Many-Valued Logics

In developing many-valued logics, we reject the assumption that there are only two truth values—and we explore the possibility that some sentences may be neither true nor false. (Actually it turns out that there are many possibilities because there are many many-valued logics, in fact, infinitely many!) Among the reasons that have been given historically for rejecting the two-valuedness assumption are the beliefs that statements about the future, statements involving vague predicates, or statements about quantum-mechanical properties are not always either true or false. Most many-valued logics begin by rejecting the law of excluded middle $A \lor \neg A$, though there are exceptions. The number of values ranges from three to various infinite sets. The interpretations of the further values vary widely from author to author, as do the motivations for introducing the additional values.

Emil Post was one of the first to study many-valued logics, but his motivation seems to have been entirely formal. The other major founder of many-valued logic was Lukasiewicz. He sketched the idea of a many-valued logic in 1920 and published a systematic account in 1930. (Both are reprinted in Borkowski, 1970.) Unlike Post, Lukasiewicz introduced three-valued logic for philosophical reasons, to provide a more appropriate representation for the indeterminacy of the future. He apparently was led to this both by a historical concern, studying Aristotle’s discussion of necessity, particularly his sea battle example, and by a quite contemporary concern about how to accommodate the indeterminism of modern physics within logic.

Aristotle’s sea battle argument is as follows:

1. If there will be a sea battle tomorrow, then necessarily there will be a sea battle tomorrow.
2. If there will not be a sea battle tomorrow, then necessarily there will not be a sea battle tomorrow.
3. Either there will or there will not be a sea battle tomorrow.
4. Therefore, either there will necessarily be a sea battle tomorrow or there will necessarily not be a sea battle tomorrow.

Aristotle suggested that the third premise—the law of excluded middle, $A \lor \neg A$—should be rejected when $A$ is a statement about a future contingency. Thus the motivation, if not the details, of many-valued logic is as ancient as the study of logic itself. Lukasiewicz developed this idea into a systematic logic.
In all of his later work, Lukasiewicz used 1 for truth, 0 for falsity, and intermediate values for other truth values. Most but not all writers use this convention. Of course it is one thing to decide that ½ is your third truth value and another thing to give a philosophical explanation of it. For Lukasiewicz the intermediate value is “indeterminate”. Given this understanding, the most natural three-valued generalization of the two-valued truth tables is the following, in which negation reverses the truth value.

<table>
<thead>
<tr>
<th>A</th>
<th>~A</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>½</td>
<td>½</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

Conjunction takes the minimum value of the conjuncts, while disjunction takes the maximum value of the disjuncts.

<table>
<thead>
<tr>
<th>A &amp; B</th>
<th>1</th>
<th>½</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>½</td>
<td>0</td>
</tr>
<tr>
<td>½</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>A v B</th>
<th>1</th>
<th>½</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>½</td>
<td>1</td>
<td>½</td>
<td>½</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>½</td>
<td>0</td>
</tr>
</tbody>
</table>

For example, the conjunction of a true sentence and an indeterminate sentence would seem to be indeterminate. It could become true if the indeterminacy were resolved in favor of truth or false if the indeterminacy were resolved in favor of falsity.

Note that when all the components of a sentence formed from these connectives are assigned value ½ the entire sentence has value ½. If we introduce the conditional as ~A v B, as is often done in two-valued logic, then the conditionals would also have this property and there would be no sentences which are logical truths. More specifically, since the identification of the conditional with ~A v B makes A ⊃ A equivalent to the law of excluded middle, A ⊃ A would not be a logical truth.

Instead of using that traditional, oft questioned, equivalence, Lukasiewicz defined the conditional thus.

<table>
<thead>
<tr>
<th>A ⊃ B</th>
<th>1</th>
<th>½</th>
<th>0</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>½</td>
<td>0</td>
</tr>
<tr>
<td>½</td>
<td>1</td>
<td>1</td>
<td>½</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>
One way of describing this table is that the conditional is false only in the case of \( T \supset F \) and is indeterminate only in two cases—\( T \supset I \) and \( I \supset F \). A rationale for these choices is that if A were true and B indeterminate, then the conditional \( A \supset B \) could be true if B were to be true and false if B were to be false. The choice of the value 1 when both constituents have value ½ is required if \( A \supset A \) is to be logically true.

Equivalence can be defined as usual: \( A \equiv B \) iff \( (A \supset B) \land (B \supset A) \). (In Łukasiewicz’ presentation of his system, he used only negation and the conditional, having noted that \( A \lor B \) can be defined as \( (A \supset B) \supset B \) and then \( A \land B \) can be defined by using the usual DeMorgan’s principle.)

**Exercise 1.** The truth table for the biconditional is specified by the definition above. Give a simpler, more direct explanation of the truth table.

In two-valued logic we define a sentence to be logically true iff it is true in all interpretations. When we have more than two truth values, we must indicate which subset of the values are the designated values, i.e., those that are truth-like. Our definition now becomes: A is a **logical truth** iff it has a designated value in all interpretations.

Since Łukasiewicz’ motivation was to deny excluded middle, he chose only 1 as a designated value. This achieves the purpose of rendering excluded middle not a logical truth. It has one somewhat counterintuitive consequence though, which is that under an interpretation in which both constituents are assigned value ½, \( A \land \neg A \) has the same truth value as \( A \lor \neg A \). Issues of the indeterminacy of the future are now generally studied within the framework of tense logic. Aristotle’s argument is generally regarded as fallacious, but Łukasiewicz’ innovations have opened the possibilities for a variety of other systems and ideas.

**Finite-valued systems with more than three values**

The Łukasiewicz three-valued generalization can be systematically carried further. The n-valued generalization consists of taking the values \( i/n-1 \) for \( 0 \leq i \leq n-1 \). For example, four-valued logic would have the values 0, 1/3, 2/3, and 1. Conjunction will take the minimum value of the conjuncts, and disjunction the maximum value of the disjuncts. The value of a negation is 1 minus the value of the sentence being negated. For the conditional \( A \supset B \) we have two cases, using \( V(A) \) for the value of A:

\[
V(A \supset B) = \begin{cases} 1, & \text{if } V(A) \text{ is less than or equal to } V(B), \\ [1 - V(A)] + V(B), & \text{otherwise.} \end{cases}
\]

In all of the Łukasiewicz systems the only designated value is 1. Excluded middle will not be logically true in any of these systems, though in the even-valued systems excluded middle is always truer than the contradiction \( A \land \neg A \). Systems with more than one designated value were mentioned by Post, and this variation on Łukasiewicz systems was studied by Slupecki and others.
Four-valued logic was proposed for modal logic, the values being “necessarily true”, “contingently true”, “contingently false”, and “necessarily false”. The Lukasiewicz definitions of the usual connectives can be used and a modal operator added. While these truth tables have some uses, they have been superseded by the possible worlds approach to modal logic.

**Infinite-valued systems**

The Lukasiewicz n-valued generalization can be systematically carried further still—Lukasiewicz also studied the cases where the set of truth values consists of all rational numbers (fractions) in the interval [0,1] and where the values consist of all real numbers (infinite decimal expansions) in the same interval. Conjunction, disjunction, negation, and the conditional (with the two cases) are the same as for finite-valued systems with more than three values. The set of logical truths in the three-valued logic is a subset of those in our traditional logic; if a sentence can be shown to be false in a two-valued interpretation, then that interpretation also works in the three-valued system.

One question to ask is whether all these values make a difference. We already know part of the answer. A v ~A is a two-valued logical truth but not a three-valued one. But this does not give us an answer for the other systems. Before addressing this question, let’s generalize our observation about excluded middle. If we think about the truth tables for ~, &, and v, we can see that when the input value(s) are ½, the output value is always ½. Thus a simple recursive proof (that we won’t bother giving) shows that, in an assignment in which all atoms receive value ½, any sentence without conditionals and biconditionals is not a logical truth, because it receives value ½ on that assignment.

**Theorem:** All logical truths of Lukasiewicz systems contain conditionals or biconditionals.

This means that if we are looking for sentences that discriminate between the n-valued systems then we need to look at conditionals or biconditionals. One candidate for a form of sentence is [A ⊃ B] v [B ⊃ C]. This is a two-valued logical truth but not a three-valued one, because we can assign the values 1, ½, and 0 to the respective atoms and give the sentence a value of ½. Generalizing, sentences of the form

\[ [A_1 \supset A_2] \lor [A_2 \supset A_3] \lor [A_3 \supset A_4] \lor \ldots \lor [A_n \supset A_{n+1}] \]

will be logical truths in an n-valued system but not in a system with at least n+1 values.

**Example.** Show whether the following sentences are three-valued truths. If so, are they n-valued truths for all finite n?

\((A \land \neg B) \supset \neg(A \supset B)\)
Solution. This is not a logical truth in any system with more than two values. If we assign both A and B the same intermediate value, then \( \sim B \) will have an intermediate value and so \((A \& \sim B)\) will have an intermediate value. But since A and B have the same value, \((A \supset B)\) will have the value 1 and \(\sim(A \supset B)\) will have the value 0. So, the whole conditional will have an intermediate value, not 1.

\[\[(A \supset C) \& (B \supset C)\] \supset [(A \lor B) \supset C]\]

Solution. We will give an argument that this sentence is true for all finite-valued logics. If the values of A and B are both less than or equal to that of C, then the right conditional has value 1 and consequently the whole sentence does too. Thus it can only be less than 1 if A or B has value greater than C. Suppose A is greater than C and greater than or equal to B. Then Max(A, B) = A and the right conditional has the same value as \((A \supset C)\). But since \(A > B\), \(1-A < 1-B\), and so the value of the left conditional will also be the value of \((A \supset C)\) because that is the minimum value on that side.

Exercise 2. All of these sentences are two-valued logical truths; show which are three-valued truths. For three-valued truths, show which are n-valued truths for all finite n.

1. \(~\sim A \equiv A\)
2. \((A \supset B) \supset (\sim A \lor B)\)
3. \((\sim A \lor B) \supset (A \supset B)\)
4. \(~(A \lor B) \equiv (\sim A \& \sim B)\)
5. \(A \supset (\sim A \supset B)\)
6. \(A \supset [(A \supset B) \supset B]\)
7. \((A \& \sim A) \supset B\)
8. \([A \supset (B \lor C)] \supset [(A \supset B) \lor C]\)
9. \((A \equiv B) \lor (A \equiv C) \lor (A \equiv D) \lor (B \equiv C) \lor (B \equiv D) \lor (C \equiv D)\)
10. \([A \lor B] \& \sim A \supset B\)

References: