ be a subspace of a finite dimensional vector space $\mathbb{V}$ over $\mathbb{F}$. Then, prove that there exists a subspace $\mathbb{W}_2$ of $\mathbb{V}$ such that*
$$\mathbb{W}_1 \cap \mathbb{W}_2 = \{\mathbf{0}\}, \mathbb{W}_1 + \mathbb{W}_2 = \mathbb{V} \text{ and } \dim(\mathbb{W}_2) = \dim(\mathbb{V}) - \dim(\mathbb{W}_1).$$
   Also, prove that for each $\mathbf{v} \in \mathbb{V}$ there exist unique vectors $\mathbf{w}_1 \in \mathbb{W}_1$ and $\mathbf{w}_2 \in \mathbb{W}_2$ with $\mathbf{v} = \mathbf{w}_1 + \mathbf{w}_2$. The subspace $\mathbb{W}_2$ is called the **complementary subspace** *of $\mathbb{W}_1$ in $\mathbb{V}$ and we write $\mathbb{V} = \mathbb{W}_1 \oplus \mathbb{W}_2$.*

7. *Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$. If $\mathbb{W}_1$ and $\mathbb{W}_2$ are two subspaces of $\mathbb{V}$ such that $\mathbb{W}_1 \cap \mathbb{W}_2 = \{\mathbf{0}\}$ and $\dim(\mathbb{W}_1) + \dim(\mathbb{W}_2) = \dim(\mathbb{V})$ then prove that $\mathbb{W}_1 + \mathbb{W}_2 = \mathbb{V}$.*

---

8. *Consider the vector space $\mathcal{C}([-\pi,\ \pi])$ over $\mathbb{R}$. For each $n \in \mathbb{N}$, define $\mathbf{e}_n(x) = \sin(nx)$. Then, prove that $S = \{\mathbf{e}_n \mid n \in \mathbb{N}\}$ is linearly independent. [Hint: Need to show that every finite subset of S is linearly independent. So, on the contrary assume that there exists $\ell \in \mathbb{N}$ and functions $\mathbf{e}_{k_1}, \ldots, \mathbf{e}_{k_\ell}$ such that $\alpha_1 \mathbf{e}_{k_1} + \cdots + \alpha_\ell \mathbf{e}_{k_\ell} = \mathbf{0}$, for some $\alpha_t \neq 0$ with $1 \le t \le \ell$. But, the above system is equivalent to looking at $\alpha_1 \sin(k_1 x) + \cdots + \alpha_\ell \sin(k_\ell x) = \mathbf{0}$ for all $x \in [-\pi,\ \pi]$. Now in the integral*
$$\int_{-\pi}^{\pi} \sin(mx)\left(\alpha_1 \sin(k_1 x) + \cdots + \alpha_\ell \sin(k_\ell x)\right)\ dx = \int_{-\pi}^{\pi} \sin(mx)\mathbf{0}\ dx = 0$$
   replace m with k_i's to show that $\alpha_i = 0$, for all $i, 1 \le i \le \ell$. This gives the required contradiction.]

9. *Is the set $\{1, \sin(x), \cos(x), \sin(2x), \cos(2x), \sin(3x), \cos(3x), \ldots\}$ a linearly subset of the vector space $\mathcal{C}([-\pi,\ \pi], \mathbb{R})$ over $\mathbb{R}$?*

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10. *Find a basis of* $\mathbb{R}^3$ *containing the vector* $(1,1,-2)^T$ *and* $(1,2,-1)^T$.

11. *Determine a basis and dimension of* $\mathbb{W} = \{(x,y,z,w)^T \in \mathbb{R}^4 \mid x+y-z+w=0\}$.

12. *Find a basis of* $\mathbb{V} = \{(x,y,z,u) \in \mathbb{R}^4 \mid x-y-z=0, x+z-u=0\}$.

13. *Let* $A = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. *Find a basis of* $\mathbb{V} = \{\mathbf{x} \in \mathbb{R}^5 \mid A\mathbf{x} = \mathbf{0}\}$.

14. *Let* $\mathbf{u}^T = (1,1,-2), \mathbf{v}^T = (-1,2,3)$ *and* $\mathbf{w}^T = (1,10,1)$. *Find a basis of* $LS(\mathbf{u},\mathbf{v},\mathbf{w})$. *Determine a geometrical representation of* $LS(\mathbf{u},\mathbf{v},\mathbf{w})$.

15. *Is the set* $\mathbb{W} = \{p(x) \in \mathbb{R}[x;4] \mid p(-1) = p(1) = 0\}$ *a subspace of* $\mathbb{R}[x;4]$*? If yes, find its dimension.*

3.5 Fundamental Subspaces Associated with a Matrix

In this section, we will study results that are intrinsic to the understanding of linear algebra from the point of view of matrices. For the sake of clarity, we will also restrict our attention to matrices with real entries. So, we start with defining the four fundamental subspaces associated with a matrix.

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Definition 3.5.1. Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, we define the four fundamental subspaces associated with A as

1. $\textsc{Col}(A) = \{A\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^n\}$ is a subspace of $\mathbb{R}^m$, called the **Column space**, and is the linear span of the columns of A.
2. $\textsc{Row}(A) = \textsc{Col}(A^T) = \{A^T\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^m\}$ is a subspace of $\mathbb{R}^n$, called the **row space** of A and is the linear span of the rows of A.
3. $\textsc{Null}(A) = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$, called the **Null space** of A.
4. $\textsc{Null}(A^T) = \{\mathbf{x} \in \mathbb{R}^m \mid A^T\mathbf{x} = \mathbf{0}\}$, also called the left-null space.

Exercise **3.5.2.** *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$. *Then prove that*

1. $\textsc{Null}(A)$ *and* $\textsc{Row}(A)$ *are subspaces of* $\mathbb{R}^n$.
2. $\textsc{Null}(A^T)$ *and* $\textsc{Col}(A)$ *are subspaces of* $\mathbb{R}^m$.

Example 3.5.3.

1. Compute the fundamental subspaces for $A = \begin{bmatrix} 1 & 1 & 1 & -2 \\ 1 & 2 & -1 & 1 \\ 1 & -2 & 7 & -11 \end{bmatrix}$.

   Solution: Verify the following

   (a) $\textsc{Row}(A) = \{(x,y,z,u)^T \in \mathbb{R}^4 \mid 3x-2y=z, 5x-3y+u=0\}$.

   (b) $\textsc{Col}(A) = \{(x,y,z)^T \in \mathbb{R}^3 \mid 4x-3y-z=0\}$.

(c) $\text{NULL}(A) = \{(x, y, z, u)^T \in \mathbb{R}^4 \mid x + 3z - 5u = 0, y - 2z + 3u = 0\}$.

(d) $\text{NULL}(A^T) = \{(x, y, z)^T \in \mathbb{R}^3 \mid x + 4z = 0, y - 3z = 0\}$.

2. Let $A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & -1 & 1 & 2 \\ 2 & 0 & 1 & 1 \end{bmatrix}$. Then, verify that

(a) $\text{COL}(A) = \{\mathbf{x} = (x_1, x_2, x_3)^T \in \mathbb{R}^3 \mid x_1 + x_2 - x_3 = 0\}$.

(b) $\text{ROW}(A) = \{\mathbf{x} = (x_1, x_2, x_3, x_4)^T \in \mathbb{R}^4 \mid x_1 - x_2 - 2x_3 = 0, x_1 - 3x_2 - 2x_4 = 0\}$.

(c) $\text{NULL}(A) = LS(\{(1, -1, -2, 0)^T, (1, -3, 0, -2)^T\})$.

(d) $\text{NULL}(A^T) = LS((1, 1, -1)^T)$.

Remark 3.5.4. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, in Example 3.5.3, observe that the direction ratios of normal vectors of* $\text{COL}(A)$ *matches with vector in* $\text{NULL}(A^T)$*. Similarly, the direction ratios of normal vectors of* $\text{ROW}(A)$ *matches with vectors in* $\text{NULL}(A)$*. Are these true in the general setting?*

EXERCISE **3.5.5.** *1. For the matrices given below, determine the four fundamental spaces. Further, find the dimensions of all the vector subspaces so obtained.*

$$A = \begin{bmatrix} 1 & 2 & 1 & 3 & 2 \\ 0 & 2 & 2 & 2 & 4 \\ 2 & -2 & 4 & 0 & 8 \\ 4 & 2 & 5 & 6 & 10 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 4 & 0 & 6 \\ -1 & 0 & -2 & 5 \\ -3 & -5 & 1 & -4 \\ -1 & -1 & 1 & 2 \end{bmatrix}.$$

2. Let $A = [X\ Y]$. Then, determine the condition under which $\text{COL}(X) = \text{COL}(Y)$.

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The next result is a re-writing of the results on system of linear equations. The readers are advised to provide the proof for clarity.

Lemma 3.5.6. *Let $A \in M_{m \times n}(\mathbb{C})$ and let E be an elementary matrix. If*

1. *$B = EA$ then* $\text{NULL}(A) = \text{NULL}(B)$, $\text{ROW}(A) = \text{ROW}(B)$*. Thus, the dimensions of the corresponding spaces are equal.*
2. *$B = AE$ then* $\text{NULL}(A^T) = \text{NULL}(B^T)$, $\text{COL}(A) = \text{COL}(B)$*. Thus, the dimensions of the corresponding spaces are equal.*

Let $\mathbb{W}_1$ and $\mathbb{W}_1$ be two subspaces of a vector space $\mathbb{V}$ over $\mathbb{F}$. Then, recall that (see Exercise 3.2.16.4d) $\mathbb{W}_1 + \mathbb{W}_2 = \{\mathbf{u} + \mathbf{v} \mid \mathbf{u} \in \mathbb{W}_1, \mathbf{v} \in \mathbb{W}_2\} = LS(\mathbb{W}_1 \cup \mathbb{W}_2)$ is the smallest subspace of $\mathbb{V}$ containing both $\mathbb{W}_1$ and $\mathbb{W}_2$. We now state a result similar to a result in Venn diagram that states $\mid A \mid + \mid B \mid = \mid A \cup B \mid + \mid A \cap B \mid$, whenever the sets A and B are finite (for a proof, see Appendix 9.4.1).

Theorem 3.5.7. *Let V be a finite dimensional vector space over $\mathbb{F}$. If $\mathbb{W}_1$ and $\mathbb{W}_2$ are two subspaces of V then*

$$\dim(\mathbb{W}_1) + \dim(\mathbb{W}_2) = \dim(\mathbb{W}_1 + \mathbb{W}_2) + \dim(\mathbb{W}_1 \cap \mathbb{W}_2). \tag{3.5.1}$$

For better understanding, we give an example for finite subsets of $\mathbb{R}^n$. The example uses Theorem 3.3.14 to obtain bases of $LS(S)$, for different choices S. The readers are advised to see Example 3.3.14 before proceeding further.

Example 3.5.8. Let $\mathbb{V} = \{(v, w, x, y, z)^T \in \mathbb{R}^5 \mid v + x + z = 3y\}$ and $\mathbb{W} = \{(v, w, x, y, z)^T \in \mathbb{R}^5 \mid w - x = z, v = y\}$. Find bases of $\mathbb{V}$ and $\mathbb{W}$ containing a basis of $\mathbb{V} \cap \mathbb{W}$.

Solution: One can first find a basis of $\mathbb{V} \cap \mathbb{W}$ and then heuristically add a few vectors to get bases for $\mathbb{V}$ and $\mathbb{W}$, separately.

Alternatively, First find bases of $\mathbb{V}, \mathbb{W}$ and $\mathbb{V} \cap \mathbb{W}$, say $\mathcal{B}_V, \mathcal{B}_W$ and $\mathcal{B}$. Now, consider $S = \mathcal{B} \cup \mathcal{B}_V$. This set is linearly dependent. So, obtain a linearly independent subset of S that contains all the elements of $\mathcal{B}$. Similarly, do for $T = \mathcal{B} \cup \mathcal{B}_W$.

So, we first find a basis of $\mathbb{V} \cap \mathbb{W}$. Note that $(v, w, x, y, z)^T \in \mathbb{V} \cap \mathbb{W}$ if v, w, x, y and z satisfy $v + x - 3y + z = 0$, $w - x - z = 0$ and $v = y$. The solution of the system is given by

$$(v, w, x, y, z)^T = (y, 2y, x, y, 2y - x)^T = y(1, 2, 0, 1, 2)^T + x(0, 0, 1, 0, -1)^T.$$

Thus, $\mathcal{B} = \{(1, 2, 0, 1, 2)^T, (0, 0, 1, 0, -1)^T\}$ is a basis of $\mathbb{V} \cap \mathbb{W}$. Similarly, a basis of $\mathbb{V}$ is given by $\mathcal{C} = \{(-1, 0, 1, 0, 0)^T, (0, 1, 0, 0, 0)^T, (3, 0, 0, 1, 0)^T, (-1, 0, 0, 0, 1)^T\}$ and that of W is given by $\mathcal{D} = \{(1, 0, 0, 1, 0)^T, (0, 1, 1, 0, 0)^T, (0, 1, 0, 0, 1)^T\}$. To find the required basis form a matrix whose rows are the vectors in $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ (see below) and apply row operations other than E_{ij}. Then, after a few row operations, we get

$$\left[\begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ \hline -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ \hline 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ \hline 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccccc} 1 & 2 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 & -1 \\ \hline 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Thus, a required basis of $\mathbb{V}$ is $\{(1, 2, 0, 1, 2)^T, (0, 0, 1, 0, -1)^T, (0, 1, 0, 0, 0)^T, (0, 0, 0, 1, 3)^T\}$. Similarly, a required basis of W is $\{(1, 2, 0, 1, 2)^T, (0, 0, 1, 0, -1)^T, (0, 1, 0, 0, 1)^T\}$.

EXERCISE **3.5.9.**

1. *Give an example to show that if A and B are equivalent then* $\text{Col}(A)$ *need not equal* $\text{Col}(B)$.

2. *Let* $\mathbb{V} = \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x + y - z + w = 0, x + y + z + w = 0, x + 2y = 0\}$ *and* $W = \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x - y - z + w = 0, x + 2y - w = 0\}$ *be two subspaces of* $\mathbb{R}^4$. *Think of a method to find bases and dimensions of* $\mathbb{V}$, $\mathbb{W}$, $\mathbb{V} \cap W$ *and* $\mathbb{V} + \mathbb{W}$.

3. *Let* $\mathbb{W}_1$ *and* $\mathbb{W}_2$ *be two subspaces of a vector space* $\mathbb{V}$. *If* $\dim(W_1) + \dim(W_2) > \dim(\mathbb{V})$, *then prove that* $\dim(\mathbb{W}_1 \cap \mathbb{W}_2) \geq 1$.

3.6 Fundamental Theorem of Linear Algebra and Applications

We start with proving the rank-nullity theorem and give some of it's consequences.

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Theorem 3.6.1 (Rank-Nullity Theorem)**.** *Let $A \in M_{m\times n}(\mathbb{C})$. Then,*

$$\dim(\textsc{Col}(A)) + \dim(\textsc{Null}(A)) = n. \tag{3.6.2}$$

Proof. Let $\dim(\textsc{Null}(A)) = r \le n$ and let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be a basis of $\textsc{Null}(A)$. Since $\mathcal{B}$ is a linearly independent set in $\mathbb{R}^n$, extend it to get $\mathcal{C} = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ as a basis of $\mathbb{R}^n$. Then,

$$\begin{aligned} \textsc{Col}(A) &= LS(A\mathcal{B}) = LS(A\mathbf{u}_1, \ldots, A\mathbf{u}_n) \\ &= LS(\mathbf{0}, \ldots, \mathbf{0}, A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n) = LS(A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n). \end{aligned}$$

So, $\mathcal{D} = \{A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n\}$ spans $\textsc{Col}(A)$. We further need to show that $\mathcal{D}$ is linearly independent. So, consider the homogeneous linear system given below in the unknowns $\alpha_1, \ldots, \alpha_{n-r}$.

$$\alpha_1 A\mathbf{u}_{r+1} + \cdots + \alpha_{n-r} A\mathbf{u}_n = \mathbf{0} \Leftrightarrow A(\alpha_1 \mathbf{u}_{r+1} + \cdots + \alpha_{n-r}\mathbf{u}_n) = \mathbf{0} \tag{3.6.3}$$

Thus, $\alpha_1 \mathbf{u}_{r+1} + \cdots + \alpha_{n-r}\mathbf{u}_n \in \textsc{Null}(A) = LS(\mathcal{B})$. Therefore, there exist scalars $\beta_i,\ 1 \le i \le r$, such that $\sum\limits_{i=1}^{n-r} \alpha_i \mathbf{u}_{r+i} = \sum\limits_{j=1}^{r} \beta_j \mathbf{u}_j$. Or equivalently,

$$\beta_1 \mathbf{u}_1 + \cdots + \beta_r \mathbf{u}_r - \alpha_1 \mathbf{u}_{r+1} - \cdots - \alpha_{n-r}\mathbf{u}_n = \mathbf{0}. \tag{3.6.4}$$

Equation (3.6.4) is a linear system in vectors from $\mathcal{C}$ with α_i's and β_j's as unknowns. As $\mathcal{C}$ is a linearly independent set, the only solution of Equation (3.6.4) is

$$\alpha_i = 0, \text{ for } 1 \le i \le n-r \text{ and } \beta_j = 0, \text{ for } 1 \le j \le r.$$

DRAFT

In other words, we have shown that the only solution of Equation (3.6.3) is the trivial solution. Hence, $\{A\mathbf{u}_{r+1}, \ldots, A\mathbf{u}_n\}$ is a basis of $\textsc{Col}(A)$. Thus, the required result follows. ∎

Theorem 3.6.1 is part of what is known as the fundamental theorem of linear algebra (see Theorem 3.6.5). The following are some of the consequences of the rank-nullity theorem. The proofs are left as an exercise for the reader.

Exercise **3.6.2.**

1. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$.*
   (a) *If $n > m$ then the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions,*
   (b) *If $n < m$ then there exists $\mathbf{b} \in \mathbb{R}^m \setminus \{\mathbf{0}\}$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent.*
2. *The following statements are equivalent for an $m \times n$ matrix A.*
   (a) *Rank $(A) = k$.*
   (b) *There exist a set of k rows of A that are linearly independent.*
   (c) *There exist a set of k columns of A that are linearly independent.*
   (d) $\dim(\textsc{Col}(A)) = k$.
   (e) *There exists a $k \times k$ submatrix B of A with $\det(B) \neq 0$. Further, the determinant of every $(k+1) \times (k+1)$ submatrix of A is zero.*

(f) *There exists a linearly independent subset* $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ *of* $\mathbb{R}^m$ *such that the system* $A\mathbf{x} = \mathbf{b}_i$, *for* $1 \le i \le k$, *is consistent.*

(g) $\dim(\textsc{Null}(A)) = n - k$.

3. *Let* $A \in \mathbb{M}_n(\mathbb{R})$ *and define a function* $f : \mathbb{R}^n \to \mathbb{R}^n$ *by* $f(\mathbf{x}) = A\mathbf{x}$. *Then, the following statements are equivalent.*

   (a) *f is one-one.*

   (b) *f is onto.*

   (c) *f is invertible.*

4. *Let* $A = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$. *Then, verify that* $\textsc{Null}(A) = \textsc{Col}(A)$. *Can such examples exist in* $\mathbb{R}^n$ *for n odd? What about n even? Further, verify that* $\mathbb{R}^2 \neq \textsc{Null}(A) + \textsc{Col}(A)$. *Does it contradict the rank-nullity theorem?*

5. *Determine a* 2×2 *matrix A of rank 1 such that* $\mathbb{R}^2 = \textsc{Null}(A) + \textsc{Col}(A)$.

We end this section by proving the fundamental theorem of linear algebra. We start with the following result.

Lemma 3.6.3. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$. *Then,* $\textsc{Null}(A) = \textsc{Null}(A^TA)$.

Proof. Clearly, $\textsc{Null}(A) \subseteq \textsc{Null}(A^TA)$ as $A\mathbf{x} = \mathbf{0}$ implies $(A^TA)\mathbf{x} = A^T(A\mathbf{x}) = \mathbf{0}$.

So, let $\mathbf{x} \in \textsc{Null}(A^TA)$. Then, $(A^TA)\mathbf{x} = \mathbf{0}$ implies $(A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^TA^TA\mathbf{x} = \mathbf{x}^T\mathbf{0} = 0$. Thus, $A\mathbf{x} = \mathbf{0}$ and the required result follows. ∎

DRAFT

Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$. Then $\mathbf{u}$ is said to be **orthogonal** to $\mathbf{v}$ if $\mathbf{u}^T\mathbf{v} = 0$ (dot product of vectors in $\mathbb{R}^n$). Further, for $S \subseteq \mathbb{R}^n$, the **orthogonal complement** of S, denoted $S^\perp$, is defined as

$$S^\perp = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}^T\mathbf{s} = 0 \text{ for all } \mathbf{s} \in S\}.$$

The readers are required to prove the following lemma.

Lemma 3.6.4. *Consider the vector space* $\mathbb{R}^n$. *Then, for* $S \subseteq \mathbb{R}^n$ *prove that*

1. $S^\perp$ *is a subspace of* $\mathbb{R}^n$.

2. $S^\perp = (LS(S))^\perp$.

3. $(S^\perp)^\perp = S^\perp$ *if and only if S is a subspace of* $\mathbb{R}^n$.

4. *Let* $\mathbb{W}$ *be a subspace of* $\mathbb{R}^n$. *Then, there exists a subspace* $\mathbb{V}$ *of* $\mathbb{R}^n$ *such that*

   (a) $\mathbb{R}^n = \mathbb{W} \oplus \mathbb{V}$. *Or equivalently,* $\mathbb{W}$ *and* $\mathbb{V}$ *are complementary subspaces.*

   (b) $\mathbf{v}^T\mathbf{u} = 0$, *for every* $\mathbf{u} \in \mathbb{W}$ *and* $\mathbf{v} \in \mathbb{V}$. *This, further implies that* $\mathbb{W}$ *and* $\mathbb{V}$ *are also orthogonal to each other. Such spaces are called* **orthogonal complements**.

Theorem 3.6.5 (Fundamental Theorem of Linear Algebra)**.** *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$. *Then,*

1. $\dim(\textsc{Null}(A)) + \dim(\textsc{Col}(A)) = n$.

2. $\textsc{Null}(A) = \left(\textsc{Col}(A^T)\right)^{\perp}$ *and* $\textsc{Null}(A^T) = \left(\textsc{Col}(A)\right)^{\perp}$.

3. $\dim(\textsc{Col}(A)) = \dim(\textsc{Col}(A^T))$. *Or equivalently, Row-rank*(A) = *Column-rank*(A).

Proof. PART 1: Proved in Theorem 3.6.1.

PART 2: To show: $\textsc{Null}(A) \subseteq \textsc{Col}(A^T)^{\perp}$. Equivalently, need to show that for each $\mathbf{x} \in \textsc{Null}(A)$ and $\mathbf{u} \in \textsc{Col}(A^T), \mathbf{u}^T\mathbf{x} = 0$. As $\mathbf{u} \in \textsc{Col}(A^T)$ there exists $\mathbf{y} \in \mathbb{R}^m$ such that $\mathbf{u} = A^T\mathbf{y}$. Further, $\mathbf{x} \in \textsc{Null}(A)$ implies $A\mathbf{x} = \mathbf{0}$. Thus, we see that

$$\mathbf{u}^T\mathbf{x} = \left(A^T\mathbf{y}\right)^T\mathbf{x} = (\mathbf{y}^TA)\mathbf{x} = \mathbf{y}^T(A\mathbf{x}) = \mathbf{y}^T\mathbf{0} = 0.$$

Hence, $\textsc{Null}(A) \subseteq \textsc{Col}(A^T)^{\perp}$.

We now show that $\textsc{Col}(A^T)^{\perp} \subseteq \textsc{Null}(A)$. Let $\mathbf{z} \in \textsc{Col}(A^T)^{\perp} \subseteq \mathbb{R}^n$. Then, for every $\mathbf{y} \in \mathbb{R}^m$, $A^T\mathbf{y} \in \textsc{Col}(A^T)$ and hence $(A^T\mathbf{y})^T\mathbf{z} = 0$. In particular, for $\mathbf{y} = A\mathbf{z} \in \mathbb{R}^m$, we have

$$0 = (A^T\mathbf{y})^T\mathbf{z} = \mathbf{y}^TA\mathbf{z} = \mathbf{y}^T\mathbf{y} \Leftrightarrow \mathbf{y} = \mathbf{0}.$$

Thus $A\mathbf{z} = \mathbf{0}$ and $\mathbf{z} \in \textsc{Null}(A)$. This completes the proof of the first equality in Part 2. A similar argument gives the second equality.

PART 3: Note that, using the rank-nullity theorem we have

$$\dim(\text{Col}(A)) = n - \dim(\text{Null}(A)) = n - \dim\left(\left(\text{Col}(A^T)\right)^{\perp}\right) = n - \left(n - \dim\left(\text{Col}(A^T)\right)\right).$$

Thus, $\dim(\text{Col}(A)) = \dim\left(\text{Col}(A^T)\right)$.

Hence the proof of the fundamental theorem is complete. ■

DRAFT

Remark 3.6.6. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, Theorem 3.6.5.2 implies the following:*

1. $\textsc{Null}(A) = \left(\textsc{Col}(A^T)\right)^{\perp}$. *This is just stating the usual fact that if $\mathbf{x} \in \textsc{Null}(A)$ then $A\mathbf{x} = \mathbf{0}$. Hence, the dot product of every row of A with $\mathbf{x}$ equals 0.*

2. $\mathbf{R}^n = \textsc{Null}(A) \oplus \textsc{Col}(A^T)$. *Further,* $\textsc{Null}(A)$ *is orthogonal complement of* $\textsc{Col}(A^T)$.

3. $\mathbf{R}^m = \textsc{Null}(A^T) \oplus \textsc{Col}(A)$. *Further,* $\textsc{Null}(A^T)$ *is orthogonal complement of* $\textsc{Col}(A)$.

As an implication of last two parts of Theorem 3.6.5, we show the existence of an invertible function $f : \textsc{Col}(A^T) \to \textsc{Col}(A)$.

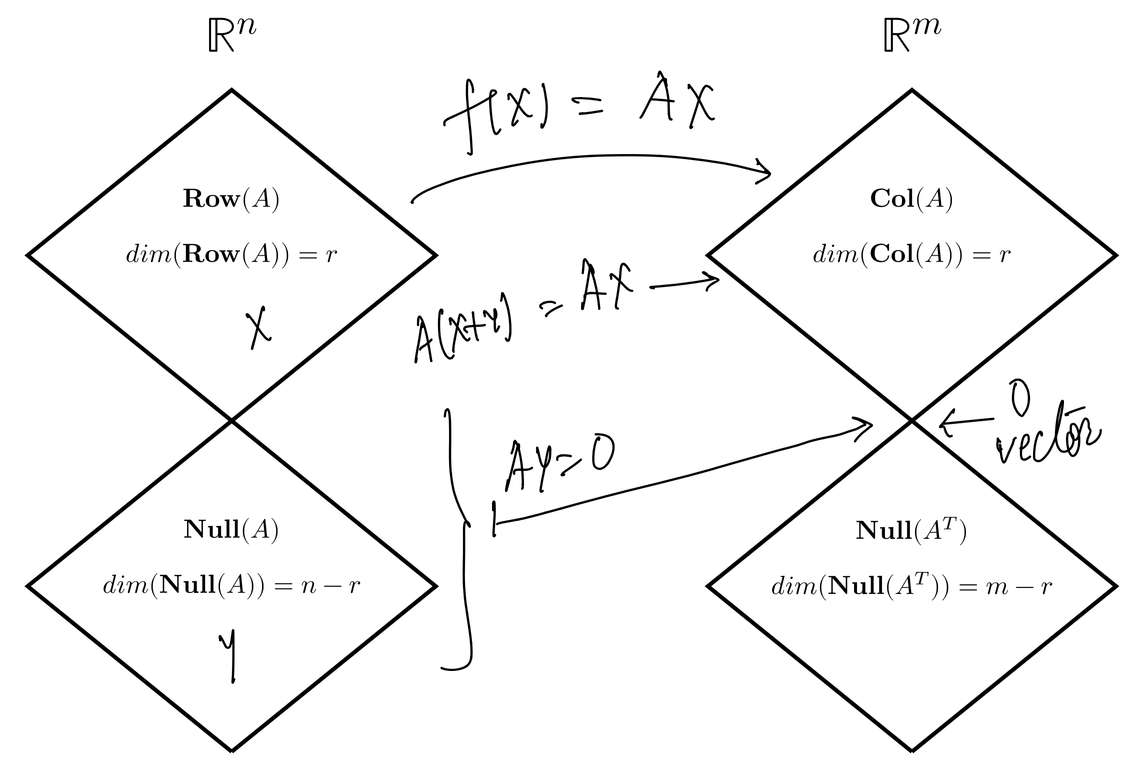
Corollary 3.6.7. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, the function $f : \textsc{Col}(A^T) \to \textsc{Col}(A)$ defined by $f(\mathbf{x}) = A\mathbf{x}$ is invertible.*

Proof. Let us first show that f is one-one. So, let $\mathbf{x}, \mathbf{y} \in \textsc{Col}(A^T)$ such that $f(\mathbf{x}) = f(\mathbf{y})$. Hence, $A\mathbf{x} = A\mathbf{y}$. Thus $\mathbf{x} - \mathbf{y} \in \textsc{Null}(A) = (\textsc{Col}(A^T))^{\perp}$ (by Theorem 3.6.5.2). Therefore, $\mathbf{x} - \mathbf{y} \in (\textsc{Col}(A^T))^{\perp} \cap \textsc{Col}(A^T) = \{\mathbf{0}\}$. Thus $\mathbf{x} = \mathbf{y}$ and hence f is one-one.

We now show that f is onto. So, let $\mathbf{z} \in \textsc{Col}(A)$. To find $\mathbf{y} \in \textsc{Col}(A^T)$ such that $f(\mathbf{y}) = \mathbf{z}$.

As $\mathbf{z} \in \textsc{Col}(A)$ there exists $\mathbf{w} \in \mathbb{R}^n$ with $\mathbf{z} = A\mathbf{w}$. But $\textsc{Null}(A)$ and $\textsc{Col}(A^T)$ are complementary subspaces and hence, there exists unique vectors, $\mathbf{w}_1 \in \textsc{Null}(A)$ and $\mathbf{w}_2 \in \textsc{Col}(A^T)$, such that $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$. Thus, $\mathbf{z} = A\mathbf{w}$ implies

$$\mathbf{z} = A\mathbf{w} = A(\mathbf{w}_1 + \mathbf{w}_2) = A\mathbf{w}_1 + A\mathbf{w}_2 = \mathbf{0} + A\mathbf{w}_2 = A\mathbf{w}_2 = f(\mathbf{w}_2),$$



$A_{m\times n}$ with **Rank**$(A) = r$

DRAFT

for $\mathbf{w}_2 \in \textsc{Col}(A^T$. Thus, the required result follows. ∎

The readers should look at Example 3.5.3 and Remark 3.5.4. We give one more example.

Example 3.6.8. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix}$. Then, verify that

1. $\{(0,1,1)^T, (1,1,2)^T\}$ is a basis of $\textsc{Col}(A)$.
2. $\{(1,1,-1)^T\}$ is a basis of $\textsc{Null}(A^T)$.
3. $\textsc{Null}(A^T) = (\textsc{Col}(A))^\perp$.

For more information related with the fundamental theorem of linear algebra the interested readers are advised to see the article "The Fundamental Theorem of Linear Algebra, Gilbert Strang, *The American Mathematical Monthly*, Vol. 100, No. 9, Nov., 1993, pp. 848 - 855." The diagram 3.6 has been taken from the above paper. It also explains Corollary 3.6.7.

Exercise **3.6.9.** *1. Find subspaces $\mathbb{W}_1 \neq \{\mathbf{0}\}$ and $\mathbb{W}_2 \neq \{\mathbf{0}\}$ in $\mathbb{R}^3$ such that they are orthogonal but they are not orthogonal complement of each other.*

2. Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Prove that $\textsc{Col}(A^T) = \textsc{Col}(A^TA)$. Thus, Rank$(A) = n$ if and only if Rank$(A^TA) = n$. [Hint: Use the rank-nullity theorem and/ or Lemma 3.6.3]

3. Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then, for every

(a) $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} \in \textsc{Col}(A^T)$ and $\mathbf{v} \in \textsc{Null}(A)$ are unique.

(b) $\mathbf{y} \in \mathbb{R}^m$, $\mathbf{y} = \mathbf{w} + \mathbf{z}$, where $\mathbf{w} \in \textsc{Col}(A)$ and $\mathbf{z} \in \textsc{Null}(A^T)$ are unique.

4. *Let* $A, B \in \mathbb{M}_n(\mathbb{R})$ *such that* A *is idempotent and* $AB = BA = \mathbf{0}$*. Then prove that* $\textsc{Col}(A + B) = \textsc{Col}(A) + \textsc{Col}(B)$*.*

5. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$*. Then, a matrix* $G \in \mathbb{M}_{n,m}(\mathbb{R})$ *is a g-inverse of* A *if and only if for any* $\mathbf{b} \in \textsc{Col}(A)$*, the vector* $\mathbf{y} = G\mathbf{b}$ *is a solution of the system* $A\mathbf{y} = \mathbf{b}$*.*

6. *Let* $A \in \mathbb{M}_{m,n}(\mathbb{R})$*. If* $G \in \mathbb{M}_{n,m}(\mathbb{R})$ *is a g-inverse of* A *then, for any* $\mathbf{b} \in \textsc{Col}(A)$ *the solution set of the system* $A\mathbf{y} = \mathbf{b}$ *is given by* $G\mathbf{b} + (I - GA)\mathbf{z}$*, for any arbitrary vector* $\mathbf{z}$*.*

3.7 Summary

In this chapter, we defined vector spaces over $\mathbb{F}$. The set $\mathbb{F}$ was either $\mathbb{R}$ or $\mathbb{C}$. To define a vector space, we start with a non-empty set $\mathbb{V}$ of vectors and $\mathbb{F}$ the set of scalars. We also needed to do the following:

1. first define vector addition and scalar multiplication and

2. then verify the conditions in Definition 3.1.1.

If all conditions in Definition 3.1.1 are satisfied then $\mathbb{V}$ is a vector space over $\mathbb{F}$. If $\mathbb{W}$ was a non-empty subset of a vector space $\mathbb{V}$ over $\mathbb{F}$ then for $\mathbb{W}$ to be a space, we only need to check whether the vector addition and scalar multiplication inherited from that in $\mathbb{V}$ hold in $\mathbb{W}$.

We then learnt linear combination of vectors and the linear span of vectors. It was also shown that the linear span of a subset S of a vector space $\mathbb{V}$ is the smallest subspace of $\mathbb{V}$ containing S. Also, to check whether a given vector $\mathbf{v}$ is a linear combination of $\mathbf{u}_1, \ldots, \mathbf{u}_n$, we needed to solve the linear system $c_1\mathbf{u}_1 + \cdots + c_n\mathbf{u}_n = \mathbf{v}$ in the variables $c_1, \ldots, c_n$. Or equivalently, the system $A\mathbf{x} = \mathbf{b}$, where in some sense $A[:, i] = \mathbf{u}_i$, $1 \le i \le n$, $\mathbf{x}^T = [c_1, \ldots, c_n]$ and $\mathbf{b} = \mathbf{v}$. It was also shown that the geometrical representation of the linear span of $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is equivalent to finding conditions in the entries of $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ was always consistent.

DRAFT

Then, we learnt linear independence and dependence. A set $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is linearly independent set in the vector space $\mathbb{V}$ over $\mathbb{F}$ if the homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution in $\mathbb{F}$. Else S is linearly dependent, whereas before the columns of A correspond to the vectors $\mathbf{u}_i$'s.

We then talked about the maximal linearly independent set (coming from the homogeneous system) and the minimal spanning set (coming from the non-homogeneous system) and culminating in the notion of the basis of a finite dimensional vector space $\mathbb{V}$ over $\mathbb{F}$. The following important results were proved.

1. A linearly independent set can be extended to form a basis of $\mathbb{V}$.

2. Any two bases of $\mathbb{V}$ have the same number of elements.

This number was defined as the dimension of $\mathbb{V}$, denoted $\dim(\mathbb{V})$.

Now let $A \in \mathbb{M}_n(\mathbb{R})$. Then, combining a few results from the previous chapter, we have the following equivalent conditions.

1. A is invertible.

2. The homogeneous system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.
3. $\mathrm{RREF}(A) = I_n$.
4. A is a product of elementary matrices.
5. The system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$.
6. The system $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$.
7. $\mathrm{Rank}(A) = n$.
8. $\det(A) \neq 0$.
9. $\textsc{Col}(A^T) = \textsc{Row}(A) = \mathbb{R}^n$.
10. Rows of A form a basis of $\mathbb{R}^n$.
11. $\textsc{Col}(A) = \mathbb{R}^n$.
12. Columns of A form a basis of $\mathbb{R}^n$.
13. $\textsc{Null}(A) = \{\mathbf{0}\}$.

DRAFT

Chapter 4

Inner Product Spaces

4.1 Definition and Basic Properties

Recall the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$. It satisfies the following properties.

1. $\mathbf{u} \cdot (a\,\mathbf{v} + \mathbf{w}) = a\,\mathbf{v} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{w}$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and $a \in \mathbb{R}$.
2. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$, for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$.
3. $\mathbf{u} \cdot \mathbf{u} \geq 0$ for all $\mathbf{u} \in \mathbb{R}^3$. Further, equality holds if and only if $\mathbf{u} = \mathbf{0}$.

The dot product helped us to compute the length of vectors and talk of perpendicularity of vectors. This enabled us to rephrase geometrical problems in $\mathbb{R}^2$ and $\mathbb{R}^3$ in the language of vectors. We now generalize the idea of dot product to achieve similar goal for a general vector space over $\mathbb{R}$ or $\mathbb{C}$.

DRAFT

Definition 4.1.1. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. An **inner product** over $\mathbb{V}$, denoted by $\langle\ ,\ \rangle$, is a map from $\mathbb{V} \times \mathbb{V}$ to $\mathbb{F}$ satisfying

1. $\langle a\,\mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = a\,\langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$, for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{V}$ and $a \in \mathbb{F}$,
2. $\langle \mathbf{u}, \mathbf{v} \rangle = \overline{\langle \mathbf{v}, \mathbf{u} \rangle}$, the complex conjugate of $\langle \mathbf{u}, \mathbf{v} \rangle$, for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ and
3. $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$ for all $\mathbf{u} \in \mathbb{V}$. Furthermore, equality holds if and only if $\mathbf{u} = \mathbf{0}$.

Remark 4.1.2. *Using the definition of inner product, we immediately observe that*

1. $\langle \mathbf{u}, \mathbf{0} \rangle = \langle \mathbf{u}, \mathbf{0} + \mathbf{0} \rangle = \langle \mathbf{u}, \mathbf{0} \rangle + \langle \mathbf{u}, \mathbf{0} \rangle$. *Thus,* $\langle \mathbf{u}, \mathbf{0} \rangle = 0$, *for all* $\mathbf{u} \in \mathbb{V}$.
2. $\langle \mathbf{v}, \alpha\,\mathbf{w} \rangle = \overline{\langle \alpha\,\mathbf{w}, \mathbf{v} \rangle} = \overline{\alpha}\,\overline{\langle \mathbf{w}, \mathbf{v} \rangle} = \overline{\alpha}\,\langle \mathbf{v}, \mathbf{w} \rangle$, *for all* $\alpha \in \mathbb{F}$ *and* $\mathbf{v}, \mathbf{w} \in \mathbb{V}$.
3. *If* $\langle \mathbf{u}, \mathbf{v} \rangle = 0$ *for all* $\mathbf{v} \in \mathbb{V}$ *then in particular* $\langle \mathbf{u}, \mathbf{u} \rangle = 0$. *Hence* $\mathbf{u} = \mathbf{0}$.

Definition 4.1.3. Let $\mathbb{V}$ be a vector space with an inner product $\langle\ ,\ \rangle$. Then, $(\mathbb{V}, \langle\ ,\ \rangle)$ is called an **inner product space** (in short, IPS).

Example 4.1.4. Examples 1 to 4 that appear below are called the **standard inner product** or the **dot product**. Whenever an inner product is not clearly mentioned, it will be assumed to be the standard inner product.

1. For $\mathbf{u} = (u_1, \ldots, u_n)^T, \mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{R}^n$ define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1v_1 + \cdots + u_nv_n = \mathbf{v}^T\mathbf{u}$. Then, $\langle \ , \ \rangle$ is indeed an inner product and hence $\big(\mathbb{R}^n, \langle \ , \ \rangle\big)$ is an IPS.

2. For $\mathbf{u} = (u_1, \ldots, u_n)^T, \mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbb{C}^n$ define $\langle \mathbf{u}, \mathbf{v} \rangle = u_1\overline{v_1} + \cdots + u_n\overline{v_n} = \mathbf{v}^*\mathbf{u}$. Then, $\big(\mathbb{C}^n, \langle \ , \ \rangle\big)$ is an IPS.

3. For $A, B \in M_n(\mathbb{R})$, define $\langle A, B \rangle = \mathsf{tr}(\mathsf{B}^\mathsf{T}\mathsf{A})$. Then,
$$\begin{aligned}\langle A + B, C\rangle &= \mathsf{tr}\big(\mathsf{C}^\mathsf{T}(\mathsf{A}+\mathsf{B})\big) = \mathsf{tr}(\mathsf{C}^\mathsf{T}\mathsf{A}) + \mathsf{tr}(\mathsf{C}^\mathsf{T}\mathsf{B}) = \langle \mathsf{A}, \mathsf{C}\rangle + \langle \mathsf{B}, \mathsf{C}\rangle \text{ and}\\ \langle A, B\rangle &= \mathsf{tr}(\mathsf{B}^\mathsf{T}\mathsf{A}) = \mathsf{tr}(\ (\mathsf{B}^\mathsf{T}\mathsf{A})^\mathsf{T}\ ) = \mathsf{tr}(\mathsf{A}^\mathsf{T}\mathsf{B}) = \langle \mathsf{B}, \mathsf{A}\rangle.\end{aligned}$$
If $A = [a_{ij}]$ then $\langle A, A\rangle = \mathsf{tr}(\mathsf{A}^\mathsf{T}\mathsf{A}) = \sum\limits_{i=1}^{n}(\mathsf{A}^\mathsf{T}\mathsf{A})_{ii} = \sum\limits_{i,j=1}^{n} a_{ij}a_{ij} = \sum\limits_{i,j=1}^{n} a_{ij}^2$ and therefore, $\langle A, A\rangle > 0$ for all non-zero matrix A.

4. Consider the real vector space $\mathcal{C}[-1,\ 1]$ and define $\langle f, g\rangle = \int\limits_{-1}^{1} f(x)g(x)dx$. Then,
   (a) $\langle \mathbf{f}, \mathbf{f}\rangle = \int\limits_{-1}^{1} \mid \mathbf{f}(x) \mid^2 dx \geq 0$ as $\mid \mathbf{f}(x) \mid^2 \geq 0$. Further, $\langle \mathbf{f}, \mathbf{f}\rangle = 0$ if and only if $\mathbf{f} \equiv \mathbf{0}$ as f is continuous.
   (b) $\langle \mathbf{g}, \mathbf{f}\rangle = \int\limits_{-1}^{1} \mathbf{g}(x)\mathbf{f}(x)dx = \int\limits_{-1}^{1} \mathbf{g}(x)\mathbf{f}(x)dx = \int\limits_{-1}^{1} \mathbf{f}(x)\mathbf{g}(x)dx = \langle \mathbf{f}, \mathbf{g}\rangle$.
   (c) $\langle \mathbf{f} + \mathbf{g}, \mathbf{h}\rangle = \int\limits_{-1}^{1} (\mathbf{f} + \mathbf{g})(x)\mathbf{h}(x)dx = \int\limits_{-1}^{1} [\mathbf{f}(x)\mathbf{h}(x) + \mathbf{g}(x)\mathbf{h}(x)]dx = \langle \mathbf{f}, \mathbf{h}\rangle + \langle \mathbf{g}, \mathbf{h}\rangle$.
   (d) $\langle \alpha\mathbf{f}, \mathbf{g}\rangle = \int\limits_{-1}^{1} (\alpha\mathbf{f}(x))\mathbf{g}(x)dx = \alpha \int\limits_{-1}^{1} \mathbf{f}(x)\mathbf{g}(x)dx = \alpha\langle \mathbf{f}, \mathbf{g}\rangle$.

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5. For $\mathbf{x} = (x_1, x_2)^T, \mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$ and $A = \begin{bmatrix} 4 & -1 \\ -1 & 2 \end{bmatrix}$, define $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^T A\mathbf{x}$. Then, $\langle \ , \ \rangle$ is an inner product as $\langle \mathbf{x}, \mathbf{x}\rangle = (x_1 - x_2)^2 + 3x_1^2 + x_2^2$.

6. Fix $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ with $a, c > 0$ and $ac > b^2$. Then, $\langle \mathbf{x}, \mathbf{y}\rangle = \mathbf{y}^T A\mathbf{x}$ is an inner product on $\mathbb{R}^2$ as $\langle \mathbf{x}, \mathbf{x}\rangle = ax_1^2 + 2bx_1x_2 + cx_2^2 = a\left[x_1 + \frac{bx_2}{a}\right]^2 + \frac{1}{a}\left[ac - b^2\right]x_2^2$.

7. Verify that for $\mathbf{x} = (x_1, x_2, x_3)^T, \mathbf{y} = (y_1, y_2, y_3)^T \in \mathbb{R}^3$, $\langle \mathbf{x}, \mathbf{y}\rangle = 10x_1y_1 + 3x_1y_2 + 3x_2y_1 + 2x_2y_2 + x_2y_3 + x_3y_2 + x_3y_3$ defines an inner product.

EXERCISE **4.1.5.** *For $\mathbf{x} = (x_1, x_2)^T, \mathbf{y} = (y_1, y_2)^T \in \mathbb{R}^2$, we define three maps that satisfy at least one condition out of the three conditions for an inner product. Determine the condition which is not satisfied. Give reasons for your answer.*

1. $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1$.

2. $\langle \mathbf{x}, \mathbf{y}\rangle = x_1^2 + y_1^2 + x_2^2 + y_2^2$.

3. $\langle \mathbf{x}, \mathbf{y}\rangle = x_1y_1^3 + x_2y_2^3$.

As $\langle \mathbf{u}, \mathbf{u}\rangle > 0$, for all $\mathbf{u} \neq \mathbf{0}$, we use inner product to define the length/ norm of a vector.

Definition 4.1.6. Let $\mathbb{V}$ be an inner product space over $\mathbb{F}$. Then, for any vector $\mathbf{u} \in \mathbb{V}$, we define the **length (norm)** of $\mathbf{u}$, denoted $\|\mathbf{u}\| = \sqrt{\langle \mathbf{u}, \mathbf{u} \rangle}$, the positive square root. A vector of norm 1 is called a **unit vector**. Thus, $\dfrac{\mathbf{u}}{\|\mathbf{u}\|}$ is called the **unit vector in the direction** of $\mathbf{u}$.

Example 4.1.7. 1. Let $\mathbb{V}$ be an IPS and $\mathbf{u} \in \mathbb{V}$. Then, for any scalar α, $\|\alpha \mathbf{u}\| = |\alpha| \cdot \|\mathbf{u}\|$.

2. Let $\mathbf{u} = (1, -1, 2, -3)^T \in \mathbb{R}^4$. Then, $\|\mathbf{u}\| = \sqrt{1+1+4+9} = \sqrt{15}$. Thus, $\frac{1}{\sqrt{15}}\mathbf{u}$ and $-\frac{1}{\sqrt{15}}\mathbf{u}$ are unit vectors in the direction of $\mathbf{u}$.

Exercise **4.1.8.** *1. Let $\mathbf{u} = (-1, 1, 2, 3, 7)^T \in \mathbb{R}^5$. Find all $\alpha \in \mathbb{R}$ such that $\|\alpha \mathbf{u}\| = 1$.*

2. Let $\mathbf{u} = (-1, 1, 2, 3, 7)^T \in \mathbb{C}^5$. Find all $\alpha \in \mathbb{C}$ such that $\|\alpha \mathbf{u}\| = 1$.

3. Let $\mathbf{u} = (1, 2)^T, \mathbf{v} = (2, -1)^T \in \mathbb{R}^2$. Then, does there exist an inner product in $\mathbb{R}^2$ such that $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1$ and $\langle \mathbf{u}, \mathbf{v} \rangle = 0$? [Hint: Let $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ and define $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{y}^T A \mathbf{x}$. Use given conditions to get a linear system of 3 equations in the variables a, b, c.]

4. Prove that under the standard inner product in $\mathbb{M}_{m,n}(\mathbb{R})$,

$$\|A\|^2 = \mathsf{tr}(\mathsf{A}^\mathsf{T}\mathsf{A}) = \sum_{\mathsf{k}=1}^{\mathsf{m}} \|\mathsf{A}[\mathsf{k},:]^\mathsf{T}\|^2 = \sum_{\ell=1}^{\mathsf{n}} \|\mathsf{A}[:,\ell]\|^2, \text{ for all } \mathsf{A} \in \mathbb{M}_{\mathsf{m},\mathsf{n}}(\mathbb{R}).$$

4.2 Cauchy-Schwartz Inequality

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A very useful and a fundamental inequality, commonly called the Cauchy-Schwartz inequality, is a generalization of $|\mathbf{x} \cdot \mathbf{y}| \le \|\mathbf{x}\| \cdot \|\mathbf{y}\|$, and is proved next.

Theorem 4.2.1 (Cauchy- Schwartz inequality)**.** *Let $\mathbb{V}$ be an inner product space over $\mathbb{F}$. Then, for any $\mathbf{u}, \mathbf{v} \in \mathbb{V}$*

$$|\ \langle \mathbf{u}, \mathbf{v} \rangle\ | \ \le \|\mathbf{u}\|\ \|\mathbf{v}\|. \tag{4.2.1}$$

Moreover, equality holds in Inequality (4.2.1) if and only if $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. In particular, if $\mathbf{u} \neq \mathbf{0}$ then $\mathbf{v} = \left\langle \mathbf{v}, \dfrac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$.

Proof. If $\mathbf{u} = \mathbf{0}$ then Inequality (4.2.1) holds. Hence, let $\mathbf{u} \neq \mathbf{0}$. Then, by Definition 4.1.1.3, $\langle \lambda\mathbf{u} + \mathbf{v}, \lambda\mathbf{u} + \mathbf{v} \rangle \ge 0$ for all $\lambda \in \mathbb{F}$ and $\mathbf{v} \in \mathbb{V}$. In particular, for $\lambda = -\dfrac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}$, we have

$$\begin{aligned} 0 &\le \langle \lambda\mathbf{u} + \mathbf{v}, \lambda\mathbf{u} + \mathbf{v} \rangle = \lambda\overline{\lambda}\|\mathbf{u}\|^2 + \lambda\langle \mathbf{u}, \mathbf{v} \rangle + \overline{\lambda}\langle \mathbf{v}, \mathbf{u} \rangle + \|\mathbf{v}\|^2 \\ &= \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\frac{\overline{\langle \mathbf{v}, \mathbf{u} \rangle}}{\|\mathbf{u}\|^2}\|\mathbf{u}\|^2 - \frac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\langle \mathbf{u}, \mathbf{v} \rangle - \frac{\overline{\langle \mathbf{v}, \mathbf{u} \rangle}}{\|\mathbf{u}\|^2}\langle \mathbf{v}, \mathbf{u} \rangle + \|\mathbf{v}\|^2 = \|\mathbf{v}\|^2 - \frac{|\ \langle \mathbf{v}, \mathbf{u} \rangle\ |^2}{\|\mathbf{u}\|^2}. \end{aligned}$$

Or, in other words $|\ \langle \mathbf{v}, \mathbf{u} \rangle\ |^2 \le \|\mathbf{u}\|^2\|\mathbf{v}\|^2$ and the proof of the inequality is over.

Now, note that equality holds in Inequality (4.2.1) if and only if $\langle \lambda\mathbf{u} + \mathbf{v}, \lambda\mathbf{u} + \mathbf{v} \rangle = 0$, or equivalently, $\lambda\mathbf{u} + \mathbf{v} = \mathbf{0}$. Hence, $\mathbf{u}$ and $\mathbf{v}$ are linearly dependent. Moreover,

$$0 = \langle \mathbf{0}, \mathbf{u} \rangle = \langle \lambda\mathbf{u} + \mathbf{v}, \mathbf{u} \rangle = \lambda\langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{v}, \mathbf{u} \rangle$$

implies that $\mathbf{v} = -\lambda\mathbf{u} = \dfrac{\langle \mathbf{v}, \mathbf{u} \rangle}{\|\mathbf{u}\|^2}\mathbf{u} = \left\langle \mathbf{v}, \dfrac{\mathbf{u}}{\|\mathbf{u}\|} \right\rangle \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$. ∎

As an immediate corollary, the following hold.

Corollary 4.2.2. *Prove the following results.*

1. $\left(\sum\limits_{i=1}^{n} \mathbf{x}_i \mathbf{y}_i\right)^2 = |\mathbf{x} \cdot \mathbf{y}|^2 \leq \|\mathbf{x}\|^2 \cdot \|\mathbf{y}\|^2 = \left(\sum\limits_{i=1}^{n} \mathbf{x}_i^2\right)\left(\sum\limits_{i=1}^{n} \mathbf{y}_i^2\right)$, *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.
2. $\|A\mathbf{x}\| \leq \|A\| \cdot \|\mathbf{x}\|$, *for all* $A \in \mathbb{M}_{m,n}(\mathbb{R})$ *and* $\mathbf{x} \in \mathbb{R}^n$.

Proof. We will just prove the second part. Note that

$$\begin{aligned} \|A\mathbf{x}\|^2 &= \sum_{k=1}^{m} |\,(A\mathbf{x})_k\,|^2 = \sum_{k=1}^{m} |\,(A[k,:]\,\mathbf{x})\,|^2 = \sum_{k=1}^{n} |\langle \mathbf{x}, A[k,:]^T\rangle|^2 \\ &\leq \sum_{k=1}^{m} \|\mathbf{x}\|^2 \cdot \|A[k,:]^T\|^2 = \|\mathbf{x}\|^2 \sum_{k=1}^{m} \|A[k,:]^T\|^2 = \|\mathbf{x}\|^2 \|A\|^2 \end{aligned}$$

■

EXERCISE **4.2.3.**
1. *Let* $a, b \in \mathbb{R}$ *with* $a, b > 0$. *Then, prove that* $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 4$. *In general, for* $1 \leq i \leq n$, *let* $a_i \in \mathbb{R}$ *with* $a_i > 0$. *Then* $\left(\sum\limits_{i=1}^{n} a_i\right)\left(\sum\limits_{i=1}^{n} \frac{1}{a_i}\right) \geq n^2$.
2. *Prove that* $|\, z_1 + \cdots + z_n \,| \leq \sqrt{n(\,|\,z_1\,|^2 + \cdots + \,|\,z_n\,|^2)}$, *for* $z_1, \ldots, z_n \in \mathbb{C}$. *When does the equality hold?*
3. *Let* $\mathbb{V}$ *be an* IPS. *If* $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ *with* $\|\mathbf{u}\| = 1, \|\mathbf{v}\| = 1$ *and* $\langle \mathbf{u}, \mathbf{v}\rangle = 1$ *then prove that* $\mathbf{u} = \alpha \mathbf{v}$ *for some* $\alpha \in \mathbb{F}$. *Is* $\alpha = 1$?

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Let $\mathbb{V}$ be a real vector space. Then, for $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, the Cauchy-Schwartz inequality implies that $-1 \leq \dfrac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\;\|\mathbf{v}\|} \leq 1$. This together with the properties of the cosine function is used to define the angle between two vectors in a real inner product space.

Definition 4.2.4. Let $\mathbb{V}$ be a real vector space. If $\theta \in [0,\ \pi]$ is the angle between $\mathbf{u}, \mathbf{v} \in \mathbb{V} \setminus \{\mathbf{0}\}$ then we define

$$\cos\theta = \frac{\langle \mathbf{u}, \mathbf{v}\rangle}{\|\mathbf{u}\|\;\|\mathbf{v}\|}.$$

Example 4.2.5.
1. Take $(1,0)^T, (1,1)^T \in \mathbb{R}^2$. Then $\cos\theta = \frac{1}{\sqrt{2}}$. So $\theta = \pi/4$.
2. Take $(1,1,0)^T, (1,1,1)^T \in \mathbb{R}^3$. Then, angle between them, say $\beta = \cos^{-1}\frac{2}{\sqrt{6}}$.
3. Angle depends on the IP. Take $\langle \mathbf{x}, \mathbf{y}\rangle = 2\mathbf{x}_1\mathbf{y}_1 + \mathbf{x}_1\mathbf{y}_2 + \mathbf{x}_2\mathbf{y}_1 + \mathbf{x}_2\mathbf{y}_2$ on $\mathbb{R}^2$. Then, angle between $(1,0)^T, (1,1)^T \in \mathbb{R}^2$ equals $\cos^{-1}\frac{3}{\sqrt{10}}$.
4. As $\langle \mathbf{x}, \mathbf{y}\rangle = \langle \mathbf{y}, \mathbf{x}\rangle$ for any real vector space, the angle between $\mathbf{x}$ and $\mathbf{y}$ is same as the angle between $\mathbf{y}$ and $\mathbf{x}$.

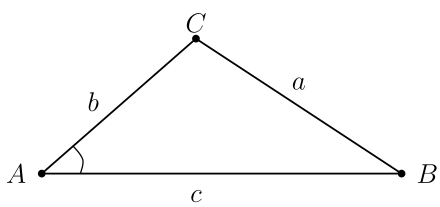

Figure 4.1: Triangle with vertices A, B and C

We will now prove that if A, B and C are the vertices of a triangle (see Figure 4.1) and a, b and c, respectively, are the lengths of the corresponding sides then $\cos(A) = \dfrac{b^2+c^2-a^2}{2bc}$. This in turn implies that the angle between vectors has been rightly defined.

Lemma 4.2.6. *Let A, B and C be the vertices of a triangle (see Figure 4.1) with corresponding side lengths a, b and c, respectively, in a real inner product space $\mathbb{V}$ then*

$$\cos(A) = \frac{b^2+c^2-a^2}{2bc}.$$

Proof. Let $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$ be the coordinates of the vertices A, B and C, respectively, of the triangle ABC. Then, $\vec{AB} = \mathbf{u}, \vec{AC} = \mathbf{v}$ and $\vec{BC} = \mathbf{v} - \mathbf{u}$. Thus, we need to prove that

$$\cos(A) = \frac{\|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2}{2\|\mathbf{v}\|\|\mathbf{u}\|} \Leftrightarrow \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2 = 2\,\|\mathbf{v}\|\,\|\mathbf{u}\|\,\cos(A).$$

Now, by definition $\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - 2\langle \mathbf{v}, \mathbf{u}\rangle$ and hence $\|\mathbf{v}\|^2 + \|\mathbf{u}\|^2 - \|\mathbf{v}-\mathbf{u}\|^2 = 2\,\langle \mathbf{u}, \mathbf{v}\rangle$. As $\langle \mathbf{v}, \mathbf{u}\rangle = \|\mathbf{v}\|\,\|\mathbf{u}\|\,\cos(A)$, the required result follows. ■

EXERCISE **4.2.7.** *Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then prove that*

1. $\langle \mathbf{x}, \mathbf{y}\rangle = 0 \Longleftrightarrow \|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ *(**Pythagoras Theorem**).*
   Solution: *Use $\|\mathbf{x}-\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle$ to get the required result follows.*

2. $\|\mathbf{x}\| = \|\mathbf{y}\| \Longleftrightarrow \langle \mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = 0$ *($\mathbf{x}$ and $\mathbf{y}$ form adjacent sides of a rhombus as the diagonals $\mathbf{x}+\mathbf{y}$ and $\mathbf{x}-\mathbf{y}$ are orthogonal).*
   Solution: *Use $\langle \mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle = \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2$ to get the required result follows.*

3. $4\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2$ *(**polarization identity in** $\mathbb{R}^n$).*
   Solution: *Just expand the right hand side to get the required result follows.*

4. $\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2$ *(**parallelogram law**: the sum of squares of the lengths of the diagonals of a parallelogram equals twice the sum of squares of the lengths of its sides).*
   Solution: *Just expand the left hand side to get the required result follows.*

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4.3 Normed Linear Space

In the last two sections, we have used the idea of inner product to define the norm/ length of a vector. This idea was used to get the Cauchy-Schwartz inequality, the basic back ground for defining the angle between two vectors. The question arises 'does every norm come from an inner product'. To understand it, we first state the properties that a norm must enjoy. We only look at linear spaces which are vector spaces over $\mathbb{R}$ or $\mathbb{C}$.

Definition 4.3.1. Let $\mathbb{V}$ be a linear space.

1. Then, a **norm** on $\mathbb{V}$ is a function $f(\mathbf{x}) = \|\mathbf{x}\|$ from $\mathbb{V}$ to $\mathbb{R}$ such that
   (a) $\|\mathbf{x}\| \geq 0$ for all $\mathbf{x} \in \mathbb{V}$ and if $\|\mathbf{x}\| = 0$ then $\mathbf{x} = \mathbf{0}$.

(b) $\|\alpha\mathbf{x}\| = \mid \alpha \mid \ \|\mathbf{x}\|$ for all $\alpha \in \mathbb{F}$ and $\mathbf{x} \in \mathbb{V}$.

(c) $\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ (triangle inequality).

2. A linear space with a norm on it is called a **normed linear space** (NLS).

Remark 4.3.2. *1. Let $\mathbb{V}$ be an* IPS. *Is it true that $f(\mathbf{x}) = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$ is a norm?*

Proof. Yes, $f(\mathbf{x})$ indeed defines an inner product. The readers should verify the first two conditions. For the third condition, using the Cauchy-Schwartz inequality, we get

$$\begin{aligned} f(\mathbf{x}+\mathbf{y})^2 &= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{x}\rangle + \langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{x}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle \\ &\leq \|\mathbf{x}\|^2 + \|\mathbf{x}\| \cdot \|\mathbf{y}\| + \|\mathbf{x}\| \cdot \|\mathbf{y}\| + \|\mathbf{y}\|^2 = (f(\mathbf{x}) + f(\mathbf{y}))^2. \end{aligned}$$

Thus, $f(\mathbf{x}) = \sqrt{\langle \mathbf{x}, \mathbf{x}\rangle}$ is a norm, called the norm **induced** *by the inner product $\langle \cdot, \cdot \rangle$.*

2. If $\|\cdot\|$ is a norm in $\mathbb{V}$ then $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}-\mathbf{y}\|$, for $\mathbf{x}, \mathbf{y} \in \mathbb{V}$, defines a **distance function**.

Proof. To see this, note that

(a) $d(\mathbf{x}, \mathbf{x}) = 0$, for each $\mathbf{x} \in \mathbb{V}$.

(b) using the triangle inequality, for any $\mathbf{z} \in \mathbb{V}$, we have

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}-\mathbf{y}\| = \|\,(\mathbf{x}-\mathbf{z}) - (\mathbf{y}-\mathbf{z})\,\| \leq \|\,(\mathbf{x}-\mathbf{z})\,\| + \|\,(\mathbf{y}-\mathbf{z})\,\| = d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y}).$$

Theorem 4.3.3. *Let $\mathbb{V}$ be a normed linear space and $\mathbf{x}, \mathbf{y} \in \mathbb{V}$. Then $\big|\|\mathbf{x}\| - \|\mathbf{y}\|\big| \leq \|\mathbf{x}-\mathbf{y}\|$.*

Proof. As $\|\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}+\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\| + \|\mathbf{y}\|$ one has $\|\mathbf{x}\| - \|\mathbf{y}\| \leq \|\mathbf{x}-\mathbf{y}\|$. Similarly, one obtains $\|\mathbf{y}\| - \|\mathbf{x}\| \leq \|\mathbf{y}-\mathbf{x}\| = \|\mathbf{x}-\mathbf{y}\|$. Combining the two, the required result follows. ∎

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EXERCISE **4.3.4.** *1. Let $\mathbb{V}$ be a complex* IPS. *Then,*

$$4\langle \mathbf{x}, \mathbf{y}\rangle = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + i\|\mathbf{x}+i\mathbf{y}\|^2 - i\|\mathbf{x}-i\mathbf{y}\|^2 \quad (\textbf{Polarization Identity}).$$

2. Consider the complex vector space $\mathbb{C}^n$. If $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ then prove that

(a) If $\mathbf{x} \neq \mathbf{0}$ then $\|\mathbf{x}+i\mathbf{x}\|^2 = \|\mathbf{x}\|^2 + \|i\mathbf{x}\|^2$, even though $\langle \mathbf{x}, i\mathbf{x}\rangle \neq 0$.

(b) $\langle \mathbf{x}, \mathbf{y}\rangle = 0$ whenever $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ and $\|\mathbf{x}+i\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|i\mathbf{y}\|^2$.

3. Let $A \in \mathbb{M}_n(\mathbb{C})$ satisfy $\|A\mathbf{x}\| \leq \|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{C}^n$. Then, prove that if $\alpha \in \mathbb{C}$ with $\mid \alpha \mid > 1$ then $A - \alpha I$ is invertible.

The next result is stated without proof as the proof is beyond the scope of this book.

Theorem 4.3.5. *Let $\|\cdot\|$ be a norm on a normed linear space $\mathbb{V}$. Then the norm $\|\cdot\|$ is induced by some inner product if and only if $\|\cdot\|$ satisfies the* PARALLELOGRAM LAW*:*

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2.$$

We now define a norm which doesn't come from an inner product.

Example 4.3.6. For $\mathbf{x} = (x_1, x_2)^T \in \mathbb{R}^2$, we define $\|\mathbf{x}\| = |\mathbf{x}_1| + |\mathbf{x}_2|$. Verify that $\|\mathbf{x}\|$ is indeed a norm. But, for $\mathbf{x} = \mathbf{e}_1$ and $\mathbf{y} = \mathbf{e}_2$, $2(\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2) = 4$ whereas

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = \|(1,1)^T\|^2 + \|(1,-1)^T\|^2 = (|1| + |1|)^2 + (|1| + |-1|)^2 = 8.$$

So the parallelogram law fails. Thus, $\|\mathbf{x}\|$ is not induced by any inner product in $\mathbb{R}^2$.

EXERCISE **4.3.7.** *Does there exist an inner product in $\mathbb{R}^2$ such that $\|\mathbf{x}\| = \max\{|x_1|, |x_2|\}$?*

4.4 Orthogonality in Inner Product Space

We come back to the study of Inner product spaces the topic which is a building block for most of the applications. To start with, we give the definition of orthogonality of two vectors.

Definition 4.4.1. Let $\mathbb{V}$ be an inner product space over $\mathbb{F}$. Then,

1. the vectors $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ are called **orthogonal/perpendicular** if $\langle \mathbf{u}, \mathbf{v} \rangle = 0$.
2. Let $S \subseteq \mathbb{V}$. Then, the **orthogonal complement** of S in $\mathbb{V}$, denoted $S^\perp$, equals

$$S^\perp = \{\mathbf{v} \in \mathbb{V} : \ \langle \mathbf{v}, \mathbf{w} \rangle = 0, \quad \text{for all} \quad \mathbf{w} \in S\}.$$

Example 4.4.2. 1. $\mathbf{0}$ is orthogonal to every vector as $\langle \mathbf{0}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathbb{V}$.

2. If $\mathbb{V}$ is a vector space over $\mathbb{R}$ or $\mathbb{C}$ then $\mathbf{0}$ is the only vector that is orthogonal to itself.

3. Let $\mathbb{V} = \mathbb{R}$.

   (a) If $S = \{0\}$ then, $S^\perp = \mathbb{R}$.

   (b) If $S = \mathbb{R}$ then, $S^\perp = \{0\}$.

   (c) Let S be any subset of $\mathbb{R}$ containing a non-zero real number. Then $S^\perp = \{0\}$.

4. Let $\mathbf{u} = (1,2)^T$. What is $\mathbf{u}^\perp$ in $\mathbb{R}^2$?
   Solution: By definition, $\mathbf{u}^\perp = \{(x,y)^T \in \mathbb{R}^2 \mid \ x + 2y = 0\}$. Thus, $\mathbf{u}^\perp$ is the $\text{NULL}(\mathbf{u})$. Note that $\mathbf{u}^\perp = LS\left((2,-1)^T\right)$. Further, observe that for any vector $\mathbf{x} \in \mathbb{R}^2$,

$$\mathbf{x} = \langle \mathbf{x}, \mathbf{u} \rangle \frac{\mathbf{u}}{\|\mathbf{u}\|^2} + \left(\mathbf{x} - \langle \mathbf{x}, \mathbf{u} \rangle \frac{\mathbf{u}}{\|\mathbf{u}\|^2}\right) = \frac{x_1 + 2x_2}{5}(1,2)^T + \frac{2x_1 - x_2}{5}(2,-1)^T$$

   is a decomposition of $\mathbf{x}$ into two vectors, one parallel to $\mathbf{u}$ and the other parallel to $\mathbf{u}^\perp$.

5. Fix $\mathbf{u} = (1,1,1,1)^T, \mathbf{v} = (1,1,-1,0)^T \in \mathbb{R}^4$. Determine $\mathbf{z}, \mathbf{w} \in \mathbb{R}^4$ such that $\mathbf{u} = \mathbf{z} + \mathbf{w}$ with the condition that $\mathbf{z}$ is parallel to $\mathbf{v}$ and $\mathbf{w}$ is orthogonal to $\mathbf{v}$.
   Solution: As $\mathbf{z}$ is parallel to $\mathbf{v}$, $\mathbf{z} = k\mathbf{v} = (k,k,-k,0)^T$, for some $k \in \mathbb{R}$. Since $\mathbf{w}$ is orthogonal to $\mathbf{v}$ the vector $\mathbf{w} = (a,b,c,d)^T$ satisfies $a + b - c = 0$. Thus, $c = a + b$ and

$$(1,1,1,1)^T = \mathbf{u} = \mathbf{z} + \mathbf{w} = (k,k,-k,0)^T + (a,b,a+b,d)^T.$$

   Comparing the corresponding coordinates, gives the linear system $d = 1$, $a + k = 1$, $b + k = 1$ and $a + b - k = 1$ in the unknowns a, b, d and k. Thus, solving for a, b, d and k gives $\mathbf{z} = \frac{1}{3}(1,1,-1,0)^T$ and $\mathbf{w} = \frac{1}{3}(2,2,4,3)^T$.

6. **Apollonius' Identity:** Let the length of the sides of a triangle be $a, b, c \in \mathbb{R}$ and that of the median be $d \in \mathbb{R}$. If the median is drawn on the side with length a then prove that $b^2 + c^2 = 2\left(d^2 + \left(\frac{a}{2}\right)^2\right)$.

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7. Let $P = (1,1,1)^T$, $Q = (2,1,3)^T$ and $R = (-1,1,2)^T$ be three vertices of a triangle in $\mathbb{R}^3$. Compute the angle between the sides PQ and PR.

   Solution: Method 1: Note that $\vec{PQ} = (2,1,3)^T - (1,1,1)^T = (1,0,2)^T$, $\vec{PR} = (-2,0,1)^T$ and $\vec{RQ} = (-3,0,-1)^T$. As $\langle \vec{PQ}, \vec{PR} \rangle = 0$, the angle between the sides PQ and PR is $\frac{\pi}{2}$.

   Method 2: $\|PQ\| = \sqrt{5}$, $\|PR\| = \sqrt{5}$ and $\|QR\| = \sqrt{10}$. As $\|QR\|^2 = \|PQ\|^2 + \|PR\|^2$, by Pythagoras theorem, the angle between the sides PQ and PR is $\frac{\pi}{2}$.

EXERCISE **4.4.3.** *1. Let $\mathbb{V}$ be an IPS.*

   (a) If $S \subseteq \mathbb{V}$ then $S^{\perp}$ is a subspace of $\mathbb{V}$ and $S^{\perp} = (LS(S))^{\perp}$.

   (b) Furthermore, if $\mathbb{V}$ is finite dimensional then $S^{\perp}$ and $LS(S)$ are complementary. Thus, $\mathbb{V} = LS(S) \oplus S^{\perp}$. Equivalently, $\langle \mathbf{u}, \mathbf{w} \rangle = 0$, for all $\mathbf{u} \in LS(S)$ and $\mathbf{w} \in S^{\perp}$.

2. *Find $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{v}, \mathbf{w}$ and $(1,-1,-2)^T$ are mutually orthogonal.*

3. *Let $\mathbb{W} = \{(x,y,z,w)^T \in \mathbb{R}^4 : x+y+z-w = 0\}$. Find a basis of $\mathbb{W}^{\perp}$.*

4. *Determine $\mathbb{W}^{\perp}$, where $\mathbb{W} = \{A \in \mathbb{M}_n(\mathbb{R}) \mid A^T = A\}$.*

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5. *Consider $\mathbb{R}^3$ with the standard inner product. Find*

   (a) $S^{\perp}$ for $S = \{(1,1,1)^T, (0,1,-1)^T\}$.

   (b) k such that $\cos^{-1}(\langle \mathbf{u}, \mathbf{v} \rangle) = \pi/3$, where $\mathbf{u} = (1,-1,1)^T$ and $\mathbf{v} = (1,k,1)^T$.

   (c) vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ such that $\mathbf{v}, \mathbf{w}$ and $\mathbf{u} = (1,1,1)^T$ are mutually orthogonal.

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6. *Consider $\mathbb{R}^3$ with the standard inner product. Find the plane containing*

   (a) $(1,1-1)$ with $(a,b,c) \neq \mathbf{0}$ as the normal vector.

   (b) $(2,-2,1)^T$ and perpendicular to the line $\ell = \{(t-1, 3t+2, t+1) : t \in \mathbb{R}\}$.

   (c) the lines $(1,2,-2) + t(1,1,0)$ and $(1,2,-2) + t(0,1,2)$.

   (d) $(1,1,2)^T$ and orthogonal to the line $\ell\{(2+t, 3, 1-t) : t \in \mathbb{R}\}$.

7. *Let $P = (3,0,2)^T, Q = (1,2,-1)^T$ and $R = (2,-1,1)^T$ be three points in $\mathbb{R}^3$. Then,*

   (a) find the area of the triangle with vertices P, Q and R.

   (b) find the area of the parallelogram built on vectors $\vec{PQ}$ and $\vec{QR}$.

   (c) find a non-zero vector orthogonal to the plane of the above triangle.

   (d) find all vectors $\mathbf{x}$ orthogonal to $\vec{PQ}$ and $\vec{QR}$ with $\|\mathbf{x}\| = \sqrt{2}$.

   (e) the volume of the parallelepiped built on vectors $\vec{PQ}$ and $\vec{QR}$ and $\mathbf{x}$, where $\mathbf{x}$ is one of the vectors found in Part (d). Do you think the volume would be different if you choose the other vector $\mathbf{x}$?

8. *Let p_1 be a plane containing the point $A = (1,2,3)^T$ and the vector $(2,-1,1)^T$ as its normal. Then,*

   (a) find the equation of the plane p_2 that is parallel to p_1 and contains $(-1,2,-3)^T$.

(b) *calculate the distance between the planes* p_1 *and* p_2.

9. *In the parallelogram* $ABCD$, $AB\|DC$ *and* $AD\|BC$ *and* $A = (-2, 1, 3)^T, B = (-1, 2, 2)^T$ *and* $C = (-3, 1, 5)^T$. *Find the*

   (a) *coordinates of the point* D,

   (b) *cosine of the angle* BCD.

   (c) *area of the triangle* ABC

   (d) *volume of the parallelepiped determined by* AB, AD *and* $(0, 0, -7)^T$.

4.4.1 Properties of Orthonormal Vectors

We start with the definition of an orthonormal set.

Definition 4.4.4. Let $\mathbb{V}$ be an IPS. Then, a non-empty set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \subseteq \mathbb{V}$ is called an **orthogonal set** if $\mathbf{v}_i$ and $\mathbf{v}_j$ are **mutually orthogonal**, for $1 \le i \ne j \le n$, *i.e.*,

$$\langle \mathbf{u}_i, \mathbf{u}_j \rangle = 0, \text{ for } 1 \le i < j \le n.$$

Further, if $\|\mathbf{v}_i\| = 1$, for $1 \le i \le n$, Then S is called an **orthonormal set**. If S is also a basis of $\mathbb{V}$ then S is called an **orthonormal basis** of $\mathbb{V}$.

Example 4.4.5. 1. A few orthonormal sets in $\mathbb{R}^2$ are

$$\{(1,0)^T, (0,1)^T\}, \{\frac{1}{\sqrt{2}}(1,1)^T, \frac{1}{\sqrt{2}}(1,-1)^T\} \text{ and } \{\frac{1}{\sqrt{5}}(2,1)^T, \frac{1}{\sqrt{5}}(1,-2)^T\}.$$

DRAFT

2. Let $S = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ be the standard basis of $\mathbb{R}^n$. Then, S is an orthonormal set as

   (a) $\|\mathbf{e}_i\| = 1$, for $1 \le i \le n$.

   (b) $\langle \mathbf{e}_i, \mathbf{e}_j \rangle = 0$, for $1 \le i \ne j \le n$.

3. The set $\left\{ \left[\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]^T, \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]^T, \left[\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right]^T \right\}$ is an orthonormal set in $\mathbb{R}^3$.

4. Recall that $\langle f(x), g(x) \rangle = \int\limits_{-\pi}^{\pi} f(x)\overline{g(x)}dx$ defines the standard inner product in $\mathcal{C}[-\pi, \pi]$. Consider $S = \{\mathbf{1}\} \cup \{\mathbf{e}_m \mid m \ge 1\} \cup \{\mathbf{f}_n \mid n \ge 1\}$, where $\mathbf{1}(x) = 1$, $\mathbf{e}_m(x) = \cos(mx)$ and $\mathbf{f}_n(x) = \sin(nx)$, for all $m, n \ge 1$ and for all $x \in [-\pi, \pi]$. Then,

   (a) S is a linearly independent set.

   (b) $\|\mathbf{1}\|^2 = 2\pi$, $\|\mathbf{e}_m\|^2 = \pi$ and $\|\mathbf{f}_n\|^2 = \pi$.

   (c) the functions in S are orthogonal.

   Hence, $\left\{\dfrac{\mathbf{1}}{\sqrt{2\pi}}\right\} \cup \left\{\dfrac{1}{\sqrt{\pi}}\mathbf{e}_m \mid m \ge 1\right\} \cup \left\{\dfrac{1}{\sqrt{\pi}}\mathbf{f}_n \mid n \ge 1\right\}$ is an orthonormal set in $\mathcal{C}[-\pi, \pi]$.

We now prove the most important initial result of this section.

Theorem 4.4.6. *Let* $S = \{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ *be an orthonormal subset of an* IPS $\mathbb{V}(\mathbb{F})$.

1. *Then S is a linearly independent subset of $\mathbb{V}$.*

2. *Suppose $\mathbf{v} \in LS(S)$ with $\mathbf{v} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{u}_i$, for some α_i's in $\mathbb{F}$. Then,*
   (a) $\alpha_i = \langle \mathbf{v}, \mathbf{u}_i \rangle$.
   (b) $\|\mathbf{v}\|^2 = \| \sum\limits_{i=1}^{n} \alpha_i \mathbf{u}_i \|^2 = \sum\limits_{i=1}^{n} \mid \alpha_i \mid^2$.

3. *Let $\mathbf{z} \in \mathbb{V}$ and $\mathbf{y} = \sum\limits_{i=1}^{n} \langle \mathbf{z}, \mathbf{u}_i \rangle \mathbf{u}_i$.*
   (a) *Then $\mathbf{z} = \mathbf{y} + (\mathbf{z} - \mathbf{y})$ with $\langle \mathbf{z} - \mathbf{y}, \mathbf{y} \rangle = 0$, i.e., $\mathbf{z} - \mathbf{y} \in LS(S)^{\perp}$.*
   (b) **Pythagoras Theorem:** $\|\mathbf{z}\|^2 = \|\mathbf{y}\|^2 + \|\mathbf{z} - \mathbf{y}\|^2$.
   (c) *Thus, $\mathbf{y}$ is the nearest vector in $LS(S)$. That is, if $\mathbf{w} \in LS(S)$ with $\mathbf{w} \neq \mathbf{y}$ then*
   $$\|\mathbf{z} - \mathbf{w}\|^2 = \|\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{w}\|^2 = \|\mathbf{z} - \mathbf{y}\|^2 + \|\mathbf{y} - \mathbf{w}\|^2 > \|\mathbf{z} - \mathbf{y}\|^2.$$

4. *Let $\dim(\mathbb{V}) = n$. Then $\langle \mathbf{v}, \mathbf{u}_i \rangle = 0$ for all $i = 1, 2, \ldots, n$ if and only if $\mathbf{v} = \mathbf{0}$.*

Proof. Part 1: Consider the linear system $c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n = \mathbf{0}$ in the unknowns $c_1, \ldots, c_n$. As $\langle \mathbf{0}, \mathbf{u} \rangle = 0$ and $\langle \mathbf{u}_j, \mathbf{u}_i \rangle = 0$, for all $j \neq i$, we have

$$0 = \langle \mathbf{0}, \mathbf{u}_i \rangle = \langle c_1 \mathbf{u}_1 + \cdots + c_n \mathbf{u}_n, \mathbf{u}_i \rangle = \sum_{j=1}^{n} c_j \langle \mathbf{u}_j, \mathbf{u}_i \rangle = c_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle = c_i.$$

Hence $c_i = 0$, for $1 \leq i \leq n$. Thus, the above linear system has only the trivial solution. So, the set S is linearly independent.

DRAFT

Part 2: Note that $\langle \mathbf{v}, \mathbf{u}_i \rangle = \langle \sum\limits_{j=1}^{n} \alpha_j \mathbf{u}_j, \mathbf{u}_i \rangle = \sum\limits_{j=1}^{n} \alpha_j \langle \mathbf{u}_j, \mathbf{u}_i \rangle = \alpha_i \langle \mathbf{u}_i, \mathbf{u}_i \rangle = \alpha_i$. This completes Sub-part (a). For Sub-part (b), we have

$$\begin{aligned} \|\mathbf{v}\|^2 &= \| \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \|^2 = \left\langle \sum_{i=1}^{n} \alpha_i \mathbf{u}_i, \sum_{i=1}^{n} \alpha_i \mathbf{u}_i \right\rangle = \sum_{i=1}^{n} \alpha_i \left\langle \mathbf{u}_i, \sum_{j=1}^{n} \alpha_j \mathbf{u}_j \right\rangle \\ &= \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{n} \overline{\alpha_j} \langle \mathbf{u}_i, \mathbf{u}_j \rangle = \sum_{i=1}^{n} \alpha_i \overline{\alpha_i} \langle \mathbf{u}_i, \mathbf{u}_i \rangle = \sum_{i=1}^{n} \mid \alpha_i \mid^2. \end{aligned}$$

Part 3: Note that for $1 \leq i \leq n$,

$$\begin{aligned} \langle \mathbf{z} - \mathbf{y}, \mathbf{u}_i \rangle &= \langle \mathbf{z}, \mathbf{u}_i \rangle - \langle \mathbf{y}, \mathbf{u}_i \rangle = \langle \mathbf{z}, \mathbf{u}_i \rangle - \left\langle \sum_{j=1}^{n} \langle \mathbf{z}, \mathbf{u}_j \rangle \mathbf{u}_j, \mathbf{u}_i \right\rangle \\ &= \langle \mathbf{z}, \mathbf{u}_i \rangle - \sum_{j=1}^{n} \langle \mathbf{z}, \mathbf{u}_j \rangle \langle \mathbf{u}_j, \mathbf{u}_i \rangle = \langle \mathbf{z}, \mathbf{u}_i \rangle - \langle \mathbf{z}, \mathbf{u}_i \rangle = 0. \end{aligned}$$

So, $\mathbf{z} - \mathbf{y} \in \mathrm{LS}(S)^{\perp}$. As $\mathbf{y} \in \mathrm{LS}(S)$, $\langle \mathbf{z} - \mathbf{y}, \mathbf{y} \rangle = 0$ and

$$\|\mathbf{z}\|^2 = \|\mathbf{y} + (\mathbf{z} - \mathbf{y})\|^2 = \|\mathbf{y}\|^2 + \|\mathbf{z} - \mathbf{y}\|^2 \geq \|\mathbf{y}\|^2.$$

Further, $\mathbf{w}, \mathbf{y} \in \mathrm{LS}(S)$ implies $\mathbf{w} - \mathbf{y} \in \mathrm{LS}(S)$. Hence $\langle \mathbf{z} - \mathbf{y}, \mathbf{w} - \mathbf{y} \rangle = 0$ and

$$\|\mathbf{z} - \mathbf{w}\|^2 = \|\mathbf{z} - \mathbf{y} + \mathbf{y} - \mathbf{w}\|^2 = \|\mathbf{z} - \mathbf{y}\|^2 + \|\mathbf{y} - \mathbf{w}\|^2 > \|\mathbf{z} - \mathbf{y}\|^2.$$

Part 4: Follows directly using Part 2b as $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ is a basis of $\mathbb{V}$. ■

A rephrasing of Theorem 4.4.6.2b gives a generalization of the Pythagoras theorem, popularly known as the Parseval's formula. The proof is left as an exercise for the reader.

Theorem 4.4.7. *Let $\mathbb{V}$ be an with an orthonormal basis $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$. Then*

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i \rangle \overline{\langle \mathbf{y}, \mathbf{v}_i \rangle}, \qquad \text{for each } \mathbf{x}, \mathbf{y} \in \mathbb{V}.$$

Furthermore, if $\mathbf{x} = \mathbf{y}$ then $\|\mathbf{x}\|^2 = \sum\limits_{i=1}^{n} |\langle \mathbf{x}, \mathbf{v}_i \rangle|^2$ (generalizing the **Pythagoras Theorem***).*

We have another corollary of Theorem 4.4.6 which talks about an orthogonal set.

Theorem 4.4.8 (Bessel's Inequality)**.** *Let $\mathbb{V}$ be an IPS with $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ as an orthogonal set. Then, for each $\mathbf{z} \in \mathbb{V}$, $\sum\limits_{k=1}^{n} \frac{|\langle \mathbf{z}, \mathbf{v}_k \rangle|^2}{\|\mathbf{v}_k\|^2} \le \|\mathbf{z}\|^2$. Equality holds if and only if $\mathbf{z} = \sum\limits_{k=1}^{n} \frac{\langle \mathbf{z}, \mathbf{v}_k \rangle}{\|\mathbf{v}_k\|^2} \mathbf{v}_k$.*

Proof. For $1 \le k \le n$, define $\mathbf{u}_k = \dfrac{\mathbf{v}_k}{\|\mathbf{v}_k\|}$ and use Theorem 4.4.6.4 to get the required result. ■

Remark 4.4.9. *Using Theorem 4.4.6, we see that if $\mathcal{B} = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ is an ordered orthonormal basis of an IPS $\mathbb{V}$ then*

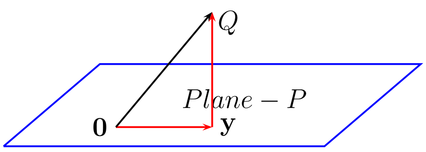
$$[\mathbf{u}]_{\mathcal{B}} = \begin{bmatrix} \langle \mathbf{u}, \mathbf{v}_1 \rangle \\ \vdots \\ \langle \mathbf{u}, \mathbf{v}_n \rangle \end{bmatrix}, \qquad \text{for each } \mathbf{u} \in \mathbb{V}.$$

DRAFT

Thus, to get the coordinates of a vector with respect to an orthonormal ordered basis, we just need to compute the inner product with basis vectors.

To proceed further with the applications of the above ideas, we pose a question for better understanding.

Example 4.4.10. Which point on the plane P is closest to the point, say Q?



Solution: Let $\mathbf{y}$ be the foot of the perpendicular from Q on P. Thus, by Pythagoras Theorem (see Theorem 4.4.6.3c), $\mathbf{y}$ is unique. So, the question arises: how do we find $\mathbf{y}$?

Note that $\overrightarrow{\mathbf{y}Q}$ gives a normal vector of the plane P. Hence, $\overrightarrow{Q} = \mathbf{y} + \overrightarrow{\mathbf{y}Q}$. So, need to decompose $\overrightarrow{Q}$ into two vectors such that one of them lies on the plane P and the other is orthogonal to the plane.

Thus, we see that given $\mathbf{u}, \mathbf{v} \in \mathbb{V} \setminus \{\mathbf{0}\}$, we need to find two vectors, say $\mathbf{y}$ and $\mathbf{z}$, such that $\mathbf{y}$ is parallel to $\mathbf{u}$ and $\mathbf{z}$ is perpendicular to $\mathbf{u}$. Thus, $\mathbf{y} = \mathbf{u}\cos(\theta)$ and $\mathbf{z} = \mathbf{u}\sin(\theta)$, where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

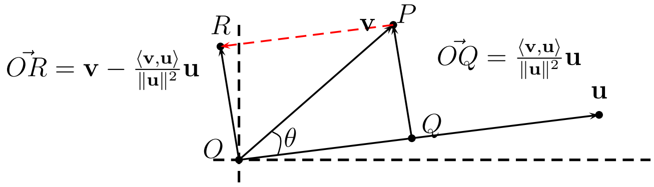


Figure 4.2: Decomposition of vector $\mathbf{v}$

We do this as follows (see Figure 4.2). Let $\hat{\mathbf{u}} = \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$ be the unit vector in the direction of $\mathbf{u}$. Then, using trigonometry, $\cos(\theta) = \frac{\|\vec{OQ}\|}{\|\vec{OP}\|}$. Hence $\|\vec{OQ}\| = \|\vec{OP}\| \cos(\theta)$. Now using Definition 4.2.4, $\|\vec{OQ}\| = \|\mathbf{v}\| \left|\frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{v}\|\,\|\mathbf{u}\|}\right| = \left|\frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{u}\|}\right|$, where the absolute value is taken as the length/norm is a positive quantity. Thus,

$$\vec{OQ} = \|\vec{OQ}\|\, \hat{\mathbf{u}} = \left\langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\right\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|}.$$

Hence, $\mathbf{y} = \vec{OQ} = \left\langle \mathbf{v}, \dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|}$ and $\mathbf{z} = \mathbf{v} - \left\langle \mathbf{v}, \dfrac{\mathbf{u}}{\|\mathbf{u}\|}\right\rangle \dfrac{\mathbf{u}}{\|\mathbf{u}\|}$. In literature, the vector $\mathbf{y} = \vec{OQ}$ is called the **orthogonal projection** of $\mathbf{v}$ on $\mathbf{u}$, denoted $\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})$. Thus,

$$\mathrm{Proj}_{\mathbf{u}}(\mathbf{v}) = \left\langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\right\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|} \quad \text{and} \quad \|\mathrm{Proj}_{\mathbf{u}}(\mathbf{v})\| = \|\vec{OQ}\| = \left|\frac{\langle \mathbf{v},\mathbf{u}\rangle}{\|\mathbf{u}\|}\right|. \tag{4.4.2}$$

DRAFT

Moreover, the distance of $\mathbf{u}$ from the point P equals $\|\vec{OR}\| = \|\vec{PQ}\| = \left\|\mathbf{v} - \langle \mathbf{v}, \frac{\mathbf{u}}{\|\mathbf{u}\|}\rangle \frac{\mathbf{u}}{\|\mathbf{u}\|}\right\|$.

Example 4.4.11. 1. Determine the foot of the perpendicular from the point $(1, 2, 3)$ on the XY-plane.

Solution: Verify that the required point is $(1, 2, 0)$?

2. Determine the foot of the perpendicular from the point $Q = (1, 2, 3, 4)$ on the plane generated by $(1, 1, 0, 0), (1, 0, 1, 0)$ and $(0, 1, 1, 1)$.

   Answer: (x, y, z, w) lies on the plane $x-y-z+2w = 0 \Leftrightarrow \langle(1, -1, -1, 2), (x, y, z, w)\rangle = 0$.

   So, the required point equals

$$\begin{aligned}(1, 2, 3, 4) - \langle(1, 2, 3, 4), \frac{1}{\sqrt{7}}(1, -1, -1, 2)\rangle \frac{1}{\sqrt{7}}(1, -1, -1, 2) \\ = \quad (1, 2, 3, 4) - \frac{4}{7}(1, -1, -1, 2) = \frac{1}{7}(3, 18, 25, 20).\end{aligned}$$

3. Determine the projection of $\mathbf{v} = (1, 1, 1, 1)^T$ on $\mathbf{u} = (1, 1, -1, 0)^T$.

   Solution: By Equation (4.4.2), we have $\mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) = \langle \mathbf{v}, \mathbf{u}\rangle \dfrac{\mathbf{u}}{\|\mathbf{u}\|^2} = \frac{1}{3}(1, 1, -1, 0)^T$ and $\mathbf{w} = (1, 1, 1, 1)^T - \mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) = \frac{1}{3}(2, 2, 4, 3)^T$ is orthogonal to $\mathbf{u}$.

4. Let $\mathbf{u} = (1, 1, 1, 1)^T, \mathbf{v} = (1, 1, -1, 0)^T, \mathbf{w} = (1, 1, 0, -1)^T \in \mathbb{R}^4$. Write $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, where $\mathbf{v}_1$ is parallel to $\mathbf{u}$ and $\mathbf{v}_2$ is orthogonal to $\mathbf{u}$. Also, write $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3$ such that $\mathbf{w}_1$ is parallel to $\mathbf{u}$, $\mathbf{w}_2$ is parallel to $\mathbf{v}_2$ and $\mathbf{w}_3$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}_2$.

   Solution: Note that

(a) $\mathbf{v}_1 = \text{Proj}_{\mathbf{u}}(\mathbf{v}) = \langle \mathbf{v}, \mathbf{u} \rangle \frac{\mathbf{u}}{\|\mathbf{u}\|^2} = \frac{1}{4}\mathbf{u} = \frac{1}{4}(1,1,1,1)^T$ is parallel to $\mathbf{u}$.

(b) $\mathbf{v}_2 = \mathbf{v} - \frac{1}{4}\mathbf{u} = \frac{1}{4}(3,3,-5,-1)^T$ is orthogonal to $\mathbf{u}$.

Note that $\text{Proj}_{\mathbf{u}}(\mathbf{w})$ is parallel to $\mathbf{u}$ and $\text{Proj}_{\mathbf{v}_2}(\mathbf{w})$ is parallel to $\mathbf{v}_2$. Hence, we have

(a) $\mathbf{w}_1 = \text{Proj}_{\mathbf{u}}(\mathbf{w}) = \langle \mathbf{w}, \mathbf{u} \rangle \frac{\mathbf{u}}{\|\mathbf{u}\|^2} = \frac{1}{4}\mathbf{u} = \frac{1}{4}(1,1,1,1)^T$ is parallel to $\mathbf{u}$,

(b) $\mathbf{w}_2 = \text{Proj}_{\mathbf{v}_2}(\mathbf{w}) = \langle \mathbf{w}, \mathbf{v}_2 \rangle \frac{\mathbf{v}_2}{\|\mathbf{v}_2\|^2} = \frac{7}{44}(3,3,-5,-1)^T$ is parallel to $\mathbf{v}_2$ and

(c) $\mathbf{w}_3 = \mathbf{w} - \mathbf{w}_1 - \mathbf{w}_2 = \frac{3}{11}(1,1,2,-4)^T$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}_2$.

4.5 Gram-Schmidt Orthonormalization Process

In view of the importance of Theorem 4.4.6, we inquire into the question of extracting an orthonormal basis from a given basis. The process of extracting an orthonormal basis from a finite linearly independent set is called the **Gram-Schmidt Orthonormalization process**. We first consider a few examples. Note that Theorem 4.4.6 also gives us an algorithm for doing so, *i.e.*, from the given vector subtract all the orthogonal projections/components. If the new vector is nonzero then this vector is orthogonal to the previous ones. The proof follows directly from Theorem 4.4.6 but we give it again for the sake of completeness.

Theorem 4.5.1 (Gram-Schmidt Orthogonalization Process)**.** *Let* $\mathbb{V}$ *be an* IPS*. If* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is a set of linearly independent vectors in* $\mathbb{V}$ *then there exists an orthonormal set* $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ *in* $\mathbb{V}$*. Furthermore,* $LS(\mathbf{w}_1, \ldots, \mathbf{w}_i) = LS(\mathbf{v}_1, \ldots, \mathbf{v}_i)$*, for* $1 \le i \le n$*.*

DRAFT

Proof. Note that for orthonormality, we need $\|\mathbf{w}_i\| = 1$, for $1 \le i \le n$ and $\langle \mathbf{w}_i, \mathbf{w}_j \rangle = 0$, for $1 \le i \ne j \le n$. Also, by Corollary 3.3.11.2, $\mathbf{v}_i \notin LS(\mathbf{v}_1, \ldots, \mathbf{v}_{i-1})$, for $2 \le i \le n$, as $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set. We are now ready to prove the result by induction.

Step 1: Define $\mathbf{w}_1 = \dfrac{\mathbf{v}_1}{\|\mathbf{v}_1\|}$ then $LS(\mathbf{v}_1) = LS(\mathbf{w}_1)$.

Step 2: Define $\mathbf{u}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1$. Then, $\mathbf{u}_2 \ne \mathbf{0}$ as $\mathbf{v}_2 \notin LS(\mathbf{v}_1)$. So, let $\mathbf{w}_2 = \dfrac{\mathbf{u}_2}{\|\mathbf{u}_2\|}$.

Note that $\{\mathbf{w}_1, \mathbf{w}_2\}$ is orthonormal and $LS(\mathbf{w}_1, \mathbf{w}_2) = LS(\mathbf{v}_1, \mathbf{v}_2)$.

Step 3: For induction, assume that we have obtained an orthonormal set $\{\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}$ such that $LS(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}) = LS(\mathbf{w}_1, \ldots, \mathbf{w}_{k-1})$. Now, note that $\mathbf{u}_k = \mathbf{v}_k - \sum\limits_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i \rangle \mathbf{w}_i = \mathbf{v}_k - \sum\limits_{i=1}^{k-1} \text{Proj}_{\mathbf{w}_i}(\mathbf{v}_k) \ne \mathbf{0}$ as $\mathbf{v}_k \notin LS(\mathbf{v}_1, \ldots, \mathbf{v}_{k-1})$. So, let us put $\mathbf{w}_k = \dfrac{\mathbf{u}_k}{\|\mathbf{u}_k\|}$. Then, $\{\mathbf{w}_1, \ldots, \mathbf{w}_k\}$ is orthonormal as $\|\mathbf{w}_k\| = 1$ and

$$\begin{aligned}
\|\mathbf{u}_k\| \langle \mathbf{w}_k, \mathbf{w}_1 \rangle &= \langle \mathbf{u}_k, \mathbf{w}_1 \rangle = \langle \mathbf{v}_k - \sum_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i \rangle \mathbf{w}_i, \mathbf{w}_1 \rangle = \langle \mathbf{v}_k, \mathbf{w}_1 \rangle - \langle \sum_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i \rangle \mathbf{w}_i,\ \mathbf{w}_1 \rangle \\
&= \langle \mathbf{v}_k, \mathbf{w}_1 \rangle - \sum_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i \rangle \langle \mathbf{w}_i,\ \mathbf{w}_1 \rangle = \langle \mathbf{v}_k, \mathbf{w}_1 \rangle - \langle \mathbf{v}_k, \mathbf{w}_1 \rangle = \mathbf{0}.
\end{aligned}$$

Similarly, $\langle \mathbf{w}_k, \mathbf{w}_i \rangle = 0$, for $2 \le i \le k-1$. Clearly, $\mathbf{w}_k = \mathbf{u}_k / \|\mathbf{u}_k\| \in LS(\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}, \mathbf{v}_k)$. So, $\mathbf{w}_k \in LS(\mathbf{v}_1, \ldots, \mathbf{v}_k)$.

As $\mathbf{v}_k = \|\mathbf{u}_k\|\mathbf{w}_k + \sum\limits_{i=1}^{k-1} \langle \mathbf{v}_k, \mathbf{w}_i\rangle \mathbf{w}_i$, we get $\mathbf{v}_k \in LS(\mathbf{w}_1, \ldots, \mathbf{w}_k)$. Hence, by the principle of mathematical induction $LS(\mathbf{w}_1, \ldots, \mathbf{w}_k) = LS(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ and the required result follows. ∎

We now illustrate the Gram-Schmidt process with a few examples.

Example 4.5.2. 1. Let $S = \{(1,-1,1,1),(1,0,1,0),(0,1,0,1)\} \subseteq \mathbb{R}^4$. Find an orthonormal set T such that $LS(S) = LS(T)$.

Solution: As we just require $LS(S) = LS(T)$, we can order the vectors as per our convenience. So, let $\mathbf{v}_1 = (1,0,1,0)^T, \mathbf{v}_2 = (0,1,0,1)^T$ and $\mathbf{v}_3 = (1,-1,1,1)^T$. Then, $\mathbf{w}_1 = \frac{1}{\sqrt{2}}(1,0,1,0)^T$. As $\langle \mathbf{v}_2, \mathbf{w}_1\rangle = 0$, we get $\mathbf{w}_2 = \frac{1}{\sqrt{2}}(0,1,0,1)^T$. For the third vector, let $\mathbf{u}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{w}_1\rangle\mathbf{w}_1 - \langle \mathbf{v}_3, \mathbf{w}_2\rangle\mathbf{w}_2 = (0,-1,0,1)^T$. Thus, $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0,-1,0,1)^T$.

2. Let $S = \left\{\mathbf{v}_1 = \begin{bmatrix}2 & 0 & 0\end{bmatrix}^T, \mathbf{v}_2 = \begin{bmatrix}\frac{3}{2} & 2 & 0\end{bmatrix}^T, \mathbf{v}_3 = \begin{bmatrix}\frac{1}{2} & \frac{3}{2} & 0\end{bmatrix}^T, \mathbf{v}_4 = \begin{bmatrix}1 & 1 & 1\end{bmatrix}^T\right\}$. Find an orthonormal set T such that $LS(S) = LS(T)$.

Solution: Take $\mathbf{w}_1 = \frac{\mathbf{v}_1}{\|\mathbf{v}_1\|} = \begin{bmatrix}1 & 0 & 0\end{bmatrix}^T = \mathbf{e}_1$. For the second vector, consider $\mathbf{u}_2 = \mathbf{v}_2 - \frac{3}{2}\mathbf{w}_1 = \begin{bmatrix}0 & 2 & 0\end{bmatrix}^T$. So, put $\mathbf{w}_2 = \frac{\mathbf{u}_2}{\|\mathbf{u}_2\|} = \begin{bmatrix}0 & 1 & 0\end{bmatrix}^T = \mathbf{e}_2$.

For the third vector, let $\mathbf{u}_3 = \mathbf{v}_3 - \sum\limits_{i=1}^{2}\langle \mathbf{v}_3, \mathbf{w}_i\rangle\mathbf{w}_i = (0,0,0)^T$. So, $\mathbf{v}_3 \in LS((\mathbf{w}_1, \mathbf{w}_2))$. Or equivalently, the set $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly dependent.

So, for again computing the third vector, define $\mathbf{u}_4 = \mathbf{v}_4 - \sum\limits_{i=1}^{2}\langle \mathbf{v}_4, \mathbf{w}_i\rangle\mathbf{w}_i$. Then, $\mathbf{u}_4 = \mathbf{v}_4 - \mathbf{w}_1 - \mathbf{w}_2 = \mathbf{e}_3$. So $\mathbf{w}_4 = \mathbf{e}_3$. Hence, $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

DRAFT

3. Find an orthonormal set in $\mathbb{R}^3$ containing $(1,2,1)^T$.

Solution: Let $(x,y,z)^T \in \mathbb{R}^3$ with $\big\langle (1,2,1),(x,y,z)\big\rangle = 0$. Thus,

$$(x,y,z) = (-2y-z, y, z) = y(-2,1,0) + z(-1,0,1).$$

Observe that $(-2,1,0)$ and $(-1,0,1)$ are orthogonal to $(1,2,1)$ but are themselves not orthogonal.

METHOD 1: Apply Gram-Schmidt process to $\{\frac{1}{\sqrt{6}}(1,2,1)^T, (-2,1,0)^T, (-1,0,1)^T\} \subseteq \mathbb{R}^3$.

METHOD 2: Valid only in $\mathbb{R}^3$ using the cross product of two vectors.

In either case, verify that $\{\frac{1}{\sqrt{6}}(1,2,1), \frac{-1}{\sqrt{5}}(2,-1,0), \frac{-1}{\sqrt{30}}(1,2,-5)\}$ is the required set.

We now state the following result without proof.

Corollary 4.5.3. *Let $\mathbb{V} \neq \{\mathbf{0}\}$ be a finite dimensional* IPS. *Then*

1. *$\mathbb{V}$ has an orthonormal basis.*
2. *any linearly independent set S in $\mathbb{V}$ can be extended to form an orthonormal basis of $\mathbb{V}$.*

Remark 4.5.4. *Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\} \neq \{\mathbf{0}\}$ be a non-empty subset of a finite dimensional vector space $\mathbb{V}$. Then, we observe the following.*

1. *If S is linearly independent then we obtain an orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\}$ of $LS(S)$.*

2. *If S is linearly dependent then as in Example 4.5.2.2, there will be stages at which the vector $\mathbf{u}_k = \mathbf{0}$. Thus, we will obtain an orthonormal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ of $LS(S)$, but note that $m < n$.*

3. *a re-arrangement of elements of S then we may obtain another orthonormal basis of $LS(\mathbf{v}_1, \ldots, \mathbf{v}_n)$. But, observe that the size of the two bases will be the same.*

EXERCISE **4.5.5.** 1. *Let $(\mathbb{V}, \langle\ ,\ \rangle)$ be an n-dimensional* IPS. *If $\mathbf{u} \in \mathbb{V}$ with $\|\mathbf{u}\| = 1$ then give reasons for the following statements.*

   (a) *Let $S^{\perp} = \{\mathbf{v} \in \mathbb{V} \mid \langle \mathbf{v}, \mathbf{u}\rangle = 0\}$. Then,* $\dim(S^{\perp}) = n - 1$.

   (b) *Let $0 \neq \beta \in \mathbb{F}$. Then $S = \{\mathbf{v} \in \mathbb{V} : \langle \mathbf{v}, \mathbf{u}\rangle = \beta\}$ is not a subspace of $\mathbb{V}$.*

   (c) *Let $\mathbf{v} \in \mathbb{V}$. Then $\mathbf{v} = \mathbf{v}_0 + \langle \mathbf{v}, \mathbf{u}\rangle\mathbf{u}$ for a vector $\mathbf{v}_0 \in S^{\perp}$. Thus $\mathbb{V} = LS(\mathbf{u}, S^{\perp})$.*

2. *Let $\mathbb{V}$ be an* IPS *with $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ as a basis. Then, prove that $\mathcal{B}$ is orthonormal if and only if for each $\mathbf{x} \in \mathbb{V}$, $\mathbf{x} = \sum\limits_{i=1}^{n} \langle \mathbf{x}, \mathbf{v}_i\rangle \mathbf{v}_i$. [Hint: Since $\mathcal{B}$ is a basis, each $\mathbf{x} \in \mathbb{V}$ has a unique linear combination in terms of $\mathbf{v}_i$'s.]*

3. *Let S be a subset of $\mathbb{V}$ having 101 elements. Suppose that the application of the Gram-Schmidt process yields $\mathbf{u}_5 = \mathbf{0}$. Does it imply that $LS(\mathbf{v}_1, \ldots, \mathbf{v}_5) = LS(\mathbf{v}_1, \ldots, \mathbf{v}_4)$? Give reasons for your answer.*

4. *Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be an orthonormal set in $\mathbb{R}^n$. For $1 \le k \le n$, define $A_k = \sum\limits_{i=1}^{k} \mathbf{v}_i \mathbf{v}_i^T$. Then prove that $A_k^T = A_k$ and $A_k^2 = A_k$. Thus, A_k's are projection matrices. Further, show that* $\mathsf{Rank}(A_k) = k$.

5. *Determine an orthonormal basis of $\mathbb{R}^4$ containing $(1, -2, 1, 3)^T$ and $(2, 1, -3, 1)^T$.*

DRAFT

6. *Let $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\| = 1$.*

   (a) *Then prove that $\{\mathbf{x}\}$ can be extended to form an orthonormal basis of $\mathbb{R}^n$.*

   (b) *Let the extended basis be $\{\mathbf{x}, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ and $\mathcal{B} = [\mathbf{e}_1, \ldots, \mathbf{e}_n]$ the standard ordered basis of $\mathbb{R}^n$. Prove that $A = \begin{bmatrix} [\mathbf{x}]_{\mathcal{B}}, & [\mathbf{x}_2]_{\mathcal{B}}, & \ldots, & [\mathbf{x}_n]_{\mathcal{B}} \end{bmatrix}$ is an orthogonal matrix.*

7. *Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n, n \ge 1$ with $\|\mathbf{u}\| = \|\mathbf{w}\| = 1$. Prove that there exists an orthogonal matrix P such that $P\mathbf{v} = \mathbf{w}$. Prove also that A can be chosen such that* $\det(P) = 1$.

4.6 QR Decomposition

In this section, we study the QR-decomposition of a matrix $A \in \mathbb{M}_n(\mathbb{R})$. The decomposition is obtained by applying the Gram-Schmidt Orthogonalization process to the columns of the matrix A. Thus, the set $\{A[:,1], \ldots, A[:,n]\}$ of the columns of A are taken as the collection of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

If $\mathsf{Rank}(A) = n$ then the columns of A are linearly independent and the application of the Gram-Schmidt process gives us vectors $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\} \subseteq \mathbb{R}^n$ such that the matrix $Q = \begin{bmatrix} \mathbf{w}_1 & \cdots & \mathbf{w}_n \end{bmatrix}$ is an orthogonal matrix. Further the condition

$$LS(A[:,1], \ldots, A[:,k]) = LS(\mathbf{w}_1, \ldots, \mathbf{w}_k), \qquad \text{for } 1 \le k \le n,$$

in the Gram-Schmidt process implies that $A[:,k] = LS(\mathbf{w}_1,\ldots,\mathbf{w}_k)$, for $1 \le k \le n$. Hence, there exist $\alpha_{jk} \in \mathbb{R}, 1 \le j \le k$, such that $A[:,k] = [\mathbf{w}_1,\ldots,\mathbf{w}_k]\begin{bmatrix}\alpha_{1k}\\ \vdots\\ \alpha_{kk}\end{bmatrix}$. Thus $A = QR$, where

$$Q = \begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_n\end{bmatrix} \text{ and } R = \begin{bmatrix}\alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n}\\ 0 & \alpha_{22} & \cdots & \alpha_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \alpha_{nn}\end{bmatrix}.$$

This decomposition is stated next.

Theorem 4.6.1 (QR Decomposition)**.** *Let $A \in \mathbb{M}_n(\mathbb{R})$ be a matrix with $\mathsf{Rank}(A) = n$. Then, there exist matrices Q and R such that Q is orthogonal and R is upper triangular with $A = QR$. Furthermore, the diagonal entries of R can be chosen to be positive. Also, in this case, the decomposition is unique.*

Proof. The argument before the statement of the theorem gives us $A = QR$, with

1. Q being an orthogonal matrix (see Exercise 5.8.8.5) and
2. R being an upper triangular matrix.

Thus, this completes the proof of the first part. Note that

1. $\alpha_{ii} \neq 0$, for $1 \le i \le n$, as $A[:,1] \neq \mathbf{0}$ and $A[:,i] \notin LS(\mathbf{w}_1,\ldots,\mathbf{w}_{i-1})$ as A has full column rank.
2. if $\alpha_{ii} < 0$, for some $i, 1 \le i \le n$ then we can replace $\mathbf{v}_i$ in Q by $-\mathbf{v}_i$ to get new matrices Q and R with the added condition that the diagonal entries of R are positive.

Uniqueness: Suppose $Q_1R_1 = Q_2R_2$ for some orthogonal matrices Q_i's and upper triangular matrices R_i's with positive diagonal entries. As Q_i's and R_i's are invertible, we get $Q_2^{-1}Q_1 = R_2R_1^{-1}$. As product of upper triangular matrices is also upper triangular (see Exercise 2) the matrix $R_2R_1^{-1}$ is an upper triangular matrix. Similarly, $Q_2^{-1}Q_1$ is an orthogonal matrix.

So, the matrix $R_2R_1^{-1}$ is an orthogonal upper triangular matrix. Hence $R_2R_1^{-1} = I_n$. So, $R_2 = R_1$ and therefore $Q_2 = Q_1$. ■

Remark 4.6.2. *Note that in the proof of Theorem 4.6.1, we just used the idea that $A[:,i] \in LS(\mathbf{w}_1,\ldots,\mathbf{w}_i)$ to get the scalars α_{ji}, for $1 \le j \le i$. As $\{\mathbf{w}_1,\ldots,\mathbf{w}_i\}$ is an orthonormal set*

$$\alpha_{ji} = \langle A[:,i], \mathbf{w}_j\rangle, \text{ for } 1 \le j \le i.$$

So, it is quite easy to compute the entries of the upper triangular matrix R.

Now, let A be an $m \times n$ matrix with $\text{Rank}(A) = r$. Then, by Remark 4.5.4, we obtain an orthonormal set $\{\mathbf{w}_1,\ldots,\mathbf{w}_r\} \subseteq \mathbb{R}^n$ such that

$$LS(A[:,1],\ldots,A[:,j]) = LS(\mathbf{w}_1,\ldots,\mathbf{w}_i), \quad \text{for } 1 \le i \le j \le n.$$

Hence, proceeding on the lines of the above theorem, one has the following result.

DRAFT

Theorem 4.6.3 (Generalized QR Decomposition)**.** *Let A be an $m \times n$ matrix with* $\mathsf{Rank}(A) = r$. *Then $A = QR$, where*

1. $Q = [\mathbf{w}_1, \ldots, \mathbf{w}_r]$ *is an $m \times r$ matrix with $Q^T Q = I_r$,*
2. $LS(A[:,1], \ldots, A[:,j]) = LS(\mathbf{v}_1, \ldots, \mathbf{v}_i)$*, for $1 \le i \le j \le n$ and*
3. *R is an $r \times n$ matrix with $Rank(R) = r$.*

We look at a few examples to understand it better.

Example 4.6.4. 1. Let $A = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Find an orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.

Solution: From Example 4.5.2, we know that $\mathbf{w}_1 = \frac{1}{\sqrt{2}}(1,0,1,0)^T$, $\mathbf{w}_2 = \frac{1}{\sqrt{2}}(0,1,0,1)^T$ and $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0,-1,0,1)^T$. We now compute $\mathbf{w}_4$. If $\mathbf{v}_4 = (2,1,1,1)^T$ then

$$\mathbf{u}_4 = \mathbf{v}_4 - \langle \mathbf{v}_4, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{v}_4, \mathbf{w}_2 \rangle \mathbf{w}_2 - \langle \mathbf{v}_4, \mathbf{w}_3 \rangle \mathbf{w}_3 = \frac{1}{2}(1,0,-1,0)^T.$$

Thus, $\mathbf{w}_4 = \frac{1}{\sqrt{2}}(-1,0,1,0)^T$. Hence, we see that $A = QR$ with

$$Q = \begin{bmatrix} \mathbf{w}_1, \ldots, \mathbf{w}_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} \text{ and } R = \begin{bmatrix} \sqrt{2} & 0 & \sqrt{2} & -\frac{3}{\sqrt{2}} \\ 0 & \sqrt{2} & 0 & -\sqrt{2} \\ 0 & 0 & \sqrt{2} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

DRAFT

2. Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -1 & 0 & -2 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$. Find a 4×3 matrix Q satisfying $Q^T Q = I_3$ and an upper triangular matrix R such that $A = QR$.

Solution: Let us apply the Gram-Schmidt orthonormalization process to the columns of A. As $\mathbf{v}_1 = (1,-1,1,1)^T$, we get $\mathbf{w}_1 = \frac{1}{2}\mathbf{v}_1$. Let $\mathbf{v}_2 = (1,0,1,0)^T$. Then,

$$\mathbf{u}_2 = \mathbf{v}_2 - \langle \mathbf{v}_2, \mathbf{w}_1 \rangle \mathbf{w}_1 = (1,0,1,0)^T - \mathbf{w}_1 = \frac{1}{2}(1,1,1,-1)^T.$$

Hence, $\mathbf{w}_2 = \frac{1}{2}(1,1,1,-1)^T$. Let $\mathbf{v}_3 = (1,-2,1,2)^T$. Then,

$$\mathbf{u}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{v}_3, \mathbf{w}_2 \rangle \mathbf{w}_2 = \mathbf{v}_3 - 3\mathbf{w}_1 + \mathbf{w}_2 = \mathbf{0}.$$

So, we again take $\mathbf{v}_3 = (0,1,0,1)^T$. Then,

$$\mathbf{u}_3 = \mathbf{v}_3 - \langle \mathbf{v}_3, \mathbf{w}_1 \rangle \mathbf{w}_1 - \langle \mathbf{v}_3, \mathbf{w}_2 \rangle \mathbf{w}_2 = \mathbf{v}_3 - 0\mathbf{w}_1 - 0\mathbf{w}_2 = \mathbf{v}_3.$$

So, $\mathbf{w}_3 = \frac{1}{\sqrt{2}}(0,1,0,1)^T$. Hence,

$$Q = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{-1}{2} & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{-1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \text{ and } R = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

The readers are advised to check the following:

(a) $\text{Rank}(A) = 3$,

(b) $A = QR$ with $Q^T Q = I_3$, and

(c) R is a 3×4 upper triangular matrix with $\text{Rank}(R) = 3$.

Remark 4.6.5. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ with Rank$(A) = n$.*

1. *Then $A^T A$ is invertible (see Exercise 3.6.9.3).*
2. *By Theorem 4.6.3, there exist matrices $Q \in \mathbb{M}_{m,n}(\mathbb{R})$ and $R \in \mathbb{M}_{n,n}(\mathbb{R})$ such that $A = QR$.*
3. *Further, the columns of Q form an orthonormal set and hence $Q^T Q = I_n$.*
4. *Furthermore, Rank$(R) = n$ as Rank$(A) = n$. Thus R is invertible. Hence $R^T R$ is invertible and $(R^T R)^{-1} = R^{-1}(R^T)^{-1}$.*
5. *So, if $Q = [\mathbf{w}_1, \ldots, \mathbf{w}_n]$ then*
$$A(A^T A)^{-1} A^T = QR(R^T R)^{-1} R^T Q^T = (QR)(R^{-1}(R^T)^{-1}) R^T Q^T = QQ^T.$$
6. *Thus $P = A(A^T A)^{-1} A^T = QQ^T = [\mathbf{w}_1, \ldots, \mathbf{w}_n] \begin{bmatrix} \mathbf{w}_1^T \\ \vdots \\ \mathbf{w}_n^T \end{bmatrix} = \sum\limits_{i=1}^{n} \mathbf{w}_i \mathbf{w}_i^T$ is the projection matrix that projects on $\text{Col}(A)$ (see Exercise 4.5.5.4).*

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4.7 Summary

In the previous chapter, we learnt that if $\mathbb{V}$ is vector space over $\mathbb{F}$ with $\dim(\mathbb{V}) = n$ then $\mathbb{V}$ basically looks like $\mathbb{F}^n$. Also, any subspace of $\mathbb{F}^n$ is either $\text{Col}(A)$ or $\text{Null}(A)$ or both, for some matrix A with entries from $\mathbb{F}$.

So, we started this chapter with inner product, a generalization of the dot product in $\mathbb{R}^3$ or $\mathbb{R}^n$. We used the inner product to define the length/norm of a vector. The norm has the property that "the norm of a vector is zero if and only if the vector itself is the zero vector". We then proved the Cauchy-Schwartz Inequality which helped us in defining the angle between two vector. Thus, one can talk of geometrical problems in $\mathbb{R}^n$ and proved some geometrical results.

We then independently defined the notion of a norm in $\mathbb{R}^n$ and showed that a norm is induced by an inner product if and only if the norm satisfies the parallelogram law (sum of squares of the diagonal equals twice the sum of square of the two non-parallel sides).

The next subsection dealt with the fundamental theorem of linear algebra where we showed that if $A \in \mathbb{M}_{m,n}(\mathbb{C})$ then

1. $\dim(\text{Null}(A)) + \dim(\text{Col}(A)) = n$.
2. $\text{Null}(A) = \big(\text{Col}(A^*)\big)^\perp$ and $\text{Null}(A^*) = \big(\text{Col}(A)\big)^\perp$.
3. $\dim(\text{Col}(A)) = \dim(\text{Col}(A^*))$.

We then saw that having an orthonormal basis is an asset as determining the coordinates of a vector boils down to computing the inner product.

So, the question arises, how do we compute an orthonormal basis? This is where we came across the Gram-Schmidt Orthonormalization process. This algorithm helps us to determine an orthonormal basis of $LS(S)$ for any finite subset S of a vector space. This also lead to the QR-decomposition of a matrix.

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Chapter 5

Linear Transformations

5.1 Definitions and Basic Properties

Recall that understanding functions, their domain, co-domain and their properties, such as one-one, onto etc. played an important role. So, in this chapter, we study functions over vector spaces that preserve the operations of vector addition and scalar multiplication.

Definition 5.1.1. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ with vector operations $+, \cdot$ in $\mathbb{V}$ and $\oplus, \odot$ in $\mathbb{W}$. A function (map) $f : \mathbb{V} \rightarrow \mathbb{W}$ is called a **linear transformation** if for all $\alpha \in \mathbb{F}$ and $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ the function f satisfies

$$f(\alpha \cdot \mathbf{u}) = \alpha \odot f(\mathbf{u}) \text{ and } f(\mathbf{u} + \mathbf{v}) = f(\mathbf{u}) \oplus f(\mathbf{v}). \tag{5.1.1}$$

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By $\mathcal{L}(\mathbb{V}, \mathbb{W})$, we denote the set of all linear transformations from $\mathbb{V}$ to $\mathbb{W}$. In particular, if $\mathbb{W} = \mathbb{V}$ then the linear transformation f is called a **linear operator** and the corresponding set of linear operators is denoted by $\mathcal{L}(\mathbb{V})$.

Even though, in the definition above, we have differentiated between the vector addition and scalar multiplication for domain and co-domain, we will not differentiate them in the book unless necessary.

Equation (5.1.1) just states that the two operations, namely, taking the image (apply f) and doing 'vector space operations (vector addition and scalar multiplication) commute, *i.e.*, first apply vector operations ($\mathbf{u} + \mathbf{v}$ or $\alpha \mathbf{v}$) and then look at their images $f(\mathbf{u} + \mathbf{v})$ or $f(\alpha \mathbf{v})$) is same as first computing the images $(f(\mathbf{u}), f(\mathbf{v}))$ and then compute vector operations ($f(\mathbf{u}) + f(\mathbf{v})$ and $\alpha f(\mathbf{v})$). Or equivalently, we look at only those functions which preserve vector operations.

Definition 5.1.2. Let $g, h \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then g and h are said to be **equal** if $g(\mathbf{x}) = h(\mathbf{x})$, for all $\mathbf{x} \in \mathbb{V}$.

We now give examples of linear transformations.

Example 5.1.3. 1. Let $\mathbb{V}$ be a vector space. Then, the maps $\mathrm{Id}, \mathbf{0} \in \mathcal{L}(\mathbb{V})$, where

(a) $\mathrm{Id}(\mathbf{v}) = \mathbf{v}$, for all $\mathbf{v} \in \mathbb{V}$, is commonly called the **identity operator**.

(b) $\mathbf{0}(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v} \in \mathbb{V}$, is commonly called the **zero operator**.

2. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$. Then, $\mathbf{0} \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, where $\mathbf{0}(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v} \in \mathbb{V}$, is commonly called the **zero transformation**.

3. The map $f(\mathbf{x}) = 7\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}$, is an element of $\mathcal{L}(\mathbb{R})$ as
$$f(a\mathbf{x}) = 7(a\mathbf{x}) = a(7\mathbf{x}) = af(\mathbf{x}) \text{ and } f(\mathbf{x}+\mathbf{y}) = 7(\mathbf{x}+\mathbf{y}) = 7\mathbf{x} + 7\mathbf{y} = f(\mathbf{x}) + f(\mathbf{y}).$$

4. Let $\mathbb{V}, \mathbb{W}$ and $\mathbb{Z}$ be vector spaces over $\mathbb{F}$. Then, for any $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ and $S \in \mathcal{L}(\mathbb{W}, \mathbb{Z})$, the map $S \circ T \in \mathcal{L}(\mathbb{V}, \mathbb{Z})$, defined by $(S \circ T)(\mathbf{v}) = S\big(T(\mathbf{v})\big)$ for all $\mathbf{v} \in \mathbb{V}$, is called the **composition** of maps. Observe that for each $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ and $\alpha, \beta \in \mathbb{R}$,
$$\begin{aligned} (S \circ T)(\alpha\mathbf{v} + \beta\mathbf{u}) &= S\big(T(\alpha\mathbf{v} + \beta\mathbf{u})\big) = S\big(\alpha T(\mathbf{v}) + \beta f(\mathbf{u})\big) \\ &= \alpha S\big(T(\mathbf{v})\big) + \beta S\big(T(\mathbf{u})\big) = \alpha(S \circ T)(\mathbf{v}) + \beta(S \circ T)(\mathbf{u}). \end{aligned}$$
Hence $S \circ T$, in short ST, is an element of $\mathcal{L}(\mathbb{V}, \mathbb{Z})$.

5. Fix $\mathbf{a} \in \mathbb{R}^n$ and define $f(\mathbf{x}) = \mathbf{a}^T\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$. Then $f \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$. In particular, if $\mathbf{x} = [x_1, \ldots, x_n]^T$ then, for all $\mathbf{x} \in \mathbb{R}^n$,
   (a) $f(\mathbf{x}) = \sum\limits_{i=1}^{n} \mathbf{x}_i = \mathbf{1}^T\mathbf{x}$ is a linear transformation.
   (b) $f_i(\mathbf{x}) = x_i = \mathbf{e}_i^T\mathbf{x}$ is a linear transformation, for $1 \le i \le n$.

6. Define $f : \mathbb{R}^2 \to \mathbb{R}^3$ by $f\left((x, y)^T\right) = (x + y, 2x - y, x + 3y)^T$. Then $f \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$. Here $f(\mathbf{e}_1) = (1, 2, 1)^T$ and $f(\mathbf{e}_2) = (1, -1, 3)^T$.

7. Fix $A \in M_{m \times n}(\mathbb{C})$. Define $f_A(\mathbf{x}) = A\mathbf{x}$, for every $\mathbf{x} \in \mathbb{C}^n$. Then, $f_A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$. Thus, for each $A \in \mathbb{M}_{m,n}(\mathbb{C})$, there exists a linear transformation $f_A \in \mathcal{L}(\mathbb{C}^n, \mathbb{C}^m)$.

8. Define $f : \mathbb{R}^{n+1} \to \mathbb{R}[x; n]$ by $f\left((a_1, \ldots, a_{n+1})^T\right) = a_1 + a_2x + \cdots + a_{n+1}x^n$, for each $(a_1, \ldots, a_{n+1}) \in \mathbb{R}^{n+1}$. Then f is a linear transformation.

9. Fix $A \in M_n(\mathbb{C})$. Now, define $f_A : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ and $g_A : M_n(\mathbb{C}) \to \mathbb{C}$ by
$$f_A(B) = AB \text{ and } g_A(B) = \operatorname{tr}(AB), \text{ for every } B \in M_n(\mathbb{C}).$$
Then f_A and g_A are both linear transformations.

10. Is the map $T : \mathbb{R}[x; n] \to \mathbb{R}[x; n+1]$ defined by $T(f(x)) = xf(x)$, for all $f(x) \in \mathbb{R}[x; n]$ a linear transformation?

11. The maps $T, S : \mathbb{R}[x] \to \mathbb{R}[x]$ defined by $T(f(x)) = \frac{d}{dx}f(x)$ and $S(f(x)) = \int\limits_0^x f(t)dt$, for all $f(x) \in \mathbb{R}[x]$ are linear transformations. Is it true that $TS = \mathrm{Id}$? What about ST?

12. Recall the vector space $\mathbb{R}^{\mathbb{N}}$ in Example 3.1.4.7. Now, define maps $T, S : \mathbb{R}^{\mathbb{N}} \to \mathbb{R}^{\mathbb{N}}$ by $T(\{a_1, a_2, \ldots\}) = \{0, a_1, a_2, \ldots\}$ and $S(\{a_1, a_2, \ldots\}) = \{a_2, a_3, \ldots\}$. Then, T and S, commonly called the **shift operators**, are linear operators with exactly one of ST or TS as the Id map.

DRAFT

13. Recall the vector space $\mathcal{C}(\mathbb{R},\mathbb{R})$ (see Example 3.1.4.9). Define $T : \mathcal{C}(\mathbb{R},\mathbb{R}) \to \mathcal{C}(\mathbb{R},\mathbb{R})$ by $T(f)(x) = \int\limits_0^x f(t)dt$. For example, $T(\sin)(x) = \int\limits_0^x \sin(t)dt = 1-\cos(x)$, for all $x \in \mathbb{R}$. Then, verify that T is a linear transformation.

Remark 5.1.4. *Let $A \in M_n(\mathbb{C})$ and define $T_A : \mathbb{C}^n \to \mathbb{C}^n$ by $T_A(\mathbf{x}) = A\mathbf{x}$, for every $\mathbf{x} \in \mathbb{C}^n$. Then, verify that $T_A^k(\mathbf{x}) = \underbrace{(T_A \circ T_A \circ \cdots \circ T_A)}_{k \text{ times}}(\mathbf{x}) = A^k\mathbf{x}$, for any positive integer k.*

Exercise **5.1.5.** *Fix $A \in M_n(\mathbb{C})$. Then, do the following maps define linear transformations?*

1. *Define $f, g : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ by $f(B) = A^*B$ and $g(B) = BA$, for every $B \in M_n(\mathbb{C})$.*
2. *Define $h, t : M_n(\mathbb{C}) \to \mathbb{C}$ by $h(B) = \mathsf{tr}(\mathsf{A}^*\mathsf{B})$ and $t(B) = \mathsf{tr}(\mathsf{BA})$, for every $B \in M_n(\mathbb{C})$.*

We now prove that any linear transformation sends the zero vector to a zero vector.

Proposition 5.1.6. *Let $T \in \mathcal{L}(\mathbb{V},\mathbb{W})$. Suppose that $\mathbf{0}_{\mathbb{V}}$ is the zero vector in $\mathbb{V}$ and $\mathbf{0}_{\mathbb{W}}$ is the zero vector of $\mathbb{W}$. Then $T(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$.*

Proof. Since $\mathbf{0}_{\mathbb{V}} = \mathbf{0}_{\mathbb{V}} + \mathbf{0}_{\mathbb{V}}$, we get $T(\mathbf{0}_{\mathbb{V}}) = T(\mathbf{0}_{\mathbb{V}} + \mathbf{0}_{\mathbb{V}}) = T(\mathbf{0}_{\mathbb{V}}) + T(\mathbf{0}_{\mathbb{V}})$. As $T(\mathbf{0}_{\mathbb{V}}) \in \mathbb{W}$,

$$\mathbf{0}_{\mathbb{W}} + T(\mathbf{0}_{\mathbb{V}}) = T(\mathbf{0}_{\mathbb{V}}) = T(\mathbf{0}_{\mathbb{V}}) + T(\mathbf{0}_{\mathbb{V}}).$$

Hence $T(\mathbf{0}_{\mathbb{V}}) = \mathbf{0}_{\mathbb{W}}$. ∎

DRAFT

From now on $\mathbf{0}$ will be used as the zero vector of the domain and co-domain. We now consider a few more examples for better understanding.

Example 5.1.7.

1. Does there exist a linear transformation $T : \mathbb{V} \to \mathbb{W}$ such that $T(\mathbf{v}) \neq \mathbf{0}$, for all $\mathbf{v} \in \mathbb{V}$?
   Solution: No, as $T(\mathbf{0}) = \mathbf{0}$ (see Proposition 5.1.6).
2. Does there exist a linear transformation $T : \mathbb{R} \to \mathbb{R}$ such that $T(x) = x^2$, for all $x \in \mathbb{R}$?
   Solution: No, as $T(ax) = (ax)^2 = a^2x^2 = a^2T(x) \neq aT(x)$, unless $a = 0, 1$.
3. Does there exist a linear transformation $T : \mathbb{R} \to \mathbb{R}$ such that $T(5) = 10$ and $T(10) = 5$?
   Solution: No, as $T(10) = T(5+5) = T(5) + t(5) = 10 + 10 = 20 \neq 5$.
4. Does there exist a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f((x,y)^T) = (x+y, 2)^T$?
   Solution: No, as $f(\mathbf{0}) \neq \mathbf{0}$.
5. Does there exist a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ such that $f((x,y)^T) = (x+y, xy)^T$?
   Solution: No, as $f((2,2)^T) = (4,4)^T \neq 2(2,1)^T = 2f((1,1)^T)$.
6. Define a map $T : \mathbb{C} \to \mathbb{C}$ by $T(\mathbf{z}) = \overline{\mathbf{z}}$, the complex conjugate of z. Is T a linear operator over the real vector space $\mathbb{R}$?
   Solution: Yes, as for any $\alpha \in \mathbb{R}$, $T(\alpha\mathbf{z}) = \overline{\alpha\mathbf{z}} = \alpha\overline{\mathbf{z}} = \alpha T(\mathbf{z})$.

We now define the range space.

Definition 5.1.8. Let $f \in \mathcal{L}(\mathbb{V},\mathbb{W})$. Then the **range/ image** of f, denoted $\textsc{Rng}(f)$ or $\textsc{Im}(f)$, is given by $\textsc{Rng}(f) = \{f(\mathbf{x}) : \mathbf{x} \in \mathbb{V}\}$.

As an exercise, show that $\textsc{Rng}(f)$ is a subspace of $\mathbb{W}$. The next result, which is a very important result, states that a linear transformation is known if we know its image on a basis of the domain space.

Lemma 5.1.9. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ with $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots\}$ as a basis of $\mathbb{V}$. If $f \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ then T is determined if we know the set $\{f(\mathbf{v}_1), f(\mathbf{v}_2), \ldots\}$, i.e., if we know the image of f on the basis vectors of $\mathbb{V}$, or equivalently, $\textsc{Rng}(f) = LS(f(\mathbf{x}) | \mathbf{x} \in \mathcal{B})$.*

Proof. Let $\mathcal{B}$ be a basis of $\mathbb{V}$ over $\mathbb{F}$. Then, for each $\mathbf{v} \in \mathbb{V}$, there exist vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ in $\mathcal{B}$ and scalars $c_1, \ldots, c_k \in \mathbb{F}$ such that $\mathbf{v} = \sum\limits_{i=1}^{k} c_i \mathbf{u}_i$. Thus

$$T(\mathbf{v}) = f\left(\sum_{i=1}^{k} c_i \mathbf{u}_i\right) = \sum_{i=1}^{k} f(c_i \mathbf{u}_i) = \sum_{i=1}^{k} c_i T(\mathbf{u}_i).$$

Or equivalently, whenever

$$\mathbf{v} = [\mathbf{u}_1, \ldots, \mathbf{u}_k] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \text{ then } f(\mathbf{v}) = \begin{bmatrix} f(\mathbf{u}_1) & \cdots & f(\mathbf{u}_k) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}. \tag{5.1.2}$$

Thus, the image of f on $\mathbf{v}$ just depends on where the basis vectors are mapped. Equation 5.1.2 also shows that $\textsc{Rng}(f) = \{f(\mathbf{x}) : \mathbf{x} \in \mathbb{V}\} = LS(f(\mathbf{x}) | \mathbf{x} \in \mathcal{B})$. ∎

DRAFT

Example 5.1.10. Determine $\textsc{Rng}(T)$ of the following linear transformations.

1. $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^4)$, where $f((x, y, z)^T) = (x - y + z, y - z, x, 2x - 5y + 5z)^T$.
   Solution: Consider the standard basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ of $\mathbb{R}^3$. Then

$$\begin{aligned} \textsc{Rng}(f) &= LS(f(\mathbf{e}_1), T(\mathbf{e}_2), T(\mathbf{e}_3)) = LS\big((1,0,1,2)^T, (-1,1,0,-5)^T, (1,-1,0,5)^T\big) \\ &= LS\big((1,0,1,2)^T, (1,-1,0,5)^T\big) = \{\lambda(1,0,1,2)^T + \beta(1,-1,0,5)^T \mid \lambda, \beta \in \mathbb{R}\} \\ &= \{(\lambda + \beta, -\beta, \lambda, 2\lambda + 5\beta) \ : \lambda, \beta \in \mathbb{R}\} \\ &= \{(x, y, z, w)^T \in \mathbb{R}^4 \mid x + y - z = 0, 5y - 2z + w = 0\}, \end{aligned}$$

2. Let $B \in \mathbb{M}_2(\mathbb{R})$. Now, define a map $T : \mathbb{M}_2(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R})$ by $T(A) = BA - AB$, for all $A \in \mathbb{M}_2(\mathbb{R})$. Determine $\textsc{Rng}(T)$ and $\textsc{Null}(T)$.
   Solution: Recall that $\{\mathbf{e}_{ij} | 1 \le i, j \le 2\}$ is a basis of $\mathbb{M}_2(\mathbb{R})$. So,
   (a) if $B = cI_2$ then $\textsc{Rng}(T) = \{\mathbf{0}\}$.
   (b) if $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ then $T(\mathbf{e}_{11}) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$, $T(\mathbf{e}_{12}) = \begin{bmatrix} -2 & -3 \\ 0 & 2 \end{bmatrix}$, $T(\mathbf{e}_{21}) = \begin{bmatrix} 2 & 0 \\ 3 & -2 \end{bmatrix}$ and $T(\mathbf{e}_{22}) = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$. Thus, $\textsc{Rng}(T) = LS\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ -3 & 2 \end{bmatrix}\right)$.
   (c) for $B = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, verify that $\textsc{Rng}(T) = LS\left(\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -2 & 0 \\ -2 & 2 \end{bmatrix}\right)$.

Recall that by Example 5.1.3.5, for each $\mathbf{a} \in \mathbb{R}^n$, the map $T(\mathbf{x}) = \mathbf{a}^T\mathbf{x}$, for each $\mathbf{x} \in \mathbb{R}^n$, is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}$. We now show that these are the only ones.

Corollary 5.1.11. **[Reisz Representation Theorem]** *Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$. Then, there exists $\mathbf{a} \in \mathbb{R}^n$ such that $T(\mathbf{x}) = \mathbf{a}^T\mathbf{x}$.*

Proof. By Lemma 5.1.9, T is known if we know the image of T on $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, the standard basis of $\mathbb{R}^n$. So, for $1 \le i \le n$, let $T(\mathbf{e}_i) = a_i$, for some $a_i \in \mathbb{R}$. Now define $\mathbf{a} = [a_1, \ldots, a_n]^T$ and $\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$. Then, for all $\mathbf{x} \in \mathbb{R}^n$,

$$T(\mathbf{x}) = T\left(\sum_{i=1}^n x_i\mathbf{e}_i\right) = \sum_{i=1}^n x_i T(\mathbf{e}_i) = \sum_{i=1}^n x_i a_i = \mathbf{a}^T\mathbf{x}.$$

Thus, the required result follows. ∎

Example 5.1.12. In each of the examples given below, state whether a linear transformation exists or not. If yes, give at least one linear transformation. If not, then give the condition due to which a linear transformation doesn't exist.

1. Can we construct a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T((1,1)^T) = (e,2)^T$ and $T((2,1)^T) = (5,4)^T$?

   Solution: The first thing that we need to answer is "is the set $\{(1,1),(2,1)\}$ linearly independent"? The answer is 'Yes'. So, we can construct it. So, how do we do it?

   We now need to write any vector $\begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ as by definition of linear transformation

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= T\left(\alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \beta\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \alpha T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) + \beta T\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \alpha\begin{bmatrix} e \\ 2 \end{bmatrix} + \beta\begin{bmatrix} 5 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} e & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} e & 5 \\ 2 & 4 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\right)^{-1}\begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} e & 5 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} (5-e)x + (2e-5)y \\ 2x \end{bmatrix}. \end{aligned}$$

2. $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T((1,1)^T) = (1,2)^T$ and $T((1,-1)^T) = (5,10)^T$?

   Solution: Yes, as the set $\{(1,1),(1,-1)\}$ is a basis of $\mathbb{R}^2$. Write $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Then,

$$\begin{aligned} T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= T\left((BB^{-1})\begin{bmatrix} x \\ y \end{bmatrix}\right) = T\left(B\left(B^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) \\ &= \left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), \; T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right]\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 5 \\ 2 & 10 \end{bmatrix}\begin{bmatrix} \frac{x+y}{2} \\ \frac{x-y}{2} \end{bmatrix} = \begin{bmatrix} 3x-2y \\ 6x-4y \end{bmatrix}. \end{aligned}$$

DRAFT

3. $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T((1,1)^T) = (1,2)^T$ and $T((5,5)^T) = (5,11)^T$?
   Solution: Note that the set $\{(1,1),(5,5)\}$ is linearly dependent. Further, $(5,11)^T = T((5,5)^T) = 5T((1,1)^T)5(1,2)^T = (5,10)^T$ gives us a contradiction. Hence, there is no such linear transformation.

4. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1,1,1) = (1,2), T(1,2,3) = (4,3)$ and $T(2,3,4) = (7,8)$?
   Solution: Here, the set $\{(1,1,1),(1,2,3),(2,3,4)\}$ is linearly dependent and $(2,3,4) = (1,1,1) + (1,2,3)$. So, we need $T((2,3,4)) = T((1,1,1)+(1,2,3)) = T((1,1,1)) + T((1,2,3)) = (1,2)+(4,3) = (5,5)$. But, we are given $T(2,3,4) = (7,8)$, a contradiction. So, such a linear transformation doesn't exist.

5. $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $T((1,1)^T) = (1,2)^T$ and $T((5,5)^T) = (5,10)^T$?
   Solution: Yes, as $(5,10)^T = T((5,5)^T) = 5T((1,1)^T) = 5(1,2)^T = (5,10)^T$.

   To construct one such linear transformation, note that $\{(1,1)^T,(1,0)^T\}$ is a basis of $\mathbb{R}^2$. Let $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and pick $\mathbf{v} \in \mathbb{R}^2$. Now define $T((1,0)^T) = \mathbf{v} = (v_1,v_2)^T$. Then, as in the previous example, note that $B^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x-y \end{bmatrix}$ and hence

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right), T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)\right] B^{-1}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 & v_1 \\ 2 & v_2 \end{bmatrix}\begin{bmatrix} y \\ x-y \end{bmatrix} = y\begin{bmatrix} 1 \\ 2 \end{bmatrix} + (x-y)\mathbf{v}.$$

6. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ with $T(1,1,1) = (1,2), T(1,2,3) = (4,3)$ and $T(2,3,4) = (5,5)$?
   Solution: As $(2,3,4) = (1,1,1)+(1,2,3)$ and $T((2,3,4)) = T((1,1,1)+(1,2,3)) = T((1,1,1)) + T((1,2,3))$, such a linear transformation exists. To get the linear transformation, get a basis, namely $\{1,1,1),(1,2,3),\mathbf{e}_1\}$, of $\mathbb{R}^3$. Note that this basis contains $(1,1,1)$ and $(1,2,3)$. Now, define $T(\mathbf{e}_1)$ as any vector of $\mathbb{R}^2$. This give us a linear transformation satisfying the given condition.

7. $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\textsc{Null}(T) = \{\mathbf{x} \in \mathbb{R}^2 \mid T(\mathbf{x}) = \mathbf{0}\} = LS\{(1,\pi)^T\}$?
   Solution: Yes. Take $\{(1,\pi)^T, \mathbf{u}\}$ as a basis of $\mathbb{R}^2$ and define $T((1,\pi)^T) = \mathbf{0}$ and $T(\mathbf{u}) = \mathbf{u}$.

8. $T : \mathbb{M}_2(\mathbb{R}) \to \mathbb{R}^3$ such that $\textsc{Null}(T) = \{\mathbf{x} \in \mathbb{M}_2(\mathbb{R}) \mid T(\mathbf{x}) = \mathbf{0}\} = LS\left\{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right\}$?

   Solution: Yes. Take $\left\{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{21}\right\}$ as a basis of $\mathbb{M}_2(\mathbb{R})$ and define

$$T(\mathbf{e}_{11}) = \mathbf{e}_1, T(\mathbf{e}_{12}) = \mathbf{e}_2, T(\mathbf{e}_{21}) = \mathbf{e}_3 \text{ and } T\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\right) = \mathbf{0}.$$

EXERCISE **5.1.13.** 1. *Use matrices to construct linear operators $T, S : \mathbb{R}^3 \to \mathbb{R}^3$ that satisfy:*

(a) $T \neq \mathbf{0}$, $T \circ T = T^2 \neq \mathbf{0}$, $T \circ T \circ T = T^3 = \mathbf{0}$.

(b) $T \neq \mathbf{0}$, $S \neq \mathbf{0}$, $S \circ T = ST \neq \mathbf{0}$, $T \circ S = TS = \mathbf{0}$.

DRAFT

(c) $S \circ S = S^2 = T^2 = T \circ T, \;\; S \neq T$.

(d) $T \circ T = T^2 = \; Id, \;\; T \neq \; Id$.

2. Fix a positive integer p and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear operator with $T^k \neq \mathbf{0}$ for $1 \leq k \leq p$ and $T^{p+1} = \mathbf{0}$. Then prove that there exists a vector $\mathbf{x} \in \mathbb{R}^n$ such that the set $\{\mathbf{x}, T(\mathbf{x}), \ldots, T^p(\mathbf{x})\}$ is linearly independent.

3. Fix $\mathbf{x}_0 \in \mathbb{R}^n$ with $\mathbf{x}_0 \neq \mathbf{0}$. Now, define $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ by $T(\mathbf{x}_0) = \mathbf{y}_0$, for some $\mathbf{y}_0 \in \mathbb{R}^m$. Define $T^{-1}(\mathbf{y}_0) = \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{y}_0\}$. Then prove that $\mathbf{x} \in T^{-1}(\mathbf{y}_0)$ if and only if $\mathbf{x} - \mathbf{x}_0 \in T^{-1}(\mathbf{0})$. Further, $T^{-1}(\mathbf{y}_0)$ is a subspace of $\mathbb{R}^n$ if and only if $\mathbf{y}_0 = \mathbf{0}$.

4. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$. If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{V}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_n\} \subseteq \mathbb{W}$ then prove that there exists a unique $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ with $T(\mathbf{v}_i) = \mathbf{w}_i$, for $i = 1, \ldots, n$.

5. Let $\mathbb{V}$ be a vector space and let $\mathbf{a} \in \mathbb{V}$. Then the map $T_{\mathbf{a}} : \mathbb{V} \to \mathbb{V}$ defined by $T_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}$, for all $\mathbf{x} \in \mathbb{V}$ is called the **translation** map. Prove that $T_{\mathbf{a}} \in \mathcal{L}(\mathbb{V})$ if and only if $\mathbf{a} = \mathbf{0}$.

6. Prove that there exists infinitely many linear transformations $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $T((1,-1,1)^T) = (1,2)^T$ and $T((-1,1,2)^T) = (1,0)^T$?

7. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that

(a) $T((1,0,1)^T) = (1,2)^T$, $T((0,1,1)^T) = (1,0)^T$ and $T((1,1,1)^T) = (2,3)^T$?

(b) $T((1,0,1)^T) = (1,2)^T$, $T((0,1,1)^T) = (1,0)^T$ and $T((1,1,2)^T) = (2,3)^T$?

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8. Find $T \in \mathcal{L}(\mathbb{R}^3)$ for which $\textsc{Rng}(T) = LS\big((1,2,0)^T, (0,1,1)^T, (1,3,1)^T\big)$.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T((x,y,z)^T) = (2x+3y+4z, x+y+z, x+y+3z)^T$. Find the value of k for which there exists a vector $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = (9,3,k)^T$.

10. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T((x,y,z)^T) = (2x-2y+2z, -2x+5y+2z, x+y+4z)^T$. Find $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = (1,1,-1)^T$.

11. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T((x,y,z)^T) = (2x+y+3z, 4x-y+3z, 3x-2y+5z)^T$. Determine $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $T(\mathbf{x}) = 6\mathbf{x}$, $T(\mathbf{y}) = 2\mathbf{y}$ and $T(\mathbf{z}) = -2\mathbf{z}$. Is the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ linearly independent?

12. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T((x,y,z)^T) = (2x+3y+4z, -y, -3y+4z)^T$. Determine $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $T(\mathbf{x}) = 2\mathbf{x}$, $T(\mathbf{y}) = 4\mathbf{y}$ and $T(\mathbf{z}) = -\mathbf{z}$. Is the set $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ linearly independent?

13. Does there exist a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^n$ such that $T((1,1,-2)^T) = \mathbf{x}$, $T((-1,2,3)^T) = \mathbf{y}$ and $T((1,10,1)^T) = \mathbf{z}$

(a) with $\mathbf{z} = \mathbf{x} + \mathbf{y}$?

(b) with $\mathbf{z} = c\mathbf{x} + d\mathbf{y}$, for some choice of $c, d \in \mathbb{R}$?

14. For each matrix A given below, define $T \in \mathcal{L}(\mathbb{R}^2)$ by $T(\mathbf{x}) = A\mathbf{x}$. What do these linear operators signify geometrically?

(a) $A \in \left\{ \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & -\sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & \cos\left(\frac{2\pi}{3}\right) \end{bmatrix} \right\}$.

(b) $A \in \left\{ \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \right\}$.

(c) $A \in \left\{ \frac{1}{2}\begin{bmatrix} \sqrt{3} & 1 \\ 1 & -\sqrt{3} \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} 1 & \sqrt{3} \\ \sqrt{3} & -1 \end{bmatrix}, \begin{bmatrix} \cos\left(\frac{2\pi}{3}\right) & \sin\left(\frac{2\pi}{3}\right) \\ \sin\left(\frac{2\pi}{3}\right) & -\cos\left(\frac{2\pi}{3}\right) \end{bmatrix} \right\}$.

15. *Consider the space* $\mathbb{C}^3$ *over* $\mathbb{C}$. *If* $f \in \mathcal{L}(\mathbb{C}^3)$ *with* $f(\mathbf{x}) = \mathbf{x}, f(\mathbf{y}) = (1+i)\mathbf{y}$ *and* $f(\mathbf{z}) = (2+3i)\mathbf{z}$, *for* $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{C}^3 \setminus \{0\}$ *then prove that* $\{\mathbf{x}, \mathbf{y}, \mathbf{z}\}$ *forms a basis of* $\mathbb{C}^3$.

5.2 Rank-Nullity Theorem

Recall that for any $f \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, $\text{Rng}(f) = \{f(\mathbf{v}) | \mathbf{v} \in \mathbb{V}\}$ (see Definition 5.1.8). Now, in line with the ideas in Theorem 3.6.1, we define the null-space or the kernel of a linear transformation. At this stage, the readers are advised to recall Section 3.6 for clarity and similarity with the results in this section.

Definition 5.2.1. Let $f \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then the **null space** of f, denoted $\text{Null}(f)$ or $\text{Ker}(f)$, is given by $\text{Null}(f) = \{\mathbf{v} \in \mathbb{V} | f(\mathbf{v}) = \mathbf{0}\}$. In most linear algebra books, it is also called the **kernel** of f and written $\text{Ker}(f)$. Further, if $\mathbb{V}$ is finite dimensional then one writes

$$\dim(\text{Rng}(T)) = \text{Rank}(T) \text{ and } \dim(\text{Null}(T)) = \text{Nullity}(T).$$

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Example 5.2.2. 1. Define $f \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^4)$ by $f((x,y,z)^T) = (x-y+z, y-z, x, 2x-5y+5z)^T$. Then, by definition,

$$\begin{aligned} \text{Null}(f) &= \{(x,y,z)^T \in \mathbb{R}^3 : f((x,y,z)^T) = \mathbf{0}\} \\ &= \{(x,y,z)^T \in \mathbb{R}^3 : (x-y+z, y-z, x, 2x-5y+5z)^T = \mathbf{0}\} \\ &= \{(x,y,z)^T \in \mathbb{R}^3 : x-y+z = 0, y-z = 0, x = 0, 2x-5y+5z = 0\} \\ &= \{(x,y,z)^T \in \mathbb{R}^3 : y-z = 0, x = 0\} \\ &= \{(0,z,z)^T \in \mathbb{R}^3 : z \in \mathbb{R}\} = LS((0,1,1)^T) \end{aligned}$$

2. Fix $B \in \mathbb{M}_2(\mathbb{R})$. Now, define $T : \mathbb{M}_2(\mathbb{R}) \to \mathbb{M}_2(\mathbb{R})$ by $T(A) = BA - AB$, for all $A \in \mathbb{M}_2(\mathbb{R})$. **Solution:** Then $A \in \text{Null}(T)$ if and only if A commutes with B. In particular, $\{I, B, B^2, \ldots\} \subseteq \text{Null}(T)$. For example, if $B = \alpha I$, for some α then $\text{Null}(T) = \mathbb{M}_2(\mathbb{R})$.

Exercise **5.2.3.** *1. Let* $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. *Then* $\text{Null}(T)$ *is a subspace of* $\mathbb{V}$. *Furthermore, if* $\mathbb{V}$ *is finite dimensional then* $\dim(\text{Null}(T)), \dim(\text{Rng}(T)) \leq \dim(\mathbb{V})$.

2. *Define* $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^4)$ *by* $T((x,y)^T) = (x+y, x-y, 2x+y, 3x-4y)^T$. *Determine* $\text{Null}(T)$.

3. *Describe* $\text{Null}(D)$ *and* $\text{Rng}(D)$, *where* $D \in \mathcal{L}(\mathbb{R}[x;n])$ *is defined by* $(D(f))(x) = f'(x)$, *the differentiation with respect to* x. *Note that* $\text{Rng}(D) \subseteq \mathbb{R}[x;n-1]$.

4. *Define* $T \in \mathcal{L}(\mathbb{R}[x])$ *by* $(T(f))(x) = xf(x)$, *for all* $f(x) \in \mathcal{L}(\mathbb{R}[x])$. *What can you say about* $\text{Null}(T)$ *and* $\text{Rng}(T)$?

5. *Define* $T \in \mathcal{L}(\mathbb{R}^3)$ *by* $T(\mathbf{e}_1) = \mathbf{e}_1 + \mathbf{e}_3$, $T(\mathbf{e}_2) = \mathbf{e}_2 + \mathbf{e}_3$ *and* $T(\mathbf{e}_3) = -\mathbf{e}_3$. *Then*

   (a) *determine* $T((x, y, z)^T)$, *for* $x, y, z \in \mathbb{R}$.

   (b) *determine* $\text{NULL}(T)$ *and* $\text{RNG}(T)$.

   (c) *is it true that* $T^2 = Id$?

We now prove a result which is similar to Exercise 3.3.17.2.

Theorem 5.2.4. *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be vector spaces over* $\mathbb{F}$ *and let* $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$.

1. *If* $S \subseteq \mathbb{V}$ *is linearly dependent then* $T(S) = \{T(\mathbf{v}) \mid \mathbf{v} \in \mathbb{V}\}$ *is linearly dependent.*

2. *Suppose* $S \subseteq \mathbb{V}$ *such that* $T(S)$ *is linearly independent then* S *is linearly independent.*

Proof. Part 1 : As S is linearly dependent, there exist $k \in \mathbb{N}$ and $\mathbf{v}_i \in S$, for $1 \le i \le k$, such that the system $\sum\limits_{i=1}^{k} x_i \mathbf{v}_i = \mathbf{0}$, in the unknowns x_i's, has a non-trivial solution, say $x_i = a_i \in \mathbb{F}, 1 \le i \le k$. Thus $\sum\limits_{i=1}^{k} a_i \mathbf{v}_i = \mathbf{0}$. Then a_i's also give a non-trivial solution to the system $\sum\limits_{i=1}^{k} y_i T(\mathbf{v}_i) = \mathbf{0}$, where y_i's are unknown, as $\sum\limits_{i=1}^{k} a_i T(\mathbf{v}_i) = \sum\limits_{i=1}^{k} T(a_i \mathbf{v}_i) = T\left(\sum\limits_{i=1}^{k} a_i \mathbf{v}_i\right) = T(\mathbf{0}) = \mathbf{0}$. Hence the required result follows.

Part 2 : On the contrary assume that S is linearly dependent. Then by Part 1, $T(S)$ is linearly dependent, a contradiction to the given assumption that $T(S)$ is linearly independent. ∎

We now prove the rank-nullity Theorem. The proof of this result is similar to the proof of Theorem 3.6.1. We give it again for the sake of completeness.

DRAFT

Theorem 5.2.5 (Rank-Nullity Theorem). *Let* $\mathbb{V}$ *and* $\mathbb{W}$ *be vector spaces over* $\mathbb{F}$. *If* $\dim(\mathbb{V})$ *is finite and* $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ *then*

$$\text{RANK}(T) + \text{NULLITY}(T) = \dim(\text{RNG}(T)) + \dim(\text{NULL}(T)) = \dim(\mathbb{V}).$$

Proof. Let $\dim(\mathbb{V}) = n$. As $\text{NULL}(T) \subseteq \mathbb{V}$, let $\dim(\text{NULL}(T)) = k \le n$. Let $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of $\text{NULL}(T)$. We extend it to form a basis $\mathcal{C} = \{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ of $\mathbb{V}$. As $T(\mathbf{v}) = \mathbf{0}$, for all $\mathbf{v} \in \mathcal{B}$,

$$\text{RNG}(T) = LS(T(\mathbf{v}_1), \ldots, T(\mathbf{v}_k), T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)) = LS(T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)).$$

We claim that $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is linearly independent subset of $\mathbb{W}$.

So, consider the system $\sum\limits_{i=1}^{n-k} a_i T(\mathbf{v}_{k+i}) = \mathbf{0}$ in the unknown $a_1, \ldots, a_{n-k}$. Note that

$$\sum_{i=1}^{n-k} a_i T(\mathbf{v}_{k+i}) = \mathbf{0} \Leftrightarrow T\left(\sum_{i=1}^{n-k} a_i \mathbf{v}_{k+i}\right) = \mathbf{0} \Leftrightarrow \sum_{i=1}^{n-k} a_i \mathbf{v}_{k+i} \in \text{NULL}(T).$$

Hence, there exists $b_1, \ldots, b_k \in \mathbb{F}$ such that $\sum\limits_{i=1}^{n-k} a_i \mathbf{v}_{k+i} = \sum\limits_{j=1}^{k} b_j \mathbf{v}_j$. This gives a new system

$$\sum_{i=1}^{n-k} a_i \mathbf{v}_{k+i} + \sum_{j=1}^{k} (-b_j) \mathbf{v}_j = \mathbf{0},$$

in the unknowns a_i's and b_j's. As $\mathcal{C}$ is linearly independent, the new system has only the trivial solution, namely $[a_1, \ldots, a_k, -b_1, \ldots, -b_\ell]^T = \mathbf{0}$. Hence, the system $\sum\limits_{i=1}^{n-k} a_i T(\mathbf{v}_{k+i}) = \mathbf{0}$ has only the trivial solution. Thus, the set $\{T(\mathbf{v}_{k+1}), \ldots, T(\mathbf{v}_n)\}$ is linearly independent subset of $\mathbb{W}$. It also spans $\textsc{Rng}(T)$ and hence is a basis of $\textsc{Rng}(T)$. Therefore,

$$\dim(\textsc{Rng}(T)) + \dim(\textsc{Null}(T)) = k + (n-k) = n = \dim(\mathbb{V}).$$

Thus, we have proved the required result. ■

As an immediate corollary, we have the following result. The proof is left for the reader.

Corollary 5.2.6. *Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over $\mathbb{F}$ and let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. If $\dim(\mathbb{V}) = \dim(\mathbb{W})$ then the following statements are equivalent.*

1. *T is one-one.*
2. *$\textsc{Ker}(T) = \{\mathbf{0}\}$.*
3. *T is onto.*
4. *$\dim(\textsc{Rng}(T)) = \dim(\mathbb{W}) = \dim(\mathbb{V})$.*

Exercise **5.2.7.**

1. *Prove Corollary 5.2.6.*
2. *Let $\mathbb{V}$ and $\mathbb{W}$ be finite dimensional vector spaces over $\mathbb{F}$. If $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ then*
   (a) *T cannot be onto if $\dim(\mathbb{V}) < \dim(\mathbb{W})$.*
   (b) *T cannot be one-one if $\dim(\mathbb{V}) > \dim(\mathbb{W})$.*
3. *Let $A \in M_n(\mathbb{R})$ with $A^2 = A$. Define $T \in \mathcal{L}(\mathbb{R}^n)$ by $T(\mathbf{v}) = A\mathbf{v}$ for all $\mathbf{v} \in \mathbb{R}^n$. Then prove that*
   (a) *$T^2 = T$, or equivalently, $(T(Id - T))(\mathbf{x}) = \mathbf{0}$, for all $\mathbf{x} \in \mathbb{R}^n$.*
   (b) *$\textsc{Null}(T) \cap \textsc{Rng}(T) = \{\mathbf{0}\}$.*
   (c) *$\mathbb{R}^n = \textsc{Rng}(T) + \textsc{Null}(T)$.*
4. *Let $z_1, z_2, \ldots, z_k$ be k distinct complex numbers. Define $T \in \mathcal{L}(\mathbb{C}[x;n], \mathbb{C}^k)$ by $T\big(P(z)\big) = \big(P(z_1), \ldots, P(z_k)\big)^T$, for all $P(z) \in \mathbb{C}[x;n]$. Determine $\textsc{Rank}(T)$.*

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5.3 Algebra of Linear Transformations

We start with the following definition.

Definition 5.3.1. Let $\mathbb{V}, \mathbb{W}$ be vector spaces over $\mathbb{F}$ and let $S, T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then, we define the point-wise

1. **sum** of S and T, denoted $S + T$, by $(S+T)(\mathbf{v}) = S(\mathbf{v}) + T(\mathbf{v})$, for all $\mathbf{v} \in \mathbb{V}$.
2. **scalar multiplication**, denoted $c\,T$ for $c \in \mathbb{F}$, by $(c\,T)(\mathbf{v}) = c\,(T(\mathbf{v}))$, for all $\mathbf{v} \in \mathbb{V}$.

To understand the next result, consider $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ and let $\mathcal{B} = \left\{\mathbf{v}_1 = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix}0\\1\end{bmatrix}\right\}$ and $\mathcal{C} = \left\{\mathbf{w}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{w}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}, \mathbf{w}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}\right\}$ be bases of $\mathbb{R}^2$ and $\mathbb{R}^3$, respectively. Now, for $1 \le i \le 2, 1 \le j \le 3$, define elements of $\mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ by

$$f_{ji}(\mathbf{v}_k) = \begin{cases} \mathbf{w}_j, & \text{if } k = i \\ \mathbf{0}, & \text{if } k \neq i. \end{cases}$$

Then verify that the above maps correspond to the following collection of matrices?

$$f_{11} = \begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}, f_{12} = \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}, f_{21} = \begin{bmatrix}0&0\\1&0\\0&0\end{bmatrix}, f_{22} = \begin{bmatrix}0&0\\0&1\\0&0\end{bmatrix}, f_{31} = \begin{bmatrix}0&0\\0&0\\1&0\end{bmatrix}, f_{32} = \begin{bmatrix}0&0\\0&0\\0&1\end{bmatrix}.$$

Theorem 5.3.2. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$. Then $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is a vector space over $\mathbb{F}$. Furthermore, if $\dim \mathbb{V} = n$ and $\dim \mathbb{W} = m$, then $\dim \mathcal{L}(\mathbb{V}, \mathbb{W}) = mn$.*

Proof. It can be easily verified that under point-wise addition and scalar multiplication, defined above, $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is indeed a vector space over $\mathbb{F}$. We now prove the other part. So, let us assume that $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\mathcal{C} = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are bases of $\mathbb{V}$ and $\mathbb{W}$, respectively. For $1 \le i \le n, 1 \le j \le m$, we define the functions $\mathbf{f}_{ji}$ on the basis vectors of $\mathbb{V}$ by

DRAFT

$$\mathbf{f}_{ji}(\mathbf{v}_k) = \begin{cases} \mathbf{w}_j, & \text{if } k = i \\ \mathbf{0}, & \text{if } k \neq i. \end{cases}$$

For other vectors of $\mathbb{V}$, we extend the definition by linearity, *i.e.*, if $\mathbf{v} = \sum\limits_{s=1}^{n} \alpha_s \mathbf{v}_s$ then

$$\mathbf{f}_{ji}(\mathbf{v}) = \mathbf{f}_{ji}\left(\sum_{s=1}^{n} \alpha_s \mathbf{v}_s\right) = \sum_{s=1}^{n} \alpha_s \mathbf{f}_{ji}(\mathbf{v}_s) = \alpha_i \mathbf{f}_{ji}(\mathbf{v}_i) = \alpha_i \mathbf{w}_j. \tag{5.3.1}$$

Thus $\mathbf{f}_{ji} \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. We now show that $\{\mathbf{f}_{ji} | 1 \le i \le n, 1 \le j \le m\}$ is a basis of $\mathcal{L}(\mathbb{V}, \mathbb{W})$.

As a first step, we show that $\mathbf{f}_{ji}$'s are linearly independent. So, consider the linear system $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{m} c_{ji} \mathbf{f}_{ji} = \mathbf{0}$, in the unknowns c_{ji}'s, for $1 \le i \le n, 1 \le j \le m$. Using the point-wise addition and scalar multiplication, we get

$$\mathbf{0} = \mathbf{0}(\mathbf{v}_k) = \left(\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ji} \mathbf{f}_{ji}\right)(\mathbf{v}_k) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ji} \mathbf{f}_{ji}(\mathbf{v}_k) = \sum_{j=1}^{m} c_{jk} \mathbf{w}_j.$$

But, the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is linearly independent. Hence the only solution equals $c_{jk} = 0$, for $1 \le j \le m$. Now, as we vary $\mathbf{v}_k$ from $\mathbf{v}_1$ to $\mathbf{v}_n$, we see that $c_{ji} = 0$, for $1 \le j \le m$ and $1 \le i \le n$. Thus, we have proved the linear independence of $\{\mathbf{f}_{ji} | 1 \le i \le n, 1 \le j \le m\}$.

Now, let us prove that $LS\left(\{\mathbf{f}_{ji} | 1 \le i \le n, 1 \le j \le m\}\right) = \mathcal{L}(\mathbb{V}, \mathbb{W})$. So, let $f \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then, for $1 \le s \le n$, $f(\mathbf{v}_s) \in \mathbb{W}$ and hence there exists β_{ts}'s such that $f(\mathbf{v}_s) = \sum\limits_{t=1}^{m} \beta_{ts} \mathbf{w}_t$. So, if

$\mathbf{v} = \sum\limits_{s=1}^{n} \alpha_s \mathbf{v}_s \in \mathbb{V}$ then, using Equation (5.3.1), we get

$$
\begin{aligned}
f(\mathbf{v}) &= f\left(\sum_{s=1}^{n} \alpha_s \mathbf{v}_s\right) = \sum_{s=1}^{n} \alpha_s f(\mathbf{v}_s) = \sum_{s=1}^{n} \alpha_s \left(\sum_{t=1}^{m} \beta_{ts} \mathbf{w}_t\right) = \sum_{s=1}^{n}\sum_{t=1}^{m} \beta_{ts}(\alpha_s \mathbf{w}_t) \\
&= \sum_{s=1}^{n}\sum_{t=1}^{m} \beta_{ts} \mathbf{f}_{ts}(\mathbf{v}) = \left(\sum_{s=1}^{n}\sum_{t=1}^{m} \beta_{ts} \mathbf{f}_{ts}\right)(\mathbf{v}).
\end{aligned}
$$

Since the above is true for every $\mathbf{v} \in \mathbb{V}$, we get $f = \sum\limits_{s=1}^{n}\sum\limits_{t=1}^{m} \beta_{ts} \mathbf{f}_{ts}$. Thus, we conclude that $f \in LS\left(\{\mathbf{f}_{ji} | 1 \le i \le n, 1 \le j \le m\}\right)$. Hence, $LS\left(\{\mathbf{f}_{ji} | 1 \le i \le n, 1 \le j \le m\}\right) = \mathcal{L}(\mathbb{V}, \mathbb{W})$ and thus the required result follows. ∎

We now give a corollary of the rank-nullity theorem.

Corollary 5.3.3. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $\dim(\mathbb{V}) = n$. If $S, T \in \mathcal{L}(\mathbb{V})$ then*

1. $\textsc{Nullity}(T) + \textsc{Nullity}(S) \ge \textsc{Nullity}(ST) \ge \max\{\textsc{Nullity}(T), \textsc{Nullity}(S)\}$.
2. $\min\{\mathsf{Rank}(S), \mathsf{Rank}(T)\} \ge \mathsf{Rank}(ST) \ge n - \mathsf{Rank}(S) - \mathsf{Rank}(T)$.

Proof. The prove of Part 2 is omitted as it directly follows from Part 1 and Theorem 5.2.5.

Part 1 - Second Inequality: Suppose $\mathbf{v} \in \textsc{Ker}(T)$. Then

$$(ST)(\mathbf{v}) = S(T(\mathbf{v})) = S(\mathbf{0}) = \mathbf{0}$$

DRAFT

implies $\textsc{Ker}(T) \subseteq \textsc{Ker}(ST)$. Thus $\textsc{Nullity}(T) \le \textsc{Nullity}(ST)$.

By Theorem 5.2.5, $\textsc{Nullity}(S) \le \textsc{Nullity}(ST) \Leftrightarrow \mathsf{Rank}(S) \ge \mathsf{Rank}(ST)$. This holds as $\textsc{Rng}(T) \subseteq \mathbb{V}$ implies $\textsc{Rng}(ST) = S(\textsc{Rng}(T)) \subseteq S(\mathbb{V}) = \textsc{Rng}(S)$.

Part 1 - First Inequality: Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of $\textsc{Null}(T)$. Then $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} \subseteq \textsc{Null}(ST)$. So, let us extend it to get a basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{u}_1, \ldots, \mathbf{u}_\ell\}$ of $\textsc{Null}(ST)$.

Now, proceeding as in the proof of the rank-nullity theorem, implies that $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_\ell)\}$ is a linearly independent subset of $\textsc{Null}(S)$. Hence, $\textsc{Nullity}(S) \ge \ell$ and therefore, we get $\textsc{Nullity}(ST) = k + \ell \le \textsc{Nullity}(T) + \textsc{Nullity}(S)$. ∎

Before proceeding further, recall the following definition about a function.

Definition 5.3.4. Let $f : S \to T$ be any function. Then

1. a function $g : T \to S$ is called a **left inverse** of f if $(g \circ f)(x) = x$, for all $x \in S$. That is, $g \circ f = \mathrm{Id}$, the identity function on S.
2. a function $h : T \to S$ is called a **right inverse** of f if $(f \circ h)(y) = y$, for all $y \in T$. That is, $f \circ h = \mathrm{Id}$, the identity function on T.
3. f is said to be **invertible** if it has a right inverse and a left inverse.

Remark 5.3.5. *Let $f : S \to T$ be invertible. Then, it can be easily shown that any right inverse and any left inverse are the same. Thus, the inverse function is unique and is denoted by f^{-1}. It is well known that f is invertible if and only if f is both one-one and onto.*

Lemma 5.3.6. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ and let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. If T is one-one and onto then, the map $T^{-1} : \mathbb{W} \to \mathbb{V}$ is also a linear transformation. The map T^{-1} is called the* **inverse linear transform** *of T and is defined by $T^{-1}(\mathbf{w}) = \mathbf{v}$ whenever $T(\mathbf{v}) = \mathbf{w}$.*

Proof. PART 1: As T is one-one and onto, by Theorem 5.2.5, $\dim(\mathbb{V}) = \dim(\mathbb{W})$. So, by Corollary 5.2.6, for each $\mathbf{w} \in \mathbb{W}$ there exists a unique $\mathbf{v} \in \mathbb{V}$ such that $T(\mathbf{v}) = \mathbf{w}$. Thus, one defines $T^{-1}(\mathbf{w}) = \mathbf{v}$.

We need to show that $T^{-1}(\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) = \alpha_1 T^{-1}(\mathbf{w}_1) + \alpha_2 T^{-1}(\mathbf{w}_2)$, for all $\alpha_1, \alpha_2 \in \mathbb{F}$ and $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{W}$. Note that by previous paragraph, there exist unique vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{V}$ such that $T^{-1}(\mathbf{w}_1) = \mathbf{v}_1$ and $T^{-1}(\mathbf{w}_2) = \mathbf{v}_2$. Or equivalently, $T(\mathbf{v}_1) = \mathbf{w}_1$ and $T(\mathbf{v}_2) = \mathbf{w}_2$. So, $T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2) = \alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2$, for all $\alpha_1, \alpha_2 \in \mathbb{F}$. Hence, for all $\alpha_1, \alpha_2 \in \mathbb{F}$, we get

$$T^{-1}(\alpha_1\mathbf{w}_1 + \alpha_2\mathbf{w}_2) = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 = \alpha_1 T^{-1}(\mathbf{w}_1) + \alpha_2 T^{-1}(\mathbf{w}_2).$$

Thus, the required result follows. ■

Example 5.3.7.
1. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $(x, y) \rightsquigarrow (x + y, x - y)$. Then, verify that T^{-1} is given by $\rightsquigarrow \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$.
2. Let $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}[x; n-1])$ be given by $(a_1, \ldots, a_n) \rightsquigarrow \sum\limits_{i=1}^{n} a_i x^{i-1}$, for $(a_1, \ldots, a_n) \in \mathbb{R}^n$. Then, T^{-1} maps $\sum\limits_{i=1}^{n} a_i x^{i-1} \rightsquigarrow (a_1, \ldots, a_n)$, for each polynomial $\sum\limits_{i=1}^{n} a_i x^{i-1} \in \mathbb{R}[x; n-1]$. Verify that $T^{-1} \in \mathcal{L}(\mathbb{R}[x; n-1], \mathbb{R}^n)$.

DRAFT

Definition 5.3.8. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ and let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then, T is said to be **singular** if $\{\mathbf{0}\} \subsetneq \mathrm{KER}(T)$, *i.e.*, $\mathrm{KER}(T)$ contains a non-zero vector. If $\mathrm{KER}(T) = \{\mathbf{0}\}$ then, T is called **non-singular**.

Example 5.3.9. Let $T \in \mathcal{L}(\mathbb{R}^2, \mathbb{R}^3)$ be defined by $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$. Then, verify that T is non-singular. Is T invertible?

We now prove a result that relates non-singularity with linear independence.

Theorem 5.3.10. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ and let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then the following statements are equivalent.*

1. *T is one-one.*
2. *T is non-singular.*
3. *Whenever $S \subseteq \mathbb{V}$ is linearly independent then $T(S)$ is necessarily linearly independent.*

Proof. 1⇒2 On the contrary, let T be singular. Then, there exists $\mathbf{v} \neq \mathbf{0}$ such that $T(\mathbf{v}) = \mathbf{0} = T(\mathbf{0})$. This implies that T is not one-one, a contradiction.

2⇒3 Let $S \subseteq \mathbb{V}$ be linearly independent. Let if possible $T(S)$ be linearly dependent. Then, there exists $\mathbf{v}_1, \ldots, \mathbf{v}_k \in S$ and $\alpha = (\alpha_1, \ldots, \alpha_k)^T \neq \mathbf{0}$ such that $\sum\limits_{i=1}^{k} \alpha_i T(\mathbf{v}_i) = \mathbf{0}$.

Thus, $T\left(\sum\limits_{i=1}^{k}\alpha_i\mathbf{v}_i\right) = \mathbf{0}$. But T is non-singular and hence we get $\sum\limits_{i=1}^{k}\alpha_i\mathbf{v}_i = \mathbf{0}$ with $\alpha \neq \mathbf{0}$, a contradiction to S being a linearly independent set.

3⇒1 Suppose that T is not one-one. Then, there exists $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ such that $\mathbf{x} \neq \mathbf{y}$ but $T(\mathbf{x}) = T(\mathbf{y})$. Thus, we have obtained $S = \{\mathbf{x} - \mathbf{y}\}$, a linearly independent subset of $\mathbb{V}$ with $T(S) = \{\mathbf{0}\}$, a linearly dependent set. A contradiction to our assumption. Thus, the required result follows. ∎

Definition 5.3.11. Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ and let $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$. Then, T is said to be an **isomorphism** if T is one-one and onto. The vector spaces $\mathbb{V}$ and $\mathbb{W}$ are said to be **isomorphic**, denoted $\mathbb{V} \cong \mathbb{W}$, if there is an isomorphism from $\mathbb{V}$ to $\mathbb{W}$.

We now give a formal proof of the statement that every finite dimensional vector space $\mathbb{V}$ over $\mathbb{F}$ looks like $\mathbb{F}^n$, where $n = \dim(\mathbb{V})$.

Theorem 5.3.12. *Let $\mathbb{V}$ be an n-dimensional vector space over $\mathbb{F}$. Then $\mathbb{V} \cong \mathbb{F}^n$.*

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathbb{V}$ and $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$, the standard basis of $\mathbb{F}^n$. Define $T \in \mathcal{L}(\mathbb{V}, \mathbb{F}^n)$ by $T\left(\sum\limits_{i=1}^{n}\alpha_i\mathbf{v}_i\right) = \sum\limits_{i=1}^{n}\alpha_i\mathbf{e}_i$, whenever $\mathbf{v} = \sum\limits_{i=1}^{n}\mathbf{v}_i\mathbf{e}_i \in \mathbb{V}$. Then, it is easy to observe that T is one-one and onto. Hence, T is an isomorphism. ∎

As a direct application using the countability argument, one obtains the following result

Corollary 5.3.13. *The vector space $\mathbb{R}$ over $\mathbb{Q}$ is not finite dimensional. Similarly, the vector space $\mathbb{C}$ over $\mathbb{Q}$ is not finite dimensional.*

DRAFT

We now summarize the different definitions related with a linear operator on a finite dimensional vector space. The proof basically uses the rank-nullity theorem and they appear in some form in previous results. Hence, we leave the proof for the reader.

Theorem 5.3.14. *Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$ with $\dim\mathbb{V} = n$. Then the following statements are equivalent for $T \in \mathcal{L}(\mathbb{V})$.*

1. *T is one-one.*
2. $\textsc{Ker}(T) = \{\mathbf{0}\}$.
3. $\mathsf{Rank}(T) = n$.
4. *T is onto.*
5. *T is an isomorphism.*
6. *If $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{V}$ then so is $\{T(\mathbf{v}_1), \ldots, T(\mathbf{v}_n)\}$.*
7. *T is non-singular.*
8. *T is invertible.*

5.4 Ordered Bases

Let $\mathbb{V}$ be a vector space of dimension n over $\mathbb{F}$. Then Theorem 5.3.12 implies that $\mathbb{V}$ is isomorphic to $\mathbb{F}^n$. So, one should be able to visualize the elements of $\mathbb{V}$ as an n-tuple. Further, our problem may require us to look at a subspace $\mathbb{W}$ of $\mathbb{V}$ whose dimension is very small as compared to the dimension of $\mathbb{V}$ (this is generally encountered when we work with sparse matrices or whenever we do computational work). It may also be possible that a basis of $\mathbb{W}$ may not look like a standard basis of $\mathbb{F}^n$, where the coefficient of $\mathbf{e}_i$ gave the i-th component of the vector. We start with the following example. Note that we will be using 'small brackets' in place of 'braces' to represent a basis.

Example 5.4.1.

1. Let $f(x) = 1 - x^2 \in \mathbb{R}[x;2]$. If $\mathcal{B} = (1, x, x^2)$ be a basis of $\mathbb{R}[x;2]$ then, $f(x) = \begin{bmatrix} 1 & x & x^2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

2. Let $\mathbb{V} = \{(u, v, w, x, y)^T \in \mathbb{R}^5 \mid w - x = u, v = y, u + v + x = 3y\}$. Then, verify that $\mathcal{B} = \left((-1, 0, 0, 1, 0)^T, (2, 1, 2, 0, 1)^T\right) = (\mathbf{u}_1, \mathbf{u}_2)$, say, can be taken as a basis of $\mathbb{V}$. So, even though $\mathbb{V}$ is a subspace of $\mathbb{R}^5$, we just need two scalars α, β to understand any vector in $\mathbb{V}$. For example, $(7, 5, 10, 3, 5)^T = 3\mathbf{u}_1 + 5\mathbf{u}_2 = [\mathbf{u}_1, \mathbf{u}_2] \begin{bmatrix} 3 \\ 5 \end{bmatrix} = [\mathbf{u}_2, \mathbf{u}_1] \begin{bmatrix} 5 \\ 3 \end{bmatrix}$.

DRAFT

So, from Example 5.4.1 we conclude the following: Let $\mathbb{V}$ be a vector space of dimension n over $\mathbb{F}$. If we fix a basis, say, $\mathcal{B} = (\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n)$ of $\mathbb{V}$ and if $\mathbf{v} \in \mathbb{V}$ with $\mathbf{v} = \sum\limits_{i=1}^{n} \alpha_i \mathbf{u}_i \Rightarrow$

$$\mathbf{v} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n] \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = [\mathbf{u}_2, \mathbf{u}_1, \ldots, \mathbf{u}_n] \begin{bmatrix} \alpha_2 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Note the change in the first two components of the column vectors which are elements of $\mathbb{F}^n$. So, a change in the position of the vectors $\mathbf{u}_i$'s gives a change in the column vector. Hence, if we fix the order of the basis vectors $\mathbf{u}_i$'s then with respect to this order all vectors can be thought of as elements of $\mathbb{F}^n$. We use the above discussion to define an ordered basis.

Definition 5.4.2. Let $\mathbb{W}$ be a vector space over $\mathbb{F}$ with a basis $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_m\}$. Then, an **ordered basis** for $\mathbb{W}$ is a basis $\mathcal{B}$ together with a one-to-one correspondence between $\mathcal{B}$ and $\{1, 2, \ldots, m\}$. Since there is an order among the elements of $\mathcal{B}$, we write $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_m)$. The matrix $B = [\mathbf{u}_1, \ldots, \mathbf{u}_m]$ containing the basis vectors of $\mathbb{W}^m$ and is called the **basis matrix**.

Example 5.4.3. Note that for Example 5.4.1.1 $[1, x, x^2]$ is a basis matrix, whereas for Example 5.4.1.2, $[\mathbf{u}_1, \mathbf{u}_2]$ and $[\mathbf{u}_2, \mathbf{u}_1]$ are basis matrices.

Definition 5.4.4. Let $B = [\mathbf{v}_1, \ldots, \mathbf{v}_m]$ be the basis matrix corresponding to an ordered basis $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_m)$ of $\mathbb{W}$. Since $\mathcal{B}$ is a basis of $\mathbb{W}$, for each $\mathbf{v} \in \mathbb{W}$, there exist $\beta_i, 1 \le i \le m$, such that $\mathbf{v} = \sum\limits_{i=1}^{m} \beta_i \mathbf{v}_i = B \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$. The vector $\begin{bmatrix} \beta_1 \\ \vdots \\ \beta_m \end{bmatrix}$, denoted $[\mathbf{v}]_{\mathcal{B}}$, is called **the coordinate vector of v with respect to $\mathcal{B}$**. Thus,

$$\mathbf{v} = B[\mathbf{v}]_{\mathcal{B}} = [\mathbf{v}_1, \ldots, \mathbf{v}_m][\mathbf{v}]_{\mathcal{B}}, \text{ or equivalently, } \mathbf{v} = [\mathbf{v}]_{\mathcal{B}}^T \begin{bmatrix} \mathbf{v}_1 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}. \tag{5.4.1}$$

The expressions in Equation (5.4.1) are generally viewed as a symbolic expressions.

Example 5.4.5. Consider Example 5.4.1. Then for

1. $f(x) = 1 - x^2 \in \mathbb{R}[x;2]$ with $\mathcal{B} = (1, x, x^2)$ as an ordered basis of $\mathbb{R}[x;2] \Rightarrow (x)]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.
2. $(7,5,10,3,5) \in \mathbb{V} = \{(u,v,w,x,y)^T \in \mathbb{R}^5 \mid w - x = u, v = y, u + v + x = 3y\}$ with $\mathcal{B} = \left((-1,0,0,1,0)^T, (2,1,2,0,1)^T\right)$ as an ordered basis of $\mathbb{V} \Rightarrow [(7,5,10,3,5)]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$.

Remark 5.4.6. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ of dimension n. Suppose $\mathcal{B} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ is an ordered basis of $\mathbb{V}$.*

1. *Then $[\alpha\mathbf{v} + \mathbf{w}]_{\mathcal{B}} = \alpha[\mathbf{v}]_{\mathcal{B}} + [\mathbf{w}]_{\mathcal{B}}$, for all $\alpha \in \mathbb{F}$ and $\mathbf{v}, \mathbf{w} \in \mathbb{V}$.*
2. *So, once we have fixed $\mathcal{B}$, we can think of each element of $\mathbb{V}$ as a vector in $\mathbb{F}^n$. Therefore, if $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\} \subseteq \mathbb{V}$ then in place of working with S, we can work with its coordinates or equivalently with $S' = \{[\mathbf{w}_1]_{\mathcal{B}}, \ldots, [\mathbf{w}_m]_{\mathcal{B}}\}$. Hence,*
   (a) *S is linearly independent if and only if S' is linearly independent in $\mathbb{F}^n$.*
   (b) *S is linearly dependent if and only if S' is linearly dependent in $\mathbb{F}^n$.*
   (c) *a vector $\mathbf{v} \in LS(S)$ if and only if $[\mathbf{v}]_{\mathcal{B}} \in LS(S')$.*
   (d) *for any ordered basis $\mathcal{C} = (\mathbf{w}_1, \ldots, \mathbf{w}_n)$ of $\mathbb{V}$, we can work with the $n \times n$ matrix $C = \begin{bmatrix} [\mathbf{w}_1]_{\mathcal{B}} & \cdots & [\mathbf{w}_m]_{\mathcal{B}} \end{bmatrix}$. Note that C is invertible.*
   (e) *the symbolic expression $\mathbf{v} = B[\mathbf{v}]_{\mathcal{B}}$ can also be thought of as a matrix equation. Thus*

$$\mathbf{v} = B[\mathbf{v}]_{\mathcal{B}} \Leftrightarrow [\mathbf{v}]_{\mathcal{B}} = B^{-1}\mathbf{v} \text{ for every } \mathbf{v} \in \mathbb{V}. \tag{5.4.2}$$

Example 5.4.7. Consider the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \end{bmatrix} \in \mathbb{M}_{m,n}(\mathbb{R})$. If $\mathcal{B} = (\mathbf{e}_{11}, \mathbf{e}_{12}, \mathbf{e}_{13}, \ldots, \mathbf{e}_{33})$ is an ordered basis of $\mathbb{M}_3(\mathbb{R})$ then $[A]_{\mathcal{B}}^T = \begin{bmatrix} 1 & 2 & 3 & 2 & 1 & 3 & 3 & 1 & 4 \end{bmatrix}$.

Thus, a little thought implies that $\mathbb{M}_{m,n}(\mathbb{R})$ can be mapped to $\mathbb{R}^{mn}$ with respect to the ordered basis $\mathcal{B} = (\mathbf{e}_{11}, \ldots, \mathbf{e}_{1n}, \mathbf{e}_{21}, \ldots, \mathbf{e}_{2n}, \ldots, \mathbf{e}_{m1}, \ldots, \mathbf{e}_{mn})$ of $\mathbb{M}_{m,n}(\mathbb{R})$.

DRAFT

The next definition relates the coordinates of a vector with respect to two distinct ordered bases. This allows us to move from one ordered basis to another ordered basis.

Definition 5.4.8. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $\dim(\mathbb{V}) = n$. Let $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and $B = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ be basis matrices corresponding to the ordered bases $\mathcal{A}$ and $\mathcal{B}$, respectively, of $\mathbb{V}$. Thus, continuing with the symbolic expression in Equation (5.4.1), we have

$$A = [\mathbf{v}_1, \ldots, \mathbf{v}_n] = [B[\mathbf{v}_1]_{\mathcal{B}}, \ldots, B[\mathbf{v}_n]_{\mathcal{B}}] = B\,[[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_n]_{\mathcal{B}}] = B[\mathcal{A}]_{\mathcal{B}}, \tag{5.4.3}$$

where $[\mathcal{A}]_{\mathcal{B}} = [[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_n]_{\mathcal{B}}]$, is called the matrix of $\mathcal{A}$ **with respect to the ordered basis** $\mathcal{B}$ or the **change of basis matrix** from $\mathcal{A}$ to $\mathcal{B}$.

We now summarize the ideas related with ordered bases. This also helps us to understand the nomenclature 'change of basis matrix' for the matrix $[\mathcal{A}]_{\mathcal{B}}$.

Theorem 5.4.9. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $\dim(\mathbb{V}) = n$. Further, let $\mathcal{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ be two ordered bases of $\mathbb{V}$.*

1. *Then the matrix $[\mathcal{A}]_{\mathcal{B}}$ is invertible. Further, Equation (5.4.2) gives $[\mathcal{A}]_{\mathcal{B}} = B^{-1}A$.*
2. *Similarly, the matrix $[\mathcal{B}]_{\mathcal{A}}$ is invertible and $[\mathcal{B}]_{\mathcal{A}} = A^{-1}B$.*
3. *Moreover, $[\mathbf{x}]_{\mathcal{B}} = [\mathcal{A}]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{A}}$, for all $\mathbf{x} \in \mathbb{V}$, i.e., $[\mathcal{A}]_{\mathcal{B}}$ takes coordinate vector of $\mathbf{x}$ with respect to $\mathcal{A}$ to the coordinate vector of $\mathbf{x}$ with respect to $\mathcal{B}$.*
4. *Similarly, $[\mathbf{x}]_{\mathcal{A}} = [\mathcal{B}]_{\mathcal{A}}[\mathbf{x}]_{\mathcal{B}}$, for all $\mathbf{x} \in \mathbb{V}$.*
5. *Furthermore $([\mathcal{A}]_{\mathcal{B}})^{-1} = [\mathcal{B}]_{\mathcal{A}}$.*

DRAFT

Proof. Part 1: Note that using Equation (5.4.3), we see that the matrix $[\mathcal{A}]_{\mathcal{B}}$ takes a linearly independent set to another linearly independent set. Hence, by Exercise 3.3.17, the matrix $[\mathcal{A}]_{\mathcal{B}}$ is invertible, which proves Part 1. A similar argument gives Part 2.

Part 3: Using Equation (5.4.2), $[\mathbf{x}]_{\mathcal{B}} = B^{-1}\mathbf{x} = B^{-1}(AA^{-1})\mathbf{x} = (B^{-1}A)(A^{-1}\mathbf{x}) = [\mathcal{A}]_{\mathcal{B}}[\mathbf{x}]_{\mathcal{A}}$, for all $\mathbf{x} \in \mathbb{V}$. A similar argument gives Part 4 and clearly Part 5. ∎

Example 5.4.10.

1. Let $\mathbb{V} = \mathbb{R}^n$, $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ be the standard ordered basis. Then $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n] = [[\mathbf{v}_1]_{\mathcal{B}}, \ldots, [\mathbf{v}_n]_{\mathcal{B}}] = [A]_{\mathcal{B}}$.
2. Suppose $\mathcal{A} = \left((1,0,0)^T, (1,1,0)^T, (1,1,1)^T\right)$ and $\mathcal{B} = \left((1,1,1)^T, (1,-1,1)^T, (1,1,0)^T\right)$ are two ordered bases of $\mathbb{R}^3$. Then, we verify the statements in the previous result.

   (a) Using Equation (5.4.2), $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{A}} = \left(\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\right)^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x-y \\ y-z \\ z \end{bmatrix}.$

   (b) Similarly, $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -x+y+2z \\ x-y \\ 2x-2z \end{bmatrix}.$

(c) $[\mathcal{A}]_{\mathcal{B}} = \begin{bmatrix} -1/2 & 0 & 1 \\ 1/2 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}$, $[\mathcal{B}]_{\mathcal{A}} = \begin{bmatrix} 0 & 2 & 0 \\ 0 & -2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and $[\mathcal{A}]_{\mathcal{B}}[\mathcal{B}]_{\mathcal{A}} = I_3$.

Exercise **5.4.11.** *In $\mathbb{R}^3$, let $\mathcal{A} = \left((1,2,0)^T, (1,3,2)^T, (0,1,3)^T\right)$ be an ordered basis.*

1. *If $\mathcal{B} = \left((1,2,1)^T, (0,1,2)^T, (1,4,6)^T\right)$ is another ordered basis of $\mathbb{R}^3$. Then, determine $[\mathcal{A}]_{\mathcal{B}}$, $[\mathcal{B}]_{\mathcal{A}}$ and verify that $[\mathcal{A}]_{\mathcal{B}}[\mathcal{B}]_{\mathcal{A}} = I_3$.*
2. *Determine the ordered basis $\mathcal{C}$ such that $[\mathcal{A}]_{\mathcal{C}} = \begin{bmatrix} 2 & 1 & 3 \\ 3 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.*

5.5 Matrix of a linear transformation

In Example 5.1.3.7, we saw that for each $A \in M_{m\times n}(\mathbb{R})$ there exists a linear transformation $T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ given by $T(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$. In this section, we prove that if $\mathbb{V}$ and $\mathbb{W}$ are finite dimensional vector spaces over $\mathbb{F}$ with ordered bases $\mathcal{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$, respectively, then any $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ corresponds to an $m \times n$ matrix.

To understand it let $A = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$ and $B = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ be the basis matrix of $\mathcal{A}$ and $\mathcal{B}$, respectively. Thus, for any $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ and $\mathbf{v} \in \mathbb{V}$, using the symbolic expression in Equation (5.4.1), we see that $T(\mathbf{v}) = B[\mathbf{T}(\mathbf{v})]_{\mathcal{B}}$ and $\mathbf{v} = A[\mathbf{v}]_{\mathcal{A}}$. Hence, for any $\mathbf{x} \in \mathbb{V}$

$$\begin{aligned} B[\mathbf{T}(\mathbf{x})]_{\mathcal{B}} &= T(\mathbf{x}) = T([\mathbf{v}_1, \ldots, \mathbf{v}_n][\mathbf{x}]_{\mathcal{A}}) = \begin{bmatrix} T(\mathbf{v}_1) & \cdots & T(\mathbf{v}_n) \end{bmatrix} [\mathbf{x}]_{\mathcal{A}} \\ &= \begin{bmatrix} B[T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & B[T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix} [\mathbf{x}]_{\mathcal{A}} = B\begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}[\mathbf{x}]_{\mathcal{A}}. \end{aligned}$$

DRAFT

As we can think of B as an invertible matrix (see Equation (5.4.2)), we get

$$[\mathbf{T}(\mathbf{x})]_{\mathcal{B}} = [[T(\mathbf{v}_1)]_{\mathcal{B}}, \ldots, [T(\mathbf{v}_n)]_{\mathcal{B}}]\,[\mathbf{x}]_{\mathcal{A}}, \text{ for each } \mathbf{x} \in \mathbb{V}.$$

Note that the matrix $\begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix}$, denoted $T[\mathcal{A}, \mathcal{B}]$, is an $m \times n$ matrix and is unique with respect to the ordered bases $\mathcal{A}$ and $\mathcal{B}$ as the i-th column equals $[T(\mathbf{v}_i)]_{\mathcal{B}}$, for the i-th vector $\mathbf{v}_i \in \mathcal{A}$, $1 \le i \le n$. So, we immediately have the following definition and result.

Definition 5.5.1. Let $\mathcal{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be ordered bases of $\mathbb{V}$ and $\mathbb{W}$, respectively. If $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ then the matrix $T[\mathcal{A}, \mathcal{B}]$ is called the **coordinate matrix** of T or the **matrix of the linear transformation** T with respect to the bases $\mathcal{A}$ and $\mathcal{B}$, respectively.

When there is no mention of bases, we take it to be the standard ordered bases and denote the corresponding matrix by $[T]$. Also, note that for each $\mathbf{x} \in \mathbb{V}$, the matrix $T[\mathcal{A}, \mathcal{B}][\mathbf{x}]_{\mathcal{A}}$ is the coordinate vector of $T(\mathbf{x})$ with respect to the ordered basis $\mathcal{B}$ of the co-domain. Thus, the matrix $T[\mathcal{A}, \mathcal{B}]$ takes coordinate vector of the domain points to the coordinate vector of its images. The above discussion is stated as the next result.

Theorem 5.5.2. *Let $\mathcal{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$ be ordered bases of $\mathbb{V}$ and $\mathbb{W}$, respectively. If $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ then there exists a matrix $S \in M_{m\times n}(\mathbb{F})$ with*

$$S = T[\mathcal{A}, \mathcal{B}] = \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_n)]_{\mathcal{B}} \end{bmatrix} \text{ and } [T(\mathbf{x})]_{\mathcal{B}} = S\,[\mathbf{x}]_{\mathcal{A}}, \text{ for all } \mathbf{x} \in \mathbb{V}.$$

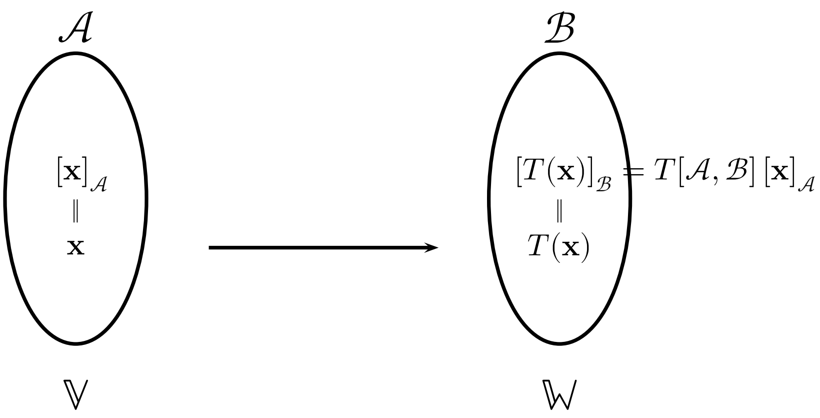

Figure 5.1: Matrix of the Linear Transformation

See Figure 5.1 for clarity on which basis occurs at which place.

Remark 5.5.3. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ with ordered bases $\mathcal{A}_1 = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{B}_1 = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$, respectively. Also, for $\alpha \in \mathbb{F}$ with $\alpha \neq 0$, let $\mathcal{A}_2 = (\alpha\mathbf{v}_1, \ldots, \alpha\mathbf{v}_n)$ and $\mathcal{B}_2 = (\alpha\mathbf{w}_1, \ldots, \alpha\mathbf{w}_m)$ be another set of ordered bases of $\mathbb{V}$ and $\mathbb{W}$, respectively. Then, for any $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$*

$$T[\mathcal{A}_2, \mathcal{B}_2] = \Big[[T(\alpha\mathbf{v}_1)]_{\mathcal{B}_2} \quad \cdots \quad [T(\alpha\mathbf{v}_n)]_{\mathcal{B}_2}\Big] = \Big[[T(\mathbf{v}_1)]_{\mathcal{B}_1} \quad \cdots \quad [T(\mathbf{v}_n)]_{\mathcal{B}_1}\Big] = T[\mathcal{A}_1, \mathcal{B}_1].$$

Thus, the same matrix can be the matrix representation of T for two different pairs of bases.

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We now give a few examples to understand the above discussion and Theorem 5.5.2.

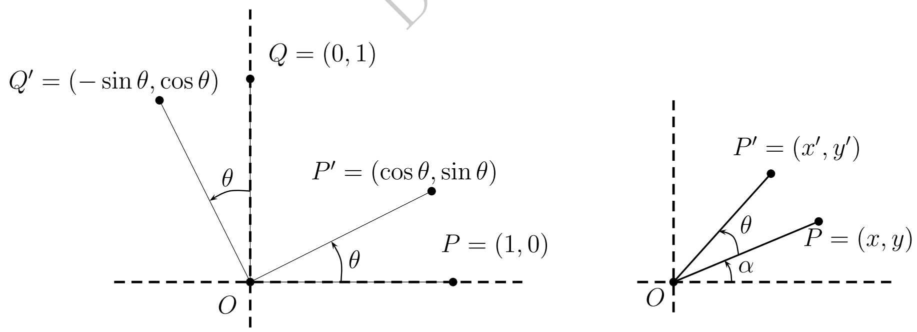

Figure 5.2: Counter-clockwise Rotation by an angle θ

Example 5.5.4. 1. Let $T \in \mathcal{L}(\mathbb{R}^2)$ represent a counter-clockwise rotation by an angle θ, for some $\theta \in [0,\ 2\pi]$. Then, using the right figure in Figure 5.2, we see that $x = OP\cos\alpha$ and $y = OP\sin\alpha$. Thus, verify that

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} OP'\cos(\alpha+\theta) \\ OP'\sin(\alpha+\theta) \end{bmatrix} = \begin{bmatrix} OP\big(\cos\alpha\cos\theta - \sin\alpha\sin\theta\big) \\ OP\big(\sin\alpha\cos\theta + \cos\alpha\sin\theta\big) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Or equivalently, using the left figure in Figure 5.2 we see that the matrix in the standard ordered basis of $\mathbb{R}^2$ equals

$$[T] = \Big[T(\mathbf{e}_1), T(\mathbf{e}_2)\Big] = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{5.5.1}$$

2. Let $T \in \mathcal{L}(\mathbb{R}^2)$ with $T((x,y)^T) = (x+y, x-y)^T$.
   (a) Then $[T] = \Big[[T(\mathbf{e}_1)] \quad [T(\mathbf{e}_2)]\Big] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

   (b) On the image space take the ordered basis as $\mathcal{B} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right)$. Then
   $$[T] = \Big[[T(\mathbf{e}_1)]_{\mathcal{B}} \quad [T(\mathbf{e}_2)]_{\mathcal{B}}\Big] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 1 \\ -1 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}.$$

   (c) In the above, let the ordered basis of the domain space be $\mathcal{A} = \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right)$. Then
   $$T[\mathcal{A},\mathcal{B}] = \left[\left[T \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \quad \left[T \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right]_{\mathcal{B}} \right] = \left[\begin{bmatrix} 0 \\ -2 \end{bmatrix}_{\mathcal{B}} \quad \begin{bmatrix} 4 \\ 2 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}.$$

3. Let $\mathcal{A} = (\mathbf{e}_1, \mathbf{e}_2)$ and $\mathcal{B} = (\mathbf{e}_1 + \mathbf{e}_2, \mathbf{e}_1 - \mathbf{e}_2)$ be two ordered bases of $\mathbb{R}^2$. Then Compute $T[\mathcal{A},\mathcal{A}]$ and $T[\mathcal{B},\mathcal{B}]$, where $T((x,y)^T) = (x+y, x-2y)^T$.
   Solution: Note that the bases matrices for the two ordered bases are $A = \mathrm{Id}_2$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, respectively. So, $A^{-1} = \mathrm{Id}_2$ and $B^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Thus,
   $$T[\mathcal{A},\mathcal{A}] = \left[\left[T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \right]_{\mathcal{A}}, \left[T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \right]_{\mathcal{A}} \right] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}_{\mathcal{A}}, \begin{bmatrix} 1 \\ -2 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \text{ and}$$
   $$T[\mathcal{B},\mathcal{B}] = \left[\left[T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) \right]_{\mathcal{B}}, \left[T\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) \right]_{\mathcal{B}} \right] = \left[\begin{bmatrix} 2 \\ -1 \end{bmatrix}_{\mathcal{B}}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & -\frac{3}{2} \end{bmatrix}$$
   as $\begin{bmatrix} 2 \\ -1 \end{bmatrix}_{\mathcal{B}} = B^{-1}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \end{bmatrix}_{\mathcal{B}} = B^{-1}\begin{bmatrix} 0 \\ 3 \end{bmatrix}$.

4. Let $T \in \mathcal{L}(\mathbb{R}^3, \mathbb{R}^2)$ be defined by $T((x,y,z)^T) = (x+y-z, x+z)^T$. Determine $[T]$.
   Solution: By definition
   $$[T] = [[T(\mathbf{e}_1)], [T(\mathbf{e}_2)], [T(\mathbf{e}_3)]] = \left[\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right] = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}.$$

5. Define $T \in \mathcal{L}(\mathbb{C}^3)$ by $T(\mathbf{x}) = \mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^3$. Note that T is the Id map. Determine the coordinate matrix with respect to the ordered basis $\mathcal{A} = \big(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\big)$ and $\mathcal{B} = \big((1,0,0),(1,1,0),(1,1,1)\big)$.
   Solution: By definition, verify that
   $$T[\mathcal{A},\mathcal{B}] = [[T(\mathbf{e}_1)]_{\mathcal{B}}, [T(\mathbf{e}_2)]_{\mathcal{B}}, [T(\mathbf{e}_3)]_{\mathcal{B}}] = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{B}}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{B}}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}_{\mathcal{B}} \right] = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$
   and
   $$T[\mathcal{B},\mathcal{A}] = \left[\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_{\mathcal{A}}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}_{\mathcal{A}}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}_{\mathcal{A}} \right] = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

DRAFT

Thus, verify that $T[\mathcal{B},\mathcal{A}]^{-1} = T[\mathcal{A},\mathcal{B}]$ and $T[\mathcal{A},\mathcal{A}] = T[\mathcal{B},\mathcal{B}] = I_3$ as the given map is indeed the identity map.

We now give a remark which relates the above ideas with respect to matrix multiplication.

Remark 5.5.5. *1. Fix $S \in \mathbb{M}_n(\mathbb{C})$ and define $T \in \mathcal{L}(\mathbb{C}^n)$ by $T(\mathbf{x}) = S\mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^n$. If $\mathcal{A}$ is the standard basis of $\mathbb{C}^n$ then $[T] = S$ as*

$$[T][:,i] = [T(\mathbf{e}_i)]_{\mathcal{A}} = [S(\mathbf{e}_i)]_{\mathcal{A}} = [S[:,i]]_{\mathcal{A}} = S[:,i], \quad \text{for } 1 \le i \le n.$$

2. Fix $S \in \mathbb{M}_{m,n}(\mathbb{C})$ and define $T \in \mathcal{L}(\mathbb{C}^n,\mathbb{C}^m)$ by $T(\mathbf{x}) = S\mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^n$. Let $\mathcal{A}$ and $\mathcal{B}$ be the standard ordered bases of $\mathbb{C}^n$ and $\mathbb{C}^m$, respectively. Then $T[\mathcal{A},\mathcal{B}] = S$ as

$$(T[\mathcal{A},\mathcal{B}])[:,i] = [T(\mathbf{e}_i)]_{\mathcal{B}} = [S\mathbf{e}_i]_{\mathcal{B}} = [S[:,i]]_{\mathcal{B}} = S[:,i], \quad \text{for } 1 \le i \le n.$$

3. Fix $S \in \mathbb{M}_n(\mathbb{C})$ and define $T \in \mathcal{L}(\mathbb{C}^n)$ by $T(\mathbf{x}) = S\mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^n$. Let $\mathcal{A} = (\mathbf{v}_1,\ldots,\mathbf{v}_n)$ and $\mathcal{B} = (\mathbf{u}_1,\ldots,\mathbf{u}_n)$ be two ordered bases of $\mathbb{C}^n$ with respective basis matrices A and B. Then

$$\begin{aligned} T[\mathcal{A},\mathcal{B}] &= \begin{bmatrix} [T(\mathbf{v}_1)]_{\mathcal{B}} & \cdots & [T(\mathbf{v}_1)]_{\mathcal{B}} \end{bmatrix} = \begin{bmatrix} B^{-1}T(\mathbf{v}_1) & \cdots & B^{-1}T(\mathbf{v}_1) \end{bmatrix} \\ &= \begin{bmatrix} B^{-1}S\mathbf{v}_1 & \cdots & B^{-1}S\mathbf{v}_1 \end{bmatrix} = B^{-1}S \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = B^{-1}SA. \end{aligned}$$

In particular, if $\mathcal{A} = \mathcal{B}$ then $T[\mathcal{A},\mathcal{A}] = A^{-1}SA$. Thus, if $S = I_n$ then

(a) $T = Id$ and $Id[\mathcal{A},\mathcal{A}] = I_n$.

(b) $Id[\mathcal{A},\mathcal{B}] = B^{-1}A$, an invertible matrix.

(c) Similarly, $Id[\mathcal{B},\mathcal{A}] = A^{-1}B$. So, $Id[\mathcal{B},\mathcal{A}] \cdot Id[\mathcal{A},\mathcal{B}] = (A^{-1}B)(B^{-1}A) = I_n$.

DRAFT

Example 5.5.6. 1. Let $T\left((x,y)^T\right) = (x+y, x-y)^T$ and $\mathcal{A} = (\mathbf{e}_1, \mathbf{e}_1 + \mathbf{e}_2)$ be the ordered basis of $\mathbb{R}^2$. Then, for $S = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $T(\mathbf{x}) = S\mathbf{x}$. Further, if $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ is the basis matrix of $\mathcal{A}$ then using Remark 5.5.5.3a, we obtain

$$T[\mathcal{A},\mathcal{A}] = A^{-1}SA = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

2. **[Finding T from $T[\mathcal{A},\mathcal{B}]$]** Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ with ordered bases $\mathcal{A}$ and $\mathcal{B}$, respectively. Suppose we are given the matrix $S = T[\mathcal{A},\mathcal{B}]$. Then to determine the corresponding $T \in \mathcal{L}(\mathbb{V},\mathbb{W})$, we go back to the symbolic expression in Equation (5.4.1) and Theorem 5.5.2. We see that

   (a) $T(\mathbf{v}) = B[T(\mathbf{v})]_{\mathcal{B}} = BT[\mathcal{A},\mathcal{B}][\mathbf{v}]_{\mathcal{A}} = BS[\mathbf{v}]_{\mathcal{A}}$.

   (b) In particular, if $\mathbb{V} = \mathbb{W} = \mathbb{F}^n$ and $\mathcal{A} = \mathcal{B}$ then $T(\mathbf{v}) = BSB^{-1}\mathbf{v}$.

   (c) Further, if $\mathcal{B}$ is the standard ordered basis then $T(\mathbf{v}) = S\mathbf{v}$.

EXERCISE **5.5.7.** *1. Relate Remark 5.5.5.3 with Theorem 5.4.9 as Id is the identity map.*

2. Verify Remark 5.5.5 from different examples in Example 5.5.4.

3. *Let $T \in \mathcal{L}(\mathbb{R}^2)$ represent the reflection about the line $y = mx$. Find $[T]$.*

4. *Let $T \in \mathcal{L}(\mathbb{R}^3)$ represent the reflection about/across the X-axis. Find $[T]$. What about the reflection across the XY-plane?*

5. *Let $T \in \mathcal{L}(\mathbb{R}^3)$ represent the counter-clockwise rotation around the positive Z-axis by an angle $\theta, 0 \leq \theta < 2\pi$. Find its matrix with respect to the standard ordered basis of $\mathbb{R}^3$. [Hint: Is $\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ the required matrix?]*

6. *Define a function $D \in \mathcal{L}(\mathbb{R}[x;n])$ by $D(f(x)) = f'(x)$. Find the matrix of D with respect to the standard ordered basis of $\mathbb{R}[x;n]$. Observe that* $\textsc{Rng}(D) \subseteq \mathbb{R}[x;n-1]$.

5.6 Similarity of Matrices

Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $\dim(\mathbb{V}) = n$ and ordered basis $\mathcal{B}$. Then any $T \in \mathcal{L}(\mathbb{V})$ corresponds to a matrix in $\mathbb{M}_n(\mathbb{F})$. Then in Remark 5.5.5.3 we have already seen that if $\mathcal{A}$ is the standard ordered basis of $\mathbb{F}^n$ and $\mathcal{B}$ is any ordered basis of $\mathbb{F}^n$ with basis matrix B then $T[\mathcal{B},\mathcal{B}] = B^{-1}T[\mathcal{A},\mathcal{A}]B$. Similarly, if $\mathcal{C}$ is any other ordered basis of $\mathbb{F}^n$ with basis matrix C then $T[\mathcal{C},\mathcal{C}] = C^{-1}T[\mathcal{A},\mathcal{A}]C$ and thus

$$T[\mathcal{C},\mathcal{C}] = C^{-1}\, T[\mathcal{A},\mathcal{A}]\, C = C^{-1}(B\, T[\mathcal{B},\mathcal{B}]\, B^{-1})C = (B^{-1}C)^{-1}\, T[\mathcal{B},\mathcal{B}]\, (B^{-1}C).$$

DRAFT

This idea can be generalized to any finite dimensional vector space. To do so, we start with the matrix of the composition of two linear transformations. This also helps us to relate matrix multiplication with composition of two functions.

$$\begin{tikzcd}
(\mathbb{V},\mathcal{B},n) \arrow[r, "T[\mathcal{B},\mathcal{C}]_{m\times n}"] \arrow[rr, bend right, "(ST)[\mathcal{B},\mathcal{D}]_{p\times n} = S[\mathcal{C},\mathcal{D}]\cdot T[\mathcal{B},\mathcal{C}]"'] & (\mathbb{W},\mathcal{C},m) \arrow[r, "S[\mathcal{C},\mathcal{D}]_{p\times m}"] & (\mathbb{Z},\mathcal{D},p)
\end{tikzcd}$$

Figure 5.3: Composition of Linear Transformations

Theorem 5.6.1 (Composition of Linear Transformations)**.** *Let $\mathbb{V}$, $\mathbb{W}$ and $\mathbb{Z}$ be finite dimensional vector spaces over $\mathbb{F}$ with ordered bases $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$, respectively. Also, let $T \in \mathcal{L}(\mathbb{V},\mathbb{W})$ and $S \in \mathcal{L}(\mathbb{W},\mathbb{Z})$. Then $S \circ T = ST \in \mathcal{L}(\mathbb{V},\mathbb{Z})$ (see Figure 5.3). Then*

$$(ST)\ [\mathcal{B},\mathcal{D}] = S[\mathcal{C},\mathcal{D}]\ \cdot\ T[\mathcal{B},\mathcal{C}].$$

Proof. Let $\mathcal{B} = (\mathbf{u}_1,\ldots,\mathbf{u}_n)$, $\mathcal{C} = (\mathbf{v}_1,\ldots,\mathbf{v}_m)$ and $\mathcal{D} = (\mathbf{w}_1,\ldots,\mathbf{w}_p)$ be the ordered bases of $\mathbb{V}, \mathbb{W}$ and $\mathbb{Z}$, respectively. As $(ST)(\mathbf{u}) \in \mathbb{Z}$, using Theorem 5.5.2, we note that

$$[(ST)(\mathbf{u})]_{\mathcal{D}} = \big[S\big(T(\mathbf{u})\big)\big]_{\mathcal{D}} = S[\mathcal{C},\mathcal{D}]\cdot[T(\mathbf{u})]_{\mathcal{C}} = S[\mathcal{C},\mathcal{D}]\cdot(T[\mathcal{B},\mathcal{C}]\cdot[\mathbf{u}]_{\mathcal{B}})\,.$$

So, for all $\mathbf{u} \in \mathbb{V}$, we get $(S[\mathcal{C},\mathcal{D}] \cdot T[\mathcal{B},\mathcal{C}])\,[\mathbf{u}]_{\mathcal{B}} = [(ST)(\mathbf{u})]_{\mathcal{D}} = (ST)[\mathcal{B},\mathcal{D}]\,[\mathbf{u}]_{\mathcal{B}}$. Hence $(ST)\,[\mathcal{B},\mathcal{D}] = S[\mathcal{C},\mathcal{D}] \cdot T[\mathcal{B},\mathcal{C}]$. ∎

As an immediate corollary of Theorem 5.6.1 we see that the matrix of the inverse linear transform is the inverse of the matrix of the linear transform, whenever the inverse exists.

Theorem 5.6.2 (Inverse of a Linear Transformation)**.** *Let $\mathbb{V}$ is a vector space with $\dim(\mathbb{V}) = n$. If $T \in \mathcal{L}(\mathbb{V})$ is invertible then for any ordered basis $\mathcal{B}$ and $\mathcal{C}$ of the domain and co-domain, respectively, one has $(T[\mathcal{C},\mathcal{B}])^{-1} = T^{-1}[\mathcal{B},\mathcal{C}]$. That is, the inverse of the coordinate matrix of T is the coordinate matrix of the inverse linear transform.*

Proof. As T is invertible, $TT^{-1} = \mathrm{Id}$. Thus, Remark 5.5.5.3 and Theorem 5.6.1 imply

$$I_n = \mathrm{Id}[\mathcal{B},\mathcal{B}] = (TT^{-1})[\mathcal{B},\mathcal{B}] = T[\mathcal{C},\mathcal{B}] \cdot T^{-1}[\mathcal{B},\mathcal{C}].$$

Hence, by definition of inverse, $T^{-1}[\mathcal{B},\mathcal{C}] = (T[\mathcal{C},\mathcal{B}])^{-1}$ and the required result follows. ∎

Exercise **5.6.3.** *Find the matrix of the linear transformations given below.*

1. *Let $\mathcal{B} = \big(\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3\big)$ be an ordered basis of $\mathbb{R}^3$. Now, define $T \in \mathcal{L}(\mathbb{R}^3)$ by $T(\mathbf{x}_1) = \mathbf{x}_2$, $T(\mathbf{x}_2) = \mathbf{x}_3$ and $T(\mathbf{x}_3) = \mathbf{x}_1$. Determine $T[\mathcal{B},\mathcal{B}]$. Is T invertible?*

2. *Let $\mathcal{B} = \big(1, x, x^2, x^3\big)$ be an ordered basis of $\mathbb{R}[x;3]$ and define $T \in \mathcal{L}(\mathbb{R}[x;3])$ by $T(1) = 1$, $T(x) = 1 + x$, $T(x^2) = (1+x)^2$ and $T(x^3) = (1+x)^3$. Prove that T is invertible. Also, find $T[\mathcal{B},\mathcal{B}]$ and $T^{-1}[\mathcal{B},\mathcal{B}]$.*

DRAFT

Let $\mathbb{V}$ be a finite dimensional vector space. Then, the next result answers the question "what happens to the matrix $T[\mathcal{B},\mathcal{B}]$ if the ordered basis $\mathcal{B}$ changes to $\mathcal{C}$?"

$$\begin{tikzcd}[column sep=huge, row sep=huge]
(\mathbb{V},\mathcal{B}) \arrow[r, "T[\mathcal{B},\mathcal{B}]"] \arrow[d, "\mathrm{Id}[\mathcal{B},\mathcal{C}]"'] \arrow[dr, bend left, "\mathrm{Id}\circ T"] \arrow[dr, bend right, "T\circ \mathrm{Id}"'] & (\mathbb{V},\mathcal{B}) \arrow[d, "\mathrm{Id}[\mathcal{B},\mathcal{C}]"] \\
(\mathbb{V},\mathcal{C}) \arrow[r, "T[\mathcal{C},\mathcal{C}]"'] & (\mathbb{V},\mathcal{C})
\end{tikzcd}$$

Figure 5.4: $T[\mathcal{C},\mathcal{C}] = \mathrm{Id}[\mathcal{B},\mathcal{C}] \cdot T[\mathcal{B},\mathcal{B}] \cdot (\mathrm{Id}[\mathcal{B},\mathcal{C}])^{-1}$ - Similarity of Matrices

Theorem 5.6.4. *Let $\mathcal{B} = (\mathbf{u}_1, \ldots, \mathbf{u}_n)$ and $\mathcal{C} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ be two ordered bases of $\mathbb{V}$ and Id the identity operator. Then, for any linear operator $T \in \mathcal{L}(\mathbb{V})$*

$$T[\mathcal{C},\mathcal{C}] = Id[\mathcal{B},\mathcal{C}] \cdot T[\mathcal{B},\mathcal{B}] \cdot (Id[\mathcal{B},\mathcal{C}])^{-1}. \tag{5.6.1}$$

Proof. As Id is the identity operator, the composite functions $(T \circ \mathrm{Id}), (\mathrm{Id} \circ T)$ from $(\mathbb{V},\mathcal{B})$ to $(\mathbb{V},\mathcal{C})$ are equal (see Figure 5.4 for clarity). Hence, their matrix representations with respect to ordered bases $\mathcal{B}$ and $\mathcal{C}$ are equal. Thus, $(T \circ \mathrm{Id})[\mathcal{B},\mathcal{C}] = T[\mathcal{B},\mathcal{C}] = (\mathrm{Id} \circ T)[\mathcal{B},\mathcal{C}]$. Thus, using Theorem 5.6.1, we get

$$\mathrm{Id}[\mathcal{B},\mathcal{C}] \cdot T[\mathcal{B},\mathcal{B}] = T[\mathcal{B},\mathcal{C}] = T[\mathcal{C},\mathcal{C}]\mathrm{Id}[\mathcal{B},\mathcal{C}].$$

Hence, using Theorem 5.6.2, the required result follows. ∎

Let $\mathbb{V}$ be a vector space and let $T \in \mathcal{L}(\mathbb{V})$. If $\dim(\mathbb{V}) = n$ then every ordered basis $\mathcal{B}$ of $\mathbb{V}$ gives an $n \times n$ matrix $T[\mathcal{B},\mathcal{B}]$. So, as we change the ordered basis, the coordinate matrix of T changes. Theorem 5.6.4 tells us that all these matrices are related by an invertible matrix. Thus, we are led to the following definitions.

Definition 5.6.5. Let $\mathbb{V}$ be a vector space with ordered bases $\mathcal{B}$ and $\mathcal{C}$. If $T \in \mathcal{L}(\mathbb{V})$ then, $T[\mathcal{C},\mathcal{C}] = \mathrm{Id}[\mathcal{B},\mathcal{C}] \cdot T[\mathcal{B},\mathcal{B}] \cdot \mathrm{Id}[\mathcal{C},\mathcal{B}]$. The matrix $\mathrm{Id}[\mathcal{B},\mathcal{C}]$ is called the **change of basis matrix** (also, see Theorem 5.4.9) from $\mathcal{B}$ to $\mathcal{C}$.

Definition 5.6.6. Let $X, Y \in \mathbb{M}_n(\mathbb{C})$. Then, X and Y are said to be **similar** if there exists a non-singular matrix P such that $P^{-1}XP = Y \Leftrightarrow X = PYP^{-1} \Leftrightarrow XP = PY$.

Example 5.6.7. Let $\mathcal{B} = \big(1+x, 1+2x+x^2, 2+x\big)$ and $\mathcal{C} = \big(1, 1+x, 1+x+x^2\big)$ be ordered bases of $\mathbb{R}[x;2]$. Then, verify that $\mathrm{Id}[\mathcal{B},\mathcal{C}]^{-1} = \mathrm{Id}[\mathcal{C},\mathcal{B}]$, as

$$\begin{aligned}\mathrm{Id}[\mathcal{C},\mathcal{B}] &= [[1]_{\mathcal{B}}, [1+x]_{\mathcal{B}}, [1+x+x^2]_{\mathcal{B}}] = \begin{bmatrix} -1 & 1 & -2 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \text{ and} \\ \mathrm{Id}[\mathcal{B},\mathcal{C}] &= [[1+x]_{\mathcal{C}}, [1+2x+x^2]_{\mathcal{C}}, [2+x]_{\mathcal{C}}] = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.\end{aligned}$$

DRAFT

Exercise **5.6.8.**
1. *Let $A \in \mathbb{M}_n(\mathbb{R})$ such that* $\mathrm{tr}(A) = 0$. *Then prove that there exists a non-singular matrix S such that $SAS^{-1} = B$ with $B = [b_{ij}]$ and $b_{ii} = 0$, for $1 \le i \le n$.*
2. *Let $\mathbb{V}$ be a vector space with* $\dim(\mathbb{V}) = n$. *Let $T \in \mathcal{L}(\mathbb{V})$ satisfy $T^{n-1} \neq \mathbf{0}$ but $T^n = \mathbf{0}$. Then, use Exercise 5.1.13.2 to get an ordered basis $\mathcal{B} = \big(\mathbf{u}, T(\mathbf{u}), \ldots, T^{n-1}(\mathbf{u})\big)$ of $\mathbb{V}$.*
   (a) *Now, prove that* $T[\mathcal{B},\mathcal{B}] = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$.
   (b) *Let $A \in \mathbb{M}_n(\mathbb{C})$ satisfy $A^{n-1} \neq \mathbf{0}$ but $A^n = \mathbf{0}$. Then, prove that A is similar to the matrix given in Part 2a.*
3. *Let $\mathcal{A}$ be an ordered basis of a vector space $\mathbb{V}$ over $\mathbb{F}$ with* $\dim(\mathbb{V}) = n$. *Then prove that the set of all possible matrix representations of T is given by (also see Definition 5.6.5)*

$$\{S \cdot T[\mathcal{A},\mathcal{A}] \cdot S^{-1} \mid S \in \mathbb{M}_n(\mathbb{F}) \textit{ is an invertible matrix}\}.$$

4. *Let* $B_1(\alpha, \beta) = \{(x, y)^T \in \mathbb{R}^2 : (x-\alpha)^2 + (y-\beta)^2 \leq 1\}$. *Then, can we get a linear transformation* $T \in \mathcal{L}(\mathbb{R}^2)$ *such that* $T(S) = W$, *where* S *and* W *are given below?*

   (a) $S = B_1(0, 0)$ *and* $W = B_1(1, 1)$.

   (b) $S = B_1(0, 0)$ *and* $W = B_1(.3, 0)$.

   (c) $S = B_1(0, 0)$ *and* $W = hull(\pm e_1, \pm e_2)$, *where hull means the* **convex hull**.

   (d) $S = B_1(0, 0)$ *and* $W = \{(x, y)^T \in \mathbb{R}^2 : x^2 + y^2/4 = 1\}$.

   (e) $S = hull(\pm e_1, \pm e_2)$ *and* W *is the interior of a pentagon.*

5. *Let* $\mathbb{V}$, $\mathbb{W}$ *be vector spaces over* $\mathbb{F}$ *with* $\dim(\mathbb{V}) = n$ *and* $\dim(\mathbb{W}) = m$ *and ordered bases* $\mathcal{B}$ *and* $\mathcal{C}$, *respectively. Define* $\mathcal{I}_{\mathcal{B},\mathcal{C}} : \mathcal{L}(\mathbb{V}, \mathbb{W}) \to \mathbb{M}_{m,n}(\mathbb{F})$ *by* $\mathcal{I}_{\mathcal{B},\mathcal{C}}(T) = T[\mathcal{B}, \mathcal{C}]$. *Show that* $\mathcal{I}_{\mathcal{B},\mathcal{C}}$ *is an isomorphism. Thus, when bases are fixed, the number of* $m \times n$ *matrices is same as the number of linear transformations.*

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6. *Define* $T \in \mathcal{L}(\mathbb{R}^3)$ *by* $T((x, y, z)^T) = (x + y + 2z, x - y - 3z, 2x + 3y + z)^T$. *Let* $\mathcal{B}$ *be the standard ordered basis and* $\mathcal{C} = \left((1, 1, 1)^T, (1, -1, 1)^T, (1, 1, 2)^T\right)$ *be another ordered basis of* $\mathbb{R}^3$. *Then find*

   (a) *matrices* $T[\mathcal{B}, \mathcal{B}]$ *and* $T[\mathcal{C}, \mathcal{C}]$.

   (b) *the matrix* P *such that* $P^{-1} T[\mathcal{B}, \mathcal{B}]\, P = T[\mathcal{C}, \mathcal{C}]$. *Note that* $P = Id[\mathcal{C}, \mathcal{B}]$.

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5.7 Orthogonal Projections and Applications

As an application of the ideas and results related with orthogonality, we would like to go back to the system of linear equations. So, recall that we started with the solution set of the linear system $A\mathbf{x} = \mathbf{b}$, for $A \in \mathbb{M}_{m,n}(\mathbb{C}), \mathbf{x} \in \mathbb{C}^n$ and $\mathbf{b} \in \mathbb{C}^m$. We saw that if $\mathbf{b} \in \textsc{Col}(A)$ then the system $A\mathbf{x} = \mathbf{b}$ is consistent and one can use the Gauss-Jordan method to get the solution set of $A\mathbf{x} = \mathbf{b}$. If the system is inconsistent can we talk of the 'best possible solution'? How do we define 'Best'?

In most practical applications, the linear systems are inconsistent due to various reasons. The reasons could be related with human error, or computational/rounding-off error or missing data or there is not enough time to solve the whole linear system. So, we need to go bound consistent linear systems. In quite a few such cases, we are interested in finding a point $\mathbf{x} \in \mathbb{R}^n$ such that the error vector, defined as $\|\mathbf{b} - A\mathbf{x}\|$ has the least norm. Thus, we consider the problem of finding $\mathbf{x}_0 \in \mathbb{R}^n$ such that

$$\|\mathbf{b} - A\mathbf{x}_0\| = \min\{\|\mathbf{b} - A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n\}, \tag{5.7.1}$$

i.e., we try to find the vector $\mathbf{x}_0 \in \mathbb{R}^n$ which is nearest to $\mathbf{b}$.

To begin with, recall the following result from Page 105.

Theorem 5.7.1 (Decomposition). *Let $\mathbb{V}$ be an IPS having $\mathbb{W}$ as a finite dimensional subspace. Suppose $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ is an orthonormal basis of $\mathbb{W}$. Then, for each $\mathbf{b} \in \mathbb{V}$, $\mathbf{y} = \sum\limits_{i=1}^{k} \langle \mathbf{b}, \mathbf{f}_i \rangle \mathbf{f}_i$ is the closest point in $\mathbb{W}$ from $\mathbf{b}$. Thus*

$$\min\{\|\mathbf{b} - A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n\} = \|\mathbf{b} - \mathbf{y}\|.$$

We now give a definition and then an implication of Theorem 5.7.1.

Definition 5.7.2. Let $\mathbb{W}$ be a finite dimensional subspace of an IPS $\mathbb{V}$. Then, by Theorem 5.7.1, for each $\mathbf{v} \in \mathbb{V}$ there exist unique vectors $\mathbf{w} \in \mathbb{W}$ and $\mathbf{u} \in \mathbb{W}^{\perp}$ with $\mathbf{v} = \mathbf{w} + \mathbf{u}$. We thus define the **orthogonal projection** of $\mathbb{V}$ onto $\mathbb{W}$, denoted $P_{\mathbb{W}}$, by

$$P_{\mathbb{W}} : \mathbb{V} \to \mathbb{W} \text{ by } P_{\mathbb{W}}(\mathbf{v}) = \mathbf{w}.$$

The vector $\mathbf{w}$ is called the **projection** of $\mathbf{v}$ on $\mathbb{W}$.

So, note that the solution $\mathbf{x}_0 \in \mathbb{R}^n$ satisfying $\|\mathbf{b} - A\mathbf{x}_0\| = \min\{\|\mathbf{b} - A\mathbf{x}\| : \mathbf{x} \in \mathbb{R}^n\}$ is the projection of $\mathbf{b}$ on the $\text{Col}(A)$.

Remark 5.7.3. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $\mathbb{W} = \text{Col}(A)$. Then, to find the orthogonal projection $P_{\mathbb{W}}(\mathbf{b})$, we can use either of the following ideas:*

1. *Determine an orthonormal basis $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ of $\text{Col}(A)$. Then $P_{\mathbb{W}}(\mathbf{b}) = \sum\limits_{i=1}^{k} \langle \mathbf{b}, \mathbf{f}_i \rangle \mathbf{f}_i$. Note that*

$$\mathbf{x}_0 = P_{\mathbb{W}}(\mathbf{b}) = \sum_{i=1}^{k} \langle \mathbf{b}, \mathbf{f}_i \rangle \mathbf{f}_i = \sum_{i=1}^{k} \mathbf{f}_i (\mathbf{f}_i^T \mathbf{b}) = \sum_{i=1}^{k} \left(\mathbf{f}_i \mathbf{f}_i^T\right) \mathbf{b} = \left(\sum_{i=1}^{k} \mathbf{f}_i \mathbf{f}_i^T\right) \mathbf{b} = P\, \mathbf{b},$$

   where $P = \sum\limits_{i=1}^{k} \mathbf{f}_i \mathbf{f}_i^T$ is called the **projection matrix** *of $\mathbb{R}^m$ onto $\text{Col}(A)$.*

2. *By Theorem 3.6.5.2, $\text{Col}(A) = \text{Null}(A^T)^{\perp}$. Hence, for $\mathbf{b} \in \mathbb{R}^m$ there exists unique $\mathbf{u} \in \text{Col}(A)$ and $\mathbf{v} \in \text{Null}(A^T)$ such that $\mathbf{b} = \mathbf{u} + \mathbf{v}$. Thus, using Definition 5.7.2 and Theorem 5.7.1, $P_{\mathbb{W}}(\mathbf{b}) = \mathbf{u}$.*

We now give another method to obtain the vector $\mathbf{x}_0$ of Equation 5.7.1.

Corollary 5.7.4. *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$. Then, $\mathbf{x}_0$ is a least square solution of $A\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}_0$ is a solution of the system $A^T A\mathbf{x} = A^T\mathbf{b}$.*

Proof. As $\mathbf{b} \in \mathbb{R}^m$, by Remark 5.7.3, there exists $\mathbf{y} \in \text{Col}(A)$ and $\mathbf{v} \in \text{Null}(A^T)$ such that $\mathbf{b} = \mathbf{y} + \mathbf{v}$ and $\min\{\|\mathbf{b} - \mathbf{w}\| \mid \mathbf{w} \in \text{Col}(A)\} = \|\mathbf{b} - \mathbf{y}\|$. As $\mathbf{y} \in \text{Col}(A)$, there exists $\mathbf{x}_0 \in \mathbb{R}^n$ such that $A\mathbf{x}_0 = \mathbf{y}$, *i.e.*,

$$\min\{\|\mathbf{b} - \mathbf{w}\| \mid \mathbf{w} \in \text{Col}(A)\} = \|\mathbf{b} - \mathbf{y}\| = \|\mathbf{b} - A\mathbf{x}_0\|.$$

Hence $(A^T A)\mathbf{x}_0 = A^T (A\mathbf{x}_0) = A^T\mathbf{y} = A^T (\mathbf{b} - \mathbf{v}) = A^T\mathbf{b} - \mathbf{0} = A^T\mathbf{b}$ (as $\mathbf{v} \in \text{Null}(A^T)$).

Conversely, let $\mathbf{x}_1 \in \mathbb{R}^n$ be a solution of $A^T A\mathbf{x} = A^T\mathbf{b}$, *i.e.*, $A^T (A\mathbf{x}_1 - \mathbf{b}) = \mathbf{0}$. To show

$$\min\{\|\mathbf{b} - A\mathbf{x}\| \mid \mathbf{x} \in \mathbb{R}^n\} = \|\mathbf{b} - A\mathbf{x}_1\|.$$

Note that $A^T(A\mathbf{x}_1 - \mathbf{b}) = \mathbf{0}$ implies

$$0 = (\mathbf{x} - \mathbf{x}_1)^T\mathbf{0} = (\mathbf{x} - \mathbf{x}_1)^T A^T(A\mathbf{x}_1 - \mathbf{b}) = (A\mathbf{x} - A\mathbf{x}_1)^T(A\mathbf{x}_1 - \mathbf{b}) = \langle A\mathbf{x}_1 - \mathbf{b}, A\mathbf{x} - A\mathbf{x}_1\rangle.$$

Thus, the vectors $\mathbf{b} - A\mathbf{x}_1$ and $A\mathbf{x}_1 - A\mathbf{x}$ are orthogonal and hence

$$\|\mathbf{b} - A\mathbf{x}\|^2 = \|\mathbf{b} - A\mathbf{x}_1 + A\mathbf{x}_1 - A\mathbf{x}\|^2 = \|\mathbf{b} - A\mathbf{x}_1\|^2 + \|A\mathbf{x}_1 - A\mathbf{x}\|^2 \geq \|\mathbf{b} - A\mathbf{x}_1\|^2.$$

Thus, $\min\{\|\mathbf{b} - A\mathbf{x}\| \mid \mathbf{x} \in \mathbb{R}^n\} = \|\mathbf{b} - A\mathbf{x}_1\|$. Hence, the required result follows. ■

We now give two examples to relate the above discussions.

Example 5.7.5. 1. Determine the projection of $(1,1,1)^T$ on $\textsc{Null}([1,1,-1])$.

Solution: Here $A = [1,1,-1]$, a basis of $\textsc{Null}(A)$ equals $\{(1,-1,0)^T, (1,0,1)^T\}$, which is not an orthonormal set. Also, a basis of $\textsc{Col}(A^T)$ equals $\{(1,1,-1)^T\}$.

(a) **Method 1:** Observe that $\{\frac{1}{\sqrt{2}}(1,-1,0)^T, \frac{1}{\sqrt{6}}(1,1,2)^T\}$ is a basis of $\textsc{Null}(A)$. Thus, the projection matrix $P = \frac{1}{2}\begin{bmatrix}1\\-1\\0\end{bmatrix}\begin{bmatrix}1 & -1 & 0\end{bmatrix} + \frac{1}{6}\begin{bmatrix}1\\1\\2\end{bmatrix}\begin{bmatrix}1 & 1 & 2\end{bmatrix} = \begin{bmatrix}2/3 & -1/3 & 1/3\\-1/3 & 2/3 & 1/3\\1/3 & 1/3 & 1/3\end{bmatrix}$ and $P\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}2/3\\2/3\\4/3\end{bmatrix}$.

DRAFT

(b) **Method 2:** Then the columns of $B = \begin{bmatrix}1 & 1 & 1\\-1 & 0 & 1\\0 & 1 & -1\end{bmatrix}$ form a basis of $\mathbb{R}^3$. Then $\mathbf{x} = \frac{1}{3}\begin{bmatrix}-2\\4\\1\end{bmatrix}$ is a solution of $B\mathbf{x} = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Thus, we see that $(1,1,1)^T = \mathbf{u} + \mathbf{v}$, where $\mathbf{u} = \frac{1}{3}(1,1,-1)^T \in \textsc{Col}(A^T)$ and $\mathbf{v} = \left(\frac{-2}{3}(1,-1,0)^T + \frac{4}{3}(1,0,1)^T\right) = \frac{2}{3}(1,1,2)^T \in \textsc{Null}(A)$. Thus, the required projection equals $\mathbf{v} = \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)^T$.

(c) **Method 3:** Since we want the projection on $\textsc{Null}(A)$. Consider $B = \begin{bmatrix}1 & 1\\-1 & 0\\0 & 1\end{bmatrix}$. Then $\textsc{Null}(A) = \textsc{Col}(B)$. Thus, we need the vector $\mathbf{x}_0$, a solution of the linear system $B^TB\mathbf{x} = B^T\begin{bmatrix}1\\1\\1\end{bmatrix}$. Or equivalently, we need the solution of $\begin{bmatrix}2 & 1\\1 & 2\end{bmatrix}\mathbf{x} = \begin{bmatrix}0\\2\end{bmatrix}$. The solution $\mathbf{x}_0 = \frac{2}{3}\begin{bmatrix}-1\\2\end{bmatrix}$. Thus, the projection vector equals $B\mathbf{x}_0 = \mathbf{v} = \left(\frac{2}{3}, \frac{2}{3}, \frac{4}{3}\right)^T$.

2. Find the foot of the perpendicular from the point $\mathbf{v} = (1,2,3,4)^T$ on the plane generated by the vectors $(1,1,0,0)^T, (1,0,1,0)^T$ and $(0,1,1,1)^T$.

(a) **Method 1:** Note that the three vectors lie on the plane $x - y - z - 2w = 0$. Then $\mathbf{r} = (1, -1, -1, 2)^T$ is the normal vector of the plane. Hence

$$\mathbf{v} - \mathrm{Proj}_{\mathbf{r}}\mathbf{v} = (1, 2, 3, 4)^T - \frac{4}{7}(1, -1, -1, 2)^T = \frac{1}{7}(3, 18, 25, 20)^T$$

is the required projection of $\mathbf{v}$.

(b) **Method 2:** Using the Gram-Schmidt process, we get

$$\mathbf{w}_1 = \frac{1}{\sqrt{2}}(1, 1, 0, 0)^T, \mathbf{w}_2 = \frac{1}{\sqrt{6}}(1, -1, 2, 0)^T, \mathbf{w}_3 = \frac{1}{\sqrt{21}}(-2, 2, 2, 3)^T$$

as an orthonormal basis of the plane generated by the vectors $(1, 1, 0, 0)^T, (1, 0, 1, 0)^T$ and $(0, 1, 1, 1)^T$. Thus, the projection matrix equals

$$P = \sum_{i=1}^{3} \mathbf{w}_i \mathbf{w}_i^T = \begin{bmatrix} 6/7 & 1/7 & 1/7 & -2/7 \\ 1/7 & 6/7 & -1/7 & 2/7 \\ 1/7 & -1/7 & 6/7 & 2/7 \\ -2/7 & 2/7 & 2/7 & 3/7 \end{bmatrix} \text{ and } P\mathbf{v} = \frac{1}{7}\begin{bmatrix} 3 \\ 18 \\ 25 \\ 20 \end{bmatrix}.$$

(c) **Method 3:** Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then we need $\mathbf{x}_0$ satisfying $(A^T A)\mathbf{x} = A^T\mathbf{b}$. Here $A^T A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $A^T\mathbf{b} = \begin{bmatrix} 3 \\ 4 \\ 9 \end{bmatrix}$. Note that $(A^T A)^{-1} = \frac{1}{7}\begin{bmatrix} 5 & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 3 \end{bmatrix}$ and hence the solution of the system $(A^T A)\mathbf{x} = A^T\mathbf{b}$ equals

DRAFT

$$\mathbf{x} = (A^T A)^{-1}(A^T\mathbf{b}) = \frac{1}{7}\begin{bmatrix} 5 & -2 & -1 \\ -2 & 5 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 3 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -2 \\ 5 \\ 20 \end{bmatrix}.$$

Thus, $A\mathbf{x} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{7}\begin{bmatrix} -2 \\ 5 \\ 20 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 3 \\ 18 \\ 25 \\ 20 \end{bmatrix}$ is the nearest vector to $\mathbf{v} = (1, 2, 3, 4)^T$.

EXERCISE **5.7.6.** *1. Let $\mathbb{W} = \{(x, y, z, w) \in \mathbb{R}^4 : x = y, z = w\}$ be a subspace of $\mathbb{R}^4$. Determine the matrix of the orthogonal projection.*

2. Let $P_{\mathbb{W}_1}$ and $P_{\mathbb{W}_2}$ be the orthogonal projections of $\mathbb{R}^2$ onto $\mathbb{W}_1 = \{(x, 0) : x \in \mathbb{R}\}$ and $\mathbb{W}_2 = \{(x, x) : x \in \mathbb{R}\}$, respectively. Note that $P_{\mathbb{W}_1} \circ P_{\mathbb{W}_2}$ is a projection onto $\mathbb{W}_1$. But, it is not an orthogonal projection. Hence or otherwise, conclude that the composition of two orthogonal projections need not be an orthogonal projection?

3. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$. Then, A is idempotent but not symmetric. Now, define $P : \mathbb{R}^2 \to \mathbb{R}^2$ by $P(\mathbf{v}) = A\mathbf{v}$, for all $\mathbf{v} \in \mathbb{R}^2$. Then,

(a) *P is idempotent.*

(b) $\text{NULL}(P) \cap \text{RNG}(P) = \text{NULL}(A) \cap \text{COL}(A) = \{\mathbf{0}\}$.

(c) $\mathbb{R}^2 = \text{NULL}(P) + \text{RNG}(P)$. *But,* $(\text{RNG}(P))^\perp = (\text{COL}(A))^\perp \neq \text{NULL}(A)$.

(d) *Since* $(\text{COL}(A))^\perp \neq \text{NULL}(A)$, *the map P is not an orthogonal projector. In this case, P is called a projection of $\mathbb{R}^2$ onto* $\text{RNG}(P)$ *along* $\text{NULL}(P)$.

4. *Find all 2×2 real matrices A such that $A^2 = A$. Hence, or otherwise, determine all projection operators of $\mathbb{R}^2$.*

5. *Let $\mathbb{W}$ be an $(n-1)$-dimensional subspace of $\mathbb{R}^n$ with ordered basis $\mathcal{B}_{\mathbb{W}} = [\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}]$. Suppose $\mathcal{B} = [\mathbf{f}_1, \ldots, \mathbf{f}_{n-1}, \mathbf{f}_n]$ is an orthogonal ordered basis of $\mathbb{R}^n$ obtained by extending $\mathcal{B}_{\mathbb{W}}$. Now, define a function $Q : \mathbb{R}^n \to \mathbb{R}^n$ by $Q(\mathbf{v}) = \langle \mathbf{v}, \mathbf{f}_n \rangle \mathbf{f}_n - \sum\limits_{i=1}^{n-1} \langle \mathbf{v}, \mathbf{f}_i \rangle \mathbf{f}_i$. Then,*

(a) *Q fixes every vector in $\mathbb{W}^\perp$.*

(b) *Q sends every vector $\mathbf{w} \in \mathbb{W}$ to $-\mathbf{w}$.*

(c) $Q \circ Q = I_n$.

The function Q is called the **reflection operator** *with respect to $\mathbb{W}^\perp$.*

6. *Let $\{\mathbf{f}_1, \ldots, \mathbf{f}_k\}$ be an orthonormal basis of a subspace $\mathbb{W}$ of $\mathbb{R}^n$. If $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is an extended orthonormal basis of $\mathbb{R}^n$, define $P_{\mathbb{W}} = \sum\limits_{i=1}^{k} \mathbf{f}_i \mathbf{f}_i^T$ and $P_{\mathbb{W}^\perp} = \sum\limits_{i=k+1}^{n} \mathbf{f}_i \mathbf{f}_i^T$. Then prove that*

(a) $I_n - P_{\mathbb{W}} = P_{\mathbb{W}^\perp}$.

(b) $(P_{\mathbb{W}})^T = P_{\mathbb{W}}$ *and* $(P_{\mathbb{W}^\perp})^T = P_{\mathbb{W}^\perp}$. *That is, $P_{\mathbb{W}}$ and $P_{\mathbb{W}^\perp}$ are symmetric.*

(c) $(P_{\mathbb{W}})^2 = P_{\mathbb{W}}$ *and* $(P_{\mathbb{W}^\perp})^2 = P_{\mathbb{W}^\perp}$. *That is, $P_{\mathbb{W}}$ and $P_{\mathbb{W}^\perp}$ are idempotent.*

(d) $P_{\mathbb{W}} \circ P_{\mathbb{W}^\perp} = P_{\mathbb{W}^\perp} \circ P_{\mathbb{W}} = \mathbf{0}$.

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5.8 Orthogonal Operator and Rigid Motion*

We now give the definition and a few properties of an orthogonal operator in $\mathbb{R}^n$.

Definition 5.8.1. A linear operator $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an **orthogonal operator** if $\|T(\mathbf{x})\| = \|\mathbf{x}\|$, for all $\mathbf{x} \in \mathbb{R}^n$.

Example 5.8.2. Prove that the following maps T are orthogonal operators.

1. Fix a unit vector $\mathbf{a} \in \mathbb{R}^n$ and define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(\mathbf{x}) = 2\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a} - \mathbf{x}$, for all $\mathbf{x} \in \mathbb{R}^n$.
Solution: Note that $\text{Proj}_{\mathbf{a}}(\mathbf{x}) = \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}$. So, $\big\langle \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a},\ \mathbf{x} - \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a} \big\rangle = 0$ and

$$\|\mathbf{x}\|^2 = \|\mathbf{x} - \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a} + \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2 = \|\mathbf{x} - \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2 + \|\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2.$$

Thus, $\|\mathbf{x} - \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2 = \|\mathbf{x}\|^2 - \|\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2$ and hence

$$\|T(\mathbf{x})\|^2 = \|(\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}) + (\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a} - \mathbf{x})\|^2 = \|\langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2 + \|\mathbf{x} - \langle \mathbf{x}, \mathbf{a} \rangle \mathbf{a}\|^2 = \|\mathbf{x}\|^2.$$

2. Fix $\theta, 0 \le \theta < 2\pi$ and define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$, for all $\mathbf{x} \in \mathbb{R}^2$.

   Solution: Note that $\|T(\mathbf{x})\| = \| \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} \| = \sqrt{x^2+y^2} = \| \begin{bmatrix} x \\ y \end{bmatrix} \|$.

We now show that an operator is orthogonal if and only if it preserves the angle.

Theorem 5.8.3. *Let $T \in \mathcal{L}(\mathbb{R}^n)$. Then, the following statements are equivalent.*

1. *T is an orthogonal operator.*
2. *$\langle T(\mathbf{x}), T(\mathbf{y}) \rangle = \langle \mathbf{x}, \mathbf{y} \rangle$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, i.e., T preserves inner product.*

Proof. $1 \Rightarrow 2$ Let T be an orthogonal operator. Then, $\|T(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}+\mathbf{y}\|^2$. So, $\|T(\mathbf{x})\|^2 + \|T(\mathbf{y})\|^2 + 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \|T(\mathbf{x}) + T(\mathbf{y})\|^2 = \|T(\mathbf{x}+\mathbf{y})\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 + 2\langle \mathbf{x}, \mathbf{y}\rangle$. Thus, using definition again $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$.

$2 \Rightarrow 1$ If $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ then T is an orthogonal operator as $\|T(\mathbf{x})\|^2 = \langle T(\mathbf{x}), T(\mathbf{x})\rangle = \langle \mathbf{x}, \mathbf{x}\rangle = \|\mathbf{x}\|^2$. ∎

As an immediate corollary, we obtain the following result.

Corollary 5.8.4. *Let $T \in \mathcal{L}(\mathbb{R}^n)$. Then, T is an orthogonal operator if and only if "for every orthonormal basis $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{R}^n$, $\{T(\mathbf{u}_1), \ldots, T(\mathbf{u}_n)\}$ is an orthonormal basis of $\mathbb{R}^n$". Thus, if $\mathcal{B}$ is an orthonormal ordered basis of $\mathbb{R}^n$ then $T[\mathcal{B}, \mathcal{B}]$ is an orthogonal matrix.*

DRAFT

Definition 5.8.5. A map $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be an **isometry or a rigid motion** if $\|T(\mathbf{x}) - T(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. That is, an isometry is distance preserving.

Observe that if T and S are two rigid motions then ST is also a rigid motion. Furthermore, it is clear from the definition that every rigid motion is invertible.

Example 5.8.6. The maps given below are rigid motions/isometry.

1. Let $\|\cdot\|$ be a norm in $\mathbb{R}^n$. If $\mathbf{a} \in \mathbb{R}^n$ then the translation map $T_{\mathbf{a}} : \mathbb{R}^n \to \mathbb{R}^n$ defined by $T_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} + \mathbf{a}$ for all $\mathbf{x} \in \mathbb{R}^n$, is an isometry/rigid motion as

$$\|T_{\mathbf{a}}(\mathbf{x}) - T_{\mathbf{a}}(\mathbf{y})\| = \|(\mathbf{x}+\mathbf{a}) - (\mathbf{y}+\mathbf{a})\| = \|\mathbf{x} - \mathbf{y}\|.$$

2. Theorem 5.8.3 implies that every orthogonal operator is an isometry.

We now prove that every rigid motion that fixes origin is an orthogonal operator.

Theorem 5.8.7. *The following statements are equivalent for any map $T : \mathbb{R}^n \to \mathbb{R}^n$.*

1. *T is a rigid motion that fixes origin.*
2. *T is linear and $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ (preserves inner product).*
3. *T is an orthogonal operator.*

Proof. We have already seen the equivalence of Part 2 and Part 3 in Theorem 5.8.3. Let us now prove the equivalence of Part 1 and Part 2/Part 3.

If T is an orthogonal operator then $T(\mathbf{0}) = \mathbf{0}$ and $\|T(\mathbf{x}) - T(\mathbf{y})\| = \|T(\mathbf{x} - \mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$. This proves Part 3 implies Part 1.

We now prove Part 1 implies Part 2. So, let T be a rigid motion that fixes $\mathbf{0}$. Thus, $T(\mathbf{0}) = \mathbf{0}$ and $\|T(\mathbf{x}) - T(\mathbf{y})\| = \|\mathbf{x} - \mathbf{y}\|$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Hence, in particular for $\mathbf{y} = \mathbf{0}$, we have $\|T(\mathbf{x})\| = \|\mathbf{x}\|$, for all $\mathbf{x} \in \mathbb{R}^n$. So,

$$\|T(\mathbf{x})\|^2 + \|T(\mathbf{y})\|^2 - 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \|T(\mathbf{x}) - T(\mathbf{y})\|^2 = \|\mathbf{x} - \mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 - 2\langle \mathbf{x}, \mathbf{y}\rangle.$$

Thus, using $\|T(\mathbf{x})\| = \|\mathbf{x}\|$, for all $\mathbf{x} \in \mathbb{R}^n$, we get $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Now, to prove T is linear, we use $\langle T(\mathbf{x}), T(\mathbf{y})\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, in 3-rd and 4-th line below to get

$$\begin{aligned}
\|T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y}))\|^2 &= \langle T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y})), T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y}))\rangle \\
&= \langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{x}+\mathbf{y})\rangle - 2\langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{x})\rangle \\
&\quad -2\langle T(\mathbf{x}+\mathbf{y}), T(\mathbf{y})\rangle + \langle T(\mathbf{x}) + T(\mathbf{y}), T(\mathbf{x}) + T(\mathbf{y})\rangle \\
&= \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle - 2\langle \mathbf{x}+\mathbf{y}, \mathbf{x}\rangle - 2\langle \mathbf{x}+\mathbf{y}, \mathbf{y}\rangle \\
&\quad +\langle T(\mathbf{x}), T(\mathbf{x})\rangle + 2\langle T(\mathbf{x}), T(\mathbf{y})\rangle + \langle T(\mathbf{y}), T(\mathbf{y})\rangle \\
&= -\langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle + \langle \mathbf{x}, \mathbf{x}\rangle + 2\langle \mathbf{x}, \mathbf{y}\rangle + \langle \mathbf{y}, \mathbf{y}\rangle = 0.
\end{aligned}$$

Thus, $T(\mathbf{x}+\mathbf{y}) - (T(\mathbf{x}) + T(\mathbf{y})) = \mathbf{0}$ and hence $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$. A similar calculation gives $T(\alpha\mathbf{x}) = \alpha T(\mathbf{x})$ and hence T is linear. ∎

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EXERCISE **5.8.8.** *1. Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, A and B are said to be*

(a) **Orthogonally Congruent** *if $B = S^T AS$, for some orthogonal matrix S.*

(b) **Unitarily Congruent** *if $B = S^* AS$, for some unitary matrix S.*

Prove that Orthogonal and Unitary congruences are equivalence relations on $\mathbb{M}_n(\mathbb{R})$ and $\mathbb{M}_n(\mathbb{C})$, respectively.

2. *Let $\mathbf{x} \in \mathbb{C}^2$. Identify it with the complex number $\mathbf{x} = \mathbf{x}_1 + i\mathbf{x}_2$. If we rotate $\mathbf{x}$ by a counterclockwise rotation $\theta, 0 \le \theta < 2\pi$ then, we have*

$$\mathbf{x}e^{i\theta} = (\mathbf{x}_1 + i\mathbf{x}_2)(\cos\theta + i\sin\theta) = \mathbf{x}_1\cos\theta - \mathbf{x}_2\sin\theta + i[\mathbf{x}_1\sin\theta + \mathbf{x}_2\cos\theta].$$

Thus, the corresponding vector in $\mathbb{R}^2$ is

$$\begin{bmatrix} x_1\cos\theta - x_2\sin\theta \\ x_1\sin\theta + x_2\cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Is the matrix, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, the matrix of the corresponding rotation? Justify.

3. *Let $A \in M_2(\mathbb{R})$ and $T(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, for $\theta \in \mathbb{R}$. Then, A is an orthogonal matrix if and only if $A = T(\theta)$ or $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} T(\theta)$, for some $\theta \in \mathbb{R}$.*

4. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then, the following statements are equivalent.*

   (a) *A is an orthogonal matrix.*

   (b) *$A^{-1} = A^T$.*

   (c) *A^T is orthogonal.*

   (d) *the columns of A form an orthonormal basis of the real vector space $\mathbb{R}^n$.*

   (e) *the rows of A form an orthonormal basis of the real vector space $\mathbb{R}^n$.*

   (f) *for any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, $\langle A\mathbf{x}, A\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ Orthogonal matrices preserve angle.*

   (g) *for any vector $\mathbf{x} \in \mathbb{R}^n$, $\|A\mathbf{x}\| = \|\mathbf{x}\|$ Orthogonal matrices preserve length.*

5. *Let $U \in \mathbb{M}_n(\mathbb{C})$. Then, prove that the following statements are equivalent.*

   (a) *U is a unitary matrix.*

   (b) *$U^{-1} = U^*$.*

   (c) *U^* is unitary.*

   (d) *the columns of U form an orthonormal basis of the complex vector space $\mathbb{C}^n$.*

   (e) *the rows of U form an orthonormal basis of the complex vector space $\mathbb{C}^n$.*

   (f) *for any two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, $\langle U\mathbf{x}, U\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ Unitary matrices preserve angle.*

   (g) *for any vector $\mathbf{x} \in \mathbb{C}^n$, $\|U\mathbf{x}\| = \|\mathbf{x}\|$ Unitary matrices preserve length.*

DRAFT

6. *If $A = [a_{ij}]$ and $B = [b_{ij}]$ are unitarily equivalent then prove that $\sum\limits_{ij} |a_{ij}|^2 = \sum\limits_{ij} |b_{ij}|^2$.*

7. *Let U be a unitary matrix and for every $\mathbf{x} \in \mathbb{C}^n$, define*

$$\|\mathbf{x}\|_1 = \max\{|\mathbf{x}_i| : \mathbf{x}^T = [\mathbf{x}_1, \ldots, \mathbf{x}_n]\}.$$

   Then, is it necessary that $\|U\mathbf{x}\|_1 = \|\mathbf{x}\|_1$?

5.9 Dual Space*

Definition 5.9.1. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Then a map $T \in \mathcal{L}(\mathbb{V}, \mathbb{F})$ is called a **linear functional** on $\mathbb{V}$.

Example 5.9.2.
1. Let $\mathbf{a} \in \mathbb{C}^n$ be fixed. Then, $T(\mathbf{x}) = \mathbf{a}^*\mathbf{x}$ is a linear function from $\mathbb{C}^n$ to $\mathbb{C}$.
2. Define $T(A) = \mathsf{tr}(\mathsf{A})$, for all $A \in \mathbb{M}_n(\mathbb{R})$. Then, T is a linear functional from $\mathbb{M}_n(\mathbb{R})$ to $\mathbb{R}$.
3. Define $T(f) = \int\limits_a^b f(t)dt$, for all $f \in \mathcal{C}([a,b],\mathbb{R})$. Then, T is a linear functional from $\mathcal{L}(\mathcal{C}([a,b],\mathbb{R})$ to $\mathbb{R}$.
4. Define $T(f) = \int\limits_a^b t^2 f(t)dt,$, for all $f \in \mathcal{C}([a,b],\mathbb{R})$. Then, T is a linear functional from $\mathcal{L}(\mathcal{C}([a,b],\mathbb{R})$ to $\mathbb{R}$.

5. Define $T : \mathbb{C}^3 \to \mathbb{C}$ by $T((x, y, z)^T) = x$. Is it a linear functional?
6. Let $\mathcal{B}$ be a basis of a vector space $\mathbb{V}$ over $\mathbb{F}$. For a fixed element $\mathbf{u} \in \mathcal{B}$, define

$$T(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{u} \\ 0 & \text{if } \mathbf{x} \in \mathcal{B} \setminus \mathbf{u}. \end{cases}$$

Now, extend T linearly to all of $\mathbb{V}$. Does, T give rise to a linear functional?

Definition 5.9.3. Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Then $\mathcal{L}(\mathbb{V}, \mathbb{F})$ is called the **dual space** of $\mathbb{V}$ and is denoted by $\mathbb{V}^*$. The **double dual space** of $\mathbb{V}$, denoted $\mathbb{V}^{**}$, is the dual space of $\mathbb{V}^*$.

We first give an immediate corollary of Theorem 5.3.12.

Corollary 5.9.4. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ with $\dim \mathbb{V} = n$ and $\dim \mathbb{W} = m$.*

1. *Then $\mathcal{L}(\mathbb{V}, \mathbb{W}) \cong \mathbb{F}^{mn}$. Moreover, $\{\mathbf{f}_{ij} | 1 \le i \le n, 1 \le j \le m\}$ is a basis of $\mathcal{L}(\mathbb{V}, \mathbb{W})$.*
2. *In particular, if $\mathbb{W} = \mathbb{F}$ then $\mathcal{L}(\mathbb{V}, \mathbb{F}) = \mathbb{V}^* \cong \mathbb{F}^n$. Moreover, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of $\mathbb{V}$ then the set $\{\mathbf{f}_i | 1 \le i \le n\}$ is a basis of $\mathbb{V}^*$, where $\mathbf{f}_i(\mathbf{v}_k) = \begin{cases} 1, & \text{if } k = i \\ \mathbf{0}, & k \ne i. \end{cases}$ The basis $\{\mathbf{f}_i | 1 \le i \le n\}$ is called the* **dual basis** *of $\mathbb{F}^n$.*

EXERCISE **5.9.5.** *Let $\mathbb{V}$ be a vector space. Suppose there exists $\mathbf{v} \in \mathbb{V}$ such that $\mathbf{f}(\mathbf{v}) = 0$, for all $\mathbf{f} \in \mathbb{V}^*$. Then prove that $\mathbf{v} = \mathbf{0}$.*

So, we see that $\mathbb{V}^*$ can be understood through a basis of $\mathbb{V}$. Thus, one can understand $\mathbb{V}^{**}$ again via a basis of $\mathbb{V}^*$. But, the question arises "can we understand it directly via the vector space $\mathbb{V}$ itself?" We answer this in affirmative by giving a canonical isomorphism from $\mathbb{V}$ to $\mathbb{V}^{**}$. To do so, for each $\mathbf{v} \in \mathbb{V}$, we define a map $L_{\mathbf{v}} : \mathbb{V}^* \to \mathbb{F}$ by $L_{\mathbf{v}}(\mathbf{f}) = \mathbf{f}(\mathbf{v})$, for each $\mathbf{f} \in \mathbb{V}^*$. Then $L_{\mathbf{v}}$ is a linear functional as

DRAFT

$$L_{\mathbf{v}}(\alpha \mathbf{f} + \mathbf{g}) = (\alpha \mathbf{f} + \mathbf{g})(\mathbf{v}) = \alpha \mathbf{f}(\mathbf{v}) + \mathbf{g}(\mathbf{v}) = \alpha L_{\mathbf{v}}(\mathbf{f}) + L_{\mathbf{v}}(\mathbf{g}).$$

So, for each $\mathbf{v} \in \mathbb{V}$, we have obtained a linear functional $L_{\mathbf{v}} \in \mathbb{V}^{**}$. Note that, if $\mathbf{v} \ne \mathbf{w}$ then, $L_{\mathbf{v}} \ne L_{\mathbf{w}}$. Indeed, if $L_{\mathbf{v}} = L_{\mathbf{w}}$ then, $L_{\mathbf{v}}(f) = L_{\mathbf{w}}(f)$, for all $f \in \mathbb{V}^*$. Thus, $f(\mathbf{v}) = f(\mathbf{w})$, for all $f \in \mathbb{V}^*$. That is, $f(\mathbf{v} - \mathbf{w}) = 0$, for each $f \in \mathbb{V}^*$. Hence, using Exercise 5.9.5, we get $\mathbf{v} - \mathbf{w} = \mathbf{0}$, or equivalently, $\mathbf{v} = \mathbf{w}$.

We use the above argument to give the required canonical isomorphism.

Theorem 5.9.6. *Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. If $\dim(\mathbb{V}) = n$ then the canonical map $T : \mathbb{V} \to \mathbb{V}^{**}$ defined by $T(\mathbf{v}) = L_{\mathbf{v}}$ is an isomorphism.*

Proof. Note that for each $\mathbf{f} \in \mathbb{V}^*$,

$$L_{\alpha \mathbf{v} + \mathbf{u}}(\mathbf{f}) = \mathbf{f}(\alpha \mathbf{v} + \mathbf{u}) = \alpha \mathbf{f}(\mathbf{v}) + \mathbf{f}(\mathbf{u}) = \alpha L_{\mathbf{v}}(\mathbf{f}) + L_{\mathbf{u}}(\mathbf{f}) = (\alpha L_{\mathbf{v}} + L_{\mathbf{u}})(\mathbf{f}).$$

Thus, $L_{\alpha \mathbf{v} + \mathbf{u}} = \alpha L_{\mathbf{v}} + L_{\mathbf{u}}$. Hence, $T(\alpha \mathbf{v} + \mathbf{u}) = \alpha T(\mathbf{v}) + T(\mathbf{u})$. Thus, T is a linear transformation. For verifying T is one-one, assume that $T(\mathbf{v}) = T(\mathbf{u})$, for some $\mathbf{u}, \mathbf{v} \in \mathbb{V}$. Then, $L_{\mathbf{v}} = L_{\mathbf{u}}$. Now, use the argument just before this theorem to get $\mathbf{v} = \mathbf{u}$. Therefore, T is one-one.

Thus, T gives an inclusion (one-one) map from $\mathbb{V}$ to $\mathbb{V}^{**}$. Further, applying Corollary 5.9.4.2 to $\mathbb{V}^*$, gives $\dim(\mathbb{V}^{**}) = \dim(\mathbb{V}^*) = n$. Hence, the required result follows. ∎

We now give a few immediate consequences of Theorem 5.9.6.

Corollary 5.9.7. *Let $\mathbb{V}$ be a vector space of dimension n with basis $\mathcal{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.*

1. *Then, a basis of $\mathbb{V}^{**}$, the double dual of $\mathbb{V}$, equals $\mathcal{D} = \{L_{\mathbf{v}_1}, \ldots, L_{\mathbf{v}_n}\}$. Thus, for each $T \in \mathbb{V}^{**}$ there exists $\mathbf{x} \in \mathbb{V}$ such that $T(\mathbf{f}) = \mathbf{f}(\mathbf{x})$, for all $\mathbf{f} \in \mathbb{V}^*$. Or equivalently, there exists $\mathbf{x} \in \mathbb{V}$ such that $T = T_{\mathbf{x}}$.*
2. *If $\mathcal{C} = \{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is the dual basis of $\mathbb{V}^*$ defined using the basis $\mathcal{B}$ (see Corollary 5.9.4.2) then $\mathcal{D}$ is indeed the dual basis of $\mathbb{V}^{**}$ obtained using the basis $\mathcal{C}$ of $\mathbb{V}^*$. Thus, each basis of $\mathbb{V}^*$ is the dual basis of some basis of $\mathbb{V}$.*

Proof. Part 1 is direct as $T : \mathbb{V} \to \mathbb{V}^{**}$ was a canonical inclusion map. For Part 2, we need to show that

$$L_{\mathbf{v}_i}(\mathbf{f}_j) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases} \quad \text{or equivalently } \mathbf{f}_j(\mathbf{v}_i) = \begin{cases} 1, & \text{if } j = i \\ 0, & \text{if } j \neq i \end{cases}$$

which indeed holds true using Corollary 5.9.4.2. ∎

Let $\mathbb{V}$ be a finite dimensional vector space. Then Corollary 5.9.7 implies that the spaces $\mathbb{V}$ and $\mathbb{V}^*$ are naturally dual to each other.

We are now ready to prove the main result of this subsection. To start with, let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$. Then, for each $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, we want to define a map $\widehat{T} : \mathbb{W}^* \to \mathbb{V}^*$. So, if $g \in \mathbb{W}^*$ then, $\widehat{T}(\mathbf{g})$ a linear functional from $\mathbb{V}$ to $\mathbb{F}$. So, we need to be evaluate $\widehat{T}(\mathbf{g})$ at an element of $\mathbb{V}$. Thus, we define $\left(\widehat{T}(\mathbf{g})\right)(\mathbf{v}) = g\left(T(\mathbf{v})\right)$, for all $\mathbf{v} \in \mathbb{V}$. Now, we note that $\widehat{T} \in \mathcal{L}(\mathbb{W}^*, \mathbb{V}^*)$, as for every $g, h \in \mathbb{W}^*$,

$$\left(\widehat{T}(\alpha\mathbf{g} + \mathbf{h})\right)(\mathbf{v}) = (\alpha\mathbf{g} + \mathbf{h})\left(T(\mathbf{v})\right) = \alpha\mathbf{g}\left(T(\mathbf{v})\right) + \mathbf{h}\left(T(\mathbf{v})\right) = \left(\alpha\widehat{T}(\mathbf{g}) + \widehat{T}(\mathbf{h})\right)(\mathbf{v}),$$

for all $\mathbf{v} \in \mathbb{V}$ implies that $\widehat{T}(\alpha\mathbf{g} + \mathbf{h}) = \alpha\widehat{T}(\mathbf{g}) + \widehat{T}(\mathbf{h})$.

DRAFT

Theorem 5.9.8. *Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces over $\mathbb{F}$ with ordered bases $\mathcal{A} = (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\mathcal{B} = (\mathbf{w}_1, \ldots, \mathbf{w}_m)$, respectively. Also, let $\mathcal{A}^* = (\mathbf{f}_1, \ldots, \mathbf{f}_n)$ and $\mathcal{B}^* = (\mathbf{g}_1, \ldots, \mathbf{g}_m)$ be the corresponding ordered bases of the dual spaces $\mathbb{V}^*$ and $\mathbb{W}^*$, respectively. Then,*

$$\widehat{T}[\mathcal{B}^*, \mathcal{A}^*] = \left(T[\mathcal{A}, \mathcal{B}]\right)^T,$$

the transpose of the coordinate matrix T.

Proof. Note that we need to compute $\widehat{T}[\mathcal{B}^*, \mathcal{A}^*] = \left[\left[\widehat{T}(\mathbf{g}_1)\right]_{\mathcal{A}^*}, \ldots, \left[\widehat{T}(\mathbf{g}_m)\right]_{\mathcal{A}^*}\right]$ and prove that it equals the transpose of the matrix $T[\mathcal{A}, \mathcal{B}]$. So, let

$$T[\mathcal{A}, \mathcal{B}] = \left[[T(\mathbf{v}_1)]_{\mathcal{B}}, \ldots, [T(\mathbf{v}_n)]_{\mathcal{B}}\right] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

Thus, to prove the required result, we need to show that

$$\left[\widehat{T}(\mathbf{g}_j)\right]_{\mathcal{A}^*} = [\mathbf{f}_1, \ldots, \mathbf{f}_n] \begin{bmatrix} a_{j1} \\ a_{j2} \\ \vdots \\ a_{jn} \end{bmatrix} = \sum_{k=1}^{n} a_{jk}\mathbf{f}_k, \text{ for } 1 \leq j \leq m. \tag{5.9.1}$$

Now, recall that the functionals $\mathbf{f}_i$'s and $\mathbf{g}_j$'s satisfy $\left(\sum\limits_{k=1}^{n} \alpha_k \mathbf{f}_k\right)(\mathbf{v}_t) = \sum\limits_{k=1}^{n} \alpha_k \left(\mathbf{f}_k(\mathbf{v}_t)\right) = \alpha_t$, for $1 \le t \le n$ and $[\mathbf{g}_j(\mathbf{w}_1), \ldots, \mathbf{g}_j(\mathbf{w}_m)] = \mathbf{e}_j^T$, a row vector with 1 at the j-th place and 0, elsewhere. So, let $B = [\mathbf{w}_1, \ldots, \mathbf{w}_m]$ and evaluate $\widehat{T}(\mathbf{g}_j)$ at $\mathbf{v}_t$'s, the elements of $\mathcal{A}$.

$$\begin{aligned}
\left(\widehat{T}(\mathbf{g}_j)\right)(\mathbf{v}_t) &= \mathbf{g}_j\left(T(\mathbf{v}_t)\right) = \mathbf{g}_j\left(B\,[T(\mathbf{v}_t)]_{\mathcal{B}}\right) = [\mathbf{g}_j(\mathbf{w}_1), \ldots, \mathbf{g}_j(\mathbf{w}_m)]\,[T(\mathbf{v}_t)]_{\mathcal{B}} \\
&= \mathbf{e}_j^T \begin{bmatrix} a_{1t} \\ a_{2t} \\ \vdots \\ a_{mt} \end{bmatrix} = a_{jt} = \left(\sum_{k=1}^{n} a_{jk}\mathbf{f}_k\right)(\mathbf{v}_t).
\end{aligned}$$

Thus, the linear functional $\widehat{T}(\mathbf{g}_j)$ and $\sum\limits_{k=1}^{n} a_{jk}\mathbf{f}_k$ are equal at $\mathbf{v}_t$, for $1 \le t \le n$, the basis vectors of $\mathbb{V}$. Hence $\widehat{T}(\mathbf{g}_j) = \sum\limits_{k=1}^{n} a_{jk}\mathbf{f}_k$ which gives Equation (5.9.1). ∎

Remark 5.9.9. *The proof of Theorem 5.9.8 also shows the following.*

1. *For each $T \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ there exists a unique map $\widehat{T} \in \mathcal{L}(\mathbb{W}^*, \mathbb{V}^*)$ such that*

$$\left(\widehat{T}(\mathbf{g})\right)(\mathbf{v}) = \mathbf{g}\left(T(\mathbf{v})\right), \text{ for each } \mathbf{g} \in \mathbb{W}^*.$$

2. *The coordinate matrices $T[\mathcal{A}, \mathcal{B}]$ and $\widehat{T}[\mathcal{B}^*, \mathcal{A}^*]$ are transpose of each other, where the ordered bases $\mathcal{A}^*$ of $\mathbb{V}^*$ and $\mathcal{B}^*$ of $\mathbb{W}^*$ correspond, respectively, to the ordered bases $\mathcal{A}$ of $\mathbb{V}$ and $\mathcal{B}$ of $\mathbb{W}$.*
3. *Thus, the results on matrices and its transpose can be re-written in the language of a vector space and its dual space.*

5.10 Summary

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Chapter 6

Eigenvalues, Eigenvectors and Diagonalizability

6.1 Introduction and Definitions

Note that we have been trying to solve the linear system $A\mathbf{x} = \mathbf{b}$. But, in most cases, we are not able to solve it because of certain restrictions. Hence in the last chapter, we looked at the nearest solution or obtained the projection of $\mathbf{b}$ on the column space of A.

DRAFT

These problems arise from the fact that either our data size is too large or there are missing informations (data is incomplete or the data has ambiguities or the data is inaccurate) or the data is coming too fast in the sense that our computational power doesn't match the speed at which data is received or it could be any other reason. So, to take care of such issues, we either work with a submatrix of A or with the matrix A^TA. We also try to concentrate on only a few important aspects depending on our past experience.

Thus, we need to find certain set of critical vectors/directions associated with the given linear system. Hence, in this chapter, all our matrices will be square matrices. They will have real numbers as entries for convenience. But, we need to work over complex numbers. Hence, we will be working with $\mathbb{M}_n(\mathbb{C})$ and $\mathbf{x} = (x_1, \ldots, x_n)^T \in \mathbb{C}^n$, for some $n \in \mathbb{N}$. Further, $\mathbb{C}^n$ will be considered only as a complex vector space. We start with an example for motivation.

Example 6.1.1. Let A be a real symmetric matrix. Consider the following problem:

$$\text{Maximize (Minimize) } \mathbf{x}^T A\mathbf{x} \text{ such that } \mathbf{x} \in \mathbb{R}^n \text{ and } \mathbf{x}^T\mathbf{x} = 1.$$

To solve this, consider the Lagrangian

$$L(\mathbf{x}, \lambda) = \mathbf{x}^T A\mathbf{x} - \lambda(\mathbf{x}^T\mathbf{x} - 1) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}x_ix_j - \lambda\left(\sum_{i=1}^{n} x_i^2 - 1\right).$$

Partially differentiating $L(\mathbf{x},\lambda)$ with respect to x_i for $1 \le i \le n$, we get

$$\begin{aligned}\frac{\partial L}{\partial x_1} &= 2a_{11}x_1 + 2a_{12}x_2 + \cdots + 2a_{1n}x_n - 2\lambda x_1,\\ \vdots &= \vdots\\ \frac{\partial L}{\partial x_n} &= 2a_{n1}x_1 + 2a_{n2}x_2 + \cdots + 2a_{nn}x_n - 2\lambda x_n.\end{aligned}$$

Therefore, to get the points of extremum, we solve for

$$\mathbf{0}^T = \left(\frac{\partial L}{\partial x_1}, \frac{\partial L}{\partial x_2}, \ldots, \frac{\partial L}{\partial x_n}\right)^T = \frac{\partial L}{\partial \mathbf{x}} = 2(A\mathbf{x} - \lambda\mathbf{x}).$$

Thus, to solve the extremal problem, we need $\lambda \in \mathbb{R}$, $\mathbf{x} \in \mathbb{R}^n$ such that $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \lambda\mathbf{x}$.

Note that we could have started with a Hermitian matrix and arrived at a similar situation. So, in previous chapters, we had looked at $A\mathbf{x} = \mathbf{b}$, where A and $\mathbf{b}$ were known. Here, we need to solve $A\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. Note that $\mathbf{0}$ is already a solution and is not of interest to us. Further, we will see that we are interested in only those solutions of $A\mathbf{x} = \lambda\mathbf{x}$ which are linearly independent. To proceed further, let us take a few examples, where we will try to look at what does the system $A\mathbf{x} = \mathbf{b}$ imply?

Example 6.1.2. 1. Let $A = \begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$, $B = \begin{bmatrix}9 & -2\\ -2 & 6\end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix}x\\ y\end{bmatrix}$.

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(a) Then A magnifies the nonzero vector $\begin{bmatrix}1\\ 1\end{bmatrix}$ three times as $A\begin{bmatrix}1\\ 1\end{bmatrix} = 3\begin{bmatrix}1\\ 1\end{bmatrix}$ and behaves by changing the direction of $\begin{bmatrix}1\\ -1\end{bmatrix}$ as $A\begin{bmatrix}1\\ -1\end{bmatrix} = -1\begin{bmatrix}1\\ -1\end{bmatrix}$. Further, the vectors $\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\begin{bmatrix}1\\ -1\end{bmatrix}$ are orthogonal.

(b) B magnifies both the vectors $\begin{bmatrix}1\\ 2\end{bmatrix}$ and $\begin{bmatrix}-2\\ 1\end{bmatrix}$ as $B\begin{bmatrix}1\\ 2\end{bmatrix} = 5\begin{bmatrix}1\\ 2\end{bmatrix}$ and $B\begin{bmatrix}2\\ -1\end{bmatrix} = 10\begin{bmatrix}2\\ -1\end{bmatrix}$. Here again, the vectors $\begin{bmatrix}1\\ 2\end{bmatrix}$ and $\begin{bmatrix}2\\ -1\end{bmatrix}$ are orthogonal.

(c) $\mathbf{x}^T A\mathbf{x} = 3\dfrac{(x+y)^2}{2} - \dfrac{(x-y)^2}{2}$. Here, the displacements occur along perpendicular lines $x+y=0$ and $x-y=0$, where $x+y = (x,y)\begin{bmatrix}1\\ 1\end{bmatrix}$ and $x-y = (x,y)\begin{bmatrix}1\\ -1\end{bmatrix}$. Whereas $\mathbf{x}^T B\mathbf{x} = 5\dfrac{(x+2y)^2}{5} + 10\dfrac{(2x-y)^2}{5}$. Here also the maximum/minimum displacements occur along the orthogonal lines $x+2y=0$ and $2x-y=0$, where $x+2y = (x,y)\begin{bmatrix}1\\ 2\end{bmatrix}$ and $2x-y = (x,y)\begin{bmatrix}2\\ -1\end{bmatrix}$.

(d) the curve $\mathbf{x}^T A\mathbf{x} = 10$ represents a hyperbola, where as the curve $\mathbf{x}^T B\mathbf{x} = 10$ represents an ellipse (see the left two curves in Figure 6.1 drawn using the package "Sagemath").

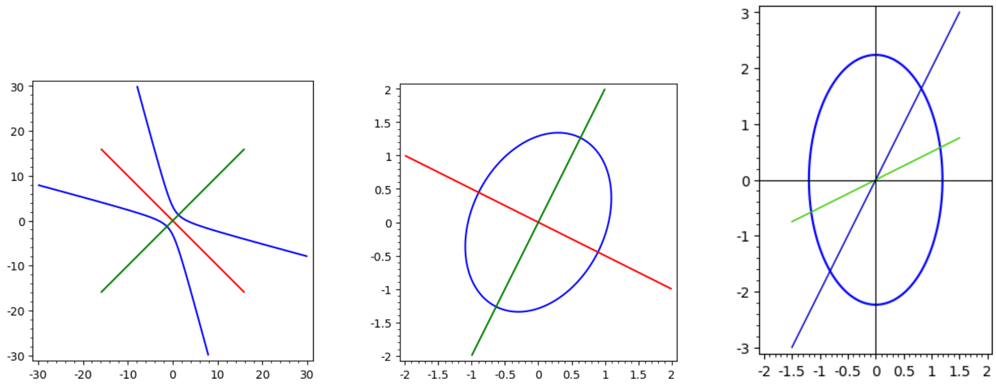

Figure 6.1: A Hyperbola and two Ellipses (first one has orthogonal axes)

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In the above two examples we looked at symmetric matrices. What if our matrix is not symmetric?

2. Let $C = \begin{bmatrix} 7 & -2 \\ 2 & 2 \end{bmatrix}$, a non-symmetric matrix. Then, does there exist a non-zero $\mathbf{x} \in \mathbb{C}^2$ which gets magnified by C?

   We need $\mathbf{x} \neq \mathbf{0}$ and $\alpha \in \mathbb{C}$ such that $C\mathbf{x} = \alpha \mathbf{x} \Leftrightarrow [\alpha I_2 - C]\mathbf{x} = \mathbf{0}$. As $\mathbf{x} \neq 0$, $[\alpha I_2 - C]\mathbf{x} = \mathbf{0}$ has a solution if and only if $\det[\alpha I - A] = 0$. But,

$$\det[\alpha I - A] = \det\left(\begin{bmatrix} \alpha - 7 & 2 \\ -2 & \alpha - 2 \end{bmatrix}\right) = \alpha^2 - 9\alpha + 18.$$

DRAFT

   So $\alpha = 6, 3$. For $\alpha = 6$, verify that $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \neq \mathbf{0}$ satisfies $C\mathbf{x} = 6\mathbf{x}$. Similarly, $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ satisfies $C\mathbf{x} = 3\mathbf{x}$. In this example,

   (a) we still have magnifications in the directions $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

   (b) the maximum/minimum displacements do not occur along the lines $2x + y = 0$ and $x + 2y = 0$ (see the third curve in Figure 6.1). Note that

$$\{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}^T A \mathbf{x} = 10\} = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}^T \begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix} \mathbf{x} = 10\},$$

   where $\begin{bmatrix} 7 & 0 \\ 0 & 2 \end{bmatrix}$ is a symmetrization of A.

   (c) the lines $2x + y = 0$ and $x + 2y = 0$ are not orthogonal.

We observe the following about the matrices A, B and C that appear above:

1. $\det(A) = -3 = 3 \times -1$, $\det(B) = 50 = 5 \times 10$ and $\det(C) = 18 = 6 \times 3$.
2. $\mathsf{tr(A)} = 2 = 3 - 1$, $\mathsf{tr(B)} = 15 = 5 + 10$ and $\det(C) = 9 = 6 + 3$.
3. The sets $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$, $\left\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}\right\}$ and $\left\{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ are linearly independent.

4. If $\mathbf{v}_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\-1\end{bmatrix}$ and $S = [\mathbf{v}_1, \mathbf{v}_2]$ then

   (a) $AS = [A\mathbf{v}_1, A\mathbf{v}_2] = [3\mathbf{v}_1, -\mathbf{v}_2] = S\begin{bmatrix}3 & 0\\0 & -1\end{bmatrix} \Leftrightarrow S^{-1}AS = \begin{bmatrix}3 & 0\\0 & -1\end{bmatrix} = \text{diag}(3, -1)$.

   (b) Let $\mathbf{u}_1 = \frac{1}{\sqrt{2}}\mathbf{v}_1$ and $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\mathbf{v}_2$. Then, $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthonormal unit vectors, *i.e.*, if $U = [\mathbf{u}_1, \mathbf{u}_2]$ then $I = UU^* = \mathbf{u}_1\mathbf{u}_1^* + \mathbf{u}_2\mathbf{u}_2^*$ and $A = 3\mathbf{u}_1\mathbf{u}_1^* - \mathbf{u}_2\mathbf{u}_2^*$.

5. If $\mathbf{v}_1 = \begin{bmatrix}1\\2\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}2\\-1\end{bmatrix}$ and $S = [\mathbf{v}_1, \mathbf{v}_2]$ then

   (a) $AS = [A\mathbf{v}_1, A\mathbf{v}_2] = [5\mathbf{v}_1, 10\mathbf{v}_2] = S\begin{bmatrix}5 & 0\\0 & 10\end{bmatrix} \Leftrightarrow S^{-1}AS = \begin{bmatrix}5 & 0\\0 & 10\end{bmatrix} = \text{diag}(3, -1)$.

   (b) Let $\mathbf{u}_1 = \frac{1}{\sqrt{5}}\mathbf{v}_1$ and $\mathbf{u}_2 = \frac{1}{\sqrt{5}}\mathbf{v}_2$. Then, $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthonormal unit vectors, *i.e.*, if $U = [\mathbf{u}_1, \mathbf{u}_2]$ then $I = UU^* = \mathbf{u}_1\mathbf{u}_1^* + \mathbf{u}_2\mathbf{u}_2^*$ and $A = 5\mathbf{u}_1\mathbf{u}_1^* + 10\mathbf{u}_2\mathbf{u}_2^*$.

6. If $\mathbf{v}_1 = \begin{bmatrix}2\\1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\2\end{bmatrix}$ and $S = [\mathbf{v}_1, \mathbf{v}_2]$ then $S^{-1}CS = \begin{bmatrix}6 & 0\\0 & 3\end{bmatrix} = \text{diag}(6, 3)$.

Thus, we see that given $A \in \mathbb{M}_n(\mathbb{C})$, the number $\lambda \in \mathbb{C}$ and $\mathbf{x} \in \mathbb{C}^n, \mathbf{x} \neq \mathbf{0}$ satisfying $A\mathbf{x} = \lambda\mathbf{x}$ have certain nice properties. For example, all of them are similar to diagonal matrices. That is, for each matrix discussed above, there exists a basis of $\mathbb{C}^2$ with respect to which the matrix representation is a diagonal matrix. To understand the ideas better, we start with the following definitions.

DRAFT

Definition 6.1.3. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then the equation

$$A\mathbf{x} = \lambda\mathbf{x} \Leftrightarrow (A - \lambda I_n)\mathbf{x} = \mathbf{0} \tag{6.1.1}$$

is called the **eigen-condition**.

1. An $\alpha \in \mathbb{C}$ is called a **characteristic value/root** or **eigenvalue** or **latent root** of A if there exists $\mathbf{x} \neq \mathbf{0}$ satisfying $A\mathbf{x} = \alpha\mathbf{x}$.

2. A $\mathbf{x} \neq \mathbf{0}$ satisfying Equation (6.1.1) is called a **characteristic vector** or **eigenvector** or **invariant/latent vector** of A corresponding to λ.

3. The tuple $(\alpha, \mathbf{x})$ with $\mathbf{x} \neq \mathbf{0}$ and $A\mathbf{x} = \alpha\mathbf{x}$ is called an **eigen-pair** or **characteristic-pair**.

4. For an eigenvalue $\alpha \in \mathbb{C}$, $\text{NULL}(A - \alpha I) = \{\mathbf{x} \in \mathbb{R}^n | A\mathbf{x} = \alpha\mathbf{x}\}$ is called the **eigen-space** or **characteristic vector space** of A corresponding to α.

Theorem 6.1.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $\alpha \in \mathbb{C}$. Then α is an eigenvalue of A if and only if $\det(A - \alpha I_n) = 0$.*

Proof. Let $B = A - \alpha I_n$. Then, by definition, α is an eigenvalue of A if any only if the system $B\mathbf{x} = \mathbf{0}$ has a non-trivial solution. By Theorem 2.6.3 this holds if and only if $\det(B) = 0$. ∎

Definition 6.1.5. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $\det(A - \lambda I)$ is a polynomial of degree n in λ and is called the **characteristic polynomial** of A, denoted $P_A(\lambda)$, or in short $P(\lambda)$.

1. The equation $P_A(\lambda) = 0$ is called the **characteristic equation** of A.
2. The multi-set (collection with multiplicities) $\{\alpha \in \mathbb{C} : P_A(\alpha) = 0\}$ is called the **spectrum** of A, denoted $\sigma(A)$. Hence, $\sigma(A)$ contains all the eigenvalues of A containing multiplicities.
3. The **Spectral Radius**, denoted $\rho(A)$, of $A \in \mathbb{M}_n(\mathbb{C})$ equals $\max\{|\alpha| : \alpha \in \sigma(A)\}$.

We thus observe the following.

Remark 6.1.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A is singular if and only if $0 \in \sigma(A)$. Further, the following statements hold.*

1. *If $\alpha \in \sigma(A)$ then*
   (a) *$\{\mathbf{0}\} \subsetneqq \text{NULL}(A - \alpha I)$. Therefore, if $\text{RANK}(A - \alpha I) = r$ then $r < n$. Hence, by Theorem 2.6.3, the system $(A - \alpha I)\mathbf{x} = \mathbf{0}$ has $n - r$ linearly independent solutions.*
   (b) *$\mathbf{v} \in \text{NULL}(A - \alpha I)$ if and only if $c\mathbf{v} \in \text{NULL}(A - \alpha I)$, for $c \neq 0$. Thus, an eigenvector $\mathbf{v}$ of A is in some sense a line $\ell = Span(\{\mathbf{v}\})$ that passes through $\mathbf{0}$ and $\mathbf{v}$ and has the property that the image of ℓ is either ℓ itself or $\mathbf{0}$.*
   (c) *If $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \text{NULL}(A - \alpha I)$ then $\sum\limits_{i=1}^{r} c_i\mathbf{x}_i \in \text{NULL}(A - \alpha I)$, for all $c_i \in \mathbb{C}$. Hence, if S is a collection of eigenvectors then, we necessarily want the set S to be* LINEARLY INDEPENDENT.
2. *$\alpha \in \sigma(A)$ if and only if α is a root of $P_A(x) \in \mathbb{C}[x]$. As $\deg(P_A(x)) = n$, A has exactly n eigenvalues in $\mathbb{C}$, including multiplicities.*
3. *Let $(\alpha, \mathbf{x})$ be an eigen-pair of $A \in \mathbb{M}_n(\mathbb{R})$. If $\alpha \in \mathbb{R}$ then $\mathbf{x} \in \mathbb{R}^n$.*
4. *Let $(\alpha, \mathbf{x})$ be an eigen-pair of A. Then $A^2\mathbf{x} = A(A\mathbf{x}) = A(\alpha\mathbf{x}) = \alpha(A\mathbf{x}) = \alpha^2\mathbf{x}$. Thus, the polynomial $f(A) = b_0 I + b_1 A + \cdots + b_k A^k$ (in A) has $(f(\alpha), \mathbf{x})$ as an eigen-pair.*

Almost all books in mathematics differentiate between characteristic value and eigenvalue as the ideas change when one moves from complex numbers to any other scalar field. We give the following example for clarity.

Remark 6.1.7. *Let $A \in \mathbb{M}_2(\mathbb{F})$. Then, A induces a map $T \in \mathcal{L}(\mathbb{F}^2)$ defined by $T(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{F}^2$. We use this idea to understand the difference.*

1. *Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then, $p_A(\lambda) = \lambda^2 + 1$.*
   (a) *A has no characteristic value in $\mathbb{R}$ as $\lambda^2 + 1 = 0$ has no root in $\mathbb{R}$.*
   (b) *A has $\pm i$ as the roots of $P(\lambda) = 0$ in $\mathbb{C}$. Hence, verify that A has $(i, (1, i)^T)$ and $(-i, (i, 1)^T)$ as eigen-pairs or characteristic-pairs.*
2. *Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$. Then $2 \pm \sqrt{3}$ are the roots of the characteristic equation.*
   (a) *Hence A has no characteristic value over $\mathbb{Q}$.*
   (b) *But A has characteristic values or eigenvalues over $\mathbb{R}$.*

DRAFT

Let us look at some more examples. Also, as stated earlier, we look at roots of the characteristic equation over $\mathbb{C}$.

Example 6.1.8. 1. Let $A = \text{diag}(d_1, \ldots, d_n)$ with $d_i \in \mathbb{C}, 1 \le i \le n$. Then, $p(\lambda) = \prod\limits_{i=1}^{n} (\lambda - d_i)$ and thus verify that $(d_1, \mathbf{e}_1), \ldots, (d_n, \mathbf{e}_n)$ are the eigen-pairs.

2. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Then, 1 is a repeated eigenvalue of A. In this case, $(A - I_2)\mathbf{x} = \mathbf{0}$ has a solution for every $\mathbf{x} \in \mathbb{C}^2$. Hence, any two LINEARLY INDEPENDENT vectors $\mathbf{x}^T, \mathbf{y}^T \in \mathbb{C}^2$ gives $(1, \mathbf{x})$ and $(1, \mathbf{y})$ as the two eigen-pairs for A. In general, if $S = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a basis of $\mathbb{C}^n$ then $(1, \mathbf{x}_1), \ldots, (1, \mathbf{x}_n)$ are eigen-pairs of I_n, the identity matrix.

3. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Then, $p(\lambda) = (1-\lambda)^2$. Hence, $\sigma(A) = \{1, 1\}$. But the complete solution of the system $(A - I_2)\mathbf{x} = \mathbf{0}$ equals $\mathbf{x} = c\mathbf{e}_1$, for $c \in \mathbb{C}$. Hence using Remark 6.1.6.2, $\mathbf{e}_1$ is an eigenvector. Therefore, 1 IS A REPEATED EIGENVALUE WHEREAS THERE IS ONLY ONE EIGENVECTOR.

4. Let $A = (a_{ij})$ be an $n \times n$ triangular matrix. Then, $p(\lambda) = \prod\limits_{i=1}^{n} (\lambda - a_{ii})$ and thus verify that $\sigma(A) = \{a_{11}, a_{22}, \ldots, a_{nn}\}$. What can you say about the eigenvectors if the diagonal entries of A are all distinct?

5. Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. Then, $\left(1+i, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$ and $\left(1-i, \begin{bmatrix} 1 \\ i \end{bmatrix}\right)$ are the eigen-pairs of A.

DRAFT

6. Let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Then, $\sigma(A) = \{0, 0, 0\}$ with $\mathbf{e}_1$ as the only eigenvector.

7. Let $A = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$. Then, $\sigma(A) = \{0, 0, 0, 0, 0\}$. Note that $A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \mathbf{0}$ implies $x_2 = 0 = x_3 = x_5$. Thus, $\mathbf{e}_1$ and $\mathbf{e}_4$ are the only eigenvectors. Note that the diagonal blocks of A are nilpotent matrices.

EXERCISE **6.1.9.** *1. Prove that the matrices A and A^T have the same set of eigenvalues. Construct a 2×2 matrix A such that the eigenvectors of A and A^T are different.*

2. Prove that $\lambda \in \mathbb{C}$ is an eigenvalue of A if and only if $\overline{\lambda} \in \mathbb{C}$ is an eigenvalue of A^*.

3. Let $A \in \mathbb{M}_n(\mathbb{R})$ be an idempotent matrix. Determine its eigenvalues and eigenvectors.

4. Let A be a nilpotent matrix. Then, prove that its eigenvalues are all 0.

5. Let $J = \mathbf{1}\mathbf{1}^T \in \mathbb{M}_n(\mathbb{C})$. Then, J is a matrix with each entry 1. Show that

(a) $(n, \mathbf{1})$ *is an eigenpair for* J.

(b) $0 \in \sigma(J)$ *with multiplicity* $n-1$. *Find a set of* $n-1$ *linearly independent eigenvectors for* $0 \in \sigma(J)$.

6. *Let* $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{R})$, *where* $a_{ij} = a$, *if* $i = j$ *and* b, *otherwise. Then, verify that* $A = (a-b)I + bJ$. *Hence, or otherwise determine the eigenvalues and eigenvectors of* J.

7. *Let* $A \in \mathbb{M}_n(\mathbb{R})$ *be a matrix of rank* 1. *Determine its eigen-pairs.*

8. *For a fixed* $\theta \in \mathbb{R}$, *find eigen-pairs of* $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ *and* $R = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$.

9. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *satisfy* $\|A\mathbf{x}\| \le \|\mathbf{x}\|$ *for all* $\mathbf{x} \in \mathbb{C}^n$. *Then prove that every eigenvalue of* A *lies between* -1 *and* 1.

10. *Let* $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ *with* $\sum\limits_{j=1}^{n} a_{ij} = a$, *for all* $1 \le i \le n$. *Then, prove that* a *is an eigenvalue of* A *with corresponding eigenvector* $\mathbf{1} = [1, 1, \ldots, 1]^T$.

11. *Let* $B \in \mathbb{M}_n(\mathbb{C})$ *and* $C \in \mathbb{M}_m(\mathbb{C})$. *Let* $Z = \begin{bmatrix} B & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix}$. *Then*

(a) $(\alpha, \mathbf{x})$ *is an eigen-pair for* B *implies* $\left(\alpha, \begin{bmatrix} \mathbf{x} \\ \mathbf{0} \end{bmatrix}\right)$ *is an eigen-pair for* Z.

(b) $(\beta, \mathbf{y})$ *is an eigen-pair for* C *implies* $\left(\beta, \begin{bmatrix} \mathbf{0} \\ \mathbf{y} \end{bmatrix}\right)$ *is an eigen-pair for* Z.

DRAFT

Definition 6.1.10. Let $A \in \mathcal{L}(\mathbb{C}^n)$. Then, a vector $\mathbf{y} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ satisfying $\mathbf{y}^* A = \lambda \mathbf{y}^*$ is called a **left eigenvector** of A for λ.

Example 6.1.11. Let $A = \begin{bmatrix} 7 & -2 \\ 2 & 2 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. Then verify that $(6, \mathbf{x})$ and $(3, \mathbf{y})$ are (right) eigen-pairs of A and $(6, \mathbf{u})$, $(3, \mathbf{v})$ are left eigen-pairs of A. Note that $\mathbf{x}^T \mathbf{v} = 0$ and $\mathbf{y}^T \mathbf{u} = 0$. This is true in general and is proved next.

Theorem 6.1.12. [**Principle of bi-orthogonality**] *Let* $(\lambda, \mathbf{x})$ *be a (right) eigen-pair and* $(\mu, \mathbf{y})$ *be a left eigen-pair of* A. *If* $\lambda \neq \mu$ *then* $\mathbf{y}$ *is orthogonal to* $\mathbf{x}$.

Proof. Verify that $\mu \mathbf{y}^* \mathbf{x} = (\mathbf{y}^* A)\mathbf{x} = \mathbf{y}^*(A\mathbf{x}) = \mathbf{y}^*(\lambda \mathbf{x}) = \lambda \mathbf{y}^* \mathbf{x}$. Thus $\mathbf{y}^* \mathbf{x} = 0$. ∎

EXERCISE **6.1.13.** 1. *Let* $A\mathbf{x} = \lambda \mathbf{x}$ *and* $\mathbf{x}^* A = \mu \mathbf{x}^*$. *Then* $\mu = \lambda$.

2. *Let* S *be a non-singular matrix such that its columns are left eigenvectors of* A. *Then, prove that the columns of* $(S^*)^{-1}$ *are right eigenvectors of* A.

Definition 6.1.14. Let $T \in \mathcal{L}(\mathbb{C}^n)$. Then $\alpha \in \mathcal{C}$ is called an **eigenvalue** of T if there exists $\mathbf{v} \in \mathbb{C}^n$ with $\mathbf{v} \neq \mathbf{0}$ such that $T(\mathbf{v}) = \alpha \mathbf{v}$.

Proposition 6.1.15. *Let* $T \in L(\mathbb{C}^n)$ *and let* $\mathcal{B}$ *be an ordered basis in* $\mathbb{C}^n$. *Then* $(\alpha, \mathbf{v})$ *is an eigen-pair of* T *if and only if* $(\alpha, [\mathbf{v}]_{\mathcal{B}})$ *is an eigen-pair of* $A = T[\mathcal{B}, \mathcal{B}]$.

Proof. By definition, $T(\mathbf{v}) = \alpha\mathbf{v}$ if and only if $[Tv]_{\mathcal{B}} = [\alpha\mathbf{v}]_{\mathcal{B}}$. Or equivalently, $\alpha \in \sigma(T)$ if and only if $A[\mathbf{v}]_{\mathcal{B}} = \alpha[\mathbf{v}]_{\mathcal{B}}$. Thus, the required result follows. ■

Thus, the spectrum of a linear operator is independent of the choice of basis.

Remark 6.1.16. *We give two examples to show that* **a linear operator on an infinite dimensional vector space need not have an eigenvalue**.

1. *Let $\mathbb{V}$ be the space of all real sequences (see Example 3.1.4.7) and define $T \in \mathcal{L}(\mathbb{V})$ by*

$$T\,(a_0, a_1, \ldots) = (0, a_1, a_2, \ldots).$$

Let if possible $(\alpha, \mathbf{x})$ be an eigen-pair of T with $\mathbf{x} = (x_1, x_2, \ldots)$. Then

$$T(\mathbf{x}) = \alpha\mathbf{x} \Leftrightarrow (0, x_1, x_2, \ldots) = \alpha(x_1, x_2, \ldots) = (\alpha x_1, \alpha x_2, \ldots).$$

So, if $\alpha \neq 0$ then $x_1 = 0$. This in turn implies $\mathbf{x} = \mathbf{0}$, a contradiction. If $\alpha = 0$ then $(0, x_1, x_2, \ldots) = (0, 0, \ldots) \Rightarrow \mathbf{x} = \mathbf{0}$, a contradiction. Hence, T doesn't have an eigenvalue.

2. *Recall the map $T \in \mathcal{L}(\mathbb{C}[x])$ defined by $T(f(x)) = xf(x)$, for all $f(x) \in \mathbb{C}[x]$.*

T has an eigen-pair $(\alpha, f(x)) \Leftrightarrow xf(x) = \alpha f(x) \Leftrightarrow (x - \alpha)f(x) = 0$. As x is an indeterminate, $f(x)$ is the zero polynomial. Hence, T cannot have an eigenvector.

We now prove the observations that $\det(A)$ is the product of eigenvalues and $\mathrm{tr}(A)$ is the sum of eigenvalues.

DRAFT

Theorem 6.1.17. *Let $\lambda_1, \ldots, \lambda_n$, not necessarily distinct, be the $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$. Then,* $\det(A) = \prod\limits_{i=1}^{n} \lambda_i$ *and* $\mathsf{tr}(\mathsf{A}) = \sum\limits_{i=1}^{n} a_{ii} = \sum\limits_{i=1}^{n} \lambda_i$.

Proof. Since $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, by definition,

$$\det(A - xI_n) = (-1)^n \prod_{i=1}^{n} (x - \lambda_i) \tag{6.1.2}$$

is an identity in x as polynomials. Therefore, by substituting $x = 0$ in Equation (6.1.2), we get $\det(A) = (-1)^n(-1)^n \prod_{i=1}^n \lambda_i = \prod_{i=1}^n \lambda_i$. Also,

$$\begin{aligned} \det(A - xI_n) &= \begin{bmatrix} a_{11} - x & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - x & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - x \end{bmatrix} && (6.1.3) \\ &= a_0 - xa_1 + \cdots + (-1)^{n-1}x^{n-1}a_{n-1} + (-1)^n x^n && (6.1.4) \end{aligned}$$

for some $a_0, a_1, \ldots, a_{n-1} \in \mathbb{C}$. Then, a_{n-1}, the coefficient of $(-1)^{n-1}x^{n-1}$, comes from the term

$$(a_{11} - x)(a_{22} - x) \cdots (a_{nn} - x).$$

So, $a_{n-1} = \sum\limits_{i=1}^{n} a_{ii} = \mathsf{tr}(\mathsf{A})$, the trace of A. Also, from Equation (6.1.2) and (6.1.4), we have

$$a_0 - xa_1 + \cdots + (-1)^{n-1}x^{n-1}a_{n-1} + (-1)^n x^n = (-1)^n \prod_{i=1}^{n} (x - \lambda_i).$$

Therefore, comparing the coefficient of $(-1)^{n-1}x^{n-1}$, we have

$$\mathsf{tr}(\mathsf{A}) = \mathsf{a}_{\mathsf{n}-1} = (-1)\left\{(-1)\sum_{\mathsf{i}=1}^{\mathsf{n}}\lambda_{\mathsf{i}}\right\} = \sum_{\mathsf{i}=1}^{\mathsf{n}}\lambda_{\mathsf{i}}.$$

Hence, we get the required result. ∎

EXERCISE **6.1.18.** *1. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, A is invertible if and only if 0 is not an eigenvalue of A.*

2. Let $A \in \mathbb{M}_n(\mathbb{R})$. Then, prove that

(a) if $\alpha \in \sigma(A)$ then $\alpha^k \in \sigma(A^k)$, for all $k \in \mathbb{N}$.

(b) if A is invertible and $\alpha \in \sigma(A)$ then $\alpha^k \in \sigma(A^k)$, for all $k \in \mathbb{Z}$.

3. Let A be a 3×3 orthogonal matrix ($AA^T = I$). If $\det(A) = 1$, then prove that there exists $\mathbf{v} \in \mathbb{R}^3 \setminus \{\mathbf{0}\}$ such that $A\mathbf{v} = \mathbf{v}$.

4. Let $A \in \mathbb{M}_{2n+1}(\mathbb{R})$ with $A^T = -A$. Then, prove that 0 is an eigenvalue of A.

6.2 Spectrum of a Matrix

Definition 6.2.1. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, for $\alpha \in \sigma(A)$

1. the **algebraic multiplicity** of α, denoted $\textsc{Alg.Mul}_\alpha(A)$, is the multiplicity of α as a root of the characteristic polynomial or the number of times $\alpha \in \sigma(A)$.
2. the **geometric multiplicity** of α, denoted $\textsc{Geo.Mul}_\alpha(A)$, equals $\dim(\textsc{Null}(A - \alpha I))$.

Example 6.2.2. 1. Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Then $\sigma(A) = \{1, 1, 1, \}$. Hence, the algebraic multiplicity of 1 is 3, *i.e.*, $\textsc{Alg.Mul}_1(A) = 3$. Verify that $\textsc{Geo.Mul}_1(A) = 1$.

2. Let $A = \left[\begin{array}{ccc|ccc|cc} 3 & 1 & 1 & & & & & \\ 0 & 3 & 1 & & \mathbf{0} & & \mathbf{0} & \\ 0 & 0 & 3 & & & & & \\ \hline & & & 2 & 1 & 0 & & \\ & \mathbf{0} & & 0 & 2 & 1 & \mathbf{0} & \\ & & & 0 & 0 & 2 & & \\ \hline & \mathbf{0} & & & \mathbf{0} & & 3 & 1 \\ & & & & & & 0 & 3 \end{array}\right]$. Then A is an upper triangular matrix and thus $\sigma(A) = \{3, 3, 3, 3, 3, 2, 2, 2\}$, $\textsc{Alg.Mul}_3(A) = 5$ and $\textsc{Alg.Mul}_2(A) = 3$. Verify that $\text{Rank}(A - 3I) = 6$, $\text{Rank}(A - 2I) = 7 \Rightarrow \textsc{Geo.Mul}_3(A) = 2$ and $\textsc{Geo.Mul}_2(A) = 1$.

We now show that for any eigenvalue α, the algebraic and geometric multiplicities do not change under similarity transformation, or equivalently, under change of basis.

Theorem 6.2.3. *Let A and B be two similar matrices. Then,*

DRAFT

1. *$\alpha \in \sigma(A)$ if and only if $\alpha \in \sigma(B)$.*
2. *for each $\alpha \in \sigma(A)$,* $\text{ALG.MUL}_\alpha(A) = \text{ALG.MUL}_\alpha(B)$ *and* $\text{GEO.MUL}_\alpha(A) = \text{GEO.MUL}_\alpha(B)$.

Proof. Since A and B are similar, there exists an invertible matrix S such that $A = SBS^{-1}$. So, $\alpha \in \sigma(A)$ if and only if $\alpha \in \sigma(B)$ as

$$\begin{aligned}\det(A - xI) &= \det(SBS^{-1} - xI) = \det\left(S(B - xI)S^{-1}\right) \\ &= \det(S)\det(B - xI)\det(A^{-1}) = \det(B - xI). \qquad (6.2.5)\end{aligned}$$

Note that Equation (6.2.5) also implies that $\text{ALG.MUL}_\alpha(A) = \text{ALG.MUL}_\alpha(B)$. We will now show that $\text{GEO.MUL}_\alpha(A) = \text{GEO.MUL}_\alpha(B)$.

So, let $Q_1 = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a basis of $\text{NULL}(A - \alpha I)$. Then, $B = SAS^{-1}$ implies that $Q_2 = \{S\mathbf{v}_1, \ldots, S\mathbf{v}_k\} \subseteq \text{NULL}(B - \alpha I)$. Since Q_1 is linearly independent and S is invertible, we get Q_2 is linearly independent. So, $\text{GEO.MUL}_\alpha(A) \le \text{GEO.MUL}_\alpha(B)$. Now, we can start with eigenvectors of B and use similar arguments to get $\text{GEO.MUL}_\alpha(B) \le \text{GEO.MUL}_\alpha(A)$. Hence the required result follows. ∎

Remark 6.2.4. 1. *Let $A = S^{-1}BS$. Then, from the proof of Theorem 6.2.3, we see that $\mathbf{x}$ is an eigenvector of A for λ if and only if $S\mathbf{x}$ is an eigenvector of B for λ.*

2. *Let A and B be two similar matrices then $\sigma(A) = \sigma(B)$. But, the converse is not true. For example, take $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.*

3. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, for any invertible matrix B, the matrices AB and $BA = B(AB)B^{-1}$ are similar. Hence, in this case the matrices AB and BA have*
   (a) *the same set of eigenvalues.*
   (b) $\text{ALG.MUL}_\alpha(AB) = \text{ALG.MUL}_\alpha(BA)$, *for each $\alpha \in \sigma(A)$.*
   (c) $\text{GEO.MUL}_\alpha(AB) = \text{GEO.MUL}_\alpha(BA)$, *for each $\alpha \in \sigma(A)$.*

DRAFT

We will now give a relation between the geometric multiplicity and the algebraic multiplicity.

Theorem 6.2.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, for $\alpha \in \sigma(A)$,* $\text{GEO.MUL}_\alpha(A) \le \text{ALG.MUL}_\alpha(A)$.

Proof. Let $\text{GEO.MUL}_\alpha(A) = k$. So, suppose that $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is an orthonormal basis of $\text{NULL}(A - \alpha I)$. Extend it to get $\{\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n\}$ as an orthonormal basis of $\mathbb{C}^n$. Put $P = [\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{v}_{k+1}, \ldots, \mathbf{v}_n]$. Then $P^* = P^{-1}$ and

$$\begin{aligned} P^*AP &= P^*\left[A\mathbf{v}_1, \ldots, A\mathbf{v}_k, A\mathbf{v}_{k+1}, \ldots, A\mathbf{v}_n\right] \\ &= \begin{bmatrix} \mathbf{v}_1^* \\ \vdots \\ \mathbf{v}_k^* \\ \mathbf{v}_{k+1}^* \\ \vdots \\ \mathbf{v}_n^* \end{bmatrix} [\alpha\mathbf{v}_1, \ldots, \alpha\mathbf{v}_k, *, \ldots, *] = \left[\begin{array}{ccc|ccc} \alpha & \cdots & 0 & * & \cdots & * \\ 0 & \ddots & 0 & * & \cdots & * \\ 0 & \cdots & \alpha & * & \cdots & * \\ \hline 0 & \cdots & 0 & * & \cdots & * \\ \vdots & & & & & \\ 0 & \cdots & 0 & * & \cdots & * \end{array}\right]. \end{aligned}$$

Now, if we denote the lower diagonal submatrix as B then $P^* = P^{-1}$ implies

$$P_A(x) = \det(A - xI) = \det(P^*AP - xI) = (\alpha - x)^k \det(B - xI). \tag{6.2.6}$$

So, $\text{ALG.MUL}_\alpha(A) = \text{ALG.MUL}_\alpha(P^*AP) \geq k = \text{GEO.MUL}_\alpha(A)$. ■

As a corollary to the above result, one obtains the following observations.

Remark 6.2.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$.*

1. *Then, for each $\alpha \in \sigma(A)$, $\dim(\text{NULL}(A - \alpha I)) \geq 1$. So, we have at least one eigenvector.*
2. *If $\text{ALG.MUL}_\alpha(A) = r$ then $\dim(\text{NULL}(A - \alpha I)) \leq r$. Thus, A may not have r linearly independent eigenvectors.*

EXERCISE **6.2.7.**

1. *Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \\ 2 & 3 & 1 \end{bmatrix}$. Then $\left(6, \mathbf{x}_1 = \frac{1}{\sqrt{3}}\mathbf{1}\right)$ is an eigen-pair of A. Let $(\mathbf{x}_1, \mathbf{e}_1, \mathbf{e}_2)$ be an ordered basis of $\mathbb{C}^3$. Put $X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{e}_1 & \mathbf{e}_2 \end{bmatrix}$. Compute $X^{-1}AX$. Can you now find the remaining eigenvalues of A?*
2. *Let $A \in \mathbb{M}_{m\times n}(\mathbb{R})$ and $B \in \mathbb{M}_{n\times m}(\mathbb{R})$.*
   (a) *If $\alpha \in \sigma(AB)$ and $\alpha \neq 0$ then*
      i. *$\alpha \in \sigma(BA)$.*
      ii. *$\text{ALG.MUL}_\alpha(AB) = \text{ALG.MUL}_\alpha(BA)$.*
      iii. *$\text{GEO.MUL}_\alpha(AB) = \text{GEO.MUL}_\alpha(BA)$.*
   (b) *If $0 \in \sigma(AB)$ and $n = m$ then $\text{ALG.MUL}_0(AB) = \text{ALG.MUL}_0(BA)$ as there are n eigenvalues, counted with multiplicity.*
   (c) *Give an example to show that $\text{GEO.MUL}_0(AB)$ need not equal $\text{GEO.MUL}_0(BA)$ even when $n = m$.*
3. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be an invertible matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y}^T A^{-1}\mathbf{x} \neq 0$. Define $B = \mathbf{x}\mathbf{y}^T A^{-1}$. Then, prove that*
   (a) *$\lambda_0 = \mathbf{y}^T A^{-1}\mathbf{x}$ is an eigenvalue of B of multiplicity 1.*
   (b) *0 is an eigenvalue of B of multiplicity $n-1$ [Hint: Use Exercise 6.2.7.2a].*
   (c) *$1 + \alpha\lambda_0$ is an eigenvalue of $I + \alpha B$ of multiplicity 1, for any $\alpha \in \mathbb{R}, \alpha \neq 0$.*
   (d) *1 is an eigenvalue of $I + \alpha B$ of multiplicity $n-1$, for any $\alpha \in \mathbb{R}, \alpha \neq 0$.*
   (e) *$\det(A + \alpha\mathbf{x}\mathbf{y}^T)$ equals $(1 + \alpha\lambda_0)\det(A)$, for any $\alpha \in \mathbb{R}, \alpha \neq 0$. This result is known as the Shermon-Morrison formula for determinant.*
4. *Let $A, B \in \mathbb{M}_2(\mathbb{R})$ such that $\det(A) = \det(B)$ and $\mathsf{tr}(\mathsf{A}) = \mathsf{tr}(\mathsf{B})$.*
   (a) *Do A and B have the same set of eigenvalues?*
   (b) *Give examples to show that the matrices A and B need not be similar.*
5. *Let $A, B \in \mathbb{M}_n(\mathbb{R})$. Also, let $(\lambda_1, \mathbf{u})$ and $(\lambda_2, \mathbf{v})$ are eigen-pairs of A and B, respectively.*

DRAFT

(a) *If* $\mathbf{u} = \alpha\mathbf{v}$ *for some* $\alpha \in \mathbb{R}$ *then* $(\lambda_1 + \lambda_2, \mathbf{u})$ *is an eigen-pair for* $A + B$.

(b) *Give an example to show that if* $\mathbf{u}$ *and* $\mathbf{v}$ *are linearly independent then* $\lambda_1 + \lambda_2$ *need not be an eigenvalue of* $A + B$.

6.3 Basic Results on Diagonalization

Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $T \in \mathcal{L}(\mathbb{C}^n)$ be defined by $T(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^n$. In this section, we first find conditions under which one can obtain a basis $\mathcal{B}$ of $\mathbb{C}^n$ such that $T[\mathcal{B}, \mathcal{B}]$ is a diagonal matrix. To start with, we have the following definition.

Definition 6.3.1. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is said to be **diagonalizable** if A is similar to a diagonal matrix. Or equivalently, $P^{-1}AP = D \Leftrightarrow AP = PD$, for some diagonal matrix D and invertible matrix P. Or equivalently, there exists an ordered basis $\mathcal{B}$ of $\mathbb{C}^n$ such that $A[\mathcal{B}, \mathcal{B}]$ is a diagonal matrix.

Example 6.3.2.

1. Let $A \in \mathbb{M}_n(\mathbb{C})$ be a diagonalizable matrix. Then, by definition, A is similar to $D = \text{diag}(d_1, \ldots, d_n)$. Thus, by Remark 6.2.4, $\sigma(A) = \sigma(D) = \{d_1, \ldots, d_n\}$.

2. Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, A cannot be diagonalized.

   Solution: A is diagonalizable implies A is similar to a diagonal matrix D with diagonal entries $\{d_1, d_2\} = \{0, 0\}$. Hence $D = \mathbf{0} \Rightarrow A = SDS^{-1} = \mathbf{0}$, a contradiction.

3. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Then, A cannot be diagonalized.

   Solution: A is diagonalizable implies A is similar to a diagonal matrix D with diagonal entries $\{d_1, d_2, d_3\} = \{2, 2, 2\}$. Hence, $D = 2I_3 \Rightarrow A = SDS^{-1} = 2I_3$, a contradiction.

4. Let $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Then, $\left(i, \begin{bmatrix} i \\ 1 \end{bmatrix}\right)$ and $\left(-i, \begin{bmatrix} -i \\ 1 \end{bmatrix}\right)$ are two eigen-pairs of A. Define $U = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$. Then, $U^*U = I_2 = UU^*$ and $U^*AU = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}$.

Theorem 6.3.3. *Let* $A \in \mathbb{M}_n(\mathbb{R})$.

1. *Let* S *be an invertible matrix such that* $S^{-1}AS = \text{diag}(d_1, \ldots, d_n)$. *Then, for* $1 \le i \le n$, *the* i*-th column of* S *is an eigenvector of* A *corresponding to* d_i.

2. *Then,* A *is diagonalizable if and only if* A *has* n *linearly independent eigenvectors.*

Proof. Let $S = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$. Then $S^{-1}AS = D$ implies $AS = SD$. Thus

$$[A\mathbf{u}_1, \ldots, A\mathbf{u}_n] = A\,[\mathbf{u}_1, \ldots, \mathbf{u}_n] = AS = SD = S\ \text{diag}(d_1, \ldots, d_n) = [d_1\mathbf{u}_1, \ldots, d_n\mathbf{u}_n]\,.$$

Or equivalently, $A\mathbf{u}_i = d_i\mathbf{u}_i$, for $1 \le i \le n$. As S is invertible, $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ are linearly independent. Hence, $(d_i, \mathbf{u}_i)$, for $1 \le i \le n$, are eigen-pairs of A. This proves Part 1 and "only if" part of Part 2.

DRAFT

Conversely, let $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ be n linearly independent eigenvectors of A corresponding to eigenvalues $\alpha_1, \ldots, \alpha_n$. Then, by Corollary 3.3.16, $S = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ is non-singular and

$$AS = [A\mathbf{u}_1, \ldots, A\mathbf{u}_n] = [\alpha_1\mathbf{u}_1, \ldots, \lambda_n\mathbf{u}_n] = [\mathbf{u}_1, \ldots, \mathbf{u}_n]\begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_n \end{bmatrix} = SD,$$

where $D = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. Therefore, $S^{-1}AS = D$. This implies A is diagonalizable. ■

The next result implies that the eigenvectors corresponding to distinct eigenvalues are linearly independent. A proof is given for clarity. A separate proof appears later in Corollary 6.3.7.

Theorem 6.3.4. *Let $(\alpha_1, \mathbf{v}_1), \ldots, (\alpha_k, \mathbf{v}_k)$ be k eigen-pairs of $A \in \mathbb{M}_n(\mathbb{C})$ with α_i's distinct. Then, $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is linearly independent.*

Proof. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be linearly dependent. Then, there exists a smallest $\ell \in \{1, \ldots, k-1\}$ and $\mathbf{c} \neq \mathbf{0}$ such that $\mathbf{v}_{\ell+1} = c_1\mathbf{v}_1 + \cdots + c_\ell\mathbf{v}_\ell$. So,

$$\alpha_{\ell+1}\mathbf{v}_{\ell+1} = \alpha_{\ell+1}c_1\mathbf{v}_1 + \cdots + \alpha_{\ell+1}c_\ell\mathbf{v}_\ell. \tag{6.3.1}$$

and

$$\alpha_{\ell+1}\mathbf{v}_{\ell+1} = A\mathbf{v}_{\ell+1} = A\left(c_1\mathbf{v}_1 + \cdots + c_\ell\mathbf{v}_\ell\right) = \alpha_1c_1\mathbf{v}_1 + \cdots + \alpha_\ell c_\ell\mathbf{v}_\ell. \tag{6.3.2}$$

Now, subtracting Equation (6.3.2) from Equation (6.3.1) gives

$$\mathbf{0} = (\alpha_{\ell+1} - \alpha_1)\,c_1\mathbf{v}_1 + \cdots + (\alpha_{\ell+1} - \alpha_\ell)\,c_\ell\mathbf{v}_\ell.$$

DRAFT

So, $\mathbf{v}_\ell \in LS(\mathbf{v}_1, \ldots, \mathbf{v}_{\ell-1})$, a contradiction to the choice of ℓ. Thus, the required result follows. ■

An immediate corollary of Theorem 6.3.3 and Theorem 6.3.4 is stated next without proof.

Corollary 6.3.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$ have n distinct eigenvalues. Then, A is diagonalizable.*

Remark 6.3.6.

1. *Let* $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 1 & 2 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & 7 \end{bmatrix}$. *Then $\sigma(A) = \{1, 2, 3, 7\}$, which are distinct. Hence, A is diagonalizable.*

2. *The converse of Theorem 6.3.4 is not true as I_n has n linearly independent eigenvectors corresponding to the eigenvalue* 1, *repeated n times.*

Corollary 6.3.7. *Let $\alpha_1, \ldots, \alpha_k$ be k distinct eigenvalues $A \in \mathbb{M}_n(\mathbb{C})$. Also, for $1 \leq i \leq k$, let $\dim(\textsc{Null}(A - \alpha_i I_n)) = n_i$. Then, A has $\sum\limits_{i=1}^{k} n_i$ linearly independent eigenvectors.*

Proof. For $1 \leq i \leq k$, let $S_i = \{\mathbf{u}_{i1}, \ldots, \mathbf{u}_{in_i}\}$ be a basis of $\textsc{Null}(A - \alpha_i I_n)$. Then, we need to prove that $\bigcup\limits_{i=1}^{k} S_i$ is linearly independent. To do so, denote $p_j(A) = \left(\prod\limits_{i=1}^{k}(A - \alpha_i I_n)\right) / (A - \alpha_j I_n)$, for $1 \leq j \leq k$. Then, note that $p_j(A)$ is a polynomial in A of degree $k-1$ and

$$p_j(A)\mathbf{u} = \begin{cases} \mathbf{0}, & \text{if } \mathbf{u} \in \textsc{Null}(A - \alpha_i I_n), \text{ for some } i \neq j \\ \prod\limits_{i \neq j}(\alpha_j - \alpha_i)\mathbf{u} & \text{if } \mathbf{u} \in \textsc{Null}(A - \alpha_j I_n) \end{cases} \tag{6.3.3}$$

So, to prove that $\bigcup\limits_{i=1}^{k} S_i$ is linearly independent, consider the linear system

$$c_{11}\mathbf{u}_{11} + \cdots + c_{1n_1}\mathbf{u}_{1n_1} + \cdots + c_{k1}\mathbf{u}_{k1} + \cdots + c_{kn_k}\mathbf{u}_{kn_k} = \mathbf{0}$$

in the variables c_{ij}'s. Now, applying the matrix $p_j(A)$ and using Equation (6.3.3), we get

$$\prod_{i \neq j}(\alpha_j - \alpha_i)\left(c_{j1}\mathbf{u}_{j1} + \cdots + c_{jn_j}\mathbf{u}_{jn_j}\right) = \mathbf{0}.$$

But $\prod\limits_{i \neq j}(\alpha_j - \alpha_i) \neq 0$ as α_i's are distinct. Hence, $c_{j1}\mathbf{u}_{j1} + \cdots + c_{jn_j}\mathbf{u}_{jn_j} = \mathbf{0}$. As S_j is a basis of $\text{NULL}(A - \alpha_j I_n)$, we get $c_{jt} = 0$, for $1 \leq t \leq n_j$. Thus, the required result follows. ∎

Corollary 6.3.8. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with distinct eigenvalues $\alpha_1, \ldots, \alpha_k$. Then, A is diagonalizable if and only if $\text{GEO.MUL}_{\alpha_i}(A) = \text{ALG.MUL}_{\alpha_i}(A)$, for each $1 \leq i \leq k$.*

Proof. Let $\text{ALG.MUL}_{\alpha_i}(A) = m_i$. Then $\sum\limits_{i=1}^{k} m_i = n$. Let $\text{GEO.MUL}_{\alpha_i}(A) = n_i$, for $1 \leq i \leq k$. Then, by Corollary 6.3.7, A has $\sum\limits_{i=1}^{k} n_i$ linearly independent eigenvectors. Also, by Theorem 6.2.5, $n_i \leq m_i$, for $1 \leq i \leq m_i$.

Now, let A be diagonalizable. Then, by Theorem 6.3.3, A has n linearly independent eigenvectors. As $n_i \leq m_i$, we get $n = \sum\limits_{i=1}^{k} n_i \leq \sum\limits_{i=1}^{k} m_i = n$. Thus $n_i = m_i, 1 \leq i \leq k$.

Now, assume that $m_i = n_i$, for $1 \leq i \leq k$. Then A has $\sum\limits_{i=1}^{k} n_i = \sum\limits_{i=1}^{k} m_i = n$ linearly independent eigenvectors. Hence by Theorem 6.3.3, A is diagonalizable. ∎

Definition 6.3.9.
1. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called **defective** if for some $\alpha \in \sigma(A)$, $\text{GEO.MUL}_{\alpha}(A) < \text{ALG.MUL}_{\alpha}(A)$.
2. A matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called **non-derogatory** if $\text{GEO.MUL}_{\alpha}(A) = 1$, for each $\alpha \in \sigma(A)$.

As a direct consequence of the above discussions, we obtain the following result.

Corollary 6.3.10. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then,*

1. *A is non-defective if and only if A is diagonalizable.*
2. *A has distinct eigenvalues if and only if A is non-derogatory and non-defective.*

Example 6.3.11. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & -1 & 1 \end{bmatrix}$. Then, $\left(1, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}\right)$ and $\left(2, \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}\right)$ are the only eigen-pairs. Hence, by Theorem 6.3.3, A is not diagonalizable.

EXERCISE **6.3.12.**
1. *A strictly upper triangular matrix is not diagonalizable.*
2. *A be diagonalizable if and only if $A + \alpha I$ is diagonalizable for every $\alpha \in \mathbb{C}$.*
3. *Let A be an $n \times n$ matrix with $\lambda \in \sigma(A)$ with $\text{ALG.MUL}_{\lambda}(A) = m$. If $\text{RANK}[A - \lambda I] \neq n - m$ then prove that A is not diagonalizable.*

DRAFT

4. *Let A and B be two similar matrices such that A is diagonalizable. Prove that B is diagonalizable.*

5. *If $\sigma(A) = \sigma(B)$ and both A and B are diagonalizable then prove that A is similar to B. Thus, they are two basis representation of the same linear transformation.*

6. *Let $A \in \mathbb{M}_n(\mathbb{R})$ and $B \in \mathbb{M}_m(\mathbb{R})$. Suppose $C = \begin{bmatrix} A & \mathbf{0} \\ \mathbf{0} & B \end{bmatrix}$. Then, prove that C is diagonalizable if and only if both A and B are diagonalizable.*

7. *Let $J = \mathbf{1}\mathbf{1}^T$ be an $n \times n$. Define $A = (a-b)I + bJ$. Is A diagonalizable?*

8. *Is the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ diagonalizable?*

9. *Let $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^5$ be a linear operator with* $\textsc{Rank}(T - I) = 3$ *and*

$$\textsc{Null}(T) = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \mid x_1 + x_4 + x_5 = 0,\ x_2 + x_3 = 0\}.$$

   (a) *Determine the eigenvalues of T?*

   (b) *For each distinct eigenvalue α of T, determine* $\textsc{Geo.Mul}_\alpha(T)$.

   (c) *Is T diagonalizable? Justify your answer.*

10. *Let $A \in \mathbb{M}_n(\mathbb{R})$ with $A \neq \mathbf{0}$ but $A^2 = \mathbf{0}$. Prove that A cannot be diagonalized.*

DRAFT

11. *Are the matrices $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$ diagonalizable?*

12. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a matrix of rank 1. Then*

   (a) *A has at most one non-zero eigenvalue of algebraic multiplicity 1.*

   (b) *find this eigenvalue and its geometric multiplicity.*

   (c) *when is A diagonalizable?*

13. *Let $\mathbf{u}, \mathbf{v} \in \mathbb{C}^n$ such that $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set. Define $A = \mathbf{u}\mathbf{v}^T + \mathbf{v}\mathbf{u}^T$.*

   (a) *Then prove that A is a symmetric matrix.*

   (b) *Then prove that* $\dim(\textsc{Ker}(A)) = n - 2$.

   (c) *Then $0 \in \sigma(A)$ and has multiplicity $n - 2$.*

   (d) *Determine the other eigenvalues of A.*

6.4 Schur's Unitary Triangularization and Diagonalizability

We now prove one of the most important results in diagonalization, called the Schur's Lemma or the Schur's unitary triangularization.

Lemma 6.4.1. **[Schur's unitary triangularization (SUT)]** *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, there exists a unitary matrix U such that A is similar to an upper triangular matrix. Further, if $A \in \mathbb{M}_n(\mathbb{R})$ and $\sigma(A)$ have real entries then U is a real orthogonal matrix.*

Proof. We prove the result by induction on n. The result is clearly true for $n = 1$. So, let $n > 1$ and assume the result to be true for $k < n$ and prove it for n.

Let $(\lambda_1, \mathbf{x}_1)$ be an eigen-pair of A with $\|\mathbf{x}_1\| = 1$. Now, extend it to form an orthonormal basis $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\}$ of $\mathbb{C}^n$ and define $X = [\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n]$. Then, X is a unitary matrix and

$$X^*AX = X^*[A\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n] = \begin{bmatrix} \mathbf{x}_1^* \\ \mathbf{x}_2^* \\ \vdots \\ \mathbf{x}_n^* \end{bmatrix} [\lambda_1\mathbf{x}_1, A\mathbf{x}_2, \ldots, A\mathbf{x}_n] = \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & B \end{bmatrix}, \tag{6.4.4}$$

where $B \in \mathbb{M}_{n-1}(\mathbb{C})$. Now, by induction hypothesis there exists a unitary matrix $U \in \mathbb{M}_{n-1}(\mathbb{C})$ such that $U^*BU = T$ is an upper triangular matrix. Define $\widehat{U} = X \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix}$. As product of unitary matrices is unitary, the matrix $\widehat{U}$ is unitary and

$$\begin{aligned} \left(\widehat{U}\right)^* A\widehat{U} &= \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U^* \end{bmatrix} X^*AX \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} = \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U^* \end{bmatrix} \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & B \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} \\ &= \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & U^*B \end{bmatrix} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & U \end{bmatrix} = \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & U^*BU \end{bmatrix} = \begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & T \end{bmatrix}. \end{aligned}$$

DRAFT

Since T is upper triangular, $\begin{bmatrix} \lambda_1 & * \\ \mathbf{0} & T \end{bmatrix}$ is upper triangular.

Further, if $A \in \mathbb{M}_n(\mathbb{R})$ and $\sigma(A)$ has real entries then $\mathbf{x}_1 \in \mathbb{R}^n$ with $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$. Now, one uses induction once again to get the required result. ∎

Remark 6.4.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, by Schur's Lemma there exists a unitary matrix U such that $U^*AU = T = [t_{ij}]$, a triangular matrix. Thus,*

$$\{\alpha_1, \ldots, \alpha_n\} = \sigma(A) = \sigma(U^*AU) = \{t_{11}, \ldots, t_{nn}\}. \tag{6.4.5}$$

Furthermore, we can get the α_i's in the diagonal of T in any prescribed order.

Definition 6.4.3. Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, A and B are said to be **unitarily equivalent/similar** if there exists a unitary matrix U such that $A = U^*BU$.

Remark 6.4.4. *We know that if two matrices are unitarily equivalent then they are necessarily similar as $U^* = U^{-1}$, for every unitary matrix U. But, similarity doesn't imply unitary equivalence (see Exercise 6.4.6.5). In numerical calculations, unitary transformations are preferred as compared to similarity transformations due to the following main reasons:*

1. *A is unitary implies $\|A\mathbf{x}\| = \|\mathbf{x}\|$. This need not be true under a similarity.*

2. *As $U^{-1} = U^*$, for a unitary matrix, unitary equivalence is computationally simpler.*

3. *Also, computation of "conjugate transpose" doesn't create round-off error in calculation.*

Example 6.4.5. Consider the two matrices $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$. Then, we show that they are similar but not unitarily similar.

Solution: Note that $\sigma(A) = \sigma(B) = \{1, 2\}$. As the eigenvalues are distinct, by Theorem 6.3.5, the matrices A and B are diagonalizable and hence there exists invertible matrices S and T such that $A = S\Lambda S^{-1}$, $B = T\Lambda T^{-1}$, where $\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$. Thus $A = ST^{-1}B(ST^{-1})^{-1}$. But $\sum |a_{ij}|^2 \neq \sum |b_{ij}|^2$. Hence by Exercise 5.8.8.6, they cannot be unitarily similar.

EXERCISE **6.4.6.**

1. *If A is unitarily similar to a triangular matrix $T = [t_{ij}]$ then* $\sum\limits_{i<j} |t_{ij}|^2 = \mathsf{tr}(\mathsf{A}^*\mathsf{A}) - \sum |\lambda_\mathsf{i}|^2$.

2. *Consider the following* 6 *matrices.*

$$M_1 = \begin{bmatrix} 2 & -1 & 3\sqrt{2} \\ 0 & 1 & \sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}, M_2 = \begin{bmatrix} 2 & 1 & 3\sqrt{2} \\ 0 & 1 & -\sqrt{2} \\ 0 & 0 & 3 \end{bmatrix}, M_3 = \begin{bmatrix} 2 & 0 & 3\sqrt{2} \\ 1 & 1 & \sqrt{2} \\ 0 & 0 & 1 \end{bmatrix},$$
$$M_4 = \begin{bmatrix} 2 & 0 & 3\sqrt{2} \\ -1 & 1 & -\sqrt{2} \\ 0 & 0 & 1 \end{bmatrix}, M_5 = \begin{bmatrix} 1 & 1 & 4 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \text{ and } M_6 = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, use the exercises given below to conclude that the upper triangular matrix obtained in the "Schur's Lemma" need not be unique.

(a) *Prove that M_1, M_2 and M_5 are unitarily equivalent.*

(b) *Prove that M_3, M_4 and M_6 are unitarily equivalent.*

(c) *Do the above results contradict Exercise 5.8.8.5c? Give reasons for your answer.*

3. *Prove that* $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$ *and* $B = \begin{bmatrix} 2 & -1 & \sqrt{2} \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *are unitarily equivalent.*

4. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, Prove that if $\mathbf{x}^*A\mathbf{x} = 0$, for all $\mathbf{x} \in \mathbb{C}^n$, then $A = \mathbf{0}$. Do these results hold for arbitrary matrices?*

5. *Show that the matrices* $A = \begin{bmatrix} 4 & 4 \\ 0 & 4 \end{bmatrix}$ *and* $B = \begin{bmatrix} 10 & 9 \\ -4 & -2 \end{bmatrix}$ *are similar. Is it possible to find a unitary matrix U such that $A = U^*BU$?*

We now use Lemma 6.4.1 to give another proof of Theorem 6.1.17.

Corollary 6.4.7. *Let $A \in \mathbb{M}_n(\mathbb{C})$. If $\alpha_1, \ldots, \alpha_n \in \sigma(A)$ then* $\det(A) = \prod\limits_{i=1}^{n} \alpha_i$ *and* $\mathsf{tr}(\mathsf{A}) = \sum\limits_{\mathsf{i}=1}^{\mathsf{n}} \alpha_\mathsf{i}$.

Proof. By Schur's Lemma there exists a unitary matrix U such that $U^*AU = T = [t_{ij}]$, a triangular matrix. By Remark 6.4.2, $\sigma(A) = \sigma(T)$. Hence, $\det(A) = \det(T) = \prod\limits_{i=1}^{n} t_{ii} = \prod\limits_{i=1}^{n} \alpha_i$ and $\mathsf{tr}(\mathsf{A}) = \mathsf{tr}(\mathsf{A}(\mathsf{UU}^*)) = \mathsf{tr}(\mathsf{U}^*(\mathsf{AU})) = \mathsf{tr}(\mathsf{T}) = \sum\limits_{\mathsf{i}=1}^{\mathsf{n}} \mathsf{t}_{\mathsf{ii}} = \sum\limits_{\mathsf{i}=1}^{\mathsf{n}} \alpha_\mathsf{i}$. ■

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6.4.1 Diagonalizability of some Special Matrices

We now use Schur's unitary triangularization Lemma to state the main theorem of this subsection. Also, recall that A is said to be a normal matrix if $AA^* = A^*A$. Further, Hermitian, skew-Hermitian and scalar multiples of Unitary matrices are examples of normal matrices.

Theorem 6.4.8. [Spectral Theorem for Normal Matrices] *Let $A \in \mathbb{M}_n(\mathbb{C})$. If A is a normal matrix then there exists a unitary matrix U such that $U^*AU = \text{diag}(\alpha_1, \ldots, \alpha_n)$.*

Proof. By Schur's Lemma there exists a unitary matrix U such that $U^*AU = T = [t_{ij}]$, a triangular matrix. Since A is a normal

$$T^*T = (U^*AU)^*(U^*AU) = U^*A^*AU = U^*AA^*U = (U^*AU)(U^*AU)^* = TT^*.$$

Thus, we see that T is an upper triangular matrix with $T^*T = TT^*$. Thus, by Exercise 1.3.13.8, T is a diagonal matrix and this completes the proof. ∎

We re-write Theorem 6.4.8 in another form to indicate that A can be decomposed into linear combination of orthogonal projectors onto eigen-spaces. Thus, it is independent of the choice of eigenvectors. This remark is also valid for Hermitian, skew-Hermitian and Unitary matrices.

Remark 6.4.9. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a normal matrix with eigen-pairs $\alpha_1, \ldots, \alpha_n$.*

1. *Then, there exists a unitary matrix $U = [\mathbf{u}_1, \ldots, \mathbf{u}_n]$ such that*
   (a) *$\mathbf{u}_i$ is an eigenvector of A for $\alpha_i, 1 \le i \le n$.*
   (b) *$I_n = U^*U = UU^* = \mathbf{u}_1\mathbf{u}_1^* + \cdots + \mathbf{u}_n\mathbf{u}_n^*$.*
   (c) *the columns of U form a set of orthonormal eigenvectors for A (use Theorem 6.3.3).*
   (d) *$A = A \cdot I_n = A(\mathbf{u}_1\mathbf{u}_1^* + \cdots + \mathbf{u}_n\mathbf{u}_n^*) = \alpha_1\mathbf{u}_1\mathbf{u}_1^* + \cdots + \alpha_n\mathbf{u}_n\mathbf{u}_n^*$.*
2. *Let the distinct eigenvalues of A be $\alpha_1, \ldots, \alpha_k$ with respective eigen-spaces $W_1, \ldots, W_k$.*
   (a) *Then each eigenvector belongs to some W_i. So, W_i's are orthogonal to each other.*
   (b) *Hence $\mathbb{C}^n = W_1 \oplus \cdots \oplus W_k$.*
   (c) *If P_{α_i} is the orthogonal projector onto W_i, $1 \le i \le k$, then $A = \alpha_1 P_1 + \cdots + \alpha_k P_k$. Thus, A depends only on the eigen-spaces and not on the computed eigenvectors.*

Theorem 6.4.8 also implies that if $A \in \mathbb{M}_n(\mathbb{C})$ is a normal matrix then after a rotation or reflection of axes (unitary transformation), the matrix A basically looks like a diagonal matrix. As a special case, we now give the spectral theorem for Hermitian matrices.

Theorem 6.4.10. [Spectral Theorem for Hermitian Matrices] *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix. Then Remark 6.4.9 holds. Further, all the eigenvalues of A are real.*

Proof. The first part is immediate from Theorem 6.4.8 as Hermitian matrices are also normal matrices. Let $(\alpha, \mathbf{x})$ be an eigen-pair. To show, α is a real number.

As $A^* = A$ and $A\mathbf{x} = \alpha\mathbf{x}$, we have $\mathbf{x}^*A = \mathbf{x}^*A^* = (A\mathbf{x})^* = (\alpha\mathbf{x})^* = \overline{\alpha}\mathbf{x}^*$. Hence,

$$\alpha\mathbf{x}^*\mathbf{x} = \mathbf{x}^*(\alpha\mathbf{x}) = \mathbf{x}^*(A\mathbf{x}) = (\mathbf{x}^*A)\mathbf{x} = (\overline{\alpha}\mathbf{x}^*)\mathbf{x} = \overline{\alpha}\mathbf{x}^*\mathbf{x}.$$

As $\mathbf{x}$ is an eigenvector, $\mathbf{x} \neq \mathbf{0}$. Hence, $\|\mathbf{x}\|^2 = \mathbf{x}^*\mathbf{x} \neq 0$. Thus $\alpha = \overline{\alpha}$, *i.e.*, $\alpha \in \mathbb{R}$. ■

As an immediate corollary of Theorem 6.4.10 and the second part of Lemma 6.4.1, we give the following result without proof.

Corollary 6.4.11. *Let $A \in \mathbb{M}_n(\mathbb{R})$ be symmetric. Then there exists an orthogonal matrix P and real numbers $\alpha_1, \ldots, \alpha_n$ such that $A = P\ diag(\alpha_1, \ldots, \alpha_n) P^T$. Or equivalently, A is diagonalizable using orthogonal matrix.*

EXERCISE **6.4.12.**

1. *Let A be a normal matrix. If all the eigenvalues of A are 0 then prove that $A = \mathbf{0}$. What happens if all the eigenvalues of A are 1?*

2. *Let A be a skew-symmetric matrix. Then A is unitarily diagonalizable and the eigenvalues of A are either zero or purely imaginary.*

3. *Characterize all normal matrices in $\mathbb{M}_2(\mathbb{R})$.*

4. *Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Then, prove that the following statements are equivalent.*
   (a) *A is normal.*
   (b) *A is unitarily diagonalizable.*
   (c) $\sum\limits_{i,j} |a_{ij}|^2 = \sum\limits_{i} |\lambda_i|^2$.
   (d) *A has n orthonormal eigenvectors.*

5. *Let A be a normal matrix with $(\lambda, \mathbf{x})$ as an eigen-pair. Then,*
   (a) *$(A^*)^k\mathbf{x}$ for $k \in \mathbb{Z}^+$ is also an eigenvector corresponding to λ.*
   (b) *$(\overline{\lambda}, \mathbf{x})$ is an eigen-pair for A^*. [Hint: Verify $\|A^*\mathbf{x} - \overline{\lambda}\mathbf{x}\|^2 = \|A\mathbf{x} - \lambda\mathbf{x}\|^2$.]*

6. *Let A be an $n \times n$ unitary matrix. Then,*
   (a) *$|\lambda| = 1$ for any eigenvalue λ of A.*
   (b) *the eigenvectors $\mathbf{x}, \mathbf{y}$ corresponding to distinct eigenvalues are orthogonal.*

7. *Let A be a 2×2 orthogonal matrix. Then, prove the following:*
   (a) *if $\det(A) = 1$ then $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, for some $\theta, 0 \leq \theta < 2\pi$. That is, A counterclockwise rotates every point in $\mathbb{R}^2$ by an angle θ.*
   (b) *if $\det A = -1$ then $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$, for some $\theta, 0 \leq \theta < 2\pi$. That is, A reflects every point in $\mathbb{R}^2$ about a line passing through origin. Determine this line. Or equivalently, there exists a non-singular matrix P such that $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.*

8. *Let A be a 3×3 orthogonal matrix. Then, prove the following:*
   (a) *if $\det(A) = 1$ then A is a rotation about a fixed axis, in the sense that A has an eigen-pair $(1, \mathbf{x})$ such that the restriction of A to the plane $\mathbf{x}^\perp$ is a two dimensional rotation in $\mathbf{x}^\perp$.*

DRAFT

(b) *if* $\det A = -1$ *then* A *corresponds to a reflection across a plane* P*, followed by a rotation about the line through origin that is orthogonal to* P*.*

9. *Let* A *be a normal matrix. Then, prove that* $\text{Rank}(A)$ *equals the number of nonzero eigenvalues of* A*.*

10. **[Equivalent characterizations of Hermitian matrices]** *Let* $A \in \mathbb{M}_n(\mathbb{C})$*. Then, the following statements are equivalent.*
    (a) *The matrix* A *is Hermitian.*
    (b) *The number* x^*Ax *is real for each* $x \in \mathbb{C}^n$*.*
    (c) *The matrix* A *is normal and has real eigenvalues.*
    (d) *The matrix* S^*AS *is Hermitian for each* $S \in \mathbb{M}_n(\mathbb{C})$*.*

6.4.2 Cayley Hamilton Theorem

Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, in Theorem 6.1.17, we saw that

$$P_A(x) = \det(xI - A) = x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + (-1)^{n-1}a_1x + (-1)^n a_0 \qquad (6.4.6)$$

for certain $a_i \in \mathbb{C}$, $0 \le i \le n-1$. Also, if α is an eigenvalue of A then $P_A(\alpha) = 0$. So, $x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + (-1)^{n-1}a_1x + (-1)^n a_0 = 0$ is satisfied by n complex numbers which are eigenvalues of A. It turns out that the expression

$$A^n - a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + (-1)^{n-1}a_1A + (-1)^n a_0 I = \mathbf{0}$$

DRAFT

holds true as a matrix identity. This is a celebrated theorem called the **Cayley Hamilton theorem**. We give a proof using Schur's unitary triangularization. To do so, we look at multiplication of certain upper triangular matrices.

Lemma 6.4.13. *Let* $A_1, \ldots, A_n \in \mathbb{M}_n(\mathbb{C})$ *be upper triangular matrices such that the* (i,i)*-th entry of* A_i *equals* 0*, for* $1 \le i \le n$*. Then* $A_1A_2\cdots A_n = \mathbf{0}$*.*

Proof. We use induction to prove that the first k columns of $A_1A_2\cdots A_k$ is $\mathbf{0}$, for $1 \le k \le n$. The result is clearly true for $k = 1$ as the first column of A_1 is $\mathbf{0}$. For clarity, we show that the first two columns of A_1A_2 is $\mathbf{0}$. Let $B = A_1A_2$. Then, using $A_1[:,1] = \mathbf{0}$ and $(A_2)_{ji} = 0$, for $i = 1, 2, j \ge 2$, we get

$$B[:,i] = A_1[:,1](A_2)_{1i} + A_1[:,2](A_2)_{2i} + \cdots + A_1[:,n](A_2)_{ni} = \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$

So, assume that the first $n-1$ columns of $C = A_1 \cdots A_{n-1}$ is $\mathbf{0}$. To show $B = CA_n = \mathbf{0}$. As $n-1$ columns of C are zero, $C[:,1](A_n)_{1i} + C[:,2](A_n)_{2i} + \cdots + C[:,n-1](A_n)_{(n-1)i} = \mathbf{0}$, for $1 \le i \le n-1$. Also $C[:,n](A_n)_{ni} = \mathbf{0}$ as the last row of $A_n = \mathbf{0}^T$. Thus

$$B[:,i] = C[:,1](A_n)_{1i} + C[:,2](A_n)_{2i} + \cdots + C[:,n](A_n)_{ni} = \mathbf{0} + \cdots + \mathbf{0} = \mathbf{0}.$$

Hence, by the induction hypothesis the required result follows. ∎

We now prove the Cayley Hamilton Theorem using Schur's unitary triangularization.

Theorem 6.4.14. **[Cayley Hamilton Theorem]** *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A satisfies its characteristic equation, i.e., if $P_A(x) = \det(xI_n - A) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^{n-1}a_1 x + (-1)^n a_0$ then*

$$A^n - a_{n-1}A^{n-1} + \cdots + (-1)^{n-1}a_1 A + (-1)^n a_0 I = \mathbf{0}$$

holds true as a matrix identity.

Proof. Let $\sigma(A) = \{\alpha_1, \ldots, \alpha_n\}$ then $P_A(x) = \prod\limits_{i=1}^{n}(x - \alpha_i)$. And, by Schur's unitary triangularization there exists a unitary matrix U such that $U^*AU = T$, an upper triangular matrix with $t_{ii} = \alpha_i$, for $1 \leq i \leq n$. Now, observe that if $A_i = T - \alpha_i I$ then the A_i's satisfy the conditions of Lemma 6.4.13. Hence

$$(T - \alpha_1 I)\cdots(T - \alpha_n I) = \mathbf{0}.$$

Therefore,

$$P_A(A) = \prod_{i=1}^{n}(A - \alpha_i I) = \prod_{i=1}^{n}(UTU^* - \alpha_i UIU^*) = U\Big[(T - \alpha_1 I)\cdots(T - \alpha_n I)\Big]U^* = U\mathbf{0}U^* = \mathbf{0}.$$

Thus, the required result follows. ∎

We now give some examples and then implications of the Cayley Hamilton Theorem.

Remark 6.4.15.

1. *Let $A = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$. Then, $P_A(x) = x^2 + 2x - 5$. Hence, verify that*

   $$A^2 + 2A - 5I_2 = \begin{bmatrix} 3 & -4 \\ -2 & 11 \end{bmatrix} + 2\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathbf{0}.$$

   Further, $A^2 = -2A + 5I$ implies $A^{-1} = \frac{1}{5}(A + 2I_2) = \frac{1}{5}\begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix}$ and

   $$A^3 = A(A^2) = A(-2A+5I) = -2A^2+5I = -2(-2A+5I)+5I = 4A-10I+5I = 4A-5I.$$

   Now, use induction to show $A^m \in LS(I, A)$, for all $m \geq 1$.

2. *Let $A = \begin{bmatrix} 3 & 1 \\ 2 & 0 \end{bmatrix}$. Then, $P_A(t) = t^2 - 3t - 2$. So, $P_A(A) = \mathbf{0} \Rightarrow A^2 = 3A + 2I$. Thus, $A^{-1} = \frac{A-3I}{2}$. Further, induction implies $A^m \in LS(I, A)$, for all $m \geq 1$.*

3. *Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then, $P_A(x) = x^2$. So, even though $A \neq \mathbf{0}$, $A^2 = \mathbf{0}$.*

4. *For $A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $P_A(x) = x^3$. So, Cayley Hamilton theorem $\Rightarrow A^3 = \mathbf{0}$. Here $A^2 = \mathbf{0}$.*

5. *For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $P_A(t) = (t-1)^3$. So $P_A(A) = \mathbf{0}$. But, observe that $q(A) = \mathbf{0}$, where $q(t) = (t-1)^2$.*

DRAFT

6. *Let $A \in \mathbb{M}_n(\mathbb{C})$ with $P_A(x) = x^n - a_{n-1}x^{n-1} + \cdots + (-1)^{n-1}a_1 x + (-1)^n a_0$.*

   (a) *Then, for any $\ell \in \mathbb{N}$, the division algorithm gives $\alpha_0, \alpha_1, \ldots, \alpha_{n-1} \in \mathbb{C}$ and a polynomial $f(x)$ with coefficients from $\mathbb{C}$ such that*

   $$x^\ell = f(x)P_A(x) + \alpha_0 + x\alpha_1 + \cdots + x^{n-1}\alpha_{n-1}.$$

   Hence, by the Cayley Hamilton theorem, $A^\ell = \alpha_0 I + \alpha_1 A + \cdots + \alpha_{n-1}A^{n-1}$.

      i. *Thus, to compute any power of A, one needs to apply the division algorithm to get α_i's and know A^i, for $1 \le i \le n-1$. This is quite helpful in numerical computation as computing powers takes much more time than division.*

      ii. *Note that $LS\left\{I, A, A^2, \ldots\right\}$ is a subspace of $\mathbb{M}_n(\mathbb{C})$. Also, $\dim\left(\mathbb{M}_n(\mathbb{C})\right) = n^2$. But, the above argument implies that $\dim\left(LS\left\{I, A, A^2, \ldots\right\}\right) \le n$.*

      iii. *In the language of graph theory, it says the following: "Let G be a graph on n vertices and A its adjacency matrix. Suppose there is no path of length $n-1$ or less from a vertex v to a vertex u in G. Then, G doesn't have a path from v to u of any length. That is, the graph G is disconnected and v and u are in different components of G."*

   (b) *Suppose A is non-singular. Then, by definition $a_0 = \det(A) \neq 0$. Hence,*

   $$A^{-1} = \frac{1}{a_0}\left[a_1 I - a_2 A + \cdots + (-1)^{n-2}a_{n-1}A^{n-2} + (-1)^{n-1}A^{n-1}\right].$$

   This matrix identity can be used to calculate the inverse.

   (c) *The above also implies that if A is invertible then $A^{-1} \in LS\left\{I, A, A^2, \ldots\right\}$. That is, A^{-1} is a linear combination of the vectors $I, A, \ldots, A^{n-1}$.*

DRAFT

EXERCISE **6.4.16.** *Miscellaneous Exercises:*

1. *Use the Cayley-Hamilton theorem to compute the inverse of following matrices:*
$\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 1 & 1 & 2 \end{bmatrix}$, $\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix}$.

2. *Let $A, B \in \mathbb{M}_2(\mathbb{C})$ such that $A = AB - BA$. Then, prove that $A^2 = \mathbf{0}$.*

3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be upper triangular matrices with the top leading principal submatrix of A of size k being $\mathbf{0}$. If $B[k+1, k+1] = 0$ then prove that the leading principal submatrix of size $k+1$ of AB is $\mathbf{0}$.*

4. *Let $B \in \mathbb{M}_{m,n}(\mathbb{C})$ and $A = \begin{bmatrix} 0 & B \\ B^T & 0 \end{bmatrix}$. Then $\left(\lambda, \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}\right)$ is an eigen-pair $\Leftrightarrow$ $\left(-\lambda, \begin{bmatrix} \mathbf{x} \\ -\mathbf{y} \end{bmatrix}\right)$ is an eigen-pair.*

5. *Let $B, C \in \mathbb{M}_n(\mathbb{R})$. Define $A = \begin{bmatrix} B & C \\ -C & B \end{bmatrix}$. Then, prove the following:*

   (a) *if s is a real eigenvalue of A with corresponding eigenvector $\begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix}$ then s is also an eigenvalue corresponding to the eigenvector $\begin{bmatrix} -\mathbf{y} \\ \mathbf{x} \end{bmatrix}$.*

(b) *if $s+it$ is a complex eigenvalue of A with corresponding eigenvector* $\begin{bmatrix} \mathbf{x}+i\mathbf{y} \\ -\mathbf{y}+i\mathbf{x} \end{bmatrix}$ *then* $s-it$ *is also an eigenvalue of A with corresponding eigenvector* $\begin{bmatrix} \mathbf{x}-i\mathbf{y} \\ -\mathbf{y}-i\mathbf{x} \end{bmatrix}$.

(c) $(s+it, \mathbf{x}+i\mathbf{y})$ *is an eigen-pair of* $B+iC$ *if and only if* $(s-it, \mathbf{x}-i\mathbf{y})$ *is an eigen-pair of* $B-iC$.

(d) $\left(s+it, \begin{bmatrix} \mathbf{x}+i\mathbf{y} \\ -\mathbf{y}+i\mathbf{x} \end{bmatrix}\right)$ *is an eigen-pair of A if and only if* $(s+it, \mathbf{x}+i\mathbf{y})$ *is an eigen-pair of* $B+iC$.

(e) $\det(A) = |\det(B+iC)|^2$.

The next section deals with quadratic forms which helps us in better understanding of conic sections in analytic geometry.

6.5 Quadratic Forms

Definition 6.5.1. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A is said to be

1. **positive semi-definite** (psd) if $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ and $\mathbf{x}^* A\mathbf{x} \geq 0$, for all $\mathbf{x} \in \mathbb{C}^n$.
2. **positive definite** (pd) if $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ and $\mathbf{x}^* A\mathbf{x} > 0$, for all $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.
3. **negative semi-definite** (nsd) if $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ and $\mathbf{x}^* A\mathbf{x} \leq 0$, for all $\mathbf{x} \in \mathbb{C}^n$.
4. **negative definite** (nd) if $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ and $\mathbf{x}^* A\mathbf{x} < 0$, for all $\mathbf{x} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$.
5. **indefinite** if $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ and there exist $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ such that $\mathbf{x}^* A\mathbf{x} < 0 < \mathbf{y}^* A\mathbf{y}$.

DRAFT

Lemma 6.5.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then A is Hermitian if and only if at least one of the following statements hold:*

1. *S^*AS is Hermitian for all $S \in \mathbb{M}_n$.*
2. *A is normal and has real eigenvalues.*
3. *$\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{C}^n$.*

Proof. Let $S \in \mathbb{M}_n$, $(S^*AS)^* = S^*A^*S = S^*AS$. Thus S^*AS is Hermitian.

Suppose $A = A^*$. Then, A is clearly normal as $AA^* = A^2 = A^*A$. Further, if $(\lambda, \mathbf{x})$ is an eigenpair then $\lambda \mathbf{x}^*\mathbf{x} = \mathbf{x}^*A\mathbf{x} \in \mathbb{R}$ implies $\lambda \in \mathbb{R}$.

For the last part, note that $\mathbf{x}^*A\mathbf{x} \in \mathbb{C}$. Thus $\overline{\mathbf{x}^*A\mathbf{x}} = (\mathbf{x}^*A\mathbf{x})^* = \mathbf{x}^*A^*\mathbf{x} = \mathbf{x}^*A\mathbf{x}$, we get $\mathrm{Im}(\mathbf{x}^*A\mathbf{x}) = 0$. Thus, $\mathbf{x}^*A\mathbf{x} \in \mathbb{R}$.

If S^*AS is Hermitian for all $S \in \mathbb{M}_n$ then taking $S = I_n$ gives A is Hermitian.

If A is normal then $A = U^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U$ for some unitary matrix U. Since $\lambda_i \in \mathbb{R}$, $A^* = (U^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U)^* = U^* \operatorname{diag}(\overline{\lambda_1}, \ldots, \overline{\lambda_n})U = U^* \operatorname{diag}(\lambda_1, \ldots, \lambda_n)U = A$. So, A is Hermitian.

If $\mathbf{x}^*A\mathbf{x} \in \mathbb{R}$ for all $\mathbf{x} \in \mathbb{C}^n$ then $a_{ii} = \mathbf{e}_i^*A\mathbf{e}_i \in \mathbb{R}$. Also, $a_{ii}+a_{jj}+a_{ij}+a_{ji} = (\mathbf{e}_i+\mathbf{e}_j)^*A(\mathbf{e}_i+\mathbf{e}_j) \in \mathbb{R}$. So, $\mathrm{Im}(a_{ij}) = -\mathrm{Im}(a_{ji})$. Similarly, $a_{ii}+a_{jj}+ia_{ij}-ia_{ji} = (\mathbf{e}_i+i\mathbf{e}_j)^*A(\mathbf{e}_i+i\mathbf{e}_j) \in \mathbb{R}$ implies that $\mathrm{Re}(a_{ij}) = \mathrm{Re}(a_{ji})$. Thus, $A = A^*$. ∎

Remark 6.5.3. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then the condition $\mathbf{x}^* A\mathbf{x} \in \mathbb{R}$, for all $\mathbf{x} \in \mathbb{C}^n$, in Definition 6.5.8 implies $A^T = A$, i.e., A is a symmetric matrix. But, when we study matrices over $\mathbb{R}$, we seldom consider vectors from $\mathbb{C}^n$. So, in such cases, we assume A is symmetric.*

Example 6.5.4. 1. Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ or $A = \begin{bmatrix} 3 & 1+i \\ 1-i & 4 \end{bmatrix}$. Then, A is positive definite.

2. Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ or $A = \begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix}$. Then, A is positive semi-definite but not positive definite.
3. Let $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ or $A = \begin{bmatrix} -2 & 1-i \\ 1+i & -2 \end{bmatrix}$. Then, A is negative definite.
4. Let $A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ or $A = \begin{bmatrix} -2 & 1-i \\ 1+i & -1 \end{bmatrix}$. Then, A is negative semi-definite.
5. Let $A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ or $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 1 \end{bmatrix}$. Then, A is indefinite.

Theorem 6.5.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, the following statements are equivalent.*

1. *A is positive semi-definite.*
2. *$A^* = A$ and each eigenvalue of A is non-negative.*
3. *$A = B^*B$ for some $B \in \mathbb{M}_n(\mathbb{C})$.*

DRAFT

Proof. $1 \Rightarrow 2$: Let A be positive semi-definite. Then, by Lemma 6.5.2, A is Hermitian. If $(\alpha, \mathbf{v})$ is an eigen-pair of A then $\alpha\|\mathbf{v}\|^2 = \alpha(\mathbf{v}^*\mathbf{v}) = \mathbf{v}^*(\alpha\mathbf{v}) = \mathbf{v}^*A\mathbf{v} \geq 0$. So, $\alpha \geq 0$.

$2 \Rightarrow 3$: Let $\sigma(A) = \{\alpha_1, \ldots, \alpha_n\}$. Then, by spectral theorem, there exists a unitary matrix U such that $U^*AU = D$ with $D = \operatorname{diag}(\alpha_1, \ldots, \alpha_n)$. As $\alpha_i \geq 0$, for $1 \leq i \leq n$, define $D^{\frac{1}{2}} = \operatorname{diag}(\sqrt{\alpha_1}, \ldots, \sqrt{\alpha_n})$. Then, $A = UD^{\frac{1}{2}}[D^{\frac{1}{2}}U^*] = B^*B$.

$3 \Rightarrow 1$: Let $A = B^*B$. Then, for $\mathbf{x} \in \mathbb{C}^n$, $\mathbf{x}^*A\mathbf{x} = \mathbf{x}^*B^*B\mathbf{x} = \|B\mathbf{x}\|^2 \geq 0$. Thus, the required result follows. ∎

A similar argument gives the next result and hence the proof is omitted.

Theorem 6.5.6. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, the following statements are equivalent.*

1. *A is positive definite.*
2. *$A^* = A$ and each eigenvalue of A is positive.*
3. *$A = B^*B$ for a non-singular matrix $B \in \mathbb{M}_n(\mathbb{C})$.*

Remark 6.5.7. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be Hermitian with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then, there exists a unitary matrix $U = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$ and a diagonal matrix $D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ such that $A = UDU^*$. Now, for $1 \leq i \leq n$, define $\alpha_i = \max\{\lambda_i, 0\}$ and $\beta_i = \min\{\lambda_i, 0\}$. Then*

1. *for $D_1 = diag(\alpha_1, \alpha_2, \ldots, \alpha_n)$, the matrix $A_1 = UD_1U^*$ is positive semi-definite.*
2. *for $D_2 = diag(\beta_1, \beta_2, \ldots, \beta_n)$, the matrix $A_2 = UD_2U^*$ is positive semi-definite.*
3. *$A = A_1 - A_2$. The matrix A_1 is generally called the positive semi-definite part of A.*

Definition 6.5.8. Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$. Then, a **sesquilinear form** in $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ is defined as $H(\mathbf{x}, \mathbf{y}) = \mathbf{y}^* A\mathbf{x}$. In particular, $H(\mathbf{x}, \mathbf{x})$, denoted $H(\mathbf{x})$, is called a **Hermitian form**. In case $A \in \mathbb{M}_n(\mathbb{R})$, $H(\mathbf{x})$ is called a **quadratic** form.

Remark 6.5.9. *Observe that*

1. *if $A = I_n$ then the bilinear/sesquilinear form reduces to the standard inner product.*
2. *$H(\mathbf{x}, \mathbf{y})$ is 'linear' in the first component and 'conjugate linear' in the second component.*
3. *the quadratic form $H(\mathbf{x})$ is a real number. Hence, for $\alpha \in \mathbb{R}$, the equation $H(\mathbf{x}) = \alpha$, represents a conic in $\mathbb{R}^n$.*

Example 6.5.10.
1. Let $A \in \mathbb{M}_n(\mathbb{R})$. Then, $f(\mathbf{x}, \mathbf{y}) = \mathbf{y}^T A\mathbf{x}$, for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$, is a bilinear form on $\mathbb{R}^n$.
2. Let $A = \begin{bmatrix} 1 & 2-i \\ 2+i & 2 \end{bmatrix}$. Then, $A^* = A$ and for $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{C}^2$, verify that
$$H(\mathbf{x}) = \mathbf{x}^* A\mathbf{x} = |x|^2 + 2|y|^2 + 2\mathrm{Re}\left((2-i)\overline{x}y\right)$$
where 'Re' denotes the real part of a complex number, is a sesquilinear form.

6.5.1 Sylvester's law of inertia

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The main idea of this section is to express $H(\mathbf{x})$ as sum or difference of squares. Since $H(\mathbf{x})$ is a quadratic in $\mathbf{x}$, replacing $\mathbf{x}$ by $c\mathbf{x}$, for $c \in \mathbb{C}$, just gives a multiplication factor by $|c|^2$. Hence, one needs to study only the normalized vectors. Let us consider Example 6.1.2 again. There we see that

$$\begin{aligned} \mathbf{x}^T A\mathbf{x} &= 3\frac{(x+y)^2}{2} - \frac{(x-y)^2}{2} = (x+2y)^2 - 3y^2, \text{ and} && (6.5.1) \\ \mathbf{x}^T B\mathbf{x} &= 5\frac{(x+2y)^2}{5} + 10\frac{(2x-y)^2}{5} = (3x - \frac{2y}{3})^2 + \frac{50y^2}{9}. && (6.5.2) \end{aligned}$$

Note that both the expressions in Equation (6.5.1) is the difference of two non-negative terms. Whereas, both the expressions in Equation (6.5.2) consists of sum of two non-negative terms. Is the number of non-negative terms, appearing in the above expressions, just a coincidence? For a better understanding, we define inertia of a Hermitian matrix.

Definition 6.5.11. Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix. The **inertia** of A, denoted $i(A)$, is the triplet $(i_+(A), i_-(A), i_0(A))$, where $i_+(A)$ is the number of positive eigenvalues of A, $i_-(A)$ is the number of negative eigenvalues of A and $i_0(A)$ is the nullity of A. The difference $i_+(A) - i_-(A)$ is called the **signature** of A.

EXERCISE **6.5.12.** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix. If the signature and the rank of A is known then prove that one can find out the inertia of A.*

To proceed with the earlier discussion, let $A \in \mathbb{M}_n(\mathbb{C})$ be Hermitian with eigenvalues $\alpha_1, \ldots, \alpha_n$. Then, by Theorem 6.4.10, $U^* AU = D = \mathrm{diag}(\alpha_1, \ldots, \alpha_n)$, for some unitary matrix

U. Let $\mathbf{x} = U\mathbf{z}$. Then, $\|\mathbf{x}\| = 1$ implies $\|\mathbf{z}\| = 1$. Thus, if $\mathbf{z} = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_n \end{bmatrix}$ then

$$H(\mathbf{x}) = \mathbf{z}^*U^*AU\mathbf{z} = \mathbf{z}^*D\mathbf{z} = \sum_{i=1}^{n} \alpha_i|\mathbf{z}_i|^2 = \sum_{i=1}^{p} |\sqrt{\alpha_i}\; \mathbf{z}_i|^2 - \sum_{i=p+1}^{r} \left|\sqrt{|\alpha_i|}\; \mathbf{z}_i\right|^2, \tag{6.5.3}$$

where $\alpha_1, \ldots, \alpha_p > 0$, $\alpha_{p+1}, \ldots, \alpha_r < 0$ and $\alpha_{r+1}, \ldots, \alpha_n = 0$, where $p = i_+(A)$ and $r - p = i_-(A)$. Thus, we see that the possible values of $H(\mathbf{x})$ seem to depend only on the positive and negative eigenvalues of A. Since U is an invertible matrix, the components $\mathbf{z}_i$'s of $\mathbf{z} = U^{-1}\mathbf{x} = U^*\mathbf{x}$ are commonly known as the **linearly independent linear forms**. Note that each $\mathbf{z}_i$ is a linear expression in the components of $\mathbf{x}$.

As a next result, we show that in any expression of $H(\mathbf{x})$ as a sum or difference of n *absolute squares of linearly independent linear forms*, the number p (respectively, $r-p$) gives the number of positive (respectively, negative) eigenvalues of A. This is popularly known as the 'Sylvester's law of inertia'.

Lemma 6.5.13. [Sylvester's Law of Inertia] *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and let $\mathbf{x} \in \mathbb{C}^n$. Then, every Hermitian form $H(\mathbf{x}) = \mathbf{x}^*A\mathbf{x}$, in n variables can be written as*

$$H(\mathbf{x}) = |\mathbf{y}_1|^2 + \cdots + |\mathbf{y}_p|^2 - |\mathbf{y}_{p+1}|^2 - \cdots - |\mathbf{y}_r|^2$$

where $\mathbf{y}_1, \ldots, \mathbf{y}_r$ are linearly independent linear forms in the components of $\mathbf{x}$ and the integers p and r satisfying $0 \le p \le r \le n$, depend only on A.

DRAFT

Proof. Equation (6.5.3) implies that $H(\mathbf{x})$ has the required form. We only need to show that p and r are uniquely determined by A. Hence, let us assume on the contrary that there exist $p, q, r, s \in \mathbb{N}$ with $p > q$ such that

$$\begin{aligned} H(\mathbf{x}) &= |\mathbf{y}_1|^2 + \cdots + |\mathbf{y}_p|^2 - |\mathbf{y}_{p+1}|^2 - \cdots - |\mathbf{y}_r|^2 && (6.5.4) \\ &= |\mathbf{z}_1|^2 + \cdots + |\mathbf{z}_q|^2 - |\mathbf{z}_{q+1}|^2 - \cdots - |\mathbf{z}_s|^2, && (6.5.5) \end{aligned}$$

where $\mathbf{y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = M\mathbf{x}$, $\mathbf{z} = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = N\mathbf{x}$ with $Y_1 = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_p \end{bmatrix}$ and $Z_1 = \begin{bmatrix} \mathbf{z}_1 \\ \vdots \\ \mathbf{z}_q \end{bmatrix}$ for some invertible matrices M and N. Now the invertibility of M and N implies $\mathbf{z} = B\mathbf{y}$, for some invertible matrix B. Decompose $B = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix}$, where B_1 is a $q \times p$ matrix. Then $\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$. As $p > q$, the homogeneous linear system $B_1Y_1 = \mathbf{0}$ has a nontrivial solution, say $\widetilde{Y_1} = \begin{bmatrix} \tilde{y_1} \\ \vdots \\ \tilde{y_p} \end{bmatrix}$ and consider $\widetilde{\mathbf{y}} = \begin{bmatrix} \widetilde{Y_1} \\ \mathbf{0} \end{bmatrix}$. Then for this choice of $\widetilde{\mathbf{y}}$, $Z_1 = \mathbf{0}$ and thus, using Equations (6.5.4) and (6.5.5), we have

$$H(\tilde{\mathbf{y}}) = |\tilde{y_1}|^2 + |\tilde{y_2}|^2 + \cdots + |\tilde{y_p}|^2 - 0 = 0 - (|z_{q+1}|^2 + \cdots + |z_s|^2).$$

Now, this can hold only if $\widetilde{Y_1} = \mathbf{0}$, a contradiction to $\widetilde{Y_1}$ being a non-trivial solution. Hence $p = q$. Similarly, the case $r > s$ can be resolved. This completes the proof of the lemma. □

Remark 6.5.14. *Since A is Hermitian,* $\textsc{Rank}(A)$ *equals the number of nonzero eigenvalues. Hence,* $\textsc{Rank}(A) = r$. *The number r is called the* **rank** *and the number $r - 2p$ is called the* **inertial degree** *of the Hermitian form $H(\mathbf{x})$.*

Do we need $*$-congruence at this stage?

We now look at another form of the Sylvester's law of inertia. We start with the following definition.

Definition 6.5.15. Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, A is said to be $*$**-congruent** (read star-congruent) to B if there exists an invertible matrix S such that $A = S^*BS$.

Theorem 6.5.16. **[Second Version: Sylvester's Law of Inertia]** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be Hermitian. Then, A is $*$-congruent to B if and only if $i(A) = i(B)$.*

Proof. By spectral theorem $U^*AU = \Lambda_A$ and $V^*BV = \Lambda_B$, for some unitary matrices U, V and diagonal matrices Λ_A, Λ_B of the form $\operatorname{diag}(+, \cdots, +, -, \cdots, -, 0, \cdots, 0)$. Thus, there exist invertible matrices S, T such that $S^*AS = D_A$ and $T^*BT = D_B$, where D_A, D_B are diagonal matrices of the form $\operatorname{diag}(1, \cdots, 1, -1, \cdots, -1, 0, \cdots, 0)$.

If $i(A) = i(B)$, then it follows that $D_A = D_B$, *i.e.*, $S^*AS = T^*BT$ and hence $A = (TS^{-1})^*B(TS^{-1})$.

Conversely, suppose that $A = P^*BP$, for some invertible matrix P, and $i(B) = (k, l, m)$. As $T^*BT = D_B$, we have, $A = P^*(T^*)^{-1}D_BT^{-1}P = (T^{-1}P)^*D_B(T^{-1}P)$. Now, let $X = (T^{-1}P)^{-1}$. Then, $A = (X^{-1})^*D_BX^{-1}$ and we have the following observations.

1. As rank and nullity do not change under similarity transformation, $i_0(A) = i_0(D_B) = m$ as $i(B) = (k, l, m)$.

2. Using $i(B) = (k, l, m)$, we also have

   $$X[:, k+1]^*AX[:, k+1] = X[:, k+1]^*\big((X^{-1})^*D_B(X^{-1})\big)X[:, k+1] = \mathbf{e}_{k+1}^*D_B\mathbf{e}_{k+1} = -1.$$

   Similarly, $X[:, k+2]^*AX[:, k+2] = \cdots = X[:, k+l]^*AX[:, k+l] = -1$. As the vectors $X[:, k+1], \ldots, X[:, k+l]$ are linearly independent, using 9.7.10, we see that A has at least l negative eigenvalues.

3. Similarly, $X[:, 1]^*AX[:, 1] = \cdots = X[:, k]^*AX[:, k] = 1$. As $X[:, 1], \ldots, X[:, k]$ are linearly independent, using 9.7.10 again, we see that A has at least k positive eigenvalues.

Thus, it now follows that $i(A) = (k, l, m)$. ■

6.5.2 Applications in Eculidean Plane

We now obtain conditions on the eigenvalues of A, corresponding to the associated quadratic form, to characterize conic sections in $\mathbb{R}^2$, with respect to the standard inner product.

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Definition 6.5.17. Let $f(x,y) = ax^2 + 2hxy + by^2 + 2fx + 2gy + c$ be a general quadratic in x and y, with coefficients from $\mathbb{R}$. Then,

$$H(\mathbf{x}) = \mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x, & y \end{bmatrix} \begin{bmatrix} a & h \\ h & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = ax^2 + 2hxy + by^2$$

is called the **associated quadratic form** of the conic $f(x,y) = 0$.

Proposition 6.5.18. *Consider the quadratic $f(x,y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$, for $a, b, c, g, f, h \in \mathbb{R}$. If $(a, b, h) \neq (0,0,0)$ then $f(x,y) = 0$ represents*

1. *a parabola or a pair of parallel lines if $ab - h^2 = 0$,*

2. *a hyperbola or a pair of perpendicular lines if $ab - h^2 < 0$,*

3. *an ellipse or a circle or a point (point of intersection of a pair of perpendicular lines) if $ab - h^2 > 0$.*

Proof. Consider the associated quadratic $ax^2 + 2hxy + by^2$ with $A = \begin{bmatrix} a & h \\ h & b \end{bmatrix}$ as the associated symmetric matrix. Then, by Corollary 6.4.11, $A = U \text{ diag}(\alpha_1, \alpha_2) U^T$, where $U = [\mathbf{u}_1, \mathbf{u}_2]$ is an orthogonal matrix, with $(\alpha_1, \mathbf{u}_1)$ and $(\alpha_2, \mathbf{u}_2)$ as eigen-pairs of A. As $(a, b, h) \neq (0,0,0)$ at least one of $\alpha_1, \alpha_2 \neq 0$. Also,

$$\mathbf{x}^T A \mathbf{x} = \begin{bmatrix} x, & y \end{bmatrix} U \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} U^T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u & v \end{bmatrix} \begin{bmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \alpha_1 u^2 + \alpha_2 v^2,$$

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where $\begin{bmatrix} u \\ v \end{bmatrix} = U^T \mathbf{x}$. The lines $u = 0, v = 0$ are the two linearly independent linear forms, which correspond to two perpendicular lines passing through the origin in the (x,y)-plane. In terms of u, v, $f(x,y)$ reduces to $f(u,v) = \alpha_1 u^2 + \alpha_2 v^2 + d_1 u + d_2 v + c$, for some choice of $d_1, d_2 \in \mathbb{R}$. We now look at different cases:

1. if $\alpha_1 = 0$ and $\alpha_2 \neq 0$ then $ab - h^2 = \det(A) = \alpha_1 \alpha_2 = 0$. In this case,

   $$f(u,v) = 0 \Leftrightarrow \alpha_2 \left(v + \frac{d_2}{2\alpha_2} \right)^2 = c_1 - d_1 u,$$

   for some $c_1 \in \mathbb{R}$.

   (a) If $d_1 = 0$, the quadratic corresponds to either the same line $v + \dfrac{d_2}{2\alpha_2} = 0$, two parallel lines or two imaginary lines, depending on whether $c_1 = 0, c_1\alpha_2 > 0$ and $c_1 \alpha_2 < 0$, respectively.

   (b) If $d_1 \neq 0$, the quadratic corresponds to a parabola of the form $V^2 = 4aU$, for some translate $U = u + \alpha$ and $V = v + \beta$.

2. If $\alpha_1 \alpha_2 < 0$ then $ab - h^2 = \det(A) = \lambda_1 \lambda_2 < 0$. If $\alpha_2 = -\beta_2 < 0$, for $\beta_2 > 0$ then the quadratic reduces to $\alpha_1 (u + d_1)^2 - \beta_2 (v + d_2)^2 = d_3$, or equivalently, to $\left(\sqrt{\alpha_1}(u + d_1) + \sqrt{\beta_2}(v + d_2)\right) \cdot \left(\sqrt{\alpha_1}(u + d_1) - \sqrt{\beta_2}(v + d_2)\right) = d_3$, for some $d_1, d_2, d_3 \in \mathbb{R}$. Thus, the quadratic corresponds to

(a) a pair of perpendicular lines $u + d_1 = 0$ and $v + d_2 = 0$ whenever $d_3 = 0$.

(b) a hyperbola with orthogonal principal axes $u + d_1 = 0$ and $v + d_2 = 0$ whenever $d_3 \neq 0$. In particular, if $d_3 > 0$ then the corresponding equation equals

$$\frac{\alpha_1(u+d_1)^2}{d_3} - \frac{\alpha_2(v+d_2)^2}{d_3} = 1.$$

3. If $\alpha_1\alpha_2 > 0$ then $ab - h^2 = \det(A) = \alpha_1\alpha_2 > 0$. Here, the quadratic reduces to $\alpha_1(u + d_1)^2 + \alpha_2(v + d_2)^2 = d_3$, for some $d_1, d_2, d_3 \in \mathbb{R}$. Thus, the quadratic corresponds to

(a) a point which is the point of intersection of the pair of orthogonal lines $u + d_1 = 0$ and $v + d_2 = 0$ if $d_3 = 0$.

(b) an empty set if $\alpha_1 d_3 < 0$.

(c) an ellipse or circle with $u + d_1 = 0$ and $v + d_2 = 0$ as the orthogonal principal axes if $\alpha_1 d_3 > 0$ with the corresponding equation

$$\frac{\alpha_1(u+d_1)^2}{d_3} + \frac{\alpha_2(v+d_2)^2}{d_3} = 1.$$

Thus, we have considered all the possible cases and the required result follows. ■

Remark 6.5.19. *Observe that the linearly independent forms* $\begin{bmatrix} u \\ v \end{bmatrix} = U^T \begin{bmatrix} x \\ y \end{bmatrix}$ *are functions of the eigenvectors* $\mathbf{u}_1$ *and* $\mathbf{u}_2$. *Further, the linearly independent forms together with the shifting of the origin give us the principal axes of the corresponding conic.*

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Example 6.5.20. 1. Let $H(\mathbf{x}) = x^2 + y^2 + 2xy$ be the associated quadratic form for a class of curves. Then, $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and the eigen-pairs are $\left(2, \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}\right)$ and $\left(0, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\right)$. In particular, for

(a) $f(x, y) = x^2 + 2xy + y^2 - 8x - 8y + 16$, we have $f(x, y) = 0 \Leftrightarrow (x + y - 4)^2 = 0$, just one line.

(b) $f(x, y) = x^2 + 2xy + y^2 - 8x - 8y$, we have $f(x, y) = 0 \Leftrightarrow (x + y - 8) \cdot (x + y) = 0$, a pair of parallel lines.

(c) $f(x, y) = x^2 + 2xy + y^2 - 6x - 10y - 3$, we have

$$\begin{aligned} f(x,y) = 0 \quad \Leftrightarrow \quad & 2\left(\frac{x+y}{\sqrt{2}}\right)^2 + 0\left(\frac{x-y}{\sqrt{2}}\right)^2 = 8\sqrt{2}\left(\frac{x+y}{\sqrt{2}}\right) - 2\sqrt{2}\left(\frac{x-y}{\sqrt{2}}\right) + 3 \\ \Leftrightarrow \quad & \left(\frac{x+y-4}{\sqrt{2}}\right)^2 = -\sqrt{2}\left(\frac{x-y-19/2}{\sqrt{2}}\right), \end{aligned}$$

a parabola with principal axes $x + y = 4$, $2x - 2y = 19$ and directrix $x - y = 10$.

2. Let $H(\mathbf{x}) = 10x^2 - 5y^2 + 20xy$ be the associated quadratic form for a class of curves. Then $A = \begin{bmatrix} 10 & 10 \\ 10 & -5 \end{bmatrix}$ and the eigen-pairs are $\left(15, \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}\right)$ and $\left(-10, \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}\right)$. So, for

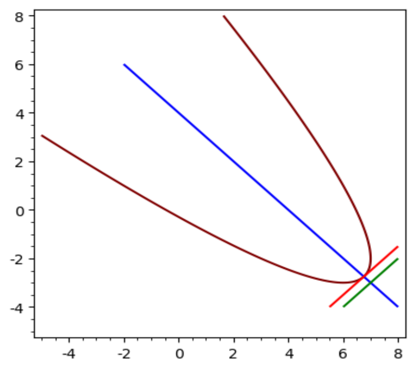

Figure 6.2: Conic $x^2 + 2xy + y^2 - 6x - 10y = 3$

(a) $f(x, y) = 10x^2 - 5y^2 + 20xy + 16x - 2y + 1$, we have $f(x, y) = 0 \Leftrightarrow 3(2x + y + 1)^2 - 2(x - 2y - 1)^2 = 0$, a pair of perpendicular lines.

(b) $f(x, y) = 10x^2 - 5y^2 + 20xy + 16x - 2y + 19$, we have

$$f(x, y) = 0 \Leftrightarrow \left(\frac{x - 2y - 1}{3}\right)^2 - \left(\frac{2x + y + 1}{\sqrt{6}}\right)^2 = 1,$$

a hyperbola.

(c) $f(x, y) = 10x^2 - 5y^2 + 20xy + 16x - 2y - 17$, we have

$$f(x, y) = 0 \Leftrightarrow \left(\frac{2x + y + 1}{\sqrt{6}}\right)^2 - \left(\frac{x - 2y - 1}{3}\right)^2 = 1,$$

a hyperbola.

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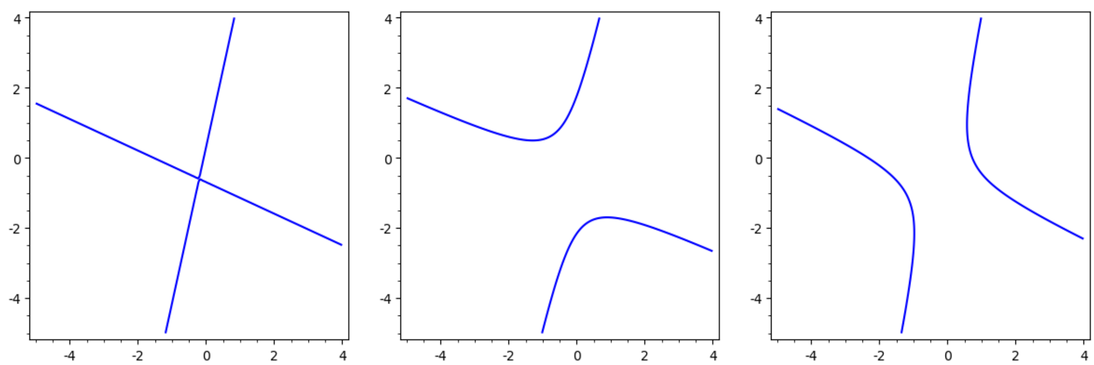

Figure 6.3: Conic $10x^2 - 5y^2 + 20xy + 16x - 2y = c$, $c = -1$, $c = -19$ and $c = 17$

3. Let $H(\mathbf{x}) = 6x^2 + 9y^2 + 4xy$ be the associated quadratic form for a class of curves. Then, $A = \begin{bmatrix} 6 & 2 \\ 2 & 9 \end{bmatrix}$, and the eigen-pairs are $\left(10, \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\right)$ and $\left(5, \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}\right)$. So, for

(a) $f(x, y) = 6x^2 + 9y^2 + 4xy + 10y - 53$, we have

$$f(x, y) = 0 \Leftrightarrow \left(\frac{x + 2y + 1}{5}\right)^2 + \left(\frac{2x - y - 1}{5\sqrt{2}}\right)^2 = 1,$$

an ellipse.

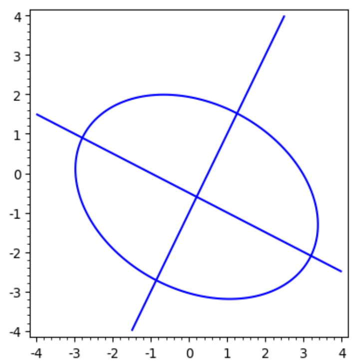

Figure 6.4: Conic $6x^2 + 9y^2 + 4xy + 10y = 53$

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EXERCISE **6.5.21.** *Sketch the graph of the following surfaces:*

1. $x^2 + 2xy + y^2 + 6x + 10y = 3$.

2. $2x^2 + 6xy + 3y^2 - 12x - 6y = 5$.

3. $4x^2 - 4xy + 2y^2 + 12x - 8y = 10$.

4. $2x^2 - 6xy + 5y^2 - 10x + 4y = 7$.

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6.5.3 Applications in Eculidean Space

As a last application, we consider a quadratic in 3 variables, namely x_1, x_2 and x_3. To do so, let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ with

$$\begin{aligned} f(x_1, x_2, x_3) &= \mathbf{x}^T A\mathbf{x} + 2\mathbf{b}^T\mathbf{x} + q \\ &= ax_1^2 + bx_2^2 + cx_3^2 + 2hx_1x_2 + 2gx_1x_3 + 2fx_2x_3 \\ &\qquad +2lx_1 + 2mx_2 + 2nx_3 + q \end{aligned} \tag{6.5.6}$$

Then, we observe the following:

1. As A is symmetric, $P^T AP = \text{diag}(\alpha_1, \alpha_2, \alpha_3)$, where $P = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$ is an orthogonal matrix and $(\alpha_i, \mathbf{u}_i)$, for $i = 1, 2, 3$ are eigen-pairs of A.

2. Let $\mathbf{y} = P^T\mathbf{x}$. Then, $f(x_1, x_2, x_3)$ reduces to

$$g(y_1, y_2, y_3) = \alpha_1 y_1^2 + \alpha_2 y_2^2 + \alpha_3 y_3^2 + 2l_1y_1 + 2l_2y_2 + 2l_3y_3 + q. \tag{6.5.7}$$

3. Depending on the values of α_i's, rewrite $g(y_1, y_2, y_3)$ to determine the center and the planes of symmetry of $f(x_1, x_2, x_3) = 0$.

Example 6.5.22. Determine the following quadrics $f(x, y, z) = 0$, where

1. $f(x,y,z) = 2x^2 + 2y^2 + 2z^2 + 2xy + 2xz + 2yz + 4x + 2y + 4z + 2$.

2. $f(x,y,z) = 3x^2 - y^2 + z^2 + 10$.

3. $f(x,y,z) = 3x^2 - y^2 + z^2 - 10$.

4. $f(x,y,z) = 3x^2 - y^2 + z - 10$.

Solution: (1) Here $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$ and $q = 2$. So, verify $P = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \end{bmatrix}$ and $P^T AP =$ diag$(4,1,1)$. Hence, $f(x,y,z) = 0$ reduces to

$$4\left(\frac{x+y+z}{\sqrt{3}}\right)^2 + \left(\frac{x-y}{\sqrt{2}}\right)^2 + \left(\frac{x+y-2z}{\sqrt{6}}\right)^2 = -(4x+2y+4z+2).$$

Or equivalently to $4\left(\frac{4(x+y+z)+5}{4\sqrt{3}}\right)^2 + \left(\frac{x-y+1}{\sqrt{2}}\right)^2 + \left(\frac{x+y-2z-1}{\sqrt{6}}\right)^2 = \frac{9}{12}$. So, the principal axes of the quadric (an ellipsoid) are $4(x+y+z) = -5, x-y = 1$ and $x+y-2z = 1$.

Part 2 Here $f(x,y,z) = 0$ reduces to $\frac{y^2}{10} - \frac{3x^2}{10} - \frac{z^2}{10} = 1$ which is the equation of a hyperboloid consisting of two sheets with center $\mathbf{0}$ and the axes x, y and z as the principal axes.

Part 3 Here $f(x,y,z) = 0$ reduces to $\frac{3x^2}{10} - \frac{y^2}{10} + \frac{z^2}{10} = 1$ which is the equation of a hyperboloid consisting of one sheet with center $\mathbf{0}$ and the axes x, y and z as the principal axes.

Part 4 Here $f(x,y,z) = 0$ reduces to $z = y^2 - 3x^2 + 10$ which is the equation of a hyperbolic paraboloid.

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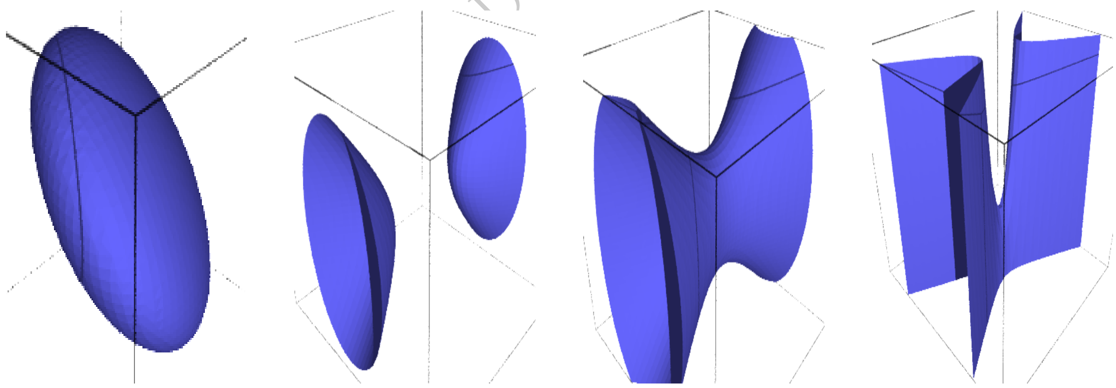

Figure 6.5: Ellipsoid, hyperboloid of two sheets and one sheet, hyperbolic paraboloid

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6.6 Singular Value Decomposition

In Theorem 6.4.10, we saw that if $A \in \mathbb{M}_n(\mathbb{C})$ is a Hermitian matrix then we can find a unitary matrix U such that $A = UDU^*$, where D is a diagonal matrix. That is, after a rotation or reflection of axes, the matrix A basically looks like a diagonal matrix. We also saw it's applications in Section 6.5. In this section, the idea is to have a similar understanding for any matrix A. We will do it over complex numbers and hence, the ideas from Theorem 6.4.10 will be used. We start with the following result.

Lemma 6.6.1. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ with $m \leq n$ and $\mathsf{Rank}A = k \leq m$. Then $A = UDV^*$, where*

1. *U is a unitary matrix and is obtained from the spectral decomposition of $AA^* = U\Lambda U^*$ with $\lambda_{11} \geq \cdots \geq \lambda_{mm} \geq 0$ are the eigenvalues of AA^*,*
2. *$D = \Lambda^{1/2}$, and*
3. *V^* is formed by taking the first k rows of U^*A and adding $m-k$ new rows so that V^* has orthonormal rows.*

If A is real, then U and V may be chosen to have real entries.

Proof. Note that AA^* is an $m \times m$ Hermitian matrix. Thus, for any $\mathbf{x} \in \mathbb{C}^m$,

$$\mathbf{x}^*(AA^*)\mathbf{x} = (\mathbf{x}^*A)(A^*\mathbf{x}) = (A^*\mathbf{x})^*(A^*\mathbf{x}) = \|A^*\mathbf{x}\|^2 \geq 0.$$

Hence, the matrix AA^* is a positive semi-definite matrix. Therefore, all it's eigenvalues are non-negative. So, by the spectral theorem, Theorem 6.4.10, $AA^* = U\Lambda U^*$, where $\lambda_{ii} \geq 0$ are in decreasing order. As $\mathsf{Rank}A = k$, $\lambda_{ii} > 0$, for $1 \leq i \leq k$ and $\lambda_{ii} = 0$, for $k+1 \leq i \leq m$. Now, let $\Sigma = [\sigma_{ij}]$ be the diagonal matrix with

$$\sigma_{ii} = \begin{cases} 1/\sqrt{\lambda_{ii}}, & \text{if } i \leq k \\ 1, & \text{otherwise.} \end{cases}$$

Then, we see that the matrix $X = \Sigma U^*A$ is an $m \times n$ matrix with

$$XX^* = (\Sigma U^*A)(A^*U\Sigma) = \Sigma U^*(U\Lambda U^*)U\Sigma = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}. \tag{6.6.8}$$

As $XX^* = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$, the first k-rows of X form an orthonormal set. Note that the first k rows of the matrix X are given by

$$X[1,:] = \frac{1}{\sqrt{\lambda_{11}}}(U^*A)[1,:], \ldots, X[k,:] = \frac{1}{\sqrt{\lambda_{kk}}}(U^*A)[k,:].$$

Or equivalently,

$$(U^*A)[1,:] = X[1,:]\sqrt{\lambda_{11}}, \ldots, (U^*A)[k,:] = X[k,:]\sqrt{\lambda_{kk}}. \tag{6.6.9}$$

Now, take these k rows of X and add $m-k$ many rows to form V^*, so that the rows of V^* are orthonormal, *i.e.*, $V^*V = I_m$. Also, using (6.6.8), we see that $(\Sigma U^*AA^*U\Sigma)_{k+1,k+1} = 0$. Thus,

$$(\Sigma U^*A)[k+1,:](A^*U\Sigma)[:,k+1] = 0 \Rightarrow (\Sigma U^*A)[k+1,:] = 0.$$

This in turn implies that $(U^*A)[k+1,:] = 0$. Similarly, $(U^*A)[j,:] = 0$, for $k+2 \le j \le m$. Thus, using (6.6.9) and the definition of the matrix V^*, we get

$$U^*A = \begin{bmatrix} X[1,:]\sqrt{\lambda_{11}} \\ \vdots \\ X[k,:]\sqrt{\lambda_{kk}} \\ \mathbf{0}^T \\ \vdots \\ \mathbf{0}^T \end{bmatrix} = \operatorname{diag}(\sqrt{\lambda_{11}}, \ldots, \sqrt{\lambda_{kk}}, 0, \ldots 0) \begin{bmatrix} X[1,:] \\ \vdots \\ X[k,:] \\ V^*[k+1,:] \\ \vdots \\ V^*[m,:] \end{bmatrix} = DV^*$$

where $D = \operatorname{diag}(\sqrt{\lambda_{11}}, \ldots, \sqrt{\lambda_{kk}}, 0, \ldots 0)$. Thus, we have $A = UDV^*$.

We already know that in spectral theorem, that if A is real symmetric, we could choose U to be a real orthogonal matrix and that makes the first k rows of V to have real entries. We can always choose the next $m-k$ vectors to also have real entries. ∎

It is important to note that

$$A^*A = (UDV^*)^*(UDV^*) = (VDU^*)(UDV^*) = VD^2V^*,$$

where $D^2 = \operatorname{diag}(\lambda_{11}, \ldots, \lambda_{kk}, 0, \ldots 0)$ are the eigenvalues of A^*A and the columns of V are the corresponding eigenvectors.

Corollary 6.6.2. **[Polar decomposition]** *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ with $m \le n$. Then $A = PW$, for some positive semi-definite matrix P with* $\mathsf{Rank}P = \mathsf{Rank}A$ *and a matrix W having orthonormal rows. In fact, $P = (AA^*)^{1/2}$.*

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Proof. By Lemma 6.6.1, $A = UDV^*$, where U is a unitary matrix which is obtained from the spectral decomposition of $AA^* = U\Lambda U^*$, $D = \Lambda^{1/2}$, and V^* has orthonormal rows. Then $A = (UDU^*)(UV^*)$. Notice that the matrix UV^* also has orthonormal rows. Note that $UDU^* = U\Lambda^{1/2}U^* = (AA^*)^{1/2}$. So, if we put $P = UDU^*$ and $W = UV^*$ then, we see that $A = PW$ with $P = (AA^*)^{1/2}$ is positive semi-definite matrix with $\mathsf{Rank}P = \mathsf{Rank}A$ and a matrix W having orthonormal rows. ∎

Corollary 6.6.3. **[Singular value decomposition]** *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ with $m \le n$ and* $\mathsf{Rank}A = k \le m$. *Then $A = UDV^*$, where*

1. *U is an $m \times m$ unitary matrix and is obtained from the spectral decomposition of $AA^* = U\Lambda U^*$ with $\lambda_{11} \ge \cdots \ge \lambda_{mm} \ge 0$ are the eigenvalues of AA^*,*
2. *$D = \begin{bmatrix} \Lambda^{1/2} & 0_{m,n-m} \end{bmatrix}$, and*
3. *V^* is formed by taking the first k rows of U^*A and adding $n-k$ new rows so that V is an $n \times n$ unitary matrix.*

If A is real, then U and V may be chosen to have real entries.

Proof. Follows from Lemma 6.6.1. ∎

Definition 6.6.4. Let $A \in M_{m,n}$. In view of Corollary 6.6.3, the values $\sqrt{\lambda_{11}}, \ldots, \sqrt{\lambda_{rr}}$, where $r = \min\{m,n\}$, are called the **singular values** of A. (Sometimes only the nonzero λ_{ii}'s are understood to be the singular values of A).

Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$. Then, by the **singular value decomposition** of A we mean writing $A = U\Sigma V^*$, where $U \in \mathbb{M}_m(\mathbb{C}), V \in \mathbb{M}_n(\mathbb{C})$ are unitary matrices and $\Sigma \in \mathbb{M}_{m,n}(\mathbb{R})$ with Σ_{ii} as the singular values of A, for $1 \le i \le \mathsf{Rank} A$ and the remaining entries of Σ being 0.

In Corollary 6.6.3, we saw that the matrix U is obtained as the unitary matrix in the spectral decomposition of AA^*, the Σ_{ii}'s are the square-root of the eigenvalues of AA^*, and V^* is formed by taking the first $r = \mathsf{Rank} A$ rows of U^*A and adding $n-k$ new rows so that V^* is a unitary matrix.

Now, let us go back to matrix multiplication and try to understand $A = U\Sigma V^*$. So, let $U = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m\end{bmatrix}$ and $V = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_m\end{bmatrix}$. Then,

$$\begin{aligned} A &= U\Sigma V^* = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_m\end{bmatrix} \begin{bmatrix} \sqrt{\lambda_{11}} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \sqrt{\lambda_{22}} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sqrt{\lambda_{33}} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \vdots \\ \mathbf{v}_n^* \end{bmatrix} \\ &= \sqrt{\lambda_{11}}\mathbf{u}_1\mathbf{v}_1^* + \sqrt{\lambda_{22}}\mathbf{u}_2\mathbf{v}_2^* + \cdots + \sqrt{\lambda_{mm}}\mathbf{u}_m\mathbf{v}_m^*. \end{aligned} \tag{6.6.10}$$

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Now, recall that if A is an $m \times n$ matrix then we can associate a linear transformation $T : \mathbb{C}^n \to \mathbb{C}^m$ such that $T(\mathbf{x}) = A\mathbf{x}$, for all $\mathbf{x} \in \mathbb{C}^n$. Thus, the singular value decomposition states that there exist unitary matrices (rotation or reflection matrices) in both the domain ($\mathbb{C}^n$, corresponds to V^*) and co-domain ($\mathbb{C}^m$, corresponds to U) such that the matrix of A with respect to these ordered bases is a diagonal matrix and the diagonal entries consist of just the singular values, including zeros.

We also note that if $r = \mathsf{Rank} A$ then $A = \sqrt{\lambda_{11}}\mathbf{u}_1\mathbf{v}_1^* + \sqrt{\lambda_{22}}\mathbf{u}_2\mathbf{v}_2^* + \cdots + \sqrt{\lambda_{rr}}\mathbf{u}_r\mathbf{v}_r^*$. Thus, $A = U_1\Sigma_1V_1^*$, where U_1 is a submatrix of U consisting of the first r orthonormal columns, Σ_1 is a diagonal matrix of non-zero singular values and V_1^* is a submatrix of V^* consisting of the first r orthonormal rows. More specifically,

$$A = \begin{bmatrix}\mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_r\end{bmatrix} \begin{bmatrix} \sqrt{\lambda_{11}} & 0 & \cdots & 0 \\ 0 & \sqrt{\lambda_{22}} & \cdots & 0 \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \sqrt{\lambda_{rr}} \end{bmatrix} \begin{bmatrix} \mathbf{v}_1^* \\ \mathbf{v}_2^* \\ \vdots \\ \mathbf{v}_r^* \end{bmatrix}.$$

Example 6.6.5. Let $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \end{bmatrix}$. Then, $AA^T = \begin{bmatrix} 6 & 3 \\ 3 & 6 \end{bmatrix}$. Thus, $AA^T = UDU^T$, where $U = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}$. Hence, $\Sigma = \begin{bmatrix} 3 & 0 & 0 \\ 0 & \sqrt{3} & 0 \end{bmatrix}$. Here,

$$U^TA = \frac{1}{\sqrt{2}}\begin{bmatrix} 3 & 3 & 0 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

Thus, $V^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} & \frac{-1}{\sqrt{3}} \end{bmatrix}$ and it's rows are the eigenvectors of $A^TA = \begin{bmatrix} 5 & 4 & 1 \\ 4 & 5 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

In actual computations, the values of m and n could be very large. Also, the largest and the smallest eigenvalues or the rows and columns of A that are of interest to us may be very small. So, in such cases, we compute the singular value decomposition to relate the above ideas or to find clusters which have maximum influence on the problem being looked. For example, in the above computation, the singular value 3 is the larger of the two singular values. So, if we are looking at the largest deviation or movement etc. then we need to concentrate on the singular value 3. Then, using equation (6.6.10), note that 3 is associated with the first column of U and the first row of V^T. Similarly, $\sqrt{3}$ is associated with the second column of U and the second row of V^T.

Note that in any computation, we need to decompose our problem into sub-problems. If the decomposition into sub-problems is possible through orthogonal decomposition then in some sense the sub-problems can be handled separately. That's how the singular value decomposition helps us in applications. This is the reason, that with slight change, SVD is also called "factor analysis" or "principal component analysis" and so on.

EXERCISE **6.6.6.** 1. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ with $m \geq n$. Then $A = WQ$, for some positive semi-definite matrix Q and a matrix W of orthonormal columns.*

2. *Let $A \in \mathbb{M}_{n,1}(\mathbb{C})$. Illustrate the polar decomposition and the singular value decompositions for $A = \mathbf{e}_i$ and for $A = \mathbf{e}_1 + 2\mathbf{e}_2 + \cdots + n\mathbf{e}_n$.*

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3. *Let $A \in \mathbb{M}_{m,n}(\mathbb{C})$ with $\mathsf{Rank} A = r$. If $d_1, \ldots, d_r$ are the non-zero singular values of A then, there exist $\Sigma \in \mathbb{M}_{m,n}(\mathbb{R})$, and unitary matrices $U \in \mathbb{M}_m(\mathbb{C})$ and $V \in \mathbb{M}_n(\mathbb{C})$ such that $A = U\Sigma V^*$, where $\Sigma = \begin{bmatrix} \Sigma_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}$ with $\Sigma_1 = diag(d_1, \ldots, d_r)$. Then, prove that $G = VDU^*$, for $D = \begin{bmatrix} \Sigma_1^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \in \mathbb{M}_{n,m}(\mathbb{C})$ is the pseudo-inverse of A.*

Chapter 7

Jordan canonical form

7.1 Jordan canonical form theorem

We start this chapter with the following theorem which generalizes the Schur Upper triangularization theorem.

Theorem 7.1.1. **[Generalized Schur's theorem]** *Let $A \in \mathbb{M}_n(\mathbb{C})$. Suppose $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A with multiplicities $m_1, \ldots, m_k$, respectively. Then, there exists a non-singular matrix W such that*

$$W^{-1}AW = \bigoplus_{i=1}^{k} T_i, \ \textit{where, } T_i \in \mathbb{M}_{m_i}(\mathbb{C}), \ \textit{for } 1 \leq i \leq k$$

and T_i's are upper triangular matrices with constant diagonal λ_i. If A has real entries with real eigenvalues then W can be chosen to have real entries.

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Proof. By Schur Upper Triangularization (see Lemma 6.4.1), there exists a unitary matrix U such that $U^*AU = T$, an upper triangular matrix with $\operatorname{diag}(T) = (\lambda_1, \ldots, \lambda_1, \ldots, \lambda_k, \ldots, \lambda_k)$.

Now, for any upper triangular matrix B, a real number α and $i < j$, consider the matrix $F(B, i, j, \alpha) = E_{ij}(-\alpha) B E_{ij}(\alpha)$, where the matrix $E_{ij}(\alpha)$ is defined in Definition 2.2.5. Then, for $1 \leq k, \ell \leq n$,

$$(F(B,i,j,\alpha))_{k\ell} = \begin{cases} B_{ij} - \alpha B_{jj} + \alpha B_{ii}, & \text{whenever } k = i, \ell = j \\ B_{i\ell} - \alpha B_{j\ell}, & \text{whenever } \ell \neq j \\ B_{kj} + \alpha B_{ki}, & \text{whenever } k \neq i \\ B_{k\ell}, & \text{otherwise.} \end{cases} \tag{7.1.1}$$

Now, using Equation (7.1.1), the diagonal entries of $F(T, i, j, \alpha)$ and T are equal and

$$(F(T,i,j,\alpha))_{ij} = \begin{cases} T_{ij}, & \text{whenever } T_{jj} = T_{ii} \\ 0, & \text{whenever } T_{jj} \neq T_{ii} \text{ and } \alpha = \frac{T_{ij}}{T_{jj} - T_{ii}}. \end{cases}$$

Thus, if we denote the matrix $F(T, i, j, \alpha)$ by T_1 then $(F(T_1, i-1, j, \alpha))_{i-1,j} = 0$, for some choice of α, whenever $(T_1)_{i-1,i-1} \neq T_{jj}$. Moreover, this operation also preserves the 0 created by

$F(T,i,j,\alpha)$ at (i,j)-th place. Similarly, $F(T_1,i,j+1,\alpha)$ preserves the 0 created by $F(T,i,j,\alpha)$ at (i,j)-th place. So, we can successively apply the following sequence of operations to get

$$T \to F(T,m_1,m_1+1,\alpha) = T_1 \to F(T_1,m_1-1,m_1+1,\beta) \to \cdots \to F(T_{m_1-1},1,m_1+1,\gamma) = T_{m_1},$$

where $\alpha,\beta,\ldots,\gamma$ are appropriately chosen and $T_{m_1}[:,m_1+1] = \lambda_2\mathbf{e}_{m_1+1}$. Thus, observe that the above operation can be applied for different choices of i and j with $i<j$ to get the required result. ■

Practice **7.1.2.** *Apply Theorem 7.1.1 to the matrix given below for better understanding.*

$$\left[\begin{array}{ccc|ccc|ccc} 1&2&3&4&5&6&7&8&9\\ 0&1&2&3&4&5&6&7&8\\ 0&0&1&2&3&4&5&6&7\\ \hline 0&0&0&2&3&4&5&6&7\\ 0&0&0&0&2&3&4&5&6\\ 0&0&0&0&0&2&3&4&5\\ \hline 0&0&0&0&0&0&3&4&5\\ 0&0&0&0&0&0&0&3&4\\ 0&0&0&0&0&0&0&0&3 \end{array}\right].$$

Definition 7.1.3. 1. Let $\lambda\in\mathbb{C}$ and k be a positive integer. Then, by the **Jordan block** $J_k(\lambda)\in\mathbb{M}_k(\mathbb{C})$, we understand the matrix

$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1\\ & & & \lambda \end{bmatrix}.$$

2. A **Jordan matrix** is a direct sum of Jordan blocks. That is, if A is a Jordan matrix having r blocks then there exist positive integers k_i's and complex numbers λ_i's (not necessarily distinct), for $1\le i\le r$ such that

$$A = J_{k_1}(\lambda_1)\oplus\cdots\oplus J_{k_r}(\lambda_r).$$

We now give some examples of Jordan matrices with diagonal entries 0.

Example 7.1.4. 1. $J_1(0) = \begin{bmatrix}0\end{bmatrix}$ is the only Jordan matrix of size 1.

2. $J_1(0)\oplus J_1(0) = \begin{bmatrix}0&0\\0&0\end{bmatrix}$ and $J_2(0) = \begin{bmatrix}0&1\\0&0\end{bmatrix}$ are Jordan matrices of size 2.

3. Even though , $J_1(0)\oplus J_2(0)$ and $J_2(0)\oplus J_1(0)$ are two Jordan matrices of size 3, we do not differentiate between them as they are similar (use permutations).

4. $J_1(0)\oplus J_1(0)\oplus J_1(0) = \begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$, $J_2(0)\oplus J_1(0) = \begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix}$ and $J_3(0) = \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ are Jordan matrices of size 3.

DRAFT

5. Observe that the number of Jordan matrices of size 4 with 0 on the diagonal are 5.

We now give some properties of the Jordan blocks. The proofs are immediate and hence left for the reader. They will be used in the proof of subsequent results.

Remark 7.1.5. **[Jordan blocks]** *Fix a positive integer k. Then,*

1. *$J_k(\lambda)$ is an upper triangular matrix with λ as an eigenvalue.*
2. *$J_k(\lambda) = \lambda I_k + J_k(0)$.*
3. *$\text{ALG.MUL}_\lambda(J_k(\lambda)) = k$.*
4. *The matrix $J_k(0)$ satisfies the following properties.*
   (a) *$\mathsf{Rank}((J_k(0)^i) = k - i$, for $1 \le i \le k$.*
   (b) *$J_k^T(0)J_k(0) = \begin{bmatrix} 0 & 0 \\ 0 & I_{k-1} \end{bmatrix}$.*
   (c) *$J_k(0)^p = 0$ whenever $p \ge k$.*
   (d) *$J_k(0)\mathbf{e}_i = \mathbf{e}_{i-1}$ for $i = 2, \ldots, k$.*
   (e) *$\Big(I - J_k^T(0)J_k(0)\Big)\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{0} \end{bmatrix} = \langle \mathbf{x}, \mathbf{e}_1 \rangle \mathbf{e}_1$.*
5. *Thus, using Remark 7.1.5.4d $\text{GEO.MUL}_\lambda(J_k(\lambda)) = 1$.*

EXERCISE **7.1.6.** 1. *Fix a positive integer k and a complex number λ. Then, prove that*
   (a) *$\mathsf{Rank}(J_k(\lambda) - \lambda I_k) = k - 1$.*
   (b) *$\mathsf{Rank}(J_k(\lambda) - \alpha I_k) = k$, whenever $\alpha \neq \lambda$. Or equivalently, for all $\alpha \neq \lambda$ the matrix $J_k(\lambda) - \alpha I_k$ is invertible.*
   (c) *for $1 \le i \le k$, $\mathsf{Rank}((J_k(\lambda) - \lambda I_k)^i) = k - i$.*
   (d) *for $\alpha \neq \lambda$, $\mathsf{Rank}((J_k(\lambda) - \alpha I_k)^i) = k$, for all i.*

2. *Let J be a Jordan matrix that contains ℓ Jordan blocks for λ. Then, prove that*
   (a) *$\mathsf{Rank}(J - \lambda I) = n - \ell$.*
   (b) *J has ℓ linearly independent eigenvectors for λ.*
   (c) *$\mathsf{Rank}(J - \lambda I) \ge \mathsf{Rank}((J - \lambda I)^2) \ge \mathsf{Rank}((J - \lambda I)^3) \ge \cdots$.*

3. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, prove that $AJ_n(\lambda) = J_n(\lambda)A$ if and only if $AJ_n(0) = J_n(0)A$.*

Definition 7.1.7. Let J be a Jordan matrix containing $J_t(\lambda)$, for some positive integer t and some complex number λ. Then, the smallest value of k for which $\mathsf{Rank}((J - \lambda I)^k)$ stops decreasing is the order of the largest Jordan block $J_k(\lambda)$ in J. This number k is called the **index of the eigenvalue** λ.

Lemma 7.1.8. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be strictly upper triangular. Then, A is similar to a direct sum of Jordan blocks. Or equivalently, there exists integers $n_1 \ge \ldots \ge n_m \ge 1$ and a non-singular matrix S such that*

$$A = S^{-1}\Big(J_{n_1}(0) \oplus \cdots \oplus J_{n_m}(0)\Big)S.$$

If $A \in \mathbb{M}_n(\mathbb{R})$ then S can be chosen to have real entries.

DRAFT

Proof. We will prove the result by induction on n. For $n=1$, the statement is trivial. So, let the result be true for matrices of size $\le n-1$ and let $A \in \mathbb{M}_n(\mathbb{C})$ be strictly upper triangular. Then, $A = \begin{bmatrix} 0 & \mathbf{a}^T \\ 0 & A_1 \end{bmatrix}$. By induction hypothesis there exists an invertible matrix S_1 such that

$$A_1 = S_1^{-1}\Big(J_{n_1}(0) \oplus \cdots \oplus J_{n_m}(0)\Big)S_1 \text{ with } \sum_{i=1}^{m} n_i = n-1.$$

Thus,

$$\begin{bmatrix} 1 & 0 \\ 0 & S_1^{-1} \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & S_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & S_1^{-1} \end{bmatrix}\begin{bmatrix} 0 & \mathbf{a}^T \\ 0 & A_1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & S_1 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{a}^T S_1 \\ 0 & S^{-1}A_1S_1 \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{a}_1^T & \mathbf{a}_2^T \\ 0 & J_{n_1}(0) & 0 \\ 0 & 0 & J \end{bmatrix},$$

where $S_1^{-1}\Big(J_{n_1}(0) \oplus \cdots \oplus J_{n_m}(0)\Big)S_1 = J_{n_1}(0) \oplus J$ and $\mathbf{a}^T S_1 = \begin{bmatrix} \mathbf{a}_1^T & \mathbf{a}_2^T \end{bmatrix}$. Now, writing J_{n_1} to mean $J_{n_1}(0)$ and using Remark 7.1.5.4e, we have

$$\begin{bmatrix} 1 & -\mathbf{a}_1^T J_{n_1}^T & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & I \end{bmatrix}\begin{bmatrix} 0 & \mathbf{a}_1^T & \mathbf{a}_2^T \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J \end{bmatrix}\begin{bmatrix} 1 & \mathbf{a}_1^T J_{n_1}^T & 0 \\ 0 & I_{n_1} & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} 0 & \langle \mathbf{a}_1, \mathbf{e}_1\rangle \mathbf{e}_1^T & \mathbf{a}_2^T \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J \end{bmatrix}.$$

So, we now need to consider two cases depending on whether $\langle \mathbf{a}_1, \mathbf{e}_1\rangle = 0$ or $\langle \mathbf{a}_1, \mathbf{e}_1\rangle \ne 0$. In the first case, A is similar to $\begin{bmatrix} 0 & 0 & \mathbf{a}_2^T \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J \end{bmatrix}$. This in turn is similar to $\begin{bmatrix} J_{n_1} & 0 & 0 \\ 0 & 0 & \mathbf{a}_2^T \\ 0 & 0 & J \end{bmatrix}$ by permuting the first row and column. At this stage, one can apply induction and if necessary do a block permutation, in order to keep the block sizes in decreasing order.

DRAFT

So, let us now assume that $\langle \mathbf{a}_1, \mathbf{e}_1\rangle \ne 0$. Then, writing $\alpha = \langle \mathbf{a}_1, \mathbf{e}_1\rangle$, we have

$$\begin{bmatrix} \frac{1}{\alpha} & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \frac{1}{\alpha}I \end{bmatrix}\begin{bmatrix} 0 & \alpha\mathbf{e}_1^T & \mathbf{a}_2^T \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J \end{bmatrix}\begin{bmatrix} \alpha & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \alpha I \end{bmatrix} = \begin{bmatrix} 0 & \mathbf{e}_1^T & \mathbf{a}_2^T \\ 0 & J_{n_1} & 0 \\ 0 & 0 & J \end{bmatrix} \equiv \begin{bmatrix} J_{n_1+1} & \mathbf{e}_1\mathbf{a}_2^T \\ 0 & J \end{bmatrix}.$$

Now, using Remark 7.1.5.4c, verify that

$$\begin{bmatrix} I & \mathbf{e}_{i+1}\mathbf{a}_2^T J^{i-1} \\ 0 & I \end{bmatrix}\begin{bmatrix} J_{n_1+1} & \mathbf{e}_i\mathbf{a}_2^T J^{i-1} \\ 0 & J \end{bmatrix}\begin{bmatrix} I & -\mathbf{e}_{i+1}\mathbf{a}_2^T J^{i-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} J_{n_1+1} & \mathbf{e}_{i+1}\mathbf{a}_2^T J^{i} \\ 0 & J \end{bmatrix}, \text{ for } i \ge 1.$$

Hence, for $p = n - n_1 - 1$, we have

$$\begin{bmatrix} I & \mathbf{e}_{p+1}\mathbf{a}_2^T J^{p-1} \\ 0 & I \end{bmatrix} \cdots \begin{bmatrix} I & \mathbf{e}_2\mathbf{a}^T \\ 0 & I \end{bmatrix}\begin{bmatrix} J_{n_1+1} & \mathbf{e}_1\mathbf{a}^T \\ 0 & J \end{bmatrix}\begin{bmatrix} I & -\mathbf{e}_2\mathbf{a}^T \\ 0 & I \end{bmatrix} \cdots \begin{bmatrix} I & -\mathbf{e}_{p+1}\mathbf{a}^T J^{p-1} \\ 0 & I \end{bmatrix} = \begin{bmatrix} J_{n_1+1} & 0 \\ 0 & J \end{bmatrix}.$$

If necessary, we need to do a block permutation, in order to keep the block sizes in decreasing order. Hence, the required result follows. ■

Practice **7.1.9.** *Convert* $\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ *to* $J_3(0)$ *and* $\begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ *to* $J_2(0) \oplus J_1(0)$.

Corollary 7.1.10. *$A \in \mathbb{M}_n(\mathbb{C})$. Then, A is similar to J, a Jordan matrix.*

Proof. Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A with algebraic multiplicities $m_1, \ldots, m_k$. By Theorem 7.1.1, there exists a non-singular matrix S such that $S^{-1}AS = \bigoplus\limits_{i=1}^{k} T_i$, where T_i is upper triangular with diagonal $(\lambda_i, \ldots, \lambda_i)$. Thus $T_i - \lambda_i I_{m_i}$ is a strictly upper triangular matrix. Thus, by Theorem 7.1.8, there exist a non-singular matrix S_i such that

$$S_i^{-1}\Big(T_i - \lambda_i I_{m_i}\Big)S_i = J(0),$$

a Jordan matrix with 0 on the diagonal and the size of the Jordan blocks decreases as we move down the diagonal. So, $S_i^{-1}T_iS_i = J(\lambda_i)$ is a Jordan matrix with λ_i on the diagonal and the size of the Jordan blocks decreases as we move down the diagonal.

Now, take $W = S\left(\bigoplus\limits_{i=1}^{k} S_i\right)$. Then, verify that $W^{-1}AW$ is a Jordan matrix. ■

Let $A \in \mathbb{M}_n(\mathbb{C})$. Suppose $\lambda \in \sigma(A)$ and J is a Jordan matrix that is similar to A. Then, for each fixed $i, 1 \le i \le n$, by $\ell_i(\lambda)$, we denote the number of Jordan blocks $J_k(\lambda)$ in J for which $k \ge i$. Then, the next result uses Exercise 7.1.6 to determine the number $\ell_i(\lambda)$.

Remark 7.1.11. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Suppose $\lambda \in \sigma(A)$ and J is a Jordan matrix that is similar to A. Then, for $1 \le k \le n$,*

$$\ell_k(\lambda) = \mathsf{Rank}(A - \lambda I)^{k-1} - \mathsf{Rank}(A - \lambda I)^k.$$

Proof. In view of Exercise 7.1.6, we need to consider only the Jordan blocks $J_k(\lambda)$, for different values of k. Hence, without loss of generality, let us assume that $J = \bigoplus\limits_{i=1}^{n} a_i J_i(\lambda)$, where a_i's are non-negative integers and J contains exactly a_i copies of the Jordan block $J_i(\lambda)$, for $1 \le i \le n$. Then, by definition and Exercise 7.1.6, we observe the following:

1. $n = \sum\limits_{i \ge 1} i a_i$.
2. $\mathsf{Rank}(J - \lambda I) = \sum\limits_{i \ge 2} (i-1)a_i$.
3. $\mathsf{Rank}((J - \lambda I)^2) = \sum\limits_{i \ge 3} (i-2)a_i$.
4. In general, for $1 \le k \le n$, $\mathsf{Rank}((J - \lambda I)^k) = \sum\limits_{i \ge k+1} (i-k)a_i$.

Thus, writing ℓ_i in place of $\ell_i(\lambda)$, we get

$$\begin{aligned}
\ell_1 &= \sum_{i\ge 1} a_i = \sum_{i \ge 1} i a_i - \sum_{i\ge 2}(i-1)a_i = n - \mathsf{Rank}(J - \lambda I), \\
\ell_2 &= \sum_{i\ge 2} a_i = \sum_{i \ge 2} (i-1) a_i - \sum_{i\ge 3}(i-2)a_i = \mathsf{Rank}(J - \lambda I) - \mathsf{Rank}((J-\lambda I)^2), \\
&\vdots \\
\ell_k &= \sum_{i\ge k} a_i = \sum_{i \ge k} (i-(k-1)) a_i - \sum_{i\ge k+1}(i-k)a_i = \mathsf{Rank}((J - \lambda I)^{k-1}) - \mathsf{Rank}((J-\lambda I)^k).
\end{aligned}$$

Now, the required result follows as rank is invariant under similarity operation and the matrices J and A are similar. ■

DRAFT

Lemma 7.1.12. **[Similar Jordan matrices]** *Let J and J' be two similar Jordan matrices of size n. Then, J is a block permutation of J'.*

Proof. For $1 \le i \le n$, let ℓ_i and ℓ_i' be, respectively, the number of Jordan blocks of J and J' of size at least i corresponding to λ. Since J and J' are similar, the matrices $(J - \lambda I)^i$ and $(J' - \lambda I)^i$ are similar for all $i, 1 \le i \le n$. Therefore, their ranks are equal for all $i \ge 1$ and hence, $\ell_i = \ell_i'$ for all $i \ge 1$. Thus the required result follows. ∎

We now state the main result of this section which directly follows from Lemma 6.4.1, Theorem 7.1.1 and Corollary 7.1.10 and hence the proof is omitted.

Theorem 7.1.13. **[Jordan canonical form theorem]** *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, A is similar to a Jordan matrix J, which is unique up to permutation of Jordan blocks. If $A \in \mathbb{M}_n(\mathbb{R})$ and has real eigenvalues then the similarity transformation matrix S may be chosen to have real entries. This matrix J is called the the Jordan canonical form of A, denoted* Jordan CF(A).

We now start with a few examples and observations.

Example 7.1.14. Let us use the idea from Lemma 7.1.11 to find the Jordan Canonical Form of the following matrices.

1. Let $A = J_4(0)^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

   Solution: Note that $\ell_1 = 4 - \mathsf{Rank}(A - 0I) = 2$. So, there are two Jordan blocks.

   Also, $\ell_2 = \mathsf{Rank}(A - 0I) - \mathsf{Rank}((A - 0I)^2) = 2$. So, there are at least 2 Jordan blocks of size 2. As there are exactly two Jordan blocks, both the blocks must have size 2. Hence, Jordan CF$(A) = J_2(0) \oplus J_2(0)$.

2. Let $A_1 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   Solution: Let $B = A_1 - I$. Then, $\ell_1 = 4 - \mathsf{Rank}(B) = 1$. So, B has exactly one Jordan block and hence A_1 is similar to $J_4(1)$.

3. $A_2 = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

   Solution: Let $C = A_2 - I$. Then, $\ell_1 = 4 - \mathsf{Rank}(C) = 2$. So, C has exactly two Jordan blocks. Also, $\ell_2 = \mathsf{Rank}(C) - \mathsf{Rank}(C^2) = 1$ and $\ell_3 = \mathsf{Rank}(C^2) - \mathsf{Rank}(C^3) = 1$. So, there is at least 1 Jordan blocks of size 3.

   Thus, we see that there are two Jordan blocks and one of them is of size 3. Also, the size of the matrix is 4. Thus, A_2 is similar to $J_3(1) \oplus J_1(1)$.

DRAFT

4. Let $A = J_4(1)^2 \oplus A_1 \oplus A_2$, where A_1 and A_2 are given in the previous exercises.

   Solution: One can directly get the answer from the previous exercises as the matrix A is already in the block diagonal form. But, we compute it again for better understanding.

   Let $B = A - I$. Then, $\ell_1 = 16 - \mathsf{Rank}(B) = 5$, $\ell_2 = \mathsf{Rank}(B) - \mathsf{Rank}(B^2) = 11 - 7 = 4$, $\ell_3 = \mathsf{Rank}(B^2) - \mathsf{Rank}(B^3) = 7 - 3 = 4$ and $\ell_4 = \mathsf{Rank}(B^3) - \mathsf{Rank}(B^4) = 3 - 0 = 3$.

   Hence, $J_4(1)$ appears thrice (as $\ell_4 = 3$ and $\ell_5 = 0$), $J_3(1)$ also appears once (as $\ell_3 - \ell_4 = 1$), $J_2(1)$ does not appear as (as $\ell_2 - \ell_3 = 0$) and $J_1(1)$ appears once (as $\ell_1 - \ell_2 = 1$). Thus, the required result follows.

Remark 7.1.15. **[Observations about** JORDAN CF(A)**]**

1. *What are the steps to find* JORDAN CFA*?*

   Ans: *Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. Now, apply the Schur Upper Triangularization Lemma (see Lemma 6.4.1) to get an upper triangular matrix, say T such that the diagonal entries of T are $\lambda_1, \ldots, \lambda_1, \lambda_2, \ldots, \lambda_2, \ldots, \lambda_k, \ldots, \lambda_k$. Now, apply similarity transformations (see Theorem 7.1.1) to get $T = \bigoplus\limits_{i=1}^{k} T_i$, where each diagonal entry of T_i is λ_i. Then, for each $i, 1 \le i \le k$, use Theorem 7.1.8 to get an invertible matrix S_i such that $S_i^{-1}(T_i - \lambda_i I)S_i = \widetilde{J_i}$, a Jordan matrix. Thus, we obtain a Jordan matrix $J_i = \widetilde{J_i} + \lambda_i I = S_i^{-1} T_i S_i$, where each diagonal entry of J_i is λ_i. Hence, $S = \bigoplus\limits_{i=1}^{k} S_i$ converts $T = \bigoplus\limits_{i=1}^{k} T_i$ into the required Jordan matrix.*

2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a diagonalizable matrix. Then, by definition, A is similar to $\bigoplus\limits_{i=1}^{n} \lambda_i$, where $\lambda_i \in \sigma(A)$, for $1 \le i \le n$. Thus,* JORDAN CF$(A) = \bigoplus\limits_{i=1}^{n} \lambda_i$*, up to a permutation of λ_i's.*

3. *In general, the computation of* JORDAN CF(A) *is not numerically stable. To understand this, let $A_\epsilon = \begin{bmatrix} \epsilon & 0 \\ 1 & 0 \end{bmatrix}$. Then, A_ϵ is diagonalizable as A has distinct eigenvalues. So,* JORDAN CF$(A_\epsilon) = \begin{bmatrix} \epsilon & 0 \\ 0 & 0 \end{bmatrix}$.

   Whereas, for $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, we know that JORDAN CF$(A) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \lim\limits_{\epsilon \to 0}$ JORDAN CF(A_ϵ). *Thus, a small change in the entries of A may change* JORDAN CF(A) *significantly.*

4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $\epsilon > 0$ be given. Then, there exists an invertible matrix S such that $S^{-1}AS = \bigoplus\limits_{i=1}^{k} J_{n_i}(\lambda_i, \epsilon)$, where $J_{n_i}(\lambda_i, \epsilon)$ is obtained from $J_{n_i}(\lambda_i)$ by replacing each off diagonal entry 1 by an ϵ. To get this, define $Di(\epsilon) = diag(1, \epsilon, \epsilon^2, \ldots, \epsilon^{n_i - 1})$, for $1 \le i \le k$. Now compute $\bigoplus\limits_{i=1}^{k} \left((Di(\epsilon))^{-1} J_{n_i}(\lambda_i) Di(\epsilon)\right)$.*

5. *Let* JORDAN CF(A) *contain ℓ Jordan blocks for λ. Then, A has ℓ linearly independent eigenvectors for λ.*

DRAFT

For if, A has at least $\ell+1$ linearly independent eigenvectors for λ, then $\dim(\textsc{Null}(A-\lambda I)) > \ell$. *So,* $\mathsf{Rank}(A-\lambda I) < n-\ell$. *But, the number of Jordan blocks for λ in A is ℓ. Thus, we must have* $\mathsf{Rank}(J-\lambda I) = n-\ell$, *a contradiction.*

6. *Let $\lambda \in \sigma(A)$. Then, by Remark 7.1.5.5,* $\textsc{Geo.Mul}_\lambda(A)$ = *the number of Jordan blocks $J_k(\lambda)$ in* $\textsc{Jordan CF}(A)$.

7. *Let $\lambda \in \sigma(A)$. Then, by Remark 7.1.5.3,* $\textsc{Alg.Mul}_\lambda(A)$ = *the sum of the sizes of all Jordan blocks $J_k(\lambda)$ in* $\textsc{Jordan CF}(A)$.

8. *Let $\lambda \in \sigma(A)$. Then,* $\textsc{Jordan CF}(A)$ *<u>does not</u> get determined by* $\textsc{Alg.Mul}_\lambda(A)$ *and* $\textsc{Geo.Mul}_\lambda(A)$. *For example,* $\begin{bmatrix}0\end{bmatrix} \oplus \begin{bmatrix}0&1\\0&0\end{bmatrix} \oplus \begin{bmatrix}0&1&0\\0&0&1\\0&0&0\end{bmatrix}$ *and* $\begin{bmatrix}0&1\\0&0\end{bmatrix} \oplus \begin{bmatrix}0&1\\0&0\end{bmatrix} \oplus \begin{bmatrix}0&1\\0&0\end{bmatrix}$ *are different* $\textsc{Jordan CF}$*s but they have the same algebraic and geometric multiplicities.*

9. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Suppose that, for each $\lambda \in \sigma(A)$, the values of* $\mathsf{Rank}(A-\lambda I)^k$, *for $k = 1, \ldots, n$ are known. Then, using Remark 7.1.11,* $\textsc{Jordan CF}(A)$ *can be computed. But, note here that finding rank is <u>numerically unstable</u> as* $\begin{bmatrix}\epsilon\end{bmatrix}$ *has rank 1 but it converges to* $\begin{bmatrix}0\end{bmatrix}$ *which has a different rank.*

Theorem 7.1.16. [**A is similar to A^T**] *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, A is similar to A^T.*

Proof. Let $K_n = \begin{bmatrix} & & 1\\ & \iddots & \\ 1 & & \end{bmatrix}$. Then, observe that $K^{-1} = K$ and $KJ_n(a)K = J_n(a)^T$, as the (i,j)-th entry of A goes to $(n-i+1, n-j+1)$-th position in KAK. Hence,

$$\left[\bigoplus K_{n_i}\right]\left[\bigoplus J_{n_i}(\lambda_i)\right]\left[\bigoplus K_{n_i}\right] = \left[\bigoplus J_{n_i}(\lambda_i)\right]^T.$$

Thus, J is similar to J^T. But, A is similar to J and hence A^T is similar to J^T and finally we get A is similar to A^T. Therefore, the required result follows. ∎

DRAFT

Exercise **7.1.17.**

1. *Let* $M = \begin{bmatrix} -2&0&-1&2&2\\ -2&2&-1&1&1\\ 1&0&2&0&-1\\ -3&0&-1&4&1\\ -4&0&-1&2&4 \end{bmatrix}$, *Null*$(M-2I) = LS\left\{\begin{bmatrix}1\\0\\0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}\right\}$ *and* $\sigma(M) = \{2,2,2,2,2\}$. *Then determine the Jordan canonical form of M.*

2. *Fix $k \in \mathbb{N}$ and let $\lambda \neq 0$.*

   (a) *Then prove that* $J_k(\lambda)^{-1} = \begin{bmatrix} 1/\lambda & -1/\lambda^2 & 1/\lambda^3 & \cdots & (-1)^{k+1}/\lambda^k\\ 0 & 1/\lambda & \ddots & \ddots & (-1)^k/\lambda^{k-1}\\ 0 & 0 & \ddots & \vdots & \\ 0 & 0 & \ddots & 1/\lambda & -1/\lambda^2\\ 0 & 0 & & 0 & 1/\lambda \end{bmatrix}$, *for all $k \geq 1$.*

   (b) *Show that* $\left(J_k(\lambda)^{-1} - \frac{1}{\lambda}I_k\right)$ *is a nilpotent matrix of index k.*

(c) *Use the previous part to conclude* Jordan CF $\left(J_k(\lambda)^{-1} - \frac{1}{\lambda} I_k\right) = J_k(0)$.

(d) *Therefore, prove that* Jordan CF $\left(J_k(\lambda)^{-1}\right) = J_k\left(\frac{1}{\lambda}\right)$.

(e) *Further, let* Jordan CF$(A) = \bigoplus\limits_{i=1}^{k} J_{n_i}(\lambda)$ *for some integers* $n_1 \geq \cdots \geq n_k \geq 1$. *Then* Jordan CF$(A^{-1}) = \bigoplus\limits_{i=1}^{k} J_{n_i}(1/\lambda)$.

(f) *Let* Jordan CF$(A) = \bigoplus\limits_{i=1}^{k} J_{n_i}(\lambda_i)$ *for some integers* $n_1 \geq \cdots \geq n_k \geq 1$ *and* $\lambda_i \neq 0$ *for* $1 \leq i \leq k$. *Then prove that* Jordan CF$(A^{-1}) = \bigoplus\limits_{i=1}^{k} J_{n_i}(1/\lambda_i)$.

7.2 Minimal polynomial

Recall that a polynomial $p(x) = a_0 + a_1x + \cdots + a_nx^n$ with $a_n = 1$ is called a monic polynomial. We now have the following definition.

Definition 7.2.1. Let $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ be a monic polynomial in t of degree n. Then, the $n \times n$ matrix $A = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & & 1 & -a_{n-1} \end{bmatrix}$, denoted $A(n : a_0, \ldots, a_{n-1})$ or Companion(P), is called the **companion matrix** of $P(t)$.

DRAFT

Remark 7.2.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ be its characteristic polynomial. Then by the Cayley Hamilton Theorem, $A^n + a_{n-1}A^{n-1} + \cdots + a_1A + a_0I = \mathbf{0}$. Hence $A^n = -(a_{n-1}A^{n-1} + \cdots + a_1A + a_0I)$.*

Suppose, there exists a vector $\mathbf{u} \in \mathbb{C}^n$ such that $\mathcal{B} = \left[\mathbf{u}, A\mathbf{u}, A^2\mathbf{u}, \ldots, A^{n-1}\mathbf{u}\right]$ is an ordered basis of $\mathbb{C}^n$. Then, $A^n\mathbf{u} = -(a_{n-1}A^{n-1}\mathbf{u} + \cdots + a_1A\mathbf{u} + a_0\mathbf{u})$ and hence the matrix of A in the basis $\mathcal{B}$ equals

$$\begin{aligned} A[\mathcal{B},\mathcal{B}] &= \left[[A\mathbf{u}]_{\mathcal{B}} \quad [A(A\mathbf{u})]_{\mathcal{B}} \quad \cdots \quad [A(A^{n-1}\mathbf{u})]_{\mathcal{B}}\right] = \left[[A\mathbf{u}]_{\mathcal{B}} \quad [A^2\mathbf{u}]_{\mathcal{B}} \quad \cdots \quad [A^n\mathbf{u}]_{\mathcal{B}}\right] \\ &= \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ 0 & 0 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & & 1 & -a_{n-1} \end{bmatrix}, \end{aligned}$$

the companion matrix of A.

Definition 7.2.3. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, the polynomial $P(t)$ is said to **annihilate** (destroy) A if $P(A) = \mathbf{0}$.

Let $P(x)$ be the characteristic polynomial of A. Then, by the Cayley-Hamilton Theorem, $P(A) = \mathbf{0}$. So, if $f(x) = P(x)g(x)$, for any multiple of $g(x)$, then $f(A) = P(A)g(A) = \mathbf{0}g(A) = \mathbf{0}$. Thus, there are infinitely many polynomials which annihilate A. In this section, we will concentrate on a monic polynomial of least positive degree that annihilates A.

Definition 7.2.4. Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, the **minimal polynomial** of A, denoted $m_A(x)$, is a monic polynomial of least positive degree satisfying $m_A(A) = \mathbf{0}$.

Theorem 7.2.5. *Let A be the companion matrix of the monic polynomial $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$. Then, $P(t)$ is both the characteristic and the minimal polynomial of A.*

Proof. Expanding $\det(tI_n - \textsc{Companion}(P))$ along the first row, we have

$$\begin{aligned}
\det(tI_n - A(n : a_0, \ldots, a_{n-1})) &= t\det(tI_{n-1} - A(n-1 : a_1, \ldots, a_{n-1})) + (-1)^{n+1}a_0(-1)^{n-1} \\
&= t^2\det(tI_{n-2} - A(n-2 : a_2, \ldots, a_{n-1})) + a_0 + a_1 t \\
&\;\;\vdots \\
&= P(t).
\end{aligned}$$

Thus, $P(t)$ is the characteristic polynomial of A and hence $P(A) = \mathbf{0}$.

We will now show that $P(t)$ is the minimal polynomial of A. To do so, we first observe that $A\mathbf{e}_1 = \mathbf{e}_2, \ldots, A\mathbf{e}_{n-1} = \mathbf{e}_n$. Thus,

$$A^k\mathbf{e}_1 = \mathbf{e}_{k+1}, \text{ for } 1 \le k \le n-1. \tag{7.2.1}$$

DRAFT

Now, Suppose we have a monic polynomial $Q(t) = t^m + b_{m-1}t^{m-1} + \cdots + b_0$, with $m < n$, such that $Q(A) = \mathbf{0}$. Then, using Equation (7.2.1), we get

$$\mathbf{0} = Q(A)\mathbf{e}_1 = A^m\mathbf{e}_1 + b_{m-1}A^{m-1}\mathbf{e}_1 + \cdots + b_0 I\mathbf{e}_1 = \mathbf{e}_{m+1} + b_{m-1}\mathbf{e}_m + \cdots + b_0\mathbf{e}_1,$$

a contradiction to the linear independence of $\{\mathbf{e}_1, \ldots, \mathbf{e}_{m+1}\} \subseteq \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. ∎

The next result gives us the existence of such a polynomial for every matrix A. To do so, recall that the well-ordering principle implies that if S is a subset of natural numbers then it contains a least element.

Lemma 7.2.6. [Existence of the minimal polynomial] *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, there exists a unique monic polynomial $m(x)$ of minimum (positive) degree such that $m(A) = \mathbf{0}$. Further, if $f(x)$ is any polynomial with $f(A) = \mathbf{0}$ then $m(x)$ divides $f(x)$.*

Proof. Let $P(x)$ be the characteristic polynomial of A. Then, $\deg(P(x)) = n$ and by the Cayley-Hamilton Theorem, $P(A) = \mathbf{0}$. So, consider the set

$$S = \{\deg(f(x)) : f(x) \text{ is a nonzero polynomial, } f(A) = \mathbf{0}\}.$$

Then, S is a non-empty subset of $\mathbb{N}$ as $n \in S$. Thus, by well-ordering principle there exists a smallest positive integer, say M, and a corresponding polynomial, say $m(x)$, such that $\deg(m(x)) = M$, $m(A) = \mathbf{0}$.

Also, without loss of generality, we can assume that $m(x)$ is monic and unique (non-uniqueness will lead to a polynomial of smaller degree in S).

Now, suppose there is a polynomial $f(x)$ such that $f(A) = \mathbf{0}$. Then, by division algorithm, there exist polynomials $q(x)$ and $r(x)$ such that $f(x) = m(x)q(x) + r(x)$, where either $r(x)$ is identically the zero polynomial of $\deg(r(x)) < M = \deg(m(x))$. As

$$\mathbf{0} = f(A) = m(A)q(A) + r(A) = \mathbf{0}q(A) + r(A) = r(A),$$

we get $r(A) = \mathbf{0}$. But, $m(x)$ was the least degree polynomial with $m(A) = \mathbf{0}$ and hence $r(x)$ is the zero polynomial. That is, $m(x)$ divides $f(x)$. ∎

As an immediate corollary, we have the following result.

Corollary 7.2.7. **[Minimal polynomial divides the characteristic polynomial]** *Let $m_A(x)$ and $P_A(x)$ be, respectively, the minimal and the characteristic polynomials of $A \in \mathbb{M}_n(\mathbb{C})$.*

1. *Then, $m_A(x)$ divides $P_A(x)$.*
2. *Further, if λ is an eigenvalue of A then $m_A(\lambda) = 0$.*

Proof. The first part following directly from Lemma 7.2.6. For the second part, let $(\lambda, \mathbf{x})$ be an eigen-pair. Then, $f(A)\mathbf{x} = f(\lambda)\mathbf{x}$, for any polynomial of f, implies that

$$m_A(\lambda)\mathbf{x} = m_A(A)\mathbf{x} = \mathbf{0}\mathbf{x} = \mathbf{0}.$$

But, $\mathbf{x} \neq \mathbf{0}$ and hence $m_A(\lambda) = 0$. Thus, the required result follows. ∎

DRAFT

we also have the following result.

Lemma 7.2.8. *Let A and B be two similar matrices. Then, they have the same minimal polynomial.*

Proof. Since A and B are similar, there exists an invertible matrix S such that $A = S^{-1}BS$. Hence, $f(A) = F(S^{-1}BS) = S^{-1}f(B)S$, for any polynomial f. Hence, $m_A(A) = \mathbf{0}$ if and only if $m_A(B) = \mathbf{0}$ and thus the required result follows. ∎

Theorem 7.2.9. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. If n_i is the size of the largest Jordan block for λ_i in $J = \text{Jordan CF}A$ then*

$$m_A(x) = \prod_{i=1}^{k}(x - \lambda_i)^{n_i}.$$

Proof. Using 7.2.7, we see that $m_A(x) = \prod\limits_{i=1}^{k}(x-\lambda_i)^{\alpha_i}$, for some α_i's with $1 \leq \alpha_i \leq \text{Alg.Mul}_{\lambda_i}(A)$. As $m_A(A) = \mathbf{0}$, using Lemma 7.2.8 we have $m_A(J) = \prod\limits_{i=1}^{k}(J - \lambda_i I)^{\alpha_i} = \mathbf{0}$. But, observe that for the Jordan block $J_{n_i}(\lambda_i)$, one has

1. $(J_{n_i}(\lambda_i) - \lambda_i I)^{\alpha_i} = \mathbf{0}$ if and only if $\alpha_i \geq n_i$, and
2. $(J_{n_m}(\lambda_m) - \lambda_i I)^{\alpha_i}$ is invertible, for all $m \neq i$.

Thus $\prod_{i=1}^{k}(J-\lambda_i I)^{n_i} = \mathbf{0}$ and $\prod_{i=1}^{k}(x-\lambda_i)^{n_i}$ divides $\prod_{i=1}^{k}(x-\lambda_i)^{\alpha_i} = m_A(x)$ and $\prod_{i=1}^{k}(x-\lambda_i)^{n_i}$ is a monic polynomial, the result follows. ∎

As an immediate consequence, we also have the following result which corresponds to the converse of the above theorem.

Theorem 7.2.10. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of A. If the minimal polynomial of A equals $\prod_{i=1}^{k}(x-\lambda_i)^{n_i}$ then n_i is the size of the largest Jordan block for λ_i in $J =$ Jordan CFA.*

Proof. It directly follows from Theorem 7.2.9. ∎

We now give equivalent conditions for a square matrix to be diagonalizable.

Theorem 7.2.11. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, the following statements are equivalent.*

1. *A is diagonalizable.*
2. *Every zero of $m_A(x)$ has multiplicity 1.*
3. *Whenever $m_A(\alpha) = 0$, for some α, then $\frac{d}{dx} m_A(x)\big|_{x=\alpha} \neq 0$.*

Proof. Part 1 ⇒ Part 2. If A is diagonalizable, then each Jordan block in $J =$ Jordan CFA has size 1. Hence, by Theorem 7.2.9, $m_A(x) = \prod_{i=1}^{k}(x-\lambda_i)$, where λ_i's are the distinct eigenvalues of A.

Part 2 ⇒ Part 3. Let $m_A(x) = \prod_{i=1}^{k}(x-\lambda_i)$, where λ_i's are the distinct eigenvalues of A. Then, $m_A(x) = 0$ if and only if $x = \lambda_i$, for some $i, 1 \leq i \leq k$. In that case, it is easy to verify that $\frac{d}{dx} m_A(x) \neq 0$, for each λ_i.

DRAFT

Part 3 ⇒ Part 1. Suppose that for each α satisfying $m_A(\alpha) = 0$, one has $\frac{d}{dx} m_A(\alpha) \neq 0$. Then, it follows that each zero of $m_A(x)$ has multiplicity 1. Also, using Corollary 7.2.7, each zero of $m_A(x)$ is an eigenvalue of A and hence by Theorem 7.2.9, the size of each Jordan block is 1. Thus, A is diagonalizable. ∎

We now have the following remarks and observations.

Remark 7.2.12.
1. *Let $f(x)$ be a monic polynomial and $A =$ Companion(f) be the companion matrix of f. Then, by Theorem 7.2.5) $f(A) = \mathbf{0}$ and no monic polynomial of smaller degree annihilates A. Thus $P_A(x) = m_A(x) = f(x)$, where $P_A(x)$ is the characteristic polynomial and $m_A(x)$, the minimal polynomial of A.*
2. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, A is similar to* Companion(f)*, for some monic polynomial f if and only if $m_A(x) = f(x)$.*

   Proof. Let $B =$ Companion (f). Then, using Lemma 7.2.8, we see that $m_A(x) = m_B(x)$. But, by Remark 7.2.12.1, we get $m_B(x) = f(x)$ and hence the required result follows.

   Conversely, assume that $m_A(x) = f(x)$. But, by Remark 7.2.12.1, $m_B(x) = f(x) = P_B(x)$, the characteristic polynomial of B. Since $m_A(x) = m_B(x)$, the matrices A and B have the same largest Jordan blocks for each eigenvalue λ. As $P_B = m_B$, we know that for each λ, there is only one Jordan block in Jordan CFB. *Thus,* Jordan CF$A =$ Jordan CFB *and hence A is similar to* Companion (f). ∎

EXERCISE **7.2.13.** *The following are some facts and questions.*

1. *Let $A \in \mathbb{M}_n(\mathbb{C})$. If $P_A(x)$ is the minimal polynomial of A then A is similar to* COMPANION (P_A) *if and only if A is nonderogatory. T/F?*
2. *Let $A, B \in \mathbb{M}_3(\mathbb{C})$ with eigenvalues $1, 2, 3$. Is it necessary that A is similar to B?*
3. *Let $A, B \in \mathbb{M}_3(\mathbb{C})$ with eigenvalues $1, 1, 3$. Is it necessary that A is similar to B?*
4. *Let $A, B \in \mathbb{M}_4(\mathbb{C})$ with the same minimal polynomial. Is it necessary that A is similar to B?*
5. *Let $A, B \in \mathbb{M}_3(\mathbb{C})$ with the same minimal polynomial. Is it necessary that A is similar to B?*
6. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be idempotent and let $J =$* JORDAN CF*A. Thus, $J^2 = J$ and hence conclude that J must be a diagonal matrix. Hence, every idempotent matrix is diagonalizable.*
7. *Let $A \in \mathbb{M}_n(\mathbb{C})$. Suppose that $m_A(x) | x(x-1)(x-2)(x-3)$. Must A be diagonalizable?*
8. *Let $A \in \mathbb{M}_9(\mathbb{C})$ be a nilpotent matrix such that $A^5 \neq \mathbf{0}$ but $A^6 = \mathbf{0}$. Determine $P_A(x)$ and $m_A(x)$.*
9. *Recall that for $A, B \in \mathbb{M}_n(\mathbb{C})$, the characteristic polynomial of AB and BA are the same. That is, $P_{AB}(x) = P_{BA}(x)$. However, they need not have the same minimal polynomial. Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ to verify that $m_{AB}(x) \neq m_{BA}(x)$.*
10. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be an invertible matrix. Then prove that if the minimal polynomial of A equals $m(x, \lambda_1, \ldots, \lambda_k)$ then the minimal polynomial of A^{-1} equals $m(x, 1/\lambda_1, \ldots, 1/\lambda_k)$.*
11. *Let λ an eigenvalue of $A \in \mathbb{M}_n(\mathbb{C})$ with two linearly independent eigenvectors. Show that there does not exist a vector $\mathbf{u} \in \mathbb{C}^n$ such that $LS\left(\mathbf{u}, A\mathbf{u}, A^2\mathbf{u}, \ldots\right) = \mathbb{C}^n$.*

DRAFT

We end this section with a method to compute the minimal polynomial of a given matrix.

Example 7.2.14. [**Computing the minimal polynomial**] Let $\lambda_1, \ldots, \lambda_k$ be the distinct eigenvalues of $A \in \mathbb{M}_n(\mathbb{C})$.

7.3 Applications of Jordan Canonical Form

In the last section, we say that the matrices if A is a square matrix then A and A^T are similar. In this section, we look at some more applications of the Jordan Canonical Form.

7.3.1 Coupled system of linear differential equations

Consider the first order Initial Value Problem (IVP) $\mathbf{x}'(t) = \begin{bmatrix} \mathbf{x}_1'(t) \\ \vdots \\ \mathbf{x}_n'(t) \end{bmatrix} = A \begin{bmatrix} \mathbf{x}_1(t) \\ \vdots \\ \mathbf{x}_n(t) \end{bmatrix} = A\mathbf{x}(t)$, with $\mathbf{x}(0) = \mathbf{0}$. If A is not a diagonal matrix then the system is called COUPLED and is hard to solve. Note that if A can be transformed to a nearly diagonal matrix, then the amount of coupling among $\mathbf{x}_i$'s can be reduced. So, let us look at $J =$ JORDAN CF$(A) = S^{-1}AS$. Then, using $S^{-1}A = JS^{-1}$. verify that the initial problem $\mathbf{x}'(t) = A\mathbf{x}(t)$ is equivalent to the equation $S^{-1}\mathbf{x}'(t) = S^{-1}A\mathbf{x}(t)$ which in turn is equivalent to $\mathbf{y}'(t) = J\mathbf{y}(t)$, where $S^{-1}\mathbf{x}(t) = \mathbf{y}(t)$ with $\mathbf{y}(0) = S^{-1}\mathbf{x}(0) = \mathbf{0}$. Therefore, if $\mathbf{y}$ is a solution to the second equation then $\mathbf{x}(t) = S\mathbf{y}$ is a solution to the initial problem.

When J is diagonalizable then solving the second is as easy as solving $\mathbf{y}_i'(t) = \lambda_i \mathbf{y}_i(t)$ for which the required solution is given by $\mathbf{y}_i(t) = \mathbf{y}_i(0)e^{\lambda_i t}$.

If J is not diagonal, then for each Jordan block, the system reduces to

$$\mathbf{y}_1'(t) = \lambda \mathbf{y}_1(t) + \mathbf{y}_2(t), \cdots, \mathbf{y}_{k-1}'(t) = \lambda \mathbf{y}_{k-1}(t) + \mathbf{y}_k(t), \;\; \mathbf{y}_k'(t) = \lambda \mathbf{y}_k(t).$$

This problem can also be solved as in this case the solution is given by $\mathbf{y}_k = c_0 e^{\lambda t}$; $\mathbf{y}_{k-1} = (c_0 t + c_1)e^{\lambda t}$ and so on.

7.3.2 Commuting matrices

Let $P(x)$ be a polynomial and $A \in \mathbb{M}_n(\mathbb{C})$. Then, $P(A)A = AP(A)$. What about the converse? That is, suppose we are given that $AB = BA$ for some $B \in \mathbb{M}_n(\mathbb{C})$. Does it necessarily imply that $B = P(A)$, for some nonzero polynomial $P(x)$? The answer is **No** as I commutes with A for every A. We start with a set of remarks.

Theorem 7.3.1. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $B \in \mathbb{M}_m(\mathbb{C})$. Then, the linear system $AX - XB = \mathbf{0}$, in the variable matrix X of size $n \times m$, has a unique solution, namely $X = \mathbf{0}$ (the trivial solution), if and only if $\sigma(A)$ and $\sigma(B)$ are disjoint.*

DRAFT

Proof. Let us assume that $\sigma(A)$ and $\sigma(B)$ are disjoint.

Since $\sigma(A)$ and $\sigma(B)$ are disjoint, the matrix $P_B(A) = \left(\prod\limits_{\lambda \in \sigma(B)} [\lambda I - A] \right)$, obtained by evaluating A at the characteristic polynomial, $P_B(t)$, of B, is invertible. So, let us look at the implication of the condition $AX = XB$. This condition implies that $A^2X = AXB = XBB = XB^2$ and hence, $P(A)X = XP(B)$, for any polynomial $P(t)$. In particular, $P_B(A)X = XP_B(B) = X\mathbf{0} = \mathbf{0}$. As $P_B(A)$ is invertible, we get $X = \mathbf{0}$.

Now, conversely assume that $AX - XB = \mathbf{0}$ has only the trivial solution $X = \mathbf{0}$. Suppose on the contrary λ is a common eigenvalue of both A and B. So, choose nonzero vectors $\mathbf{x} \in \mathbb{C}^n$ and $\mathbf{y} \in \mathbb{C}^m$ such that $(\lambda, \mathbf{x})$ is an eigen-pair of A and $(\lambda, \mathbf{y})$ is a left eigen-pair of B. Now, define $X_0 = \mathbf{x}\mathbf{y}^T$. Then, X_0 is an $n \times m$ nonzero matrix and

$$AX - XB = A\mathbf{x}\mathbf{y}^T - \mathbf{x}\mathbf{y}^T B = \lambda \mathbf{x}\mathbf{y}^T - \lambda \mathbf{x}\mathbf{y}^T = \mathbf{0}.$$

Thus, we see that if λ is a common eigenvalue of A and B then the system $AX - XB = \mathbf{0}$ has a nonzero solution X_0, a contradiction. Hence, the required result follows. ∎

Corollary 7.3.2. *Let $A \in \mathbb{M}_n(\mathbb{C}), B \in \mathbb{M}_m(\mathbb{C})$ and C be an $n \times m$ matrix. Also, assume that $\sigma(A)$ and $\sigma(B)$ are disjoint. Then, it can be easily verified that the system $AX - XB = C$, in the variable matrix X of size $n \times m$, has a unique solution, for any given C.*

Proof. Consider the linear transformation $T : \mathbb{M}_{n,m}(\mathbb{C}) \to \mathbb{M}_{n,m}(\mathbb{C})$, defined by $T(X) = AX - XB$. Then, by Theorem 7.3.1, $\text{NULL}(T) = \{\mathbf{0}\}$. Hence, by the rank-nullity theorem, T is a bijection and the required result follows. ∎

Definition 7.3.3. A square matrix A is said to be of **Toeplitz type** if each (super/sub)-diagonal of A consists of the same element. For example, $A = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ a_1 & b_1 & b_2 & b_3 \\ a_2 & a_1 & b_1 & b_2 \\ a_3 & a_2 & a_1 & b_1 \end{bmatrix}$ is a 4×4 Toeplitz type matrix. and the matrix $B = \begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ 0 & b_1 & b_2 & b_3 \\ 0 & 0 & b_1 & b_2 \\ 0 & 0 & 0 & b_1 \end{bmatrix}$ is an upper triangular Toeplitz type matrix.

EXERCISE **7.3.4.** *Let $J_n(0) \in \mathbb{M}_n(\mathbb{C})$ be the Jordan block with 0 on the diagonal.*

1. *Further, if $A \in \mathbb{M}_n(\mathbb{C})$ such that $AJ_n(0) = J_n(0)A$ then prove that A is an upper Toeplitz type matrix.*
2. *Further, if $A, B \in \mathbb{M}_n(\mathbb{C})$ are two upper Toeplitz type matrices then prove that*
   (a) *there exists $a_i \in \mathbb{C}, 1 \le i \le n$, such that $A = a_0 I + a_1 J_n(0) + \cdots + a_n J_n(0)^{n-1}$.*
   (b) *$P(A)$ is a Toeplitz matrix for any polynomial $P(t)$.*
   (c) *AB is a Toeplitz matrix.*
   (d) *if A is invertible then A^{-1} is also an upper Toeplitz type matrix.*

DRAFT

To proceed further, recall that a matrix $A \in \mathbb{M}_n(\mathbb{C})$ is called **non-derogatory** if $\text{GEO.MUL}_\alpha(A) = 1$, for each $\alpha \in \sigma(A)$ (see Definition 6.3.9).

Theorem 7.3.5. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a non-derogatory matrix. Then, the matrices A and B commute if and only if $B = P(A)$, for some polynomial $P(t)$ of degree at most $n-1$.*

Proof. If $B = P(A)$, for some polynomial $P(t)$, then A and B commute. Conversely, suppose that $AB = BA$, $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$ and let $J = \text{JORDAN CF}A = S^{-1}AS$ be the Jordan matrix of A. Then, $J = \begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_k}(\lambda_k) \end{bmatrix}$. Now, write $\overline{B} = S^{-1}BS = \begin{bmatrix} \overline{B}_{11} & \cdots & \overline{B}_{1k} \\ \vdots & \ddots & \vdots \\ \overline{B}_{k1} & \cdots & \overline{B}_{kk} \end{bmatrix}$, where $\overline{B}$ is partitioned conformally with J. Note that $AB = BA$ gives $J\overline{B} = \overline{B}J$. Thus, verify that

$$J_{n_1}(\lambda_1)\overline{B}_{12} = [JB]_{12} = [BJ]_{12} = \overline{B}_{12}J_{n_2}(\lambda_2),$$

and hence $\overline{B}_{12} = \mathbf{0}$. A similar argument gives $\overline{B}_{ij} = 0$, for all $i \ne j$. Hence, $JB = BJ$ implies $J_{n_i}(\lambda_i)\overline{B}_{ii} = \overline{B}_{ii}J_{n_i}(\lambda_i)$, for $1 \le i \le k$. Or equivalently, $J_{n_i}(0)B_{ii} = B_{ii}J_{n_i}(0)$, for $1 \le i \le k$ (using Exercise 7.1.6.3). Now, using Exercise 7.3.4.1, we see that $\overline{B}_{ii}$ is an upper triangular Toeplitz type matrix.

To proceed further, for $1 \le i \le k$, define $F_i(t) = \prod\limits_{j \ne i} (t-\lambda_j)^{n_j}$. Then, $F_i(t)$ is a polynomial with $\deg(F_i(t)) = n - n_i$ and $F_i(J_{n_j}(\lambda_j)) = \mathbf{0}$ if $j \ne i$. Also, note that $F_i(J_{n_i}(\lambda_i))$ is a nonsingular upper triangular Toeplitz type matrix. Hence, its inverse has the same form and using Exercise 7.3.4.1, the matrix $F_i(J_{n_i}(\lambda_i))^{-1}\overline{B}_{ii}$ is also a Toeplitz type upper triangular matrix. Hence,

$$F_i(J_{n_i}(\lambda_i))^{-1}\overline{B}_{ii} = c_1 I + c_2 J_{n_i}(0) + \cdots + c_{n_i} J_{n_i}(0)^{n_i - 1} = R_i(J_{n_i}(\lambda_i)), \text{ (say)}.$$

Thus, $\overline{B}_{ii} = F_i(J_{n_i}(\lambda_i))R_i(J_{n_i}(\lambda_i))$. Putting $P_i(t) = F_i(t)R_i(t)$, for $1 \le i \le k$, we see that $P_i(t)$ is a polynomial of degree at most $n-1$ with $P_i((J_{n_j}(\lambda_j)) = \mathbf{0}$, for $j \ne i$ and $P_i((J_{n_j}(\lambda_i)) = \overline{B}_{ii}$. Taking, $P = P_1 + \cdots + P_k$, we have

$$\begin{aligned} P(J) &= P_1\left(\begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_k}(\lambda_k)\end{bmatrix}\right) + \cdots + P_k\left(\begin{bmatrix} J_{n_1}(\lambda_1) & & \\ & \ddots & \\ & & J_{n_k}(\lambda_k)\end{bmatrix}\right) \\ &= \begin{bmatrix} \overline{B}_{11} & & \\ & \ddots & \\ & & 0\end{bmatrix} + \cdots + \begin{bmatrix} 0 & & \\ & \ddots & \\ & & \overline{B}_{kk}\end{bmatrix} = \overline{B}. \end{aligned}$$

Hence, $B = S\overline{B}S^{-1} = SP(J)S^{-1} = P(SJS^{-1}) = P(A)$ and the required result follows. ■

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Chapter 8

Advanced Topics on Diagonalizability and Triangularization*

8.1 More on the Spectrum of a Matrix

We start this subsection with a few definitions and examples. So, it will be nice to recall the notations used in Section 1.5 and a few results from Appendix 9.2.

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Definition 8.1.1. [Principal Minor] Let $A \in \mathbb{M}_n(\mathbb{C})$.

1. Also, let $S \subseteq [n]$. Then, $\det(A[S,S])$ is called the **Principal minor** of A corresponding to S.
2. By $\mathbf{EM}_k(A)$, we denote the sum of all $k \times k$ principal minors of A.

Definition 8.1.2. [Elementary Symmetric Functions] Let k be a positive integer. Then, the kth **elementary symmetric function** of the numbers $r_1, \ldots, r_n$ is $S_k(r_1, \ldots, r_n)$ and is defined as

$$S_k(r_1, \ldots, r_n) = \sum_{i_1 < \cdots < i_k} r_{i_1} \cdots r_{i_k}.$$

Example 8.1.3. Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 8 & 7 & 6 \\ 5 & 4 & 3 & 2 \end{bmatrix}$. Then, note that

1. $\mathrm{EM}_1(A) = 1 + 6 + 7 + 2 = 16$ and $\mathrm{EM}_2(A) = \det A(\{1,2\},\{1,2\}) + \det A(\{1,3\},\{1,3\}) + \det A(\{1,4\},\{1,4\}) + \det A(\{2,3\},\{2,3\}) + \det A(\{2,4\},\{2,4\}) + \det A(\{3,4\},\{3,4\}) = -80$.
2. $S_1(1,2,3,4) = 10$ and $S_2(1,2,3,4) = 1 \cdot (2+3+4) + 2 \cdot (3+4) + 3 \cdot 4 = 9 + 14 + 12 = 35$.

Theorem 8.1.4. *Let $A \in \mathbb{M}_n(\mathbb{C})$ and let $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$. Then,*

1. *the coefficient of* t^{n-k} *in* $P_A(t) = \prod\limits_{i=1}^{n}(t-\lambda_i)$, *the characteristic polynomial of* A, *is*

$$(-1)^k \sum_{i_1<\cdots<i_k} \lambda_{i_1}\cdots\lambda_{i_k} = (-1)^k S_k(\lambda_1,\ldots,\lambda_n). \tag{8.1.1}$$

2. $EM_k(A) = S_k(\lambda_1,\ldots,\lambda_n)$.

Proof. Note that by definition,

$$\begin{aligned} P_A(t) &= \prod_{i=1}^{n}(t-\lambda_i) = t^n - S_1(\lambda_1,\ldots,\lambda_n)t^{n-1} \\ &\qquad\qquad +S_2(\lambda_1,\ldots,\lambda_n)t^{n-2} - \cdots + (-1)^n S_n(\lambda_1,\ldots,\lambda_n) \quad (8.1.2)\\ &= t^n - EM_1(A)t^{n-1} + EM_2(A)t^{n-2} - \cdots + (-1)^n EM_n(A). \quad (8.1.3)\end{aligned}$$

As the second part is just a re-writing of the first, we will just prove the first part. To do so, let $B = tI - A = \begin{bmatrix} t-a_{11} & \cdots & -a_{1n} \\ & \ddots & \\ -a_{n1} & \cdots & t-a_{nn}\end{bmatrix}$. Then, using Definition 9.2.2 in Appendix, note that $\det B = \sum\limits_{\sigma} \text{sgn}\sigma \prod\limits_{i=1}^{n} b_{i\sigma(i)}$ and hence each $S \subseteq [n]$ with $|S| = n-k$ has a contribution to the coefficient of t^{n-k} in the following way:

For all $i \in S$, consider all permutations σ such that $\sigma(i) = i$. Our idea is to select a 't' from these $b_{i\sigma(i)}$. Since we do not want any more 't', we set $t = 0$ for any other diagonal position. So the contribution from S to the coefficient of t^{n-k} is $\det[-A(S|S)] = (-1)^k \det A(S|S)$. Hence the coefficient of t^{n-k} in $P_A(t)$ is

$$(-1)^k \sum_{S\subseteq[n],\,|S|=n-k} \det A(S|S) = (-1)^k \sum_{T\subseteq[n],\,|T|=k} \det A[T,T] = (-1)^k E_k(A).$$

The proof is complete in view of Equation (8.1.2). ■

As a direct application, we obtain Theorem 6.1.17 which we state again.

Corollary 8.1.5. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *and let* $\sigma(A) = \{\lambda_1,\ldots,\lambda_n\}$. *Then* $\mathsf{tr}(\mathsf{A}) = \sum_1^n \lambda_i$ *and* $\det A = \prod_1^n \lambda_i$.

Let A and B be similar matrices. Then, by Theorem 6.2.3, we know that $\sigma(A) = \sigma(B)$. Thus, as a direct consequence of Part 2 of Theorem 8.1.4 gives the following result.

Corollary 8.1.6. *Let* A *and* B *be two similar matrices of order* n. *Then,* $EM_k(A) = EM_k(B)$ *for* $1 \le k \le n$.

So, the sum of principal minors of similar matrices are equal. Or in other words, the sum of principal minors are invariant under similarity.

Corollary 8.1.7. [Derivative of Characteristic Polynomial] *Let* $A \in \mathbb{M}_n(\mathbb{C})$. *Then*

$$\frac{d}{dt}P_A(t) = P_A'(t) = \sum_{i=1}^{n} P_{A(i|i)}(t).$$

Proof. For $1 \le i \le n$, let us denote $A(i|i)$ by A_i. Then, using Equation (8.1.3), we have

$$\begin{aligned}\sum_{i=1}^{n} P_{A_i}(t) &= \sum_i t^{n-1} - \sum_i EM_1(A_i)t^{n-2} + \cdots + (-1)^{n-1}\sum_i EM_{n-1}(A_i)\\ &= nt^{n-1} - (n-1)EM_1(A)t^{n-2} + (n-2)EM_2(A)t^{n-3} - \cdots + (-1)^{n-1}EM_{n-1}(A)\\ &= P_A'(t).\end{aligned}$$

Which gives us the desired result. ∎

Corollary 8.1.8. *Let $A \in \mathbb{M}_n(\mathbb{C})$. If* $\text{ALG.MUL}_\alpha(A) = 1$ *then* $\mathsf{Rank}[A - \lambda I] = n - 1$.

Proof. As $\text{ALG.MUL}_\alpha(A) = 1$, $P_A(t) = (t - \lambda)q(t)$, where $q(t)$ is a polynomial with $q(\lambda) \neq 0$. Thus $P_A'(t) = q(t) + (t - \lambda)q'(t)$. Hence, $P_A'(\lambda) = q(\lambda) \neq 0$. Thus, by Corollary 8.1.7, $\sum_i P_{A(i|i)}(\lambda) = P_A'(\lambda) \neq 0$. Hence, there exists $i, 1 \le i \le n$ such that $P_{A(i|i)}(\lambda) \neq 0$. That is, $\det[A(i|i) - \lambda I] \neq 0$ or $\mathsf{Rank}[A - \lambda I] = n - 1$. ∎

Remark 8.1.9. *Converse of Corollary 8.1.8 is false. Note that for the matrix* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $\mathsf{Rank}[A - 0I] = 1 = 2 - 1 = n - 1$, *but* 0 *has multiplicity* 2 *as a root of* $P_A(t) = 0$.

As an application of Corollary 8.1.7, we have the following result.

We now relate the multiplicity of an eigenvalue with the spectrum of a principal sub-matrix.

Theorem 8.1.10. **[Multiplicity and Spectrum of a Principal Sub-Matrix]** *Let $A \in \mathbb{M}_n(\mathbb{C})$ and k be a positive integer. Then 1 ⇒ 2 ⇒ 3, where*

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1. $\text{GEO.MUL}_\lambda(A) \ge k$.
2. *If B is a principal sub-matrix of A of size $m > n - k$ then $\lambda \in \sigma(B)$.*
3. $\text{ALG.MUL}_\lambda(A) \ge k$.

Proof. Part 1⇒ Part 2. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ be linearly independent eigenvectors for λ and let B be a principal sub-matrix of A of size $m > n - k$. Without loss, we may write $A = \begin{bmatrix} B & * \\ * & * \end{bmatrix}$. Let us partition the $\mathbf{x}_i$'s , say $\mathbf{x}_i = \begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix}$, such that

$$\left[\begin{array}{c|c} B & * \\ * & * \end{array}\right] \begin{bmatrix} \mathbf{x}_{i1} \\ \mathbf{x}_{i2} \end{bmatrix} = \lambda \begin{bmatrix} x_{i1} \\ x_{i2} \end{bmatrix}, \text{ for } 1 \le i \le k.$$

As $m > n - k$, the size of $\mathbf{x}_{i2}$ is less than k. Thus, the set $\{\mathbf{x}_{12}, \ldots, \mathbf{x}_{k2}\}$ is linearly dependent (see Corollary 3.3.9). So, there is a nonzero linear combination $\mathbf{y} = \begin{bmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{bmatrix}$ of $\mathbf{x}_1, \ldots, \mathbf{x}_k$ such that $\mathbf{y}_2 = \mathbf{0}$. Notice that $\mathbf{y}_1 \neq \mathbf{0}$ and $B\mathbf{y}_1 = \lambda\mathbf{y}_1$.

Part 2⇒ Part 3. By Corollary 8.1.7, we know that $P_A'(t) = \sum\limits_{i=1}^{n} P_{A(i|i)}(t)$. As $A(i|i)$ is of size $n - 1$, we get $P_{A(i|i)}(\lambda) = 0$, for all $i = 1, 2, \ldots, n$. Thus, $P_A'(\lambda) = 0$. A similar argument now applied to each of the $A(i|i)$'s, gives $P_A^{(2)}(\lambda) = 0$, where $P_A^{(2)}(t) = \frac{d}{dt}P_A'(t)$. Proceeding on above lines, we finally get $P_A^{(i)}(\lambda) = 0$, for $i = 0, 1, \ldots, k - 1$. This implies that $\text{ALG.MUL}_\lambda(A) \ge k$. ∎

Definition 8.1.11. **[Moments]** Fix a positive integer n and let $\alpha_1, \ldots, \alpha_n$ be n complex numbers. Then, for a positive integer k, the sum $\sum\limits_{i=1}^{n} \alpha_i^k$ is called the k**-th moment** of the numbers $\alpha_1, \ldots, \alpha_n$.

Theorem 8.1.12. **[Newton's identities]** *Let $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ have zeros $\lambda_1, \ldots, \lambda_n$, counted with multiplicities. Put $\mu_k = \sum\limits_{i=1}^{n} \lambda_i^k$. Then, for $1 \le k \le n$,*

$$k\, a_{n-k} + \mu_1 a_{n-k+1} + \cdots + \mu_{k-1} a_{n-1} + \mu_k = 0. \tag{8.1.4}$$

That is, the first n moments of the zeros determine the coefficients of $P(t)$.

Proof. For simplicity of expression, let $a_n = 1$. Then, using Equation (8.1.4), we see that $k = 1$ gives us $a_{n-1} = -\mu_1$. To compute a_{n-2}, put $k = 2$ in Equation (8.1.4) to verify that $a_{n-2} = \frac{-\mu_2 + \mu_1^2}{2}$. This process can be continued to get all the coefficients of $P(t)$. Now, let us prove the n given equations.

Define $f(t) = \sum\limits_i \frac{1}{t-\lambda_i} = \frac{P'(t)}{P(t)}$ and take $|t| > \max\limits_i |\lambda_i|$. Then, the left hand side can be re-written as

$$f(t) = \sum_{i=1}^{n} \frac{1}{t-\lambda_i} = \sum_{i=1}^{n} \frac{1}{t\left(1 - \dfrac{\lambda_i}{t}\right)} = \sum_{i=1}^{n} \left[\frac{1}{t} + \frac{\lambda_i}{t^2} + \cdots\right] = \frac{n}{t} + \frac{\mu_1}{t^2} + \cdots. \tag{8.1.5}$$

Thus, using $P'(t) = f(t)P(t)$, we get

$$n a_n t^{n-1} + (n-1)a_{n-1}t^{n-2} + \cdots + a_1 = P'(t) = \left[\frac{n}{t} + \frac{\mu_1}{t^2} + \cdots\right]\left[a_n t^n + \cdots + a_0\right].$$

Now, equating the coefficient of t^{n-k-1} on both sides, we get

$$(n-k)a_{n-k} = n a_{n-k} + \mu_1 a_{n-k+1} + \cdots + \mu_k a_n, \text{ for } 0 \le k \le n-1$$

which is the required Newton's identity. ■

Remark 8.1.13. *Let $P(t) = a_n t^n + \cdots + a_1 t + a_0$ with $a_n = 1$. Thus, we see that we need not find the zeros of $P(t)$ to find the k-th moments of the zeros of $P(t)$. It can directly be computed recursively using the Newton's identities.*

EXERCISE **8.1.14.** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, prove that A and B have the same eigenvalues if and only if $\mathsf{tr}(\mathsf{A}^\mathsf{k}) = \mathsf{tr}(\mathsf{B}^\mathsf{k})$, for $k = 1, \ldots, n$.*

8.2 Methods for Tridiagonalization and Diagonalization

Let $\mathcal{G} = \{A \in \mathbb{M}_n(\mathbb{C}) : A^*A = I\}$. Then, using Exercise 5.8.8, we see that

1. for every $A, B \in \mathcal{G}$, $AB \in \mathcal{G}$.
2. for every $A, B, C \in \mathcal{G}$, $(AB)C = A(BC)$.
3. I_n is the identity element of $\mathcal{G}$. That is, for any $A \in \mathcal{G}$, $AI_n = A = I_nA$.
4. for every $A \in \mathcal{G}$, $A^{-1} \in \mathcal{G}$.

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Thus, the set $\mathcal{G}$ forms a group with respect to multiplication. We now define this group.

Definition 8.2.1. **[Unitary Group]** Let $\mathcal{G} = \{A \in \mathbb{M}_n(\mathbb{C}) : A^*A = I\}$. Then, $\mathcal{G}$ forms a multiplicative group. This group is called the **unitary group**.

Proposition 8.2.2. **[Selection Principle of Unitary Matrices]** *Let $\{U_k : k \geq 1\}$ be a sequence of unitary matrices. Viewing them as elements of $\mathbb{C}^{n^2}$, let us assume that "for any $\epsilon > 0$, there exists a positive integer N such that $\|U_k - U\| < \epsilon$, for all $k \geq N$". That is, the matrices U_k's converge to U as elements in $\mathbb{C}^{n^2}$. Then, U is also a unitary matrix.*

Proof. Let $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ be an unitary matrix. Then $\sum\limits_{i,j=1}^{n} |a_{ij}|^2 = \mathsf{tr}(\mathsf{A}^*\mathsf{A}) = \mathsf{n}$. Thus, the set of unitary matrices is a compact subset of $\mathbb{C}^{n^2}$. Hence, any sequence of unitary matrices has a convergent subsequence (Bolzano-Weierstrass Theorem), whose limit is again unitary. Thus, the required result follows. ■

For a unitary matrix U, we know that $U^{-1} = U^*$. Our next result gives a necessary and sufficient condition on an invertible matrix A so that the matrix A^{-1} is similar to A^*.

Theorem 8.2.3. **[Generalizing a Unitary Matrix]** *Let A be an invertible matrix. Then A^{-1} is similar to A^* if and only if there exists an invertible matrix B such that $A = B^{-1}B^*$.*

Proof. Suppose $A = B^{-1}B^*$, for some invertible matrix B. Then

$$A^* = B(B^{-1})^* = B(B^{-1})^*BB^{-1} = B(B^{-1}B^*)^{-1}B^{-1} = BA^{-1}B^{-1}.$$

Conversely, let $A^* = SA^{-1}S^{-1}$, for some invertible matrix S. Need to show, $A = S^{-1}S^*$.

We first show that there exists a nonsingular Hermitian H_θ such that $A^{-1} = H_\theta^{-1}A^*H_\theta$, for some $\theta \in \mathbb{R}$.

Note that for any $\theta \in \mathbb{R}$, if we put $S_\theta = e^{i\theta}S$ then

$$S_\theta A^{-1}S_\theta^{-1} = A^* \text{ and } S_\theta = A^*S_\theta A.$$

Now, define $H_\theta = S_\theta + S_\theta^*$. Then, H_θ is a Hermitian matrix and $H_\theta = A^*H_\theta A$. Furthermore, there are infinitely many choices of $\theta \in \mathbb{R}$ such that $\det H_\theta = 0$. To see this, let us choose a $\theta \in \mathbb{R}$ such that H_θ is singular. Hence, there exists $\mathbf{x} \neq \mathbf{0}$ such that $H_\theta\mathbf{x} = \mathbf{0}$. So,

$$S_\theta\mathbf{x} = -S_\theta^*\mathbf{x} = -e^{-i\theta}S^*\mathbf{x}. \text{ Or equivalently, } -e^{2i\theta}\mathbf{x} = S^{-1}S^*\mathbf{x}.$$

That is, $-e^{2i\theta} \in \sigma(S^{-1}S^*)$. Thus, if we choose $\theta_0 \in \mathbb{R}$ such that $-e^{2i(\theta_0)} \notin \sigma(S^{-1}S^*)$ then $H_(\theta_0)$ is nonsingular.

To get our result, we finally choose $B = \beta(\alpha I - A^*)H_(\theta_0)$ such that $\beta \neq 0$ and $\alpha = e^{i\gamma} \notin \sigma(A^*)$.

Note that with α and β chosen as above, B is invertible. Furthermore,

$$BA = \alpha\beta H_(\theta_0)A - \beta A^*H_(\theta_0)A = \alpha\beta H_(\theta_0)A - \beta H_(\theta_0) = \beta H_(\theta_0)(\alpha A - I).$$

As we need, $BA = B^*$, we get $\beta H_(\theta_0)(\alpha A - I) = \overline{\beta}H_(\theta_0)(\overline{\alpha}I - A)$ and thus, we need $\overline{\beta} = -\beta\alpha$, which holds true if $\beta = e^{i(\pi-\gamma)/2}$. Thus, the required result follows. ■

EXERCISE **8.2.4.** *Suppose that A is similar to a unitary matrix. Then, prove that A^{-1} is similar to A^*.*

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8.2.1 Plane Rotations

Definition 8.2.5. [**Plane Rotations**] For a fixed positive integer n, consider the vector space $\mathbb{R}^n$ with standard basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Also, for $1 \le i, j \le n$, let $E_{i,j} = \mathbf{e}_i \mathbf{e}_j^T$. Then, for $\theta \in \mathbb{R}$ and $1 \le i, j \le n$, a **plane rotation**, denoted $U(\theta; i, j)$, is defined as

$$U(\theta;i,j) = I - E_{i,i} - E_{j,j} + [E_{i,i} + E_{j,j}]\cos\theta - E_{i,j}\sin\theta + E_{j,i}\sin\theta.$$

That is, $U(\theta;i,j) = \begin{array}{c@{}l} \begin{bmatrix} 1 & & & & & & \\ & \ddots & & & & & \\ & & \cos\theta & & -\sin\theta & & \\ & & & \ddots & & & \\ & & \sin\theta & & \cos\theta & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{bmatrix} & \begin{array}{l} \\ \\ \leftarrow i\text{-th row} \\ \\ \leftarrow j\text{-th row} \\ \\ \\ \end{array} \end{array}$, where the unmentioned diagonal entries are 1 and the unmentioned off-diagonal entries are 0.

Remark 8.2.6. *Note the following about the matrix $U(\theta; i, j)$, where $\theta \in \mathbb{R}$ and $1 \le i, j \le n$.*

1. *$U(\theta; i, j)$ are orthogonal.*
2. *Geometrically $U(\theta; i, j)\mathbf{x}$ rotates $\mathbf{x}$ by the angle θ in the ij-plane.*
3. *Geometrically $(U(\theta; i, j))^T \mathbf{x}$ rotates $\mathbf{x}$ by the angle $-\theta$ in the ij-plane.*
4. *If $\mathbf{y} = U(\theta; i, j)\mathbf{x}$ then the coordinates of $\mathbf{y}$ are given by*
   (a) $\mathbf{y}_i = \mathbf{x}_i \cos\theta - \mathbf{x}_j \sin\theta$,
   (b) $\mathbf{y}_j = \mathbf{x}_i \sin\theta + \mathbf{x}_j \cos\theta$, *and*
   (c) for $l \ne i, j$, $\mathbf{y}_l = \mathbf{x}_l$.
5. *Thus, for $\mathbf{x} \in \mathbb{R}^n$, the choice of θ for which $\mathbf{y}_j = 0$, where $\mathbf{y} = U(\theta; i, j)\mathbf{x}$ equals*
   (a) $\theta = 0$, *whenever* $\mathbf{x}_j = 0$. *That is,* $U(0; i, j) = I$.
   (b) $\theta = \cot^{-1}\left(-\frac{\mathbf{x}_i}{\mathbf{x}_j}\right)$, *whenever* $\mathbf{x}_j \ne 0$.
6. [**Geometry**] *Imagine standing at $\mathbf{1} = (1, 1, 1)^T \in \mathbb{R}^3$. We want to apply a plane rotation U, so that $\mathbf{v} = U^T\mathbf{1}$ with $\mathbf{v}_2 = 0$. That is, the final point is on the xz-plane.*

   Then, we can either apply a plane rotation along the xy-plane or the yz-plane. For the xy-plane, we need the plane $z = 1$ (xy plane lifted by 1). This plane contains the vector $\mathbf{1}$. Imagine moving the tip of $\vec{\mathbf{1}}$ on this plane. Then this locus corresponds to a circle that lies on the plane $z = 1$, has radius $\sqrt{2}$ and is centred at $(0, 0, 1)$. That is, we draw the circle $x^2 + y^2 = 1$ on the xy-plane and then lifted it up by so that it lies on the plane $z = 1$. Thus, note that the xz-plane cuts this circle at two points. These two points of intersections give us the two choices for the vector $\mathbf{v}$ (see Figure 8.1). A similar calculation can be done for the yz-plane.

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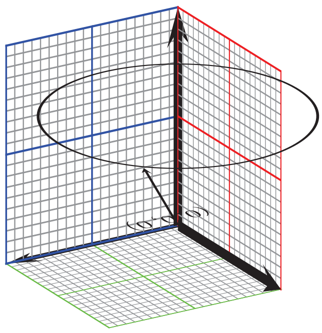

Figure 8.1: Geometry of plane rotations in $\mathbb{R}^3$

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7. *In general, in $\mathbb{R}^n$, suppose that we want to apply plane rotation to $\mathbf{a}$ along the x_1x_2-plane so that the resulting vector has 0 in the 2-nd coordinate. In that case, our circle on x_1x_2-plane has radius $r = \sqrt{\mathbf{a}_1^2 + \mathbf{a}_2^2}$ and it gets translated by $\begin{bmatrix} 0, & 0, & \mathbf{a}_3, & \cdots & \mathbf{a}_n \end{bmatrix}^T$. So, there are two points $\mathbf{x}$ on this circle with $\mathbf{x}_2 = 0$ and they are $\begin{bmatrix} \pm r, & 0, & a_3, & \cdots & a_n \end{bmatrix}^T$.*

8. *Consider three mutually orthogonal unit vectors, say $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Then, $\mathbf{x}$ can be brought to $\mathbf{e}_1$ by two plane rotations, namely by an appropriate $U(\theta_1; 1, 3)$ and $U(\theta_2; 1, 2)$. Thus,*

$$U(\theta_2; 1, 2)U(\theta_1; 1, 3)\mathbf{x} = \mathbf{e}_1.$$

In this process, the unit vectors $\mathbf{y}$ and $\mathbf{z}$, get shifted to say,

$$\hat{\mathbf{y}} = U(\theta_2; 1, 2)U(\theta_1; 1, 3)\mathbf{y} \ \text{ and } \ \hat{\mathbf{z}} = U(\theta_2; 1, 2)U(\theta_1; 1, 3)\mathbf{z}.$$

As unitary transformations preserve angles, note that $\hat{\mathbf{y}}(1) = \hat{\mathbf{z}}(1) = 0$. Now, we can apply an appropriate plane rotation $U(\theta_3; 2, 3)$ so that $U(\theta_3; 2, 3)\hat{\mathbf{y}} = \mathbf{e}_2$. Since $\mathbf{e}_3$ is the only unit vector in $\mathbb{R}^3$ orthogonal to both $\mathbf{e}_1$ and $\mathbf{e}_2$, it follows that $U(\theta_3; 2, 3)\hat{\mathbf{z}} = \mathbf{e}_3$. Thus,

$$I = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \end{bmatrix} = U(\theta_3; 2, 3)U(\theta_2; 1, 2)U(\theta_1; 1, 3)\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{z} \end{bmatrix}.$$

Hence, any real orthogonal matrix $A \in \mathbb{M}_3(\mathbb{R})$ is a product of three plane rotations.

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We are now ready to give another method to get the QR-decomposition of a square matrix (see Theorem 4.6.1 that uses the Gram-Schmidt Orthonormalization Process).

Proposition 8.2.7. [QR Factorization Revisited: Square Matrix] *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then there exists a real orthogonal matrix Q and an upper triangular matrix R such that $A = QR$.*

Proof. We start by applying the plane rotations to A so that the positions $(2, 1), (3, 1), \ldots, (n, 1)$ of A become zero. This means, if $a_{21} = 0$, we multiply by I. Otherwise, we use the plane rotation $U(\theta; 1, 2)$, where $\theta = \cot^{-1}(-a_{11}/a_{21})$. Then, we apply a similar technique to A so that the

$(3,1)$ entry of A becomes 0. Note that this plane rotation doesn't change the $(2,1)$ entry of A. We continue this process till all the entry in the first column of A, except possibly the $(1,1)$ entry, is zero.

We then apply the plane rotations to make positions $(3,2),(4,2),\ldots,(n,2)$ zero. Observe that this does not disturb the zeros in the first column. Thus, continuing the above process a finite number of times give us the required result. ∎

Lemma 8.2.8. **[QR Factorization Revisited: Rectangular Matrix]** *Let $A \in \mathbb{M}_{m,n}(\mathbb{R})$. Then there exists a real orthogonal matrix Q and a matrix $R \in \mathbb{M}_{m,n}(\mathbb{R})$ in upper triangular form such that $A = QR$.*

Proof. If $\mathsf{Rank}A < m$, add some columns to A to get a matrix, say $\tilde{A}$ such that $\mathsf{Rank}\tilde{A} = m$. Now suppose that $\tilde{A}$ has k columns. For $1 \le i \le k$, let $\mathbf{v}_i = \tilde{A}[:,i]$. Now, apply the Gram-Schmidt Orthonormalization Process to $\{\mathbf{v}_1,\ldots,\mathbf{v}_k\}$. For example, suppose the result is a sequence of k vectors $\mathbf{w}_1, 0, \mathbf{w}_2, 0, 0, \ldots, 0, \mathbf{w}_m, 0, \ldots, 0$, where $Q = \begin{bmatrix}\mathbf{w}_1 & \cdots & \mathbf{w}_m\end{bmatrix}$ is real orthogonal. Then $\tilde{A}[:,1]$ is a linear combination of $\mathbf{w}_1$, $\tilde{A}[:,2]$ is also a linear combination of $\mathbf{w}_1$, $\tilde{A}[:,3]$ is a linear combination of $\mathbf{w}_1, \mathbf{w}_2$ and so on. In general, for $1 \le s \le k$, the column $\tilde{A}[:,s]$ is a linear combination of $\mathbf{w}_i$-s in the list that appear up to the s-th position. Thus, $\tilde{A}[:,s] = \sum\limits_{i=1}^{m} \mathbf{w}_i r_{is}$, where $r_{is} = 0$ for all $i > s$. That is, $\tilde{A} = QR$, where $R = [r_{ij}]$. Now, remove the extra columns of $\tilde{A}$ and the corresponding columns in R to get the required result. ∎

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Note that Proposition 8.2.7 is also valid for any complex matrix. In this case the matrix Q will be unitary. This can also be seen from Theorem 4.6.1 as we need to apply the Gram-Schmidt Orthonormalization Process to vectors in $\mathbb{C}^n$.

To proceed further recall that a matrix $A = [a_{ij}] \in \mathbb{M}_n(\mathbb{C})$ is called a **tri-diagonal** matrix if $a_{ij} = 0$, whenever $|i-j| > 1, 1 \le i,j \le n$.

Proposition 8.2.9. **[Tridiagonalization of a Real Symmetric Matrix: Given's Method]** *Let A be a real symmetric. Then, there exists a real orthogonal matrix Q such that QAQ^T is a tri-diagonal matrix.*

Proof. If $a_{31} \neq 0$, then put $U_1 = U(\theta_1; 2,3)$, where $\theta_1 = \cot^{-1}(-a_{21}/a_{31})$. Notice that $U_1^T[:,1] = \mathbf{e}_1$ and so

$$(U_1 A U_1^T)[:,1] = (U_1 A)[:,1].$$

We already know that $U_1 A[3,1] = 0$. Hence, $U_1 A U_1^T$ is a real symmetric matrix with $(3,1)$-th entry 0. Now, proceed to make the $(4,1)$-th entry of $U_1 A$ equal to 0. To do so, take $U_2 = U(\theta_2; 2,4)$. Notice that $U_2^T(:,1) = \mathbf{e}_1$ and so

$$(U_2(U_1 A U_1^T)U_2^T)[:,1) = (U_2 U_1 A U_1^T)[:,1].$$

But by our choice of the plane rotation U_2, we have $U_2(U_1 A U_1^T)(4,1) = 0$. Furthermore, as $U_2[3,:] = \mathbf{e}_3^T$, we have

$$(U_2 U_1 A U_1^T)[3,1] = U_2[3,:](U_1 A U_1^T)[:,1] = (U_1 A U_1^T)[3,1] = 0.$$

That is, the previous zeros are preserved.

Continuing this way, we can find a real orthogonal matrix Q such that QAQ^T is tri-diagonal. ∎

Proposition 8.2.10. **[Almost Diagonalization of a Real Symmetric Matrix: Jacobi method]** *Let $A \in M_n(\mathbb{R})$ be real symmetric. Then there exists a real orthogonal matrix S, a product of plane rotations, such that SAS^T is almost a diagonal matrix.*

Proof. The idea is to reduce the off-diagonal entries of A to 0 as much as possible. So, we start with choosing $i \neq j$) such that $i < j$ and $|a_{ij}|$ is maximum. Now, put

$$\theta = \frac{1}{2}\cot^{-1}\frac{a_{ii} - a_{jj}}{2a_{ij}}, \quad U = U(\theta; i, j), \quad \text{and} \quad B = U^T AU.$$

Then, for all $l, k \neq i, j$, we see that

$$\begin{aligned}
b_{lk} &= U^T[l,:]AU[:,k] = \mathbf{e}_l^T A\mathbf{e}_k = a_{lk} \\
b_{ik} &= U^T[i,:]AU[:,k] = (\cos\theta\mathbf{e}_i^T + \sin\theta\mathbf{e}_j^T)A\mathbf{e}_k = a_{ik}\cos\theta + a_{jk}\sin\theta \\
b_{lj} &= U^T[l,:]AU[:,j] = \mathbf{e}_l^T A(-\sin\theta\mathbf{e}_i + \cos\theta\mathbf{e}_j) = -a_{li}\sin\theta + a_{lj}\cos\theta \\
b_{ij} &= U^T[i,:]AU[:,j] = (\cos\theta\mathbf{e}_i^T + \sin\theta\mathbf{e}_j^T)A(-\sin\theta\mathbf{e}_i + \cos\theta\mathbf{e}_j) \\
&= \sin(2\theta)\tfrac{a_{jj}-a_{ii}}{2} + a_{ij}\cos(2\theta) = 0
\end{aligned}$$

Thus, using the above, we see that whenever $l, k \neq i, j$, $a_{lk}^2 = b_{lk}^2$ and for $l \neq i, j$, we have

$$b_{il}^2 + b_{lj}^2 = a_{il}^2 + a_{lj}^2.$$

DRAFT

As U is unitary and $B = U^T AU$, we get $\sum |a_{ij}|^2 = \sum |b_{ij}|^2$. Further, $b_{ij} = 0$ implies that

$$a_{ii}^2 + 2a_{ij}^2 + a_{jj}^2 = b_{ii}^2 + 2b_{ij}^2 + b_{jj}^2 = b_{ii}^2 + b_{jj}^2.$$

As the rest of the diagonal entries have not changed, we observe that the sum of the squares of the off-diagonal entries have reduced by $2a_{ij}^2$. Thus, a repeated application of the above process makes the matrix "close to diagonal". ∎

8.2.2 Householder Matrices

We will now look at another class of unitary matrices, commonly called the Householder matrices (see Exercise 1.5.5.8).

Definition 8.2.11. **[Householder Matrix]** Let $\mathbf{w} \in \mathbb{C}^n$ be a unit vector. Then, the matrix $U_{\mathbf{w}} = I - 2\mathbf{w}\mathbf{w}^*$ is called a **Householder matrix**.

Remark 8.2.12. *We observe the following about the Householder matrix $U_{\mathbf{w}}$.*

1. *$U_{\mathbf{w}} = I - 2\mathbf{w}\mathbf{w}^*$ is the sum of two Hermitian matrices and hence is also Hermitian.*
2. *$U_{\mathbf{w}}U_{\mathbf{w}}^* = (I - 2\mathbf{w}\mathbf{w}^*)(I - 2\mathbf{w}\mathbf{w}^*)^T = I - 2\mathbf{w}\mathbf{w}^* - 2\mathbf{w}\mathbf{w}^* + 4\mathbf{w}\mathbf{w}^* = I$. Or equivalently, verify that $\|U_{\mathbf{w}}\mathbf{x}\| = \|\mathbf{x}\|$, for all $\mathbf{x} \in \mathbb{C}^n$. So $U_{\mathbf{w}}$ is unitary.*
3. *If $\mathbf{x} \in \mathbf{w}^\perp$ then $U_{\mathbf{w}}\mathbf{x} = \mathbf{x}$.*

4. *If* $\mathbf{x} = c\mathbf{w}$, *for some* $c \in \mathbb{C}$, *then* $U_{\mathbf{w}}\mathbf{x} = -\mathbf{x}$.
5. *Thus, if* $\mathbf{v} \in \mathbb{C}^n$ *then we know that* $\mathbf{v} = \mathbf{x} + \mathbf{y}$, *where* $\mathbf{x} \in \mathbf{w}^{\perp}$ *and* $\mathbf{y} = c\mathbf{w}$, *for some* $c \in \mathbb{C}$. *In this case,* $U_{\mathbf{w}}\mathbf{v} = U_{\mathbf{w}}(\mathbf{x} + \mathbf{y}) = \mathbf{x} - \mathbf{y}$.
6. *Geometrically,* $U_{\mathbf{w}}\mathbf{v}$ *reflects the vector* $\mathbf{v}$ *along the vector* $\mathbf{w}^{\perp}$. *Thus,* $U_{\mathbf{w}}$ *is a reflection matrix along* $\mathbf{w}^{\perp}$ *(see Definition 1.4.1.7).*

Example 8.2.13. In $\mathbb{R}^2$, let $\mathbf{w} = \mathbf{e}_2$. Then $\mathbf{w}^{\perp}$ is the x-axis. The vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \mathbf{e}_1 + 2\mathbf{e}_2$, where $\mathbf{e}_1 \in \mathbf{w}^{\perp}$ and $2\mathbf{e}_2 \in LS(\mathbf{w})$. So

$$U_{\mathbf{w}}(\mathbf{e}_1 + 2\mathbf{e}_2) = U_{\mathbf{w}}\mathbf{v} = U_{\mathbf{w}}(\mathbf{x} + \mathbf{y}) = \mathbf{x} - \mathbf{y} = \mathbf{e}_1 - 2\mathbf{e}_2.$$

That is, the <u>reflection of $\mathbf{v}$ along the x-axis ($\mathbf{w}^{\perp}$)</u>.

Recall that if $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ with $\mathbf{x} \neq \mathbf{y}$ and $\|\mathbf{x}\| = \|\mathbf{y}\|$ then, $(\mathbf{x} + \mathbf{y}) \perp (\mathbf{x} - \mathbf{y})$. This is not true in $\mathbb{C}^n$ as can be seen from the following example. Take $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} i \\ -1 \end{bmatrix}$. Then $\left\langle \begin{bmatrix} 1+i \\ 0 \end{bmatrix}, \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \right\rangle = (1+i)^2 \neq 0$. Thus, to pick the right choice for the matrix $U_{\mathbf{w}}$, we need to be observant of the choice of the inner product space.

DRAFT

Example 8.2.14. Let $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$ with $\mathbf{x} \neq \mathbf{y}$ and $\|\mathbf{x}\| = \|\mathbf{y}\|$. Then, which $U_{\mathbf{w}}$ should be used to reflect $\mathbf{y}$ to $\mathbf{x}$?

1. **Solution in case of $\mathbb{R}^n$:** Imagine the line segment joining $\mathbf{x}$ and $\mathbf{y}$. Now, place a mirror at the midpoint and perpendicular to the line segment. Then, the reflection of $\mathbf{y}$ on that mirror is $\mathbf{x}$. So, take $\mathbf{w} = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|} \in \mathbb{R}^n$. Then,

$$\begin{aligned} U_{\mathbf{w}}\mathbf{y} &= (I - 2\mathbf{w}\mathbf{w}^T)\mathbf{y} = \mathbf{y} - 2\mathbf{w}\mathbf{w}^T\mathbf{y} = \mathbf{y} - 2\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^2}(\mathbf{x}-\mathbf{y})^T\mathbf{y} \\ &= \mathbf{y} - 2\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^2}\frac{-\|\mathbf{x}-\mathbf{y}\|^2}{2} = \mathbf{x}. \end{aligned}$$

2. **Solution in case of $\mathbb{C}^n$:** Suppose there is a unit vector $\mathbf{w} \in \mathbb{C}^n$ such that $(I - 2\mathbf{w}\mathbf{w}^*)\mathbf{y} = \mathbf{x}$. Then $\mathbf{y} - \mathbf{x} = 2\mathbf{w}\mathbf{w}^*\mathbf{y}$ and hence $\mathbf{w}^*(\mathbf{y} - \mathbf{x}) = 2\mathbf{w}^*\mathbf{w}\mathbf{w}^*\mathbf{y} = 2\mathbf{w}^*\mathbf{y}$. Thus,

$$\mathbf{w}^*(\mathbf{y} + \mathbf{x}) = 0, \text{ that is, } \mathbf{w} \perp (\mathbf{y} + \mathbf{x}). \tag{8.2.1}$$

Furthermore, again using $\mathbf{w}^*(\mathbf{y} + \mathbf{x}) = 0$, we get $\mathbf{y} - \mathbf{x} = 2\mathbf{w}\mathbf{w}^*\mathbf{y} = -2\mathbf{w}\mathbf{w}^*\mathbf{x}$. So,

$$2(\mathbf{y} - \mathbf{x}) = 2\mathbf{w}\mathbf{w}^*(\mathbf{y} - \mathbf{x}) \text{ or } \mathbf{y} - \mathbf{x} = \mathbf{w}\mathbf{w}^*(\mathbf{y} - \mathbf{x}).$$

On the other hand, using Equation (8.2.1), we get $\mathbf{w}\mathbf{w}^*(\mathbf{y} + \mathbf{x}) = 0$. So,

$$0 = [(\mathbf{y} + \mathbf{x})^*\mathbf{w}\mathbf{w}^*](\mathbf{y} - \mathbf{x}) = (\mathbf{y} + \mathbf{x})^*[\mathbf{w}\mathbf{w}^*(\mathbf{y} - \mathbf{x})] = (\mathbf{y} + \mathbf{x})^*(\mathbf{y} - \mathbf{x}).$$

Therefore, if such a $\mathbf{w}$ exists, then $(\mathbf{y} - \mathbf{x}) \perp (\mathbf{y} + \mathbf{x})$.

But, in that case, $\mathbf{w} = \frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|}$ will work as using above $\|\mathbf{x}-\mathbf{y}\|^2 = 2(\mathbf{y}-\mathbf{x})^*\mathbf{y}$ and

$$\begin{aligned} U_{\mathbf{w}}\mathbf{y} &= (I - 2\mathbf{w}\mathbf{w}^*)\mathbf{y} = \mathbf{y} - 2\mathbf{w}\mathbf{w}^*\mathbf{y} = \mathbf{y} - 2\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^2}(\mathbf{x}-\mathbf{y})^*\mathbf{y} \\ &= \mathbf{y} - 2\frac{\mathbf{x}-\mathbf{y}}{\|\mathbf{x}-\mathbf{y}\|^2}\frac{-\|\mathbf{x}-\mathbf{y}\|^2}{2} = \mathbf{x}. \end{aligned}$$

Thus, in this case, if $\langle \mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \neq 0$ then we will not find a $\mathbf{w}$ such that $U_{\mathbf{w}}\mathbf{y} = \mathbf{x}$. For example, taking $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} i \\ -1 \end{bmatrix}$, we have $\langle \mathbf{x}+\mathbf{y}, \mathbf{x}-\mathbf{y}\rangle \neq 0$.

As an application, we now prove that any real symmetric matrix can be transformed into a tri-diagonal matrix.

Proposition 8.2.15. **[Householder's Tri-Diagonalization]** *Let* $\mathbf{v} \in \mathbb{R}^{n-1}$ *and* $A = \begin{bmatrix} a & \mathbf{v}^T \\ \mathbf{v} & B \end{bmatrix} \in \mathbb{M}_n(\mathbb{R})$ *be a real symmetric matrix. Then, there exists a real orthogonal matrix* Q*, a product of Householder matrices, such that* Q^TAQ *is tri-diagonal.*

Proof. If $\mathbf{v} = \mathbf{e}_1$ then we proceed to apply our technique to the matrix B, a matrix of lower order. So, without loss of generality, we assume that $\mathbf{v} \neq \mathbf{e}_1$.

As we want Q^TAQ to be tri-diagonal, we need to find a vector $\mathbf{w} \in \mathbb{R}^{n-1}$ such that $U_{\mathbf{w}}\mathbf{v} = r\mathbf{e}_1 \in \mathbb{R}^{n-1}$, where $r = \|\mathbf{v}\| = \|U_{\mathbf{w}}\mathbf{v}\|$. Thus, using Example 8.2.14, choose the required vector $\mathbf{w} \in \mathbb{R}^{n-1}$. Then,

$$\begin{bmatrix} 1 & 0 \\ 0 & U_{\mathbf{w}} \end{bmatrix}\begin{bmatrix} a & \mathbf{v}^T \\ \mathbf{v} & B \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & U_{\mathbf{w}}^T \end{bmatrix} = \begin{bmatrix} a & \mathbf{v}^TU_{\mathbf{w}}^T \\ U_{\mathbf{w}}\mathbf{v} & U_{\mathbf{w}}BU_{\mathbf{w}}^T \end{bmatrix} = \begin{bmatrix} a & r & 0 \\ r & * & * \\ 0 & * & * \end{bmatrix} = \begin{bmatrix} a & r\mathbf{e}_1^T \\ r\mathbf{e}_1 & S \end{bmatrix},$$

where $S \in \mathbb{M}_{n-1}(\mathbb{R})$ is a symmetric matrix. Now, use induction on the matrix S to get the required result. ∎

8.2.3 Schur's Upper Triangularization Revisited

Definition 8.2.16. Let s and t be two symbols. Then, an expression of the form

$$W(s,t) = s^{m_1}t^{n_1}\ldots s^{m_k}t^{n_k} \text{ where } m_i, n_i \text{ are positive integers}$$

is called a **word in symbols** s **and** t of degree $\sum\limits_{i=1}^{k}(m_i+n_i)$.

Remark 8.2.17. **[More on Unitary Equivalence]** *Let* s *and* t *be two symbols and* $W(s,t)$ *be a word in symbols* s *and* t.

1. *Suppose* U *is a unitary matrix such that* $B = U^*AU$*. Then,* $W(A,A^*) = U^*W(B,B^*)U$*. Thus,* $\mathsf{tr}[\mathsf{W}(\mathsf{A},\mathsf{A}^*)] = \mathsf{tr}[\mathsf{W}(\mathsf{B},\mathsf{B}^*)]$.
2. *Let* A *and* B *be two matrices such that* $\mathsf{tr}[\mathsf{W}(\mathsf{A},\mathsf{A}^*)] = \mathsf{tr}[\mathsf{W}(\mathsf{B},\mathsf{B}^*)]$*, for each word* W*. Then, does it imply that* A *and* B *are unitarily equivalent? The answer is* **'yes'** *as provided by the following result. The proof is outside the scope of this book.*

DRAFT

Theorem 8.2.18. [**Specht-Pearcy**] *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ and suppose that* $\mathsf{tr}[\mathsf{W}(\mathsf{A}, \mathsf{A}^*)] = \mathsf{tr}[\mathsf{W}(\mathsf{B}, \mathsf{B}^*)]$ *holds for all words of degree less than or equal to $2n^2$. Then $B = U^*AU$, for some unitary matrix U.*

EXERCISE **8.2.19.** [**Triangularization via Complex Orthogonal Matrix need not be Possible**] *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $A = QTQ^T$, where Q is complex orthogonal matrix and T is upper triangular. Then, prove that*

1. *A has an eigenvector $\mathbf{x}$ such that $\mathbf{x}^T\mathbf{x} \neq 0$.*
2. *there is no orthogonal matrix Q such that $Q^T \begin{bmatrix} 1 & \mathbf{i} \\ \mathbf{i} & -1 \end{bmatrix} Q$ is upper triangular.*

Proposition 8.2.20. [**Matrices with Distinct Eigenvalues are Dense in $\mathbb{M}_n(\mathbb{C})$**] *Let $A \in \mathbb{M}_n(\mathbb{C})$. Then, for each $\epsilon > 0$, there exists a matrix $A(\epsilon) \in \mathbb{M}_n(\mathbb{C})$ such that $A(\epsilon) = [a(\epsilon)_{ij}]$ has distinct eigenvalues and $\sum |a_{ij} - a(\epsilon)_{ij}|^2 < \epsilon$.*

Proof. By Schur Upper Triangularization (see Lemma 6.4.1), there exists a unitary matrix U such that $U^*AU = T$, an upper triangular matrix. Now, choose α_i's such that $t_{ii} + \alpha_i$ are distinct and $\sum |\alpha_i|^2 < \epsilon$. Now, consider the matrix $A(\epsilon) = U\left(T + \text{ diag}(\alpha_1, \ldots, \alpha_n)\right)U^*$. Then, $B = A(\epsilon) - A = U \text{ diag}(\alpha_1, \ldots, \alpha_n)]U^*$ with

$$\sum_{i,j} |b_{ij}|^2 = \mathsf{tr}(\mathsf{B}^*\mathsf{B}) = \mathsf{trU} \text{ diag}(|\alpha_\mathsf{1}|^2, \ldots, |\alpha_\mathsf{n}|^2)\mathsf{U}^* = \sum_\mathsf{i} |\alpha_\mathsf{i}|^2 < \epsilon.$$

Thus, the required result follows. ∎

Before proceeding with our next result on almost diagonalizability, we look at the following example.

Example 8.2.21. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ and $\epsilon > 0$ be given. Then, determine a diagonal matrix D such that the non-diagonal entry of $D^{-1}AD$ is less than ϵ.

Solution: Choose $\alpha < \frac{\epsilon}{2}$ and define $D = \text{ diag}(1, \alpha)$. Then,

$$D^{-1}AD = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\alpha} \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \alpha \end{bmatrix} = \begin{bmatrix} 1 & 2\alpha \\ 0 & 3 \end{bmatrix}.$$

As $\alpha < \frac{\epsilon}{2}$, the required result follows.

Proposition 8.2.22. [**A matrix is Almost Diagonalizable**] *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $\epsilon > 0$ be given. Then, there exists an invertible matrix S_ϵ such that $S_\epsilon^{-1}AS_\epsilon = T$, an upper triangular matrix with $|t_{ij}| < \epsilon$, for all $i \neq j$.*

Proof. By Schur Upper Triangularization (see Lemma 6.4.1), there exists a unitary matrix U such that $U^*AU = T$, an upper triangular matrix. Now, take $t = 2 + \max\limits_{i<j} |t_{ij}|$ and choose α such that $0 < \alpha < \epsilon/t$. Then, if we take $D_\alpha = \text{ diag}(1, \alpha, \alpha^2, \ldots, \alpha^{n-1})$ and $S = UD_\alpha$, we have $S^{-1}AS = D_{\frac{1}{\alpha}}TD_\alpha = F$ (say), an upper triangular. Furthermore, note that for $i < j$, we have $|f_{ij}| = |t_{ij}|\alpha^{j-i} \leq \epsilon$. Thus, the required result follows. ∎

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8.3 Commuting Matrices and Simultaneous Diagonalization

Definition 8.3.1. **[Simultaneously Diagonalizable]** Let $A, B \in \mathbb{M}_n(\mathbb{C})$. Then, they are said to be **simultaneously diagonalizable** if there exists an invertible matrix S such that $S^{-1}AS$ and $S^{-1}BS$ are both diagonal matrices.

Since diagonal matrices commute, we have our next result.

Proposition 8.3.2. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$. If A and B are simultaneously diagonalizable then $AB = BA$.*

Proof. By definition, there exists an invertible matrix S such that $S^{-1}AS = \Lambda_1$ and $S^{-1}BS = \Lambda_2$. Hence,

$$AB = (S\Lambda_1 S^{-1}) \cdot (S\Lambda_2 S^{-1}) = S\Lambda_1\Lambda_2 S^{-1} = S\Lambda_2\Lambda_1 S^{-1} = S\Lambda_2 S^{-1} S\Lambda_1 S^{-1} = BA.$$

Thus, we have proved the required result. ∎

Theorem 8.3.3. *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be diagonalizable matrices. Then they are simultaneously diagonalizable if and only if they commute.*

Proof. One part of this theorem has already been proved in Proposition 8.3.2. For the other part, let us assume that $AB = BA$. Since A is diagonalizable, there exists an invertible matrix S such that

$$S^{-1}AS = \Lambda = \lambda_1 I \oplus \cdots \oplus \lambda_k I, \tag{8.3.1}$$

where $\lambda_1, \ldots, \lambda_k$ are the distinct eigenvalues of A. We now use the sub-matrix structure of $S^{-1}AS$ to decompose $C = S^{-1}BS$ as $C = \begin{bmatrix} C_{11} & \cdots & C_{1k} \\ & \ddots & \\ C_{k1} & \cdots & C_{kk} \end{bmatrix}$. Since $AB = BA$ and S is invertible, we have $\Lambda C = C\Lambda$. Thus,

$$\begin{bmatrix} \lambda_1 C_{11} & \cdots & \lambda_1 C_{1k} \\ & \ddots & \\ \lambda_k C_{k1} & \cdots & \lambda_k C_{kk} \end{bmatrix} = \begin{bmatrix} \lambda_1 C_{11} & \cdots & \lambda_k C_{1k} \\ & \ddots & \\ \lambda_1 C_{k1} & \cdots & \lambda_k C_{kk} \end{bmatrix}.$$

Since $\lambda_i \neq \lambda_j$ for $1 \leq i \neq j \leq k$, we have $C_{ij} = 0$, whenever $i \neq j$. Thus, the matrix $C = C_{11} \oplus \cdots \oplus C_{kk}$.

Since B is diagonalizable, the matrix C is also diagonalizable and hence the matrices C_{ii}, for $1 \leq i]lek$, are diagonalizable. So, for $1 \leq i \leq k$, there exists invertible matrices T_i's such that $T_i^{-1} C_{ii} T_i = \Lambda_i$. Put $T = T_1 \oplus \cdots \oplus T_k$. Then,

$$T^{-1}S^{-1}AST = \begin{bmatrix} T_1^{-1} & & \\ & \ddots & \\ & & T_k^{-1} \end{bmatrix} \begin{bmatrix} \lambda_1 I & & \\ & \ddots & \\ & & \lambda_k I \end{bmatrix} \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_k \end{bmatrix} = \begin{bmatrix} \lambda_1 I & & \\ & \ddots & \\ & & \lambda_k I \end{bmatrix}$$

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and

$$T^{-1}S^{-1}BST = \begin{bmatrix} T_1^{-1} & & \\ & \ddots & \\ & & T_k^{-1} \end{bmatrix} \begin{bmatrix} C_{11} & & \\ & \ddots & \\ & & C_{kk} \end{bmatrix} \begin{bmatrix} T_1 & & \\ & \ddots & \\ & & T_k \end{bmatrix} = \begin{bmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_k \end{bmatrix}.$$

Thus A and B are simultaneously diagonalizable and the required result follows. ■

Definition 8.3.4. [Commuting Family of Matrices]

1. Let $\mathcal{F} \subseteq \mathbb{M}_n(\mathbb{C})$. Then $\mathcal{F}$ is said to be a **commuting family** if each pair of matrices in $\mathcal{F}$ commutes.
2. Let $B \in \mathbb{M}_n(\mathbb{C})$ and W be a subspace of $\mathbb{C}^n$. Then, W is said to be a B**-invariant** subspace if $B\mathbf{w} \in W$, for all $\mathbf{w} \in W$ (or equivalently, $BW \subseteq W$).
3. A subspace W of $\mathbb{C}^n$ is said to be $\mathcal{F}$**-invariant** if W is B-invariant for each $B \in \mathcal{F}$.

Example 8.3.5. Let $A \in \mathbb{M}_n(\mathbb{C})$ with $(\lambda, \mathbf{x})$ as an eigenpair. Then, $W = \{c\mathbf{x} : c \in \mathbb{C}\}$ is an A-invariant subspace. Furthermore, if W is an A-invariant subspace with $\dim(W) = 1$ then verify that any non-zero vector in W is an eigenvector of A.

Theorem 8.3.6. [An A-invariant Subspace Contains an Eigenvector of A] *Let $A \in \mathbb{M}_n(\mathbb{C})$ and $W \subseteq \mathbb{C}^n$ be an A-invariant subspace of dimension at least 1. Then W contains an eigenvector of A.*

DRAFT

Proof. Let $\mathcal{B} = \{\mathbf{f}_1, \ldots, \mathbf{f}_k\} \subseteq \mathbb{C}^n$ be an ordered basis for W. Define $T : W \to W$ as $T\mathbf{v} = A\mathbf{v}$. Then $T[\mathcal{B}, \mathcal{B}] = \begin{bmatrix} [T\mathbf{f}_1]_{\mathcal{B}} & \cdots & [T\mathbf{f}_k]_{\mathcal{B}} \end{bmatrix}$ is a $k \times k$ matrix which satisfies $[T\mathbf{w}]_{\mathcal{B}} = T[\mathcal{B}, \mathcal{B}]\ [\mathbf{w}]_{\mathcal{B}}$, for all $\mathbf{w} \in W$. As $T[\mathcal{B}, \mathcal{B}] \in \mathbb{M}_k(\mathbb{C})$, it has an eigenpair, say $(\lambda, \hat{\mathbf{x}})$ with $\hat{\mathbf{x}} \in \mathbb{C}^k$. That is,

$$T[\mathcal{B}, \mathcal{B}]\hat{\mathbf{x}} = \lambda\hat{\mathbf{x}}. \tag{8.3.2}$$

Now, put $\mathbf{x} = \sum\limits_{i=1}^{k} (\hat{\mathbf{x}})_i\ \mathbf{f}_i \in \mathbb{C}^n$. Then, verify that $\mathbf{x} \in W$ and $[\mathbf{x}]_{\mathcal{B}} = \hat{\mathbf{x}}$. Thus, $T\mathbf{x} \in W$ and now using Equation (8.3.2), we get

$$T\mathbf{x} = \sum_{i=1}^{k} \left([T\mathbf{x}]_{\mathcal{B}}\right)_i \mathbf{f}_i = \sum_{i=1}^{k} \left(T[\mathcal{B}, \mathcal{B}][\mathbf{x}]_{\mathcal{B}}\right)_i \mathbf{f}_i = \sum_{i=1}^{k} \left(T[\mathcal{B}, \mathcal{B}]\hat{\mathbf{x}}\right)_i \mathbf{f}_i = \sum_{i=1}^{k} (\lambda\hat{\mathbf{x}})_i \mathbf{f}_i = \lambda \sum_{i=1}^{k} (\hat{\mathbf{x}})_i \mathbf{f}_i = \lambda\mathbf{x}.$$

So, A has an eigenvector $\mathbf{x} \in W$ corresponding to the eigenvalue λ. ■

Theorem 8.3.7. *Let $\mathcal{F} \subseteq \mathbb{M}_n(\mathbb{C})$ be a commuting family of matrices. Then, all the matrices in $\mathcal{F}$ have a common eigenvector.*

Proof. Note that $\mathbb{C}^n$ is $\mathcal{F}$-invariant. Let $W \subseteq \mathbb{C}^n$ be $\mathcal{F}$-invariant with minimum positive dimension. Let $\mathbf{y} \in W$ such that $\mathbf{y} \neq 0$. We claim that $\mathbf{y}$ is an eigenvector, for each $A \in \mathcal{F}$.

So, on the contrary assume $\mathbf{y}$ is not an eigenvector for some $A \in \mathcal{F}$. Then, by Theorem 8.3.6, W contains an eigenvector $\mathbf{x}$ of A for some eigenvalue, say λ. Define $W_0 = \{\mathbf{z} \in W : A\mathbf{z} = \lambda\mathbf{z}\}$. So W_0 is a proper subspace of W as $\mathbf{y} \in W \setminus W_0$. Also, for $\mathbf{z} \in W_0$ and $C \in \mathcal{F}$, we note that $A(C\mathbf{z}) = CA\mathbf{z} = \lambda(C\mathbf{z})$, so that $C\mathbf{z} \in W_0$. So W_0 is $\mathcal{F}$-invariant and $1 \leq \dim W_0 < \dim W$, a contradiction. ■

Theorem 8.3.8. *Let $\mathcal{F} \subseteq \mathbb{M}_n(\mathbb{C})$ be a family of diagonalizable matrices. Then $\mathcal{F}$ is commuting if and only if $\mathcal{F}$ is simultaneously diagonalizable.*

Proof. We prove the result by induction on n. The result is clearly true for $n = 1$. So, let us assume the result to be valid for all $n < m$. Now, let us assume that $\mathcal{F} \subseteq \mathbb{M}_m(\mathbb{C})$ is a family of diagonalizable matrices.

If $\mathcal{F}$ is simultaneously diagonalizable, then by Proposition 8.3.2, the family $\mathcal{F}$ is commuting. Conversely, let $\mathcal{F}$ be a commuting family. If each $A \in \mathcal{F}$ is a scalar matrix then they are simultaneously diagonalizable via I. So, let $A \in \mathcal{F}$ be a non-scalar matrix. As A is diagonalizable, there exists an invertible matrix S such that

$$S^{-1}AS = \lambda_1 I \oplus \cdots \oplus \lambda_k I, \; k \geq 2,$$

where λ_i's are distinct. Now, consider the family $\mathcal{G} = \{\hat{X} = S^{-1}XS \mid X \in \mathcal{F}\}$. As $\mathcal{F}$ is a commuting family, the set $\mathcal{G}$ is also a commuting family. So, each $\hat{X} \in \mathcal{G}$ has the form $\hat{X} = X_1 \oplus \cdots \oplus X_k$. Note that $\mathcal{H}_i = \{X_i \mid \hat{X} \in \mathcal{G}\}$ is a commuting family of diagonalizable matrices of size $< m$. Thus, by induction hypothesis, $\mathcal{H}_i$'s are simultaneously diagonalizable, say by the invertible matrices T_i's. That is, $T_i^{-1}X_iT_i = \Lambda_i$, a diagonal matrix, for $1 \leq i \leq k$. Thus, if $T = T_1 \oplus \cdots \oplus T_k$ then

$$T^{-1}S^{-1}\hat{X}ST = T^{-1}(X_1 \oplus \cdots \oplus X_k)T = T_1^{-1}X_1T_1 \oplus \cdots \oplus T_k^{-1}X_kT_k = \Lambda_1 \oplus \cdots \oplus \Lambda_k,$$

a diagonal matrix, for all $X \in \mathcal{F}$. Thus the result holds by induction. ∎

We now give prove of some parts of Exercise 6.2.7.exe:eigen:1.

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Remark 8.3.9. [$\sigma(AB)$ **and** $\sigma(BA)$] *Let $m \leq n$, $A \in \mathbb{M}_{m\times n}(\mathbb{C})$, and $B \in \mathbb{M}_{n\times m}(\mathbb{C})$. Then $\sigma(BA) = \sigma(AB)$ with $n - m$ extra 0's. In particular, if $A, B \in \mathbb{M}_n(\mathbb{C})$ then, $P_{AB}(t) = P_{BA}(t)$.*

Proof. Note that

$$\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} AB & ABA \\ B & BA \end{bmatrix} = \begin{bmatrix} I_m & A \\ 0 & I_n \end{bmatrix} \begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}.$$

Thus, the matrices $\begin{bmatrix} AB & 0 \\ B & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ B & BA \end{bmatrix}$ are similar. Hence, AB and BA have precisely the same non-zero eigenvalues. Therefore, if they have the same size, they must have the same characteristic polynomial. ∎

EXERCISE **8.3.10.** **[Miscellaneous Exercises]**

1. *Let A be nonsingular. Then, verify that $A^{-1}(AB)A = BA$. Hence, AB and BA are similar. Thus, $P_{AB}(t) = P_{BA}(t)$.*

2. *Fix a positive integer $k, 0 \leq k \leq n$. Now, define the function $f_k : \mathbb{M}_n(\mathbb{C}) \to \mathbb{C}$ by $f(A) =$ coefficient of t^k in $P_A(t)$. Prove that f_k is a continuous function.*

3. *For any matrix A, prove that there exists an $\epsilon > 0$ such that $A_\alpha = A + \alpha I$ is invertible, for all $\alpha \in (0, \epsilon)$. Thus, use the first part to conclude that for any given B, we have $P_{A_\alpha B}(t) = P_{BA_\alpha}(t)$, for all $\alpha \in (0, \epsilon)$.*

4. *Now, use continuity to argue that* $P_{AB}(t) = \lim\limits_{\alpha \to 0+} P_{A_\alpha B}(t) = \lim\limits_{\alpha \to 0+} P_{BA_\alpha}(t) = P_{BA}(t)$.
5. *Let* $\sigma(A) = \{\lambda_1, \ldots, \lambda_n\}$, $\sigma(B) = \{\mu_1, \ldots, \mu_n\}$ *and suppose that* $AB = BA$. *Then,*
   (a) *prove that there is a permutation* π *such that* $\sigma(A+B) = \{\lambda_1 + \mu_{\pi(1)}, \ldots, \lambda_n + \mu_{\pi(n)}\}$. *In particular,* $\sigma(A+B) \subseteq \sigma(A) + \sigma(B)$.
   (b) *if we further assume that* $\sigma(A) \cap \sigma(-B) = \emptyset$ *then the matrix* $A + B$ *is nonsingular.*
6. *Let* A *and* B *be two non-commuting matrices. Then, give an example to show that it is difficult to relate* $\sigma(A+B)$ *with* $\sigma(A)$ *and* $\sigma(B)$.
7. *Are the matrices* $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ *and* $B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ *simultaneously triangularizable?*
8. *Let* $\mathcal{F} \subseteq \mathbb{M}_n(\mathbb{C})$ *be a family of commuting normal matrices. Then, prove that each element of* $\mathcal{F}$ *is simultaneously unitarily diagonalizable.*
9. *Let* $A \in \mathbb{M}_n(\mathbb{C})$ *with* $A^* = A$ *and* $\mathbf{x}^* A \mathbf{x} \geq 0$, *for all* $x \in \mathbb{C}^n$. *Then prove that* $\sigma(A) \subseteq \mathbb{R}_+$ *and if* $\mathsf{tr(A)} = 0$, *then* $A = \mathbf{0}$.

8.3.1 Diagonalization and Real Orthogonal Matrix

Proposition 8.3.11. **[Triangularization: Real Matrix]** *Let* $A \in \mathbb{M}_n(\mathbb{R})$. *Then, there exists a real orthogonal matrix* Q *such that* $Q^T A Q$ *is block upper triangular, where each diagonal block is of size either* 1 *or* 2.

DRAFT

Proof. If all the eigenvalues of A are real then the corresponding eigenvectors have real entries and hence, one can use induction to get the result in this case (see Lemma 6.4.1).

So, now let us assume that A has a complex eigenvalue, say $\lambda = \alpha + i\beta$ with $\beta \neq 0$ and $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ as an eigenvector for λ. Thus, $A\mathbf{x} = \lambda\mathbf{x}$ and hence $A\overline{\mathbf{x}} = \overline{\lambda}\overline{\mathbf{x}}$. But, $\lambda \neq \overline{\lambda}$ as $\beta \neq 0$. Thus, the eigenvectors $\mathbf{x}, \overline{\mathbf{x}}$ are linearly independent and therefore, $\{\mathbf{u}, \mathbf{v}\}$ is a linearly independent set. By Gram-Schmidt Orthonormalization process, we get an ordered basis, say $\{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n\}$ of $\mathbb{R}^n$, where $LS(\mathbf{w}_1, \mathbf{w}_2) = LS(\mathbf{u}, \mathbf{v})$. Also, using the eigen-condition $A\mathbf{x} = \lambda\mathbf{x}$ gives

$$A\mathbf{w}_1 = a\mathbf{w}_1 + b\beta\mathbf{w}_2, \; A\mathbf{w}_2 = c\mathbf{w}_1 + d\mathbf{w}_2,$$

for some real numbers a, b, c and d.

Now, form a matrix $X = [\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_n]$. Then, X is a real orthogonal matrix and

$$\begin{aligned} X^*AX &= X^*[A\mathbf{w}_1, A\mathbf{w}_2, \ldots, A\mathbf{w}_n] = \begin{bmatrix} \mathbf{w}_1^* \\ \mathbf{w}_2^* \\ \vdots \\ \mathbf{w}_n^* \end{bmatrix} [a\mathbf{w}_1 + b\mathbf{w}_2, c\mathbf{w}_1 + d\mathbf{w}_2, \ldots, A\mathbf{w}_n] \\ &= \begin{bmatrix} a & b & \\ c & d & * \\ \multicolumn{2}{c}{\mathbf{0}} & B \end{bmatrix} \end{aligned} \tag{8.3.3}$$

where $B \in \mathbb{M}_{n-2}(\mathbb{R})$. Now, by induction hypothesis the required result follows. ∎

The next result is a direct application of Proposition 8.3.11 and hence the proof is omitted.

Corollary 8.3.12. **[Simultaneous Triangularization: Real Matrices]** *Let $\mathcal{F} \subseteq \mathbb{M}_n(\mathbb{R})$ be a commuting family. Then, there exists a real orthogonal matrix Q such that Q^TAQ is a block upper triangular matrix, where each diagonal block is of size either 1 or 2, for all $A \in \mathcal{F}$.*

Proposition 8.3.13. *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then the following statements are equivalent.*

1. *A is normal.*
2. *There exists a real orthogonal matrix Q such that $Q^TAQ = \bigoplus_i A_i$, where A_i's are real normal matrices of size either 1 or 2.*

Proof. $2 \Rightarrow 1$ is trivial. To prove $1 \Rightarrow 2$, recall that Proposition 8.3.11 gives the existence of a real orthogonal matrix Q such that Q^TAQ is upper triangular with diagonal blocks of size either 1 or 2. So, we can write

$$Q^TAQ = \left[\begin{array}{ccc|ccc} \lambda_1 & * & * & * & * & * \\ 0 & \ddots & * & * & * & * \\ 0 & \cdots & \lambda_p & * & * & * \\ \hline 0 & \cdots & 0 & A_{11} & \cdots & A_{1k} \\ 0 & \cdots & 0 & 0 & \ddots & * \\ 0 & \cdots & 0 & 0 & \cdots & A_{kk} \end{array}\right] = \begin{bmatrix} R & C \\ 0 & B \end{bmatrix} \text{ (say).}$$

As A is normal, $\begin{bmatrix} R & C \\ 0 & B \end{bmatrix}\begin{bmatrix} R^T & 0 \\ C^T & B^T \end{bmatrix} = \begin{bmatrix} R^T & 0 \\ C^T & B^T \end{bmatrix}\begin{bmatrix} R & C \\ 0 & B \end{bmatrix}$. Thus, $\mathsf{tr}(\mathsf{CC}^\mathsf{T}) = \mathsf{tr}(\mathsf{RR}^\mathsf{T} - \mathsf{R}^\mathsf{T}\mathsf{R}) = \mathsf{0}$. Now, using Exercise 8.3.10.9, we get $C = \mathbf{0}$. Hence, $RR^T = R^TR$ and therefore, R is a diagonal matrix.

As $B^TB = BB^T$, we have $\sum A_{1i}A_{1i}^T = A_{11}A_{11}^T$. So $\mathsf{tr}\left(\sum\limits_{2}^{\mathsf{k}} \mathsf{A}_{1\mathsf{i}}\mathsf{A}_{1\mathsf{i}}^\mathsf{T}\right) = \mathsf{0}$. Now, using Exercise 8.3.10.9 again, we have $\sum\limits_{2}^{k} A_{1i}A_{1i}^T = 0$ and so $A_{1i}A_{1i}^T = \mathbf{0}$, for all $i = 2, \ldots, k$. Thus, $A_{1i} = \mathbf{0}$, for all $i = 2, \ldots, k$. Hence, the required result follows. ∎

EXERCISE **8.3.14.** *Let $A \in \mathbb{M}_n(\mathbb{R})$. Then the following are true.*

1. *$A = -A^T$ if and only if A is real orthogonally similar to $[\bigoplus_j 0] \oplus [\bigoplus_i A_i]$, where $A_i = \begin{bmatrix} 0 & a_i \\ -a_i & 0 \end{bmatrix}$, for some real numbers a_i's.*
2. *$AA^T = I$ if and only if A is real orthogonally similar to $[\bigoplus_i \lambda_i] \oplus [\bigoplus_j A_j]$, where $\lambda_i = \pm 1$ and $A_j = \begin{bmatrix} \cos\theta_j & \sin\theta_j \\ -\sin\theta_j & \cos\theta_j \end{bmatrix}$, for some real numbers θ_i's.*

8.3.2 Convergent and nilpotent matrices

Definition 8.3.15. **[Convergent matrices]** A matrix A is called a **convergent matrix** if $A^m \to \mathbf{0}$ as $m \to \infty$.

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Remark 8.3.16. 1. *Let A be a diagonalizable matrix with $\rho(A) < 1$. Then, A is a convergent matrix.*

Proof. *Let $A = U^*$ diag$(\lambda_1, \ldots, \lambda_n)U$. As $\rho(A) < 1$, for each $i, 1 \le i \le n$, $\lambda_i^m \to 0$ as $m \to \infty$. Thus, $A^m = U^*$ diag$(\lambda_1^m, \ldots, \lambda_n^m)U \to \mathbf{0}$.* ∎

2. *Even if the matrix A is not diagonalizable, the above result holds. That is, whenever $\rho(A) < 1$, the matrix A is convergent. The converse is also true.*

Proof. *Let $J_k(\lambda) = \lambda I_k + N_k$ be a Jordan block of $J =$ Jordan CFA. Then as $N_k^k = 0$, for each fixed k, we have*

$$J_k(\lambda)^m = \lambda^m + C(m,1)\lambda^{m-1}N_k + \cdots + C(m,k-1)\lambda^{m-k+1}N_k^{k-1} \to 0, \text{ as } m \to \infty.$$

As $\lambda^m \to 0$ as $m \to \infty$, the matrix $J_k(\lambda)^m \to \mathbf{0}$ and hence J is convergent. Thus, A is a convergent matrix.

Conversely, if A is convergent, then J must be convergent. Thus each Jordan block $J_k(\lambda)$ must be convergent. Hence $|\lambda| < 1$. ∎

Theorem 8.3.17. [**Decomposition into Diagonalizable and Nilpotent Parts**] Let $A \in \mathbb{M}_n(\mathbb{C})$. Then $A = B + C$, where B is diagonalizable matrix and C is nilpotent such that $BC = CB$.

Proof. Let $J =$ Jordan CFA. Then, $J = D + N$, where $D =$ diag(J) and N is clearly a nilpotent matrix.

Now, note that $DN = ND$ as for each Jordan block $J_k(\lambda) = D_k + N_k$, we have $D_k = \lambda I$ and $N_k = J_k(0)$ so that $D_k N_k = N_k D_k$. As $J =$ Jordan CFA, there exists an invertible matrix S, such that $S^{-1}AS = J$. Hence, $A = SJS^{-1} = SDS^{-1} + SNS^{-1} = B + C$, which satisfy the required conditions. ∎

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Chapter 9

Appendix

9.1 Uniqueness of RREF

Definition 9.1.1. Fix $n \in \mathbb{N}$. Then, for each $f \in \mathcal{S}_n$, we associate an $n \times n$ matrix, denoted $P^f = [p_{ij}]$, such that $p_{ij} = 1$, whenver $f(j) = i$ and 0, otherwise. The matrix P^f is called the **Permutation matrix** corresponding to the permutation f. For example, I_2, corresponding to Id_2, and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_{12}$, corresponding to the permutation $(1,2)$, are the two permutation matrices of order 2×2.

Remark 9.1.2. *Recall that in Remark 9.2.16.1, it was observed that each permutation is a product of n transpositions, $(1,2), \ldots, (1,n)$.*

DRAFT

1. *Verify that the elementary matrix E_{ij} is the permutation matrix corresponding to the transposition (i,j) .*

2. *Thus, every permutation matrix is a product of elementary matrices E_{1j}, $1 \le j \le n$.*

3. *For $n = 3$, the permutation matrices are I_3, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = E_{23} = E_{12}E_{13}E_{12}$, $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_{12}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} = E_{12}E_{13}$, $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = E_{13}E_{12}$ and $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = E_{13}$.*

4. *Let $f \in \mathcal{S}_n$ and $P^f = [p_{ij}]$ be the corresponding permutation matrix. Since $p_{ij} = \delta_{i,j}$ and $\{f(1), \ldots, f(n)\} = [n]$, each entry of P^f is either 0 or 1. Furthermore, every row and column of P^f has exactly one nonzero entry. This nonzero entry is a 1 and appears at the position $p_{i,f(i)}$.*

5. *By the previous paragraph, we see that when a permutation matrix is multiplied to A*
   (a) *from left then it permutes the rows of A.*
   (b) *from right then it permutes the columns of A.*

6. *P is a permutation matrix if and only if P has exactly one 1 in each row and column.*
   Solution: *If P has exactly one 1 in each row and column, then P is a square matrix, say*

$n \times n$. Now, apply GJE to P. The occurrence of exactly one 1 in each row and column implies that these 1's are the pivots in each column. We just need to interchange rows to get it in RREF. So, we need to multiply by E_{ij}. Thus, GJE of P is I_n and P is indeed a product of E_{ij}'s. The other part has already been explained earlier.

We are now ready to prove Theorem 2.4.6.

Theorem 9.1.3. *Let A and B be two matrices in RREF. If they are row equivalent then $A = B$.*

Proof. Note that the matrix $A = \mathbf{0}$ if and only if $B = \mathbf{0}$. So, let us assume that the matrices $A, B \neq \mathbf{0}$. Also, the row-equivalence of A and B implies that there exists an invertible matrix C such that $A = CB$, where C is product of elementary matrices.

Since B is in RREF, either $B[:,1] = \mathbf{0}^T$ or $B[:,1] = (1,0,\ldots,0)^T$. If $B[:,1] = \mathbf{0}^T$ then $A[:,1] = CB[:,1] = C\mathbf{0} = \mathbf{0}$. If $B[:,1] = (1,0,\ldots,0)^T$ then $A[:,1] = CB[:,1] = C[:,1]$. As C is invertible, the first column of C cannot be the zero vector. So, $A[:,1]$ cannot be the zero vector. Further, A is in RREF implies that $A[:,1] = (1,0,\ldots,0)^T$. So, we have shown that if A and B are row-equivalent then their first columns must be the same.

Now, let us assume that the first $k-1$ columns of A and B are equal and it contains r pivotal columns. We will now show that the k-th column is also the same.

Define $A_k = [A[:,1],\ldots,A[:,k]]$ and $B_k = [B[:,1],\ldots,B[:,k]]$. Then, our assumption implies that $A[:,i] = B[:,i]$, for $1 \le i \le k-1$. Since, the first $k-1$ columns contain r pivotal columns, there exists a permutation matrix P such that

$$A_kP = \left[\begin{array}{cc|c} I_r & W & A[:,k] \\ \mathbf{0} & \mathbf{0} & \end{array}\right] \text{ and } B_kP = \left[\begin{array}{cc|c} I_r & W & B[:,k] \\ \mathbf{0} & \mathbf{0} & \end{array}\right].$$

DRAFT

If the k-th columns of A and B are pivotal columns then by definition of RREF, $A[:,k] = \begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} = B[:,k]$, where $\mathbf{0}$ is a vector of size r and $\mathbf{e}_1 = (1,0,\ldots,0)^T$. So, we need to consider two cases depending on whether both are non-pivotal or one is pivotal and the other is not.

As $A = CB$, we get $A_k = CB_k$ and

$$\left[\begin{array}{cc|c} I_r & W & A[:,k] \\ \mathbf{0} & \mathbf{0} & \end{array}\right] = A_kP = CB_kP = \left[\begin{array}{c|c} C_1 & C_2 \\ C_3 & C_4 \end{array}\right]\left[\begin{array}{cc|c} I_r & W & B[:,k] \\ \mathbf{0} & \mathbf{0} & \end{array}\right] = \left[\begin{array}{cc|c} C_1 & C_1W & CB[:,k] \\ C_3 & C_3W & \end{array}\right].$$

So, we see that $C_1 = I_r$, $C_3 = \mathbf{0}$ and $A[:,k] = \left[\begin{array}{c|c} I_r & C_2 \\ \mathbf{0} & C_4 \end{array}\right] B[:,k]$.

Case 1: Neither $A[:,k]$ nor $B[:,k]$ are pivotal. Then

$$\begin{bmatrix} X \\ \mathbf{0} \end{bmatrix} = A[:,k] = \left[\begin{array}{c|c} I_r & C_2 \\ \mathbf{0} & C_4 \end{array}\right] B[:,k] = \left[\begin{array}{c|c} I_r & C_2 \\ \mathbf{0} & C_4 \end{array}\right]\begin{bmatrix} Y \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} Y \\ \mathbf{0} \end{bmatrix}.$$

Thus, $X = Y$ and in this case the k-th columns are equal.

Case 2: $A[:,k]$ is pivotal but $B[:,k]$ in non-pivotal. Then

$$\begin{bmatrix} \mathbf{0} \\ \mathbf{e}_1 \end{bmatrix} = A[:,k] = \left[\begin{array}{c|c} I_r & C_2 \\ \mathbf{0} & C_4 \end{array}\right] B[:,k] = \left[\begin{array}{c|c} I_r & C_2 \\ \mathbf{0} & C_4 \end{array}\right]\begin{bmatrix} Y \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} Y \\ \mathbf{0} \end{bmatrix},$$

a contradiction as $\mathbf{e}_1 \neq \mathbf{0}$. Thus, this case cannot arise.

Therefore, combining both the cases, we get the required result. ∎

9.2 Permutation/Symmetric Groups

Definition 9.2.1. For a positive integer n, denote $[n] = \{1, 2, \ldots, n\}$. A function $f : A \to B$ is called

1. **one-one/injective** if $f(x) = f(y)$ for some $x, y \in A$ necessarily implies that $x = y$.
2. **onto/surjective** if for each $b \in B$ there exists $a \in A$ such that $f(a) = b$.
3. a **bijection** if f is both one-one and onto.

Example 9.2.2. Let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$ and $C = \{\alpha, \beta, \gamma\}$. Then, the function

1. $j : A \to B$ defined by $j(1) = a, j(2) = c$ and $j(3) = c$ is neither one-one nor onto.
2. $f : A \to B$ defined by $f(1) = a, f(2) = c$ and $f(3) = d$ is one-one but not onto.
3. $g : B \to C$ defined by $g(a) = \alpha, g(b) = \beta, g(c) = \alpha$ and $g(d) = \gamma$ is onto but not one-one.
4. $h : B \to A$ defined by $h(a) = 2, h(b) = 2, h(c) = 3$ and $h(d) = 1$ is onto.
5. $h \circ f : A \to A$ is a bijection.
6. $g \circ f : A \to C$ is neither one-one not onto.

Remark 9.2.3. *Let $f : A \to B$ and $g : B \to C$ be functions. Then, the* **composition** *of functions, denoted $g \circ f$, is a function from A to C defined by $(g \circ f)(a) = g(f(a))$. Also, if*

1. *f and g are one-one then $g \circ f$ is one-one.*
2. *f and g are onto then $g \circ f$ is onto.*

Thus, if f and g are bijections then so is $g \circ f$.

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Definition 9.2.4. A function $f : [n] \to [n]$ is called a **permutation** on n elements if f is a bijection. For example, $f, g : [2] \to [2]$ defined by $f(1) = 1, f(2) = 2$ and $g(1) = 2, g(2) = 1$ are permutations.

Exercise **9.2.5.** *Let $\mathcal{S}_3$ be the set consisting of all permutation on 3 elements. Then, prove that $\mathcal{S}_3$ has 6 elements. Moreover, they are one of the 6 functions given below.*

1. $f_1(1) = 1, f_1(2) = 2$ *and* $f_1(3) = 3$.
2. $f_2(1) = 1, f_2(2) = 3$ *and* $f_2(3) = 2$.
3. $f_3(1) = 2, f_3(2) = 1$ *and* $f_3(3) = 3$.
4. $f_4(1) = 2, f_4(2) = 3$ *and* $f_4(3) = 1$.
5. $f_5(1) = 3, f_5(2) = 1$ *and* $f_5(3) = 2$.
6. $f_6(1) = 3, f_6(2) = 2$ *and* $f_6(3) = 1$.

Remark 9.2.6. *Let $f : [n] \to [n]$ be a bijection. Then, the* **inverse** *of f, denote f^{-1}, is defined by $f^{-1}(m) = \ell$ whenever $f(\ell) = m$ for $m \in [n]$ is well defined and f^{-1} is a bijection. For example, in Exercise 9.2.5, note that $f_i^{-1} = f_i$, for $i = 1, 2, 3, 6$ and $f_4^{-1} = f_5$.*

Remark 9.2.7. *Let $\mathcal{S}_n = \{f : [n] \to [n] \ : \ \sigma \text{ is a permutation}\}$. Then, $\mathcal{S}_n$ has $n!$ elements and forms a* **group** *with respect to composition of functions, called* **product***, due to the following.*

1. *Let $f \in \mathbf{S}_n$. Then,*
   (a) *f can be written as* $f = \begin{pmatrix} 1 & 2 & \cdots & n \\ f(1) & f(2) & \cdots & f(n) \end{pmatrix}$, *called a* **two row notation**.
   (b) *f is one-one. Hence, $\{f(1), f(2), \ldots, f(n)\} = [n]$ and thus, $f(1) \in [n], f(2) \in [n] \setminus \{f(1)\}, \ldots$ and finally $f(n) = [n] \setminus \{f(1), \ldots, f(n-1)\}$. Therefore, there are n choices for $f(1)$, $n-1$ choices for $f(2)$ and so on. Hence, the number of elements in $\mathcal{S}_n$ equals $n(n-1)\cdots 2\cdot 1 = n!$.*

2. *By Remark 9.2.3, $f \circ g \in \mathcal{S}_n$, for any $f, g \in \mathbf{S}_n$.*

3. *Also associativity holds as $f \circ (g \circ h) = (f \circ g) \circ h$ for all functions f, g and h.*

4. *$\mathcal{S}_n$ has a special permutation called the* **identity** *permutation, denoted Id_n, such that $Id_n(i) = i$, for $1 \le i \le n$.*

5. *For each $f \in \mathcal{S}_n$, $f^{-1} \in \mathcal{S}_n$ and is called the* **inverse** *of f as $f \circ f^{-1} = f^{-1} \circ f = Id_n$.*

Lemma 9.2.8. *Fix a positive integer n. Then, the group $\mathcal{S}_n$ satisfies the following:*

1. *Fix an element $f \in \mathcal{S}_n$. Then, $\mathcal{S}_n = \{f \circ g : g \in \mathcal{S}_n\} = \{g \circ f : g \in \mathcal{S}_n\}$.*

2. *$\mathcal{S}_n = \{g^{-1} : g \in \mathcal{S}_n\}$.*

Proof. Part 1: Note that for each $\alpha \in \mathcal{S}_n$ the functions $f^{-1}\circ\alpha, \alpha\circ f^{-1} \in \mathcal{S}_n$ and $\alpha = f\circ(f^{-1}\circ\alpha)$ as well as $\alpha = (\alpha \circ f^{-1}) \circ f$.

Part 2: Note that for each $f \in \mathcal{S}_n$, by definition, $(f^{-1})^{-1} = f$. Hence the result holds. □

DRAFT

Definition 9.2.9. Let $f \in \mathcal{S}_n$. Then, the number of inversions of f, denoted $n(f)$, equals

$$\begin{aligned} n(f) &= |\,\{(i,j) :\ i < j,\ f(i) > f(j)\ \}\,| \\ &= |\,\{j : i+1 \le j \le n, f(j) < f(i)\}\,| \ \text{ using two row notation.} \end{aligned} \tag{9.2.1}$$

Example 9.2.10. 1. For $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$, $n(f) = \ |\,\{(1,2), (1,3), (2,3)\}\,|\ = 3$.

2. In Exercise 9.2.5, $n(f_5) = 2 + 0 = 2$.

3. Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 3 & 5 & 1 & 9 & 8 & 7 & 6 \end{pmatrix}$. Then, $n(f) = 3+1+1+1+0+3+2+1 = 12$.

Definition 9.2.11. [Cycle Notation] Let $f \in \mathcal{S}_n$. Suppose there exist $r, 2 \le r \le n$ and $i_1, \ldots, i_r \in [n]$ such that $f(i_1) = i_2, f(i_2) = i_3, \ldots, f(i_r) = i_1$ and $f(j) = j$ for all $j \ne i_1, \ldots, i_r$. Then, we represent such a permutation by $f = (i_1, i_2, \ldots, i_r)$ and call it an **r-cycle**. For example, $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 5 & 1 \end{pmatrix} = (1,4,5)$ and $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix} = (2,3)$.

Remark 9.2.12. 1. *One also write the r-cycle $(i_1, i_2, \ldots, i_r)$ as $(i_2, i_3, \ldots, i_r, i_1)$ and so on. For example, $(1,4,5) = (4,5,1) = (5,1,4)$.*

2. *The permutation* $f = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$ *is not a cycle.*

3. *Let* $f = (1, 3, 5, 4)$ *and* $g = (2, 4, 1)$ *be two cycles. Then, their product, denoted* $f \circ g$ *or* $(1, 3, 5, 4)(2, 4, 1)$ *equals* $(1, 2)(3, 5, 4)$. *The calculation proceeds as (the arrows indicate the images):*
   $1 \to 2$. *Note* $(f \circ g)(1) = f(g(1)) = f(2) = 2$.
   $2 \to 4 \to 1$ *as* $(f \circ g)(2) = f(g(2)) = f(4) = 1$. *So,* $(1, 2)$ *forms a cycle.*
   $3 \to 5$ *as* $(f \circ g)(3) = f(g(3)) = f(3) = 5$.
   $5 \to 4$ *as* $(f \circ g)(5) = f(g(5)) = f(5) = 4$.
   $4 \to 1 \to 3$ *as* $(f \circ g)(4) = f(g(4)) = f(1) = 3$. *So, the other cycle is* $(3, 5, 4)$.
4. *Let* $f = (1, 4, 5)$ *and* $g = (2, 4, 1)$ *be two permutations. Then,* $(1, 4, 5)(2, 4, 1) = (1, 2, 5)(4) = (1, 2, 5)$ *as* $1 \to 2, 2 \to 4 \to 5, 5 \to 1, 4 \to 1 \to 4$ *and*
   $(2, 4, 1)(1, 4, 5) = (1)(2, 4, 5) = (2, 4, 5)$ *as* $1 \to 4 \to 1, 2 \to 4, 4 \to 5, 5 \to 1 \to 2$.
5. *Even though* $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 3 & 2 & 5 & 1 \end{pmatrix}$ *is not a cycle, verify that it is a product of the cycles* $(1, 4, 5)$ *and* $(2, 3)$.

Definition 9.2.13. A permutation $f \in \mathcal{S}_n$ is called a **transposition** if there exist $m, r \in [n]$ such that $f = (m, r)$.

Remark 9.2.14. *Verify that*

1. $(2, 4, 5) = (2, 5)(2, 4) = (4, 2)(4, 5) = (5, 4)(5, 2) = (1, 2)(1, 5)(1, 4)(1, 2)$.
2. *in general, the* r*-cycle* $(i_1, \ldots, i_r) = (1, i_1)(1, i_r)(1, i_{r-1}) \cdots (1, i_2)(1, i_1)$.
3. *So, every* r*-cycle can be written as product of transpositions. Furthermore, they can be written using the* n *transpositions* $(1, 2), (1, 3), \ldots, (1, n)$.

DRAFT

With the above definitions, we state and prove two important results.

Theorem 9.2.15. *Let* $f \in \mathcal{S}_n$. *Then,* f *can be written as product of transpositions.*

Proof. Note that using use Remark 9.2.14, we just need to show that f can be written as product of disjoint cycles.

Consider the set $S = \{1, f(1), f^{(2)}(1) = (f \circ f)(1), f^{(3)}(1) = (f \circ (f \circ f))(1), \ldots\}$. As S is an infinite set and each $f^{(i)}(1) \in [n]$, there exist i, j with $0 \le i < j \le n$ such that $f^{(i)}(1) = f^{(j)}(1)$. Now, let j_1 be the least positive integer such that $f^{(i)}(1) = f^{(j_1)}(1)$, for some i, $0 \le i < j_1$. Then, we claim that $i = 0$.

For if, $i - 1 \ge 0$ then $j_1 - 1 \ge 1$ and the condition that f is one-one gives

$$f^{(i-1)}(1) = (f^{-1} \circ f^{(i)})(1) = f^{-1}\left(f^{(i)}(1)\right) = f^{-1}\left(f^{(j_1)}(1)\right) = (f^{-1} \circ f^{(j_1)})(1) = f^{(j_1-1)}(1).$$

Thus, we see that the repetition has occurred at the $(j_1 - 1)$-th instant, contradicting the assumption that j_1 was the least such positive integer. Hence, we conclude that $i = 0$. Thus, $(1, f(1), f^{(2)}(1), \ldots, f^{(j_1-1)}(1))$ is one of the cycles in f.

Now, choose $i_1 \in [n] \setminus \{1, f(1), f^{(2)}(1), \ldots, f^{(j_1-1)}(1)\}$ and proceed as above to get another cycle. Let the new cycle by $(i_1, f(i_1), \ldots, f^{(j_2-1)}(i_1))$. Then, using f is one-one follows that

$$\left\{1, f(1), f^{(2)}(1), \ldots, f^{(j_1-1)}(1)\right\} \cap \left\{i_1, f(i_1), \ldots, f^{(j_2-1)}(i_1)\right\} = \emptyset.$$

So, the above process needs to be repeated at most n times to get all the disjoint cycles. Thus, the required result follows. $\square$

Remark 9.2.16. *Note that when one writes a permutation as product of disjoint cycles, cycles of length* 1 *are suppressed so as to match Definition 9.2.11. For example, the algorithm in the proof of Theorem 9.2.15 implies*

1. *Using Remark 9.2.14.3, we see that every permutation can be written as product of the* n *transpositions* $(1,2),(1,3),\ldots,(1,n)$.
2. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 3 & 5 & 2 \end{pmatrix} = (1)(2,4,5)(3) = (2,4,5)$.
3. $\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 4 & 2 & 3 & 5 & 1 & 9 & 8 & 7 & 6 \end{pmatrix} = (1,4,5)(2)(3)(6,9)(7,8) = (1,4,5)(6,9)(7,8)$.

Note that $Id_3 = (1,2)(1,2) = (1,2)(2,3)(1,2)(1,3)$, as well. The question arises, is it possible to write Id_n as a product of odd number of transpositions? The next lemma answers this question in negative.

Lemma 9.2.17. *Suppose there exist transpositions* f_i, $1 \le i \le t$, *such that*

$$Id_n = f_1 \circ f_2 \circ \cdots \circ f_t,$$

then t *is even.*

Proof. We will prove the result by mathematical induction. Observe that $t \neq 1$ as Id_n is not a transposition. Hence, $t \geq 2$. If $t = 2$, we are done. So, let us assume that the result holds for all expressions in which the number of transpositions $t \le k$. Now, let $t = k+1$.

Suppose $f_1 = (m,r)$ and let $\ell, s \in [n] \setminus \{m,r\}$. Then, the possible choices for the composition $f_1 \circ f_2$ are $(m,r)(m,r) = Id_n$, $(m,r)(m,\ell) = (r,\ell)(r,m)$, $(m,r)(r,\ell) = (\ell,r)(\ell,m)$ and $(m,r)(\ell,s) = (\ell,s)(m,r)$. In the first case, f_1 and f_2 can be removed to obtain $Id_n = f_3 \circ f_4 \circ \cdots \circ f_t$, where the number of transpositions is $t-2 = k-1 < k$. So, by mathematical induction, $t-2$ is even and hence t is also even.

In the remaining cases, the expression for $f_1 \circ f_2$ is replaced by their counterparts to obtain another expression for Id_n. But in the new expression for Id_n, m doesn't appear in the first transposition, but appears in the second transposition. The shifting of m to the right can continue till the number of transpositions reduces by 2 (which in turn gives the result by mathematical induction). For if, the shifting of m to the right doesn't reduce the number of transpositions then m will get shifted to the right and will appear only in the right most transposition. Then, this expression for Id_n does not fix m whereas $Id_n(m) = m$. So, the later case leads us to a contradiction. Hence, the shifting of m to the right will surely lead to an expression in which the number of transpositions at some stage is $t-2 = k-1$. At this stage, one applies mathematical induction to get the required result. $\square$

Theorem 9.2.18. *Let* $f \in \mathcal{S}_n$. *If there exist transpositions* $g_1, \ldots, g_k$ *and* $h_1, \ldots, h_\ell$ *with*

$$f = g_1 \circ g_2 \circ \cdots \circ g_k = h_1 \circ h_2 \circ \cdots \circ h_\ell$$

then, either k *and* ℓ *are both even or both odd.*

DRAFT

Proof. As $g_1 \circ \cdots \circ g_k = h_1 \circ \cdots \circ h_\ell$ and $h^{-1} = h$ for any transposition $h \in \mathcal{S}_n$, we have

$$Id_n = g_1 \circ g_2 \circ \cdots \circ g_k \circ h_\ell \circ h_{\ell-1} \circ \cdots \circ h_1.$$

Hence by Lemma 9.2.17, $k + \ell$ is even. Thus, either k and ℓ are both even or both odd. □

Definition 9.2.19. [Even and Odd Permutation] A permutation $f \in \mathcal{S}_n$ is called an

1. **even permutation** if f can be written as product of even number of transpositions.
2. **odd permutation** if f can be written as a product of odd number of transpositions.

Definition 9.2.20. Observe that if f and g are both even or both odd permutations, then $f \circ g$ and $g \circ f$ are both even. Whereas, if one of them is odd and the other even then $f \circ g$ and $g \circ f$ are both odd. We use this to define a function sgn : $\mathcal{S}_n \to \{1, -1\}$, called the **signature** of a permutation, by

$$\text{sgn}(f) = \begin{cases} 1 & \text{if } f \text{ is an even permutation} \\ -1 & \text{if } f \text{ is an odd permutation} \end{cases}.$$

Example 9.2.21. Consider the set $\mathcal{S}_n$. Then,

1. by Lemma 9.2.17, Id_n is an even permutation and $\text{sgn}(Id_n) = 1$.
2. a transposition, say f, is an odd permutation and hence $\text{sgn}(f) = -1$
3. using Remark 9.2.20, $\text{sgn}(f \circ g) = \text{sgn}(f) \cdot \text{sgn}(g)$ for any two permutations $f, g \in \mathcal{S}_n$.

We are now ready to define determinant of a square matrix A.

Definition 9.2.22. Let $A = [a_{ij}]$ be an $n \times n$ matrix with complex entries. Then, the **determinant** of A, denoted $\det(A)$, is defined as

$$\det(A) = \sum_{g \in \mathcal{S}_n} \text{sgn}(g) a_{1g(1)} a_{2g(2)} \ldots a_{ng(n)} = \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(g) \prod_{i=1}^{n} a_{ig(i)}. \tag{9.2.2}$$

For example, if $\mathcal{S}_2 = \{Id, f = (1,2)\}$ then for $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$, $\det(A) = \text{sgn}(Id) \cdot a_{1Id(1)} a_{2Id(2)} + \text{sgn}(f) \cdot a_{1f(1)} a_{2f(2)} = 1 \cdot a_{11} a_{22} + (-1) a_{12} a_{21} = 1 - 4 = -3$.

Observe that $\det(A)$ is a scalar quantity. Even though the expression for $\det(A)$ seems complicated at first glance, it is very helpful in proving the results related with "properties of determinant". We will do so in the next section. As another examples, we verify that this definition also matches for 3×3 matrices. So, let $A = [a_{ij}]$ be a 3×3 matrix. Then, using Equation (9.2.2),

$$\begin{aligned} \det(A) &= \sum_{\sigma \in \mathcal{S}_n} \text{sgn}(\sigma) \prod_{i=1}^{3} a_{i\sigma(i)} \\ &= \text{sgn}(f_1) \prod_{i=1}^{3} a_{if_1(i)} + \text{sgn}(f_2) \prod_{i=1}^{3} a_{if_2(i)} + \text{sgn}(f_3) \prod_{i=1}^{3} a_{if_3(i)} + \\ &\qquad \text{sgn}(f_4) \prod_{i=1}^{3} a_{if_4(i)} + \text{sgn}(f_5) \prod_{i=1}^{3} a_{if_5(i)} + \text{sgn}(f_6) \prod_{i=1}^{3} a_{if_6(i)} \\ &= a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \end{aligned}$$

DRAFT

9.3 Properties of Determinant

Theorem 9.3.1 (Properties of Determinant)**.** *Let $A = [a_{ij}]$ be an $n \times n$ matrix.*

1. *If $A[i,:] = \mathbf{0}^T$ for some i then $\det(A) = 0$.*
2. *If $B = E_i(c)A$, for some $c \neq 0$ and $i \in [n]$ then $\det(B) = c\det(A)$.*
3. *If $B = E_{ij}A$, for some $i \neq j$ then $\det(B) = -\det(A)$.*
4. *If $A[i,:] = A[j,:]$ for some $i \neq j$ then $\det(A) = 0$.*
5. *Let B and C be two $n \times n$ matrices. If there exists $m \in [n]$ such that $B[i,:] = C[i,:] = A[i,:]$ for all $i \neq m$ and $C[m,:] = A[m,:] + B[m,:]$ then $\det(C) = \det(A) + \det(B)$.*
6. *If $B = E_{ij}(c)$, for $c \neq 0$ then $\det(B) = \det(A)$.*
7. *If A is a triangular matrix then $\det(A) = a_{11} \cdots a_{nn}$, the product of the diagonal entries.*
8. *If E is an $n \times n$ elementary matrix then $\det(EA) = \det(E)\det(A)$.*
9. *A is invertible if and only if $\det(A) \neq 0$.*
10. *If B is an $n \times n$ matrix then $\det(AB) = \det(A)\det(B)$.*
11. *If A^T denotes the transpose of the matrix A then $\det(A) = \det(A^T)$.*

DRAFT

Proof. Part 1: Note that each sum in $\det(A)$ contains one entry from each row. So, each sum has an entry from $A[i,:] = \mathbf{0}^T$. Hence, each sum in itself is zero. Thus, $\det(A) = 0$.
Part 2: By assumption, $B[k,:] = A[k,:]$ for $k \neq i$ and $B[i,:] = cA[i,:]$. So,

$$\begin{aligned}
\det(B) &= \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k \neq i} b_{k\sigma(k)} \right) b_{i\sigma(i)} = \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k \neq i} a_{k\sigma(k)} \right) c a_{i\sigma(i)} \\
&= c \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{k\sigma(k)} = c\det(A).
\end{aligned}$$

Part 3: Let $\tau = (i,j)$. Then, $\operatorname{sgn}(\tau) = -1$, by Lemma 9.2.8, $\mathcal{S}_n = \{\sigma \circ \tau : \sigma \in \mathcal{S}_n\}$ and

$$\begin{aligned}
\det(B) &= \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)} = \sum_{\sigma\circ\tau \in \mathcal{S}_n} \operatorname{sgn}(\sigma \circ \tau) \prod_{i=1}^{n} b_{i,(\sigma\circ\tau)(i)} \\
&= \sum_{\sigma\circ\tau \in \mathcal{S}_n} \operatorname{sgn}(\tau) \cdot \operatorname{sgn}(\sigma) \left(\prod_{k \neq i,j} b_{k\sigma(k)} \right) b_{i(\sigma\circ\tau)(i)} b_{j(\sigma\circ\tau)(j)} \\
&= \operatorname{sgn}(\tau) \sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k \neq i,j} b_{k\sigma(k)} \right) b_{i\sigma(j)} b_{j\sigma(i)} = -\sum_{\sigma \in \mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{k\sigma(k)} \\
&= -\det(A).
\end{aligned}$$

Part 4: As $A[i,:] = A[j,:]$, $A = E_{ij}A$. Hence, by Part 3, $\det(A) = -\det(A)$. Thus, $\det(A) = 0$.

Part 5: By assumption, $C[i,:] = B[i,:] = A[i,:]$ for $i \neq m$ and $C[m,:] = B[m,:] + A[m,:]$. So,

$$\begin{aligned}
\det(C) &= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} c_{i\sigma(i)} = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{i\neq m} c_{i\sigma(i)}\right) c_{m\sigma(m)} \\
&= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{i\neq m} c_{i\sigma(i)}\right) (a_{m\sigma(m)} + b_{m\sigma(m)}) \\
&= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i\sigma(i)} + \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} b_{i\sigma(i)} = \det(A) + \det(B).
\end{aligned}$$

Part 6: By assumption, $B[k,:] = A[k,:]$ for $k \neq i$ and $B[i,:] = A[i,:] + cA[j,:]$. So,

$$\begin{aligned}
\det(B) &= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} b_{k\sigma(k)} = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k\neq i} b_{k\sigma(k)}\right) b_{i\sigma(i)} \\
&= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k\neq i} a_{k\sigma(k)}\right) (a_{i\sigma(i)} + c a_{j\sigma(j)}) \\
&= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k\neq i} a_{k\sigma(k)}\right) a_{i\sigma(i)} + c \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \left(\prod_{k\neq i} a_{k\sigma(k)}\right) a_{j\sigma(j)}) \\
&= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma) \prod_{k=1}^{n} a_{k\sigma(k)} + c\cdot 0 = \det(A). \qquad\qquad UsePart\ 4
\end{aligned}$$

DRAFT

Part 7: Observe that if $\sigma \in \mathcal{S}_n$ and $\sigma \neq Id_n$ then $n(\sigma) \geq 1$. Thus, for every $\sigma \neq Id_n$, there exists $m \in [n]$ (depending on σ) such that $m > \sigma(m)$ or $m < \sigma(m)$. So, if A is triangular, $a_{m\sigma(m)} = 0$. So, for each $\sigma \neq Id_n$, $\prod_{i=1}^{n} a_{i\sigma(i)} = 0$. Hence, $\det(A) = \prod_{i=1}^{n} a_{ii}$. the result follows.
Part 8: Using Part 7, $\det(I_n) = 1$. By definition $E_{ij} = E_{ij}I_n$ and $E_i(c) = E_i(c)I_n$ and $E_{ij}(c) = E_{ij}(c)I_n$, for $c \neq 0$. Thus, using Parts 2, 3 and 6, we get $\det(E_i(c)) = c, \det(E_{ij}) = -1$ and $\det(E_{ij}(k)) = 1$. Also, again using Parts 2, 3 and 6, we get $\det(EA) = \det(E)\det(A)$.
Part 9: Suppose A is invertible. Then, by Theorem 2.7.1, $A = E_1 \cdots E_k$, for some elementary matrices $E_1, \ldots, E_k$. So, a repeated application of Part 8 implies $\det(A) = \det(E_1)\cdots\det(E_k) \neq 0$ as $\det(E_i) \neq 0$ for $1 \leq i \leq k$.

Now, suppose that $\det(A) \neq 0$. We need to show that A is invertible. On the contrary, assume that A is not invertible. Then, by Theorem 2.7.1, $\text{Rank}(A) < n$. So, by Proposition 2.4.9, there exist elementary matrices $E_1, \ldots, E_k$ such that $E_1 \cdots E_k A = \begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}$. Therefore, by Part 1 and a repeated application of Part 8 gives

$$\det(E_1)\cdots\det(E_k)\det(A) = \det(E_1\cdots E_k A) = \det\left(\begin{bmatrix} B \\ \mathbf{0} \end{bmatrix}\right) = 0.$$

As $\det(E_i) \neq 0$, for $1 \leq i \leq k$, we have $\det(A) = 0$, a contradiction. Thus, A is invertible.
Part 10: Let A be invertible. Then, by Theorem 2.7.1, $A = E_1 \cdots E_k$, for some elementary matrices $E_1, \ldots, E_k$. So, applying Part 8 repeatedly gives $\det(A) = \det(E_1)\cdots\det(E_k)$ and

$$\det(AB) = \det(E_1\cdots E_k B) = \det(E_1)\cdots\det(E_k)\det(B) = \det(A)\det(B).$$

In case A is not invertible, by Part 9, $\det(A) = 0$. Also, AB is not invertible (AB is invertible will imply A is invertible using the rank argument). So, again by Part 9, $\det(AB) = 0$. Thus, $\det(AB) = \det(A)\det(B)$.

Part 11: Let $B = [b_{ij}] = A^T$. Then, $b_{ij} = a_{ji}$, for $1 \le i, j \le n$. By Lemma 9.2.8, we know that $\mathcal{S}_n = \{\sigma^{-1} : \ \sigma \in \mathcal{S}_n\}$. As $\sigma \circ \sigma^{-1} = Id_n$, $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma^{-1})$. Hence,

$$\begin{aligned}\det(B) &= \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} b_{i\sigma(i)} = \sum_{\sigma\in\mathcal{S}_n} \operatorname{sgn}(\sigma)\prod_{i=1}^{n} a_{\sigma(i),i} = \sum_{\sigma^{-1}\in\mathcal{S}_n} \operatorname{sgn}(\sigma^{-1})\prod_{i=1}^{n} a_{i\sigma^{-1}(i)} \\ &= \det(A).\end{aligned}$$

□

Remark 9.3.2. *1. As $\det(A) = \det(A^T)$, we observe that in Theorem 9.3.1, the condition on "row" can be replaced by the condition on "column".*

2. Let $A = [a_{ij}]$ be a matrix satisfying $a_{1j} = 0$, for $2 \le j \le n$. Let $B = A(1 \mid 1)$, the submatrix of A obtained by removing the first row and the first column. Then $\det(A) = a_{11}\det(B)$.
Proof: *Let $\sigma \in \mathcal{S}_n$ with $\sigma(1) = 1$. Then, σ has a cycle (1). So, a disjoint cycle representation of σ only has numbers $\{2, 3, \ldots, n\}$. That is, we can think of σ as an element of $\mathcal{S}_{n-1}$. Hence,*

$$\begin{aligned}\det(A) &= \sum_{\sigma\in\mathcal{S}_n} sgn(\sigma)\prod_{i=1}^{n} a_{i\sigma(i)} = \sum_{\sigma\in\mathcal{S}_n,\sigma(1)=1} sgn(\sigma)\prod_{i=1}^{n} a_{i\sigma(i)} \\ &= a_{11}\sum_{\sigma\in\mathcal{S}_n,\sigma(1)=1} sgn(\sigma)\prod_{i=2}^{n} a_{i\sigma(i)} = a_{11}\sum_{\sigma\in\mathcal{S}_{n-1}} sgn(\sigma)\prod_{i=1}^{n-1} b_{i\sigma(i)} = a_{11}\det(B).\end{aligned}$$

We now relate this definition of determinant with the one given in Definition 2.8.1.

Theorem 9.3.3. *Let A be an $n \times n$ matrix. Then, $\det(A) = \sum\limits_{j=1}^{n} (-1)^{1+j} a_{1j}\det\big(A(1 \mid j)\big)$, where recall that $A(1 \mid j)$ is the submatrix of A obtained by removing the 1^{st} row and the j^{th} column.*

Proof. For $1 \le j \le n$, define an $n \times n$ matrix $B_j = \begin{bmatrix} 0 & 0 & \cdots & a_{1j} & \cdots & 0 \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$. Also, for each matrix B_j, we define the $n \times n$ matrix C_j by

1. $C_j[:, 1] = B_j[:, j]$,
2. $C_j[:, i] = B_j[:, i-1]$, for $2 \le i \le j$ and
3. $C_j[:, k] = B_j[:, k]$ for $k \ge j + 1$.

Also, observe that B_j's have been defined to satisfy $B_1[1, :] + \cdots + B_n[1, :] = A[1, :]$ and $B_j[i, :] = A[i, :]$ for all $i \ge 2$ and $1 \le j \le n$. Thus, by Theorem 9.3.1.5,

$$\det(A) = \sum_{j=1}^{n} \det(B_j). \tag{9.3.1}$$

Let us now compute $\det(B_j)$, for $1 \le j \le n$. Note that $C_j = E_{12}E_{23}\cdots E_{j-1,j}B_j$, for $1 \le j \le n$. Then, by Theorem 9.3.1.3, we get $\det(B_j) = (-1)^{j-1}\det(C_j)$. So, using Remark 9.3.2.2 and Theorem 9.3.1.2 and Equation (9.3.1), we have

$$\det(A) = \sum_{j=1}^{n}(-1)^{j-1}\det(C_j) = \sum_{j=1}^{n}(-1)^{j+1}a_{1j}\det\big(A(1 \mid j)\big).$$

Thus, we have shown that the determinant defined in Definition 2.8.1 is valid. $\square$

9.4 Dimension of $\mathbb{W}_1 + \mathbb{W}_2$

Theorem 9.4.1. *Let $\mathbb{V}$ be a finite dimensional vector space over $\mathbb{F}$ and let $\mathbb{W}_1$ and $\mathbb{W}_2$ be two subspaces of $\mathbb{V}$. Then,*

$$\dim(\mathbb{W}_1) + \dim(\mathbb{W}_2) = \dim(\mathbb{W}_1 + \mathbb{W}_2) + \dim(\mathbb{W}_1 \cap \mathbb{W}_2). \tag{9.4.1}$$

Proof. Since $\mathbb{W}_1 \cap \mathbb{W}_2$ is a vector subspace of V, let $\mathcal{B} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ be a basis of $\mathbb{W}_1 \cap \mathbb{W}_2$. As, $\mathbb{W}_1 \cap \mathbb{W}_2$ is a subspace of both $\mathbb{W}_1$ and $\mathbb{W}_2$, let us extend the basis $\mathcal{B}$ to form a basis $\mathcal{B}_1 = \{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s\}$ of $\mathbb{W}_1$ and a basis $\mathcal{B}_2 = \{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{w}_1, \ldots, \mathbf{w}_t\}$ of $\mathbb{W}_2$.

We now prove that $\mathcal{D} = \{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{w}_1, \ldots, \mathbf{w}_s, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ is a basis of $\mathbb{W}_1 + \mathbb{W}_2$. To do this, we show that

1. $\mathcal{D}$ is linearly independent subset of $\mathbb{V}$ and
2. $LS(\mathcal{D}) = \mathbb{W}_1 + \mathbb{W}_2$.

DRAFT

The second part can be easily verified. For the first part, consider the linear system

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_r\mathbf{u}_r + \beta_1\mathbf{w}_1 + \cdots + \beta_s\mathbf{w}_s + \gamma_1\mathbf{v}_1 + \cdots + \gamma_t\mathbf{v}_t = \mathbf{0} \tag{9.4.2}$$

in the variables α_i's, β_j's and γ_k's. We re-write the system as

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_r\mathbf{u}_r + \beta_1\mathbf{w}_1 + \cdots + \beta_s\mathbf{w}_s = -(\gamma_1\mathbf{v}_1 + \cdots + \gamma_t\mathbf{v}_t).$$

Then, $\mathbf{v} = -\sum\limits_{i=1}^{s}\gamma_i\mathbf{v}_i \in LS(\mathcal{B}_1) = \mathbb{W}_1$. Also, $\mathbf{v} = \sum\limits_{j=1}^{r}\alpha_r\mathbf{u}_r + \sum\limits_{k=1}^{T}\beta_k\mathbf{w}_k$. So, $\mathbf{v} \in LS(\mathcal{B}_2) = \mathbb{W}_2$. Hence, $\mathbf{v} \in \mathbb{W}_1 \cap \mathbb{W}_2$ and therefore, there exists scalars $\delta_1, \ldots, \delta_k$ such that $\mathbf{v} = \sum\limits_{j=1}^{r}\delta_j\mathbf{u}_j$.

Substituting this representation of $\mathbf{v}$ in Equation (9.4.2), we get

$$(\alpha_1 - \delta_1)\mathbf{u}_1 + \cdots + (\alpha_r - \delta_r)\mathbf{u}_r + \beta_1\mathbf{w}_1 + \cdots + \beta_t\mathbf{w}_t = \mathbf{0}.$$

So, using Exercise 3.4.16.1, $\alpha_i - \delta_i = 0$, for $1 \le i \le r$ and $\beta_j = 0$, for $1 \le j \le t$. Thus, the system (9.4.2) reduces to

$$\alpha_1\mathbf{u}_1 + \cdots + \alpha_k\mathbf{u}_k + \gamma_1\mathbf{v}_1 + \cdots + \gamma_r\mathbf{v}_r = \mathbf{0}$$

which has $\alpha_i = 0$ for $1 \le i \le r$ and $\gamma_j = 0$ for $1 \le j \le s$ as the only solution. Hence, we see that the linear system of Equations (9.4.2) has no nonzero solution. Therefore, the set $\mathcal{D}$ is linearly independent and the set $\mathcal{D}$ is indeed a basis of $\mathbb{W}_1 + \mathbb{W}_2$. We now count the vectors in the sets $\mathcal{B}, \mathcal{B}_1, \mathcal{B}_2$ and $\mathcal{D}$ to get the required result. $\square$

9.5 When does Norm imply Inner Product

In this section, we prove the following result. A generalization of this result to complex vector space is left as an exercise for the reader as it requires similar ideas.

Theorem 9.5.1. *Let $\mathbb{V}$ be a real vector space. A norm $\|\cdot\|$ is induced by an inner product if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$, the norm satisfies*

$$\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{y}\|^2 = 2\|\mathbf{x}\|^2 + 2\|\mathbf{y}\|^2 \qquad \text{(PARALLELOGRAM LAW)}. \tag{9.5.1}$$

Proof. Suppose that $\|\cdot\|$ is indeed induced by an inner product. Then, by Exercise 4.2.7.3 the result follows.

So, let us assume that $\|\cdot\|$ satisfies the parallelogram law. So, we need to define an inner product. We claim that the function $f : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ defined by

$$f(\mathbf{x}, \mathbf{y}) = \frac{1}{4}\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2\right), \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{V}$$

satisfies the required conditions for an inner product. So, let us proceed to do so.

STEP 1: Clearly, for each $\mathbf{x} \in \mathbb{V}$, $f(\mathbf{x}, \mathbf{0}) = 0$ and $f(\mathbf{x}, \mathbf{x}) = \frac{1}{4}\|\mathbf{x}+\mathbf{x}\|^2 = \|\mathbf{x}\|^2$. Thus, $f(\mathbf{x}, \mathbf{x}) \geq 0$. Further, $f(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

STEP 2: By definition $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$.

STEP 3: Now note that $\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 = 2\left(\|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2\right)$. Or equivalently,

$$2f(\mathbf{x}, \mathbf{y}) = \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{y}\|^2, \text{ for } \mathbf{x}, \mathbf{y} \in \mathbb{V}. \tag{9.5.2}$$

Thus, for $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{V}$, we have

$$\begin{aligned} 4\left(f(\mathbf{x}, \mathbf{y}) + f(\mathbf{z}, \mathbf{y})\right) &= \|\mathbf{x}+\mathbf{y}\|^2 - \|\mathbf{x}-\mathbf{y}\|^2 + \|\mathbf{z}+\mathbf{y}\|^2 - \|\mathbf{z}-\mathbf{y}\|^2 \\ &= 2\left(\|\mathbf{x}+\mathbf{y}\|^2 + \|\mathbf{z}+\mathbf{y}\|^2 - \|\mathbf{x}\|^2 - \|\mathbf{z}\|^2 - 2\|\mathbf{y}\|^2\right) \\ &= \|\mathbf{x}+\mathbf{z}+2\mathbf{y}\|^2 + \|\mathbf{x}-\mathbf{z}\|^2 - \left(\|\mathbf{x}+\mathbf{z}\|^2 + \|\mathbf{x}-\mathbf{z}\|^2\right) - 4\|\mathbf{y}\|^2 \\ &= \|\mathbf{x}+\mathbf{z}+2\mathbf{y}\|^2 - \|\mathbf{x}+\mathbf{z}\|^2 - \|2\mathbf{y}\|^2 \\ &= 2f(\mathbf{x}+\mathbf{z}, 2\mathbf{y}) \text{ using Equation (9.5.2).} \end{aligned} \tag{9.5.3}$$

Now, substituting $\mathbf{z} = \mathbf{0}$ in Equation (9.5.3) and using Equation (9.5.2), we get $2f(\mathbf{x}, \mathbf{y}) = f(\mathbf{x}, 2\mathbf{y})$ and hence $4f(\mathbf{x}+\mathbf{z}, \mathbf{y}) = 2f(\mathbf{x}+\mathbf{z}, 2\mathbf{y}) = 4\left(f(\mathbf{x}, \mathbf{y}) + f(\mathbf{z}, \mathbf{y})\right)$. Thus,

$$f(\mathbf{x}+\mathbf{z}, \mathbf{y}) = f(\mathbf{x}, \mathbf{y}) + f(\mathbf{z}, \mathbf{y}), \text{ for all } \mathbf{x}, \mathbf{y} \in \mathbb{V}. \tag{9.5.4}$$

STEP 4: Using Equation (9.5.4), $f(\mathbf{x}, \mathbf{y}) = f(\mathbf{y}, \mathbf{x})$ and the principle of mathematical induction, it follows that $nf(\mathbf{x}, \mathbf{y}) = f(n\mathbf{x}, \mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $n \in \mathbb{N}$. Another application of Equation (9.5.4) with $f(\mathbf{0}, \mathbf{y}) = 0$ implies that $nf(\mathbf{x}, \mathbf{y}) = f(n\mathbf{x}, \mathbf{y})$, for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $n \in \mathbb{Z}$. Also, for $m \neq 0$,

$$mf\left(\frac{n}{m}\mathbf{x}, \mathbf{y}\right) = f(m\frac{n}{m}\mathbf{x}, \mathbf{y}) = f(n\mathbf{x}, \mathbf{y}) = nf(\mathbf{x}, \mathbf{y}).$$

Hence, we see that for all $\mathbf{x}, \mathbf{y} \in \mathbb{V}$ and $a \in \mathbb{Q}$, $f(a\mathbf{x}, \mathbf{y}) = af(\mathbf{x}, \mathbf{y})$.

DRAFT

STEP 5: Fix $\mathbf{u}, \mathbf{v} \in \mathbb{V}$ and define a function $g : \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} g(x) &= f(x\mathbf{u}, \mathbf{v}) - xf(\mathbf{u}, \mathbf{v}) \\ &= \frac{1}{2}\left(\|x\mathbf{u}+\mathbf{v}\|^2 - \|x\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right) - \frac{x}{2}\left(\|\mathbf{u}+\mathbf{v}\|^2 - \|\mathbf{u}\|^2 - \|\mathbf{v}\|^2\right). \end{aligned}$$

Then, by previous step $g(x) = 0$, for all $x \in \mathbb{Q}$. So, if g is a continuous function then continuity implies $g(x) = 0$, for all $x \in \mathbb{R}$. Hence, $f(x\mathbf{u}, \mathbf{v}) = xf(\mathbf{u}, \mathbf{v})$, for all $x \in \mathbb{R}$.

Note that the second term of $g(x)$ is a constant multiple of x and hence continuous. Using a similar reason, it is enough to show that $g_1(x) = \|x\mathbf{u} + \mathbf{v}\|$, for certain fixed vectors $\mathbf{u}, \mathbf{v} \in \mathbb{V}$, is continuous. To do so, note that

$$\|x_1\mathbf{u} + \mathbf{v}\| = \|(x_1 - x_2)\mathbf{u} + x_2\mathbf{u} + \mathbf{v}\| \le \|(x_1 - x_2)\mathbf{u}\| + \|x_2\mathbf{u} + \mathbf{v}\|.$$

Thus, $\Big|\|x_1\mathbf{u} + \mathbf{v}\| - \|x_2\mathbf{u} + \mathbf{v}\|\Big| \le \|(x_1 - x_2)\mathbf{u}\|$. Hence, taking the limit as $x_1 \to x_2$, we get $\lim\limits_{x_1 \to x_2} \|x_1\mathbf{u} + \mathbf{v}\| = \|x_2\mathbf{u} + \mathbf{v}\|$.

Thus, we have proved the continuity of g and hence the prove of the required result. ■

9.6 Roots of a Polynomials

The main aim of this subsection is to prove the continuous dependence of the zeros of a polynomial on its coefficients and to recall Descartes's rule of signs.

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Definition 9.6.1. **[Jordan Curves]** A **curve** in $\mathbb{C}$ is a continuous function $f : [a, b] \to \mathbb{C}$, where $[a, b] \subseteq \mathbb{R}$.

1. If the function f is one-one on $[a, b)$ and also on $(a, b]$, then it is called a **simple curve**.
2. If $f(b) = f(a)$, then it is called a **closed curve**.
3. A closed simple curve is called a **Jordan curve**.
4. The derivative (integral) of a curve $f = u+iv$ is defined component wise. If f' is continuous on $[a, b]$, we say f is a $\mathcal{C}^1$**-curve** (at end points we consider one sided derivatives and continuity).
5. A $\mathcal{C}^1$-curve on $[a, b]$ is called a **smooth curve**, if f' is never zero on (a, b).
6. A piecewise smooth curve is called a **contour**.
7. A positively oriented simple closed curve is called a **simple closed curve** such that while traveling on it the interior of the curve always stays to the left. (Camille Jordan has proved that such a curve always divides the plane into two connected regions, one of which is called the **bounded** region and the other is called the **unbounded** region. The one which is bounded is considered as the interior of the curve.)

We state the famous Rouche Theorem of complex analysis without proof.

Theorem 9.6.2. **[Rouche's Theorem]** *Let C be a positively oriented simple closed contour. Also, let f and g be two analytic functions on R_C, the union of the interior of C and the curve C itself. Assume also that $|f(x)| > |g(x)|$, for all $x \in C$. Then, f and $f + g$ have the same number of zeros in the interior of C.*

Corollary 9.6.3. **[Alen Alexanderian, The University of Texas at Austin, USA.]** *Let* $P(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_0$ *have distinct roots* $\lambda_1, \ldots, \lambda_m$ *with multiplicities* $\alpha_1, \ldots, \alpha_m$*, respectively. Take any* $\epsilon > 0$ *for which the balls* $\overline{B_\epsilon(\lambda_i)}$ *are disjoint. Then, there exists a* $\delta > 0$ *such that the polynomial* $q(t) = t^n + a'_{n-1}t^{n-1} + \cdots + a'_0$ *has exactly* α_i *roots (counting with multiplicities) in* $B_\epsilon(\lambda_i)$*, whenever* $|a_j - a'_j| < \delta$*.*

Proof. For an $\epsilon > 0$ and $1 \leq i \leq m$, let $C_i = \{z \in \mathbb{C} : |z - \lambda_i| = \epsilon\}$. Now, for each $i, 1 \leq i \leq m$, take $\nu_i = \min\limits_{z \in C_i} |p(z)|$, $\rho_i = \max\limits_{z \in C_i}[1 + |z| + \cdots + |z|^{n-1}]$ and choose $\delta > 0$ such that $\rho_i \delta < \nu_i$. Then, for a fixed j and $z \in C_j$, we have

$$|q(z) - P(z)| = |(a'_{n-1} - a_{n-1})z^{n-1} + \cdots + (a'_0 - a_0)| \leq \delta \rho_j < \nu_j \leq |p(z)|.$$

Hence, by Rouche's theorem, $p(z)$ and $q(z)$ have the same number of zeros inside C_j, for each $j = 1, \ldots, m$. That is, the zeros of $q(t)$ are within the ϵ-neighborhood of the zeros of $P(t)$. ■

As a direct application, we obtain the following corollary.

Corollary 9.6.4. *Eigenvalues of a matrix are continuous functions of its entries.*

Proof. Follows from Corollary 9.6.3. ■

Remark 9.6.5.

1. **[Sign changes in a polynomial]** *Let* $P(x) = \sum_0^n a_i x^{n-i}$ *be a real polynomial, with* $a_0 \neq 0$*. Read the coefficient from the left* $a_0, a_1, \ldots$*. We say the* SIGN CHANGES OF a_i OCCUR AT $m_1 < m_2 < \cdots < m_k$ *to mean that* a_{m_1} *is the first after* a_0 *with sign opposite to* a_0*;* a_{m_2} *is the first after* a_{m_1} *with sign opposite to* a_{m_1}*; and so on.*

2. **[Descartes' Rule of Signs]** *Let* $P(x) = \sum_0^n a_i x^{n-i}$ *be a real polynomial. Then, the maximum number of positive roots of* $P(x) = 0$ *is the number of changes in sign of the coefficients and that the maximum number of negative roots is the number of sign changes in* $P(-x) = 0$*.*

   Proof. Assume that $a_0, a_1, \cdots, a_n$ *has* $k > 0$ *sign changes. Let* $b > 0$*. Then, the coefficients of* $(x - b)P(x)$ *are*

   $$a_0, a_1 - ba_0, a_2 - ba_1, \cdots, a_n - ba_{n-1}, -ba_n.$$

   This list has at least $k + 1$ *changes of signs. To see this, assume that* $a_0 > 0$ *and* $a_n \neq 0$*. Let the sign changes of* a_i *occur at* $m_1 < m_2 < \cdots < m_k$*. Then, setting*

   $$c_0 = a_0, c_1 = a_{m_1} - ba_{m_1 - 1}, c_2 = a_{m_2} - ba_{m_2 - 1}, \cdots, c_k = a_{m_k} - ba_{m_k - 1}, c_{k+1} = -ba_n,$$

   we see that $c_i > 0$ *when* i *is even and* $c_i < 0$*, when* i *is odd. That proves the claim.*

   Now, assume that $P(x) = 0$ *has* k *positive roots* $b_1, b_2, \cdots, b_k$*. Then,*

   $$P(x) = (x - b_1)(x - b_2) \cdots (x - b_k)Q(x),$$

   where $Q(x)$ *is a real polynomial. By the previous observation, the coefficients of* $(x - b_k)Q(x)$ *has at least one change of signs, coefficients of* $(x - b_{k-1})(x - b_k)Q(x)$ *has at least two, and so on. Thus coefficients of* $P(x)$ *has at least* k *change of signs. The rest of the proof is similar.* ■

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9.7 Variational characterizations of Hermitian Matrices

Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix. Then, by Theorem 6.4.10, we know that all the eigenvalues of A are real. So, we write $\lambda_i(A)$ to mean the i-th smallest eigenvalue of A. That is, the i-th from the left in the list $\lambda_1(A) \le \lambda_2(A) \le \cdots \le \lambda_n(A)$.

Lemma 9.7.1. **[Rayleigh-Ritz Ratio]** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix. Then,*

1. $\lambda_1(A)\mathbf{x}^*\mathbf{x} \le \mathbf{x}^*A\mathbf{x} \le \lambda_n(A)\mathbf{x}^*\mathbf{x}$, *for each* $\mathbf{x} \in \mathbb{C}^n$.
2. $\lambda_1(A) = \min\limits_{\mathbf{x}\ne\mathbf{0}} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \min\limits_{\|\mathbf{x}\|=1} \mathbf{x}^*A\mathbf{x}$.
3. $\lambda_n(A) = \max\limits_{\mathbf{x}\ne 0} \frac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}} = \max\limits_{\|\mathbf{x}\|=1} \mathbf{x}^*A\mathbf{x}$.

Proof. Proof of Part 1: By spectral theorem (see Theorem 6.4.10, there exists a unitary matrix U such that $A = UDU^*$, where $D = \operatorname{diag}(\lambda_1(A), \ldots, \lambda_n(A))$ is a real diagonal matrix. Thus, the set $\{U[:,1], \ldots, U[:,n]\}$ is a basis of $\mathbb{C}^n$. Hence, for each $\mathbf{x} \in \mathbb{C}^n$, there exists **Ans** $:_i$'s (scalar) such that $\mathbf{x} = \sum \alpha_i U[:,i]$. So, note that $\mathbf{x}^*\mathbf{x} = |\alpha_i|^2$ and

$$\lambda_1(A)\mathbf{x}^*\mathbf{x} = \lambda_1(A)\sum |\alpha_i|^2 \le \sum |\alpha_i|^2\lambda_i(A) = \mathbf{x}^*A\mathbf{x} \le \lambda_n \sum |\alpha_i|^2 = \lambda_n\mathbf{x}^*\mathbf{x}.$$

For Part 2 and Part 3, take $\mathbf{x} = U[:,1]$ and $\mathbf{x} = U(:,n)$, respectively. ∎

As an immediate corollary, we state the following result.

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Corollary 9.7.2. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and $\alpha = \dfrac{\mathbf{x}^*A\mathbf{x}}{\mathbf{x}^*\mathbf{x}}$. Then, A has an eigenvalue in the interval $(-\infty, \alpha]$ and has an eigenvalue in the interval $[\alpha, \infty)$.*

We now generalize the second and third parts of Theorem 9.7.2.

Proposition 9.7.3. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix with $A = UDU^*$, where U is a unitary matrix and D is a diagonal matrix consisting of the eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Then, for any positive integer $k, 1 \le k \le n$,*

$$\lambda_k = \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp U[:,1],\ldots,U[:,k-1]}} \mathbf{x}^*A\mathbf{x} = \max_{\substack{\|\mathbf{x}\|=1 \\ x\perp U[:,n],\ldots,U[:,k+1]}} \mathbf{x}^*A\mathbf{x}.$$

Proof. Let $\mathbf{x} \in \mathbb{C}^n$ such that $\mathbf{x}$ is orthogonal to $U[,1], \ldots, U[:,k-1]$. Then, we can write $\mathbf{x} = \sum\limits_{i=k}^{n} \alpha_i U[:,i]$, for some scalars α_i's. In that case,

$$\lambda_k\mathbf{x}^*\mathbf{x} = \lambda_k \sum_{i=k}^{n} |\alpha_i|^2 \le \sum_{i=k}^{n} |\alpha_i|^2\lambda_i = \mathbf{x}^*A\mathbf{x}$$

and the equality occurs for $\mathbf{x} = U[:,k]$. Thus, the required result follows. ∎

Theorem 9.7.4. **[Courant-Fischer]** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Then,*

$$\lambda_k = \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} \mathbf{x}^*A\mathbf{x} = \min_{\mathbf{w}_n,\ldots,\mathbf{w}_{k+1}} \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_n,\ldots,\mathbf{w}_{k+1}}} \mathbf{x}^*A\mathbf{x}.$$

Proof. Let $A = UDU^*$, where U is a unitary matrix and $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$. Now, choose a set of $k-1$ linearly independent vectors from $\mathbb{C}^n$, say $S = \{\mathbf{w}_1, \ldots, \mathbf{w}_{k-1}\}$. Then, some of the eigenvectors $\{U[:,1], \ldots, U[:,k-1]\}$ may be an element of $S^\perp$. Thus, using Proposition 9.7.3, we see that

$$\lambda_k = \min_{\substack{\|\mathbf{x}\|=1, \\ \mathbf{x}\perp U[:,1],\ldots,U[:,k-1]}} \mathbf{x}^* A\mathbf{x} \geq \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\in S^\perp}} \mathbf{x}^* A\mathbf{x}.$$

Hence, $\lambda_k \geq \max\limits_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min\limits_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} \mathbf{x}^* A\mathbf{x}$, for each choice of $k-1$ linearly independent vectors. But, by Proposition 9.7.3, the equality holds for the linearly independent set $\{U[:,1], \ldots, U[:,k-1]\}$ which proves the first equality. A similar argument gives the second equality and hence the proof is omitted. ■

Theorem 9.7.5. **[Weyl Interlacing Theorem]** *Let $A, B \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrices. Then, $\lambda_k(A) + \lambda_1(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_n(B)$. In particular, if $B = P^*P$, for some matrix P, then $\lambda_k(A+B) \geq \lambda_k(A)$. In particular, for $\mathbf{z} \in \mathbb{C}^n$, $\lambda_k(A + \mathbf{z}\mathbf{z}^*) \leq \lambda_{k+1}(A)$.*

Proof. As A and B are Hermitian matrices, the matrix $A+B$ is also Hermitian. Hence, by Courant-Fischer theorem and Lemma 9.7.1.1,

$$\begin{aligned}
\lambda_k(A+B) &= \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} \mathbf{x}^*(A+B)\mathbf{x} \\
&\leq \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} [\mathbf{x}^* A\mathbf{x} + \lambda_n(B)] = \lambda_k(A) + \lambda_n(B)
\end{aligned}$$

DRAFT

and

$$\begin{aligned}
\lambda_k(A+B) &= \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} \mathbf{x}^*(A+B)\mathbf{x} \\
&\geq \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} [\mathbf{x}^* A\mathbf{x} + \lambda_1(B)] = \lambda_k(A) + \lambda_1(B).
\end{aligned}$$

If $B = P^*P$, then $\lambda_1(B) = \min\limits_{\|\mathbf{x}\|=1} \mathbf{x}^*(P^*P)\mathbf{x} = \min\limits_{\|\mathbf{x}\|=1} \|P\mathbf{x}\|^2 \geq 0$. Thus,

$$\lambda_k(A+B) \geq \lambda_k(A) + \lambda_1(B) \geq \lambda_k(A).$$

In particular, for $\mathbf{z} \in \mathbb{C}^n$, we have

$$\begin{aligned}
\lambda_k(A + \mathbf{z}\mathbf{z}^*) &= \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}}} [\mathbf{x}^* A\mathbf{x} + |\mathbf{x}^*\mathbf{z}|^2] \\
&\leq \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1},\mathbf{z}}} [\mathbf{x}^* A\mathbf{x} + |\mathbf{x}^*\mathbf{z}|^2] \\
&= \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1}} \min_{\substack{\|x\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1},\mathbf{z}}} \mathbf{x}^* A\mathbf{x} \\
&\leq \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{k-1},\mathbf{w}_k} \min_{\substack{\|x\|=1 \\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{k-1},\mathbf{w}_k}} \mathbf{x}^* A\mathbf{x} = \lambda_{k+1}(A).
\end{aligned}$$

■

Theorem 9.7.6. **[Cauchy Interlacing Theorem]** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix. Define $\hat{A} = \begin{bmatrix} A & \mathbf{y} \\ \mathbf{y}^* & a \end{bmatrix}$, for some $a \in \mathbb{R}$ and $\mathbf{y} \in \mathbb{C}^n$. Then,*

$$\lambda_k(\hat{A}) \le \lambda_k(A) \le \lambda_{k+1}(\hat{A}).$$

Proof. Note that

$$\begin{aligned} \lambda_{k+1}(\hat{A}) &= \max_{\mathbf{w}_1,\ldots,\mathbf{w}_k \in \mathbb{C}^{n+1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp \mathbf{w}_1,\ldots,\mathbf{w}_k}} \mathbf{x}^*\hat{A}\mathbf{x} \le \max_{\mathbf{w}_1,\ldots,\mathbf{w}_k \in \mathbb{C}^{n+1}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp \mathbf{w}_1,\ldots,\mathbf{w}_k}} \ \min_{\mathbf{x}_{n+1}=0} \mathbf{x}^*\hat{A}\mathbf{x} \\ &= \max_{\mathbf{w}_1,\ldots,\mathbf{w}_k \in \mathbb{C}^{n}} \min_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp \mathbf{w}_1,\ldots,\mathbf{w}_k}} \mathbf{x}^*A\mathbf{x} = \lambda_{k+1}(A) \end{aligned}$$

and

$$\begin{aligned} \lambda_{k+1}(\hat{A}) &= \min_{\mathbf{w}_1,\ldots,\mathbf{w}_{n-k} \in \mathbb{C}^{n+1}} \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp \mathbf{w}_1,\ldots,\mathbf{w}_{n-k}}} \mathbf{x}^*\hat{A}\mathbf{x} \ge \min_{\mathbf{w}_1,\ldots,\mathbf{w}_{n-k} \in \mathbb{C}^{n+1}} \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp \mathbf{w}_1,\ldots,\mathbf{w}_{n-k}}} \ \max_{\mathbf{x}_{n+1}=0} \mathbf{x}^*\hat{A}\mathbf{x} \\ &= \min_{\mathbf{w}_1,\ldots,\mathbf{w}_{n-k} \in \mathbb{C}^{n}} \max_{\substack{\|\mathbf{x}\|=1 \\ \mathbf{x}\perp \mathbf{w}_1,\ldots,\mathbf{w}_{n-k}}} \mathbf{x}^*A\mathbf{x} = \lambda_k(A) \end{aligned}$$

∎

As an immediate corollary, one has the following result.

Corollary 9.7.7. **[Inclusion principle]** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and r be a positive integer with $1 \le r \le n$. If $B_{r\times r}$ is a principal submatrix of A then, $\lambda_k(A) \le \lambda_k(B) \le \lambda_{k+n-r}(A)$.*

DRAFT

Theorem 9.7.8. **[Poincare Separation Theorem]** *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\} \subseteq \mathbb{C}^n$ be an orthonormal set for some positive integer $r, 1 \le r \le n$. If further $B = [b_{ij}]$ is an $r \times r$ matrix with $b_{ij} = \mathbf{u}_i^* A \mathbf{u}_j$, $1 \le i, j \le r$ then, $\lambda_k(A) \le \lambda_k(B) \le \lambda_{k+n-r}(A)$.*

Proof. Let us extend the orthonormal set $\{\mathbf{u}_1, \ldots, \mathbf{u}_r\}$ to an orthonormal basis, say $\{\mathbf{u}_1, \ldots, \mathbf{u}_n\}$ of $\mathbb{C}^n$ and write $U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_n \end{bmatrix}$. Then, B is a $r \times r$ principal submatrix of U^*AU. Thus, by inclusion principle, $\lambda_k(U^*AU) \le \lambda_k(B) \le \lambda_{k+n-r}(U^*AU)$. But, we know that $\sigma(U^*AU) = \sigma(A)$ and hence the required result follows. ∎

The proof of the next result is left for the reader.

Corollary 9.7.9. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and r be a positive integer with $1 \le r \le n$. Then,*

$$\lambda_1(A) + \cdots + \lambda_r(A) = \min_{U^*U=I_r} \mathsf{tr} U^*AU \quad \text{and} \quad \lambda_{n-r+1}(A) + \cdots + \lambda_n(A) = \max_{U^*U=I_r} \mathsf{tr} U^*AU.$$

Corollary 9.7.10. *Let $A \in \mathbb{M}_n(\mathbb{C})$ be a Hermitian matrix and W be a k-dimensional subspace of $\mathbb{C}^n$. Suppose, there exists a real number c such that $\mathbf{x}^*A\mathbf{x} \ge c\mathbf{x}^*\mathbf{x}$, for each $\mathbf{x} \in W$. Then, $\lambda_{n-k+1}(A) \ge c$. In particular, if $\mathbf{x}^*A\mathbf{x} > 0$, for each nonzero $x \in W$, then $\lambda_{n-k+1} > 0$. $\Bigg($Note that, a k-dimensional subspace need not contain an eigenvector of A. For example, the line $y = 2x$ does not contain an eigenvector of $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.$\Bigg)$*

Proof. Let $\{\mathbf{x}_1, \ldots, \mathbf{x}_{n-k}\}$ be a basis of $W^{\perp}$. Then,

$$\lambda_{n-k+1}(A) = \max_{\mathbf{w}_1,\ldots,\mathbf{w}_{n-k}} \min_{\substack{\|\mathbf{x}\|=1\\ \mathbf{x}\perp\mathbf{w}_1,\ldots,\mathbf{w}_{n-k}}} \mathbf{x}^*A\mathbf{x} \geq \min_{\substack{\|\mathbf{x}\|=1\\ \mathbf{x}\perp\mathbf{x}_1,\ldots,\mathbf{x}_{n-k}}} \mathbf{x}^*A\mathbf{x} \geq c.$$

Now assume that $\mathbf{x}^*A\mathbf{x} > 0$ holds for each nonzero $\mathbf{x} \in W$ and that $\lambda_{n-k+1} = 0$. Then, it follows that $\min\limits_{\substack{\|\mathbf{x}\|=1\\ \mathbf{x}\perp\mathbf{x}_1,\ldots,\mathbf{x}_{n-k}}} x^*Ax = 0$. Now, define $f : \mathbb{C}^n \to \mathbb{C}$ by $f(\mathbf{x}) = \mathbf{x}^*A\mathbf{x}$.

Then, f is a continuous function and $\min\limits_{\substack{\|\mathbf{x}\|=1\\ \mathbf{x}\in W}} f(\mathbf{x}) = 0$. Thus, f must attain its bound on the unit sphere. That is, there exists $\mathbf{y} \in W$ with $\|\mathbf{y}\| = 1$ such that $\mathbf{y}^*A\mathbf{y} = 0$, a contradiction. Thus, the required result follows. ∎

DRAFT

Index

DRAFT

DRAFT

DRAFT

DRAFT

DRAFT