DIFFRACTION OF WAVES ON SQUARE LATTICE BY SEMI-INFINITE CRACK

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Abstract. In this paper, a discrete analogue of Sommerfeld half plane diffraction is investigated. The two-dimensional problem of diffraction on a square lattice, of a plane (transverse) wave by a semi-infinite crack, is solved. The discrete Wiener–Hopf method has been used to obtain the exact solution of discrete Helmholtz equation, with input data prescribed on the crack boundary sites due to a time harmonic incident wave. It is established that there exists a unique saddle point for the diffraction integral and its properties are characterized. An asymptotic approximation of the solution in far field is provided and, for some values of the frequency it is compared with numerical solution, of the diffraction problem, using a finite grid. A low frequency approximation of the solution in integral form recovers the classical continuum solution. At sufficiently large frequency in pass band, the effects due to discreteness and anisotropy appear. The analysis is relevant to a 5-point discretization based numerical solution of the two-dimensional Helmholtz equation. Applications include scattering of an $H$-polarized electromagnetic wave by a conducting half plane, or its three dimensional acoustic equivalent.

Key words. Edge Diffraction, Semi-infinite crack, Lattice model, Discrete Wiener–Hopf method

AMS subject classifications. 78A45, 39A14, 47A40, 74S20, 74H10, 74J20, 47A68

0. Introduction. More than a century ago, the problem of diffraction of a time harmonic wave by a straight-edged semi-infinite half plane was solved by Sommerfeld [12]. Prior to this, experiments in scattering of light from a knife edge were performed by Gouy [15], and a theoretical explanation was provided by Poincaré [32], but the solution of the diffraction problem, indeed the first in the history of this problem, credits to Sommerfeld [12]. Specifically, Sommerfeld solved the two dimensional Helmholtz equation with either Dirichlet boundary condition (diffraction by ‘soft surface’) or Neumann boundary condition (diffraction by ‘hard surface’) on the half plane. It is well known that the two-dimensional problem of scattering of acoustic, or electromagnetic wave, is mathematically equivalent to that of elastic SH (horizontally polarized shear) wave in an infinite medium [13, 51]. For example, the diffraction of an acoustic wave in a fluid by a sound-reflecting (hard) half plane, is same as the SH wave diffraction by a semi-infinite crack, both belonging to the category of Neumann boundary condition. Consequently, Sommerfeld’s solution has several diverse applications.

Many methods have been applied to solve the Sommerfeld problem and its variants [31, 94, 41, 8, 55, 54, 17, 16, 43]. For instance, a method using parabolic-cylindrical coordinates, proposed a long time ago by Lamb [22], gives an alternative derivation of Sommerfeld’s solution. Out of many such systematic, and analytically powerful, methods [5, 31], the formulation of the problem as an inhomogeneous Wiener–Hopf integral equation has become a common approach [44, 51]. However, application of the Wiener–Hopf technique to realistic problems requires spatial discretization. This is further complicated by the associated numerical approximations (see, for example, [25, 11]). A simple example of spatial discretization using 5-point scheme is the diffraction due to a finite size defect in a square lattice. Several aspects of which have been analyzed and discussed in the literature (for example see [29, 35, 3]). Indeed, the problem of scattering by defects in arbitrary lattices is endowed with several distinguished contributions [25, 28], their review shall not be attempted here.
In this paper, it is shown that a discrete analogue of the Sommerfeld problem of diffraction by a semi-infinite crack, when posed on a square lattice, admits an exact solution. A discrete model of crack in a square lattice, that has been extensively used by Slepyan (see for example [40]), has been used in problem formulation. Noble’s approach for diffraction of an elastic SH wave [31] motivates the solution of this problem using discrete Wiener–Hopf method [44, 21, 13]. This amalgamation is reflected in the choice of notation. The explicit factorization of Wiener–Hopf kernel stated in this paper, a crucial step in any successful application of the Wiener–Hopf technique [31], has appeared before (see for example, [41] and [10]), though in a different form.

0.1. Outline. After a preliminary description of the model and the incident wave, a general solution of discrete Helmholtz equation is derived using the discrete Fourier transform along rows of intact lattice. The discrete Wiener–Hopf equation appears as an inhomogeneous equation between the Fourier transform of internal forces ahead of the crack and that of the crack opening displacement. The next step is a multiplicative factorization of the Wiener–Hopf kernel and additive factorization of the non-homogeneous term. The solution of discrete Wiener–Hopf equation is constructed and inversion of the discrete Fourier transform yields an integral form of the solution in lattice coordinates. Behavior of the scattered displacement in far-field is captured by employing the method of stationary phase for diffraction integrals. Existence and uniqueness of saddle point for the diffraction integral is established. A numerical solution of the problem, using a finite grid, is compared with the asymptotic approximation of the exact solution in far field. A low frequency approximation of the exact solution is discussed briefly. Concluding remarks and four short appendices on accessory definitions, details, and justifications, appear in the end.

0.2. Notation. Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. The real part, $\text{Re} \ z$, of a complex number $z \in \mathbb{C}$ is denoted by $z_1 \in \mathbb{R}$ and its imaginary part, $\text{Im} \ z$, is denoted by $z_2 \in \mathbb{R}$ (so that $z = z_1 + iz_2$). $|z|$ denotes the modulus and $\text{Arg} \ z$ denotes the argument (with branch cut along negative real axis) for $z \in \mathbb{C}$. Following [13], symbols $\pm 0, \pm i 0$ denote the limit of expression in context, after the substitution $\pm \varepsilon, \pm i \varepsilon$, as $\varepsilon \to 0$ ($\varepsilon > 0$). If $f$ is a differentiable, real or complex, function, then $f'$ denotes the derivative of $f$ with respect to its argument, and $f''$ denotes the second derivative. The discrete Fourier transform of a sequence $\{u_m\}_{m \in \mathbb{Z}}$ is denoted by $\hat{u}$. The symbol $\mathbb{T}$ denotes the unit circle (as a counterclockwise contour) in complex plane. The symbol $\hat{z}$ is exclusively used throughout as a complex variable for the discrete Fourier transform. Latin letters $C_1, C_2$, etc, denote constants in expressions, inequalities, etc. The letter $H$ stands for the Heaviside function, $H(m) = 0, m < 0$ and $H(m) = 1, m \geq 0$. The signum function is denoted by $\text{sgn}$. The real valued $\cos^{-1}$ represents the inverse of cosine function, restricted to $[0, \pi]$, defined with domain $[-1, 1]$. As its extension the complex valued $\cos^{-1}$ is defined such that it has a branch cut in the complex plane running from $-\infty$ to $-1$ and $+1$ to $+\infty$. The square root function, $\sqrt{\cdot}$, has the usual branch cut in the complex plane running from $-\infty$ to $0$. The notation for relevant physical entities is described in the main text, the nature of which, real or complex, is clear from the context.

1. Square Lattice Model. Consider an infinite square lattice, denoted by $\mathcal{S}$, of identical particles of mass $M$. Let the displacement of a particle in $\mathcal{S}$, indexed by its lattice coordinates $(x, y) \in \mathbb{Z}^2$, be denoted by $u_{x,y}$. Due to nature of the problem, $u_{x,y}$ takes values in $\mathbb{C}$. It is assumed that each particle is connected with its four...
nearest neighbors in $\mathcal{S}$ by linearly elastic identical (massless) bonds. The bonds have a spring constant $K$ and an equilibrium length $b$ (see Fig. 1.1).

\[ y=-1 \quad y=0 \quad y=1 \quad y=-3 \quad y=2 \]

\[ x=-3 \quad x=-2 \quad x=-1 \quad x=0 \quad x=1 \quad x=2 \quad x=3 \]

\[ y=-2 \quad x=-4 \]

$<$ $\Theta$

$b$

Crack

IncidentWave

ReflectedWave

DiffractedWave

**Fig. 1.1.** Square lattice $\mathcal{S}$ with a semi-infinite crack between $y = 0$ and $y = -1$ is shown schematically. Intact lattice is shown as black dots and the particles located at crack boundary $\Sigma$ (lacking a nearest neighbor bond) are shown as empty dots. Important regions (1), (2), and (3) in edge diffraction (see the paragraph preceding (2.3)) are also shown schematically.

In the context of three dimensional physical space, this model represents a square sub-lattice of particles arranged in a parent cubic lattice. Each particle's displacement is anti-plane with respect to a parallel family of such square sub-lattices. During such anti-plane displacement, each particle is assumed to interact, effectively, with only the particles in its sub-lattice plane. The constant $K$ represents effective shear force per unit length for such three dimensional problems.

A semi-infinite crack is assumed to exist between two adjacent rows of particles as shown in Fig. 1.1 where the specific rows are indexed by $y = 0$ and $y = -1$. The semi-infinite crack is modeled by assuming broken bonds between particles located at $(x, 0) \in \mathbb{Z}^2$ and $(x, -1) \in \mathbb{Z}^2$ for all negative integers $x$. This model of a crack in square lattice $\mathbb{Z}^2$ has been discussed before (see for example, the monograph [40] and several other references within). In the context of three dimensional problems, the two dimensional model represents the diffraction of a time-harmonic elastic SH wave by a semi-infinite mode III crack [14].

1.1. Equation of Motion. Let $\Sigma$ denote the set of all lattice sites that index those particles in $\mathcal{S}$ lacking at least one nearest-neighbor-bond. Indeed,

\[ \Sigma = \{ (x, y) \in \mathbb{Z}^2 : x < 0, y = 0 \text{ or } -1 \}, \quad (1.1) \]

so that it can be interpreted as the crack face (Fig. 1.1). The equation of motion of intact lattice, while suppressing the explicit dependence of $u$ on time $t$, is

\[ M \frac{d^2}{dt^2} u_{x,y} = K \Delta u_{x,y}, \quad (x, y) \in \mathbb{Z}^2 \setminus \Sigma, \quad (1.2) \]

where $\Delta u_{x,y} := u_{x+1,y} + u_{x-1,y} + u_{x,y+1} + u_{x,y-1} - 4u_{x,y}, \quad (x, y) \in \mathbb{Z}^2. \quad (1.3)$

$\Delta$ is called the discrete (5-point) Laplacian [23]. The equation of motion for particles on the crack face $\Sigma$ is analogous to (1.2) and is derived by taking into account the broken bond between $(x, 0)$ and $(x, -1)$ for all integers $x < 0$. Motivated by the three-dimensional context, let

\[ M = \rho b^2, K = \mu b, \tilde{c} = c_s, \quad \text{where} \quad c_s = \sqrt{\mu/\rho} = b \sqrt{K/M}, \quad (1.4) \]
\( \rho \) is the three dimensional mass density (assuming spacing \( b \) between parallel square sub-lattices), \( \mu \) is the linear elastic shear modulus [1], and \( c_s \) is the macroscopic shear wave speed [1]. Ignoring the decoration \( \sim \) on \( \omega \), (1.2) is expressed as

\[
\frac{d^2}{dt^2} u_{x,y} = \frac{1}{b^2} \Delta u_{x,y}, \quad (x, y) \in \mathbb{Z}^2 \setminus \Sigma. \tag{1.5}
\]

The equation of motion for particles on the crack face \( \Sigma \), in the context of definitions (1.4), is analogous to (1.5). Henceforth, the two dimensional lattice \( \mathcal{S} \) is considered, with each particle of unit mass, and, in the context of the three-dimensional model, mass density is \( 1/b^2 \) and the shear wave speed is unit. On the square lattice model described thus far, a time harmonic lattice wave is considered incident (see Fig. 1.1) and its diffraction by the crack is studied.

1.2. Incident Lattice Wave and Dispersion Relation. Suppose \( u^i \) describes the incident lattice wave with frequency \( \omega \) and a lattice wave vector \( (\kappa_x, \kappa_y) \), as the ordinary frequency (aside from a factor due to scaling of time), lies in the pass band of lattice \( \mathcal{S} \). Specifically, it is assumed that \( u^i \) is given by the expression

\[
u_{k_x, k_y} = A e^{-i(\kappa_x x + i\kappa_y y + i\omega t)}, \quad (x, y) \in \mathbb{Z}^2, \tag{1.6}
\]

where \( A \in \mathbb{C} \) is constant. By virtue of (1.5) in intact lattice, using \( u = u^i \), the triplet \( \omega, \kappa_x, \kappa_y \) must satisfy the square lattice dispersion relation, that is \( \omega^2 = \frac{1}{b^2}(\sin^2 \frac{1}{2} \kappa_x + \sin^2 \frac{1}{2} \kappa_y) \). In view of the low frequency approximation discussed later, let \( \omega \), referred to as the lattice frequency in this paper, be defined as \( \omega := b \omega \). Hence, \( \omega, \kappa_x, \) and \( \kappa_y \) satisfy the dispersion relation,

\[
\omega^2 = 4(\sin^2 \frac{1}{2} \kappa_x + \sin^2 \frac{1}{2} \kappa_y), \quad (\kappa_x, \kappa_y) \in [-\pi, \pi]^2. \tag{1.7}
\]

The lattice wave (1.6) is diffracted by the semi-infinite crack surrounded by \( \Sigma \), as shown in Fig. 1.1. In order to avoid discussion of non-decaying wavefronts, and associated technical issues, a vanishingly small amount of damping is introduced in the model, indeed a traditional choice in diffraction theory [8, 31]. With a damping coefficient \( c > 0 \), (1.5) becomes

\[
\omega = \omega_1 + i \omega_2, \quad \omega_2 > 0. \tag{1.8}
\]

Due to the assumption that \( c \ll 1 \), it follows that \( \omega_2 \ll 1 \) and (1.8) is also written as \( \omega = \omega_1 + i\theta \), where it is assumed that \( \omega_1 = b \omega \) lies in the pass band. With an abuse of notation, \( \omega \) is identified with \( \omega_1 \) and convenient statements, such as \( \omega \in (0, 2) \) or \( \omega \in (0, 2\sqrt{2}) \), repeatedly occur. For the rest of this paper, the phrase ‘zero’ damping − denotes \( \omega_2 \ll 1 \) (but note that \( \omega_2 \neq 0 \)). It is well known that, for the negative discrete Laplacian, \( -\Delta \), the spectrum is (absolutely continuous) \([0, 8] \), but there is an exceptional set \( \{0, 4, 8\} \) in \([0, 8] \) where the limiting absorption principle fails [35]. In the case, when \( \omega_2 = 0 \) and the evolution in time is not necessarily harmonic, the uniqueness of ‘steady-state’ solution is achieved by selecting the required particular
solution based on the causality principle (see [40] §3.3.2 and [5]). However, in the considered ‘zero’ damping, the selected solution is the same. Thus, the frequency pass band for $\omega$ is $[0,2\sqrt{2}]$, and in view of the interpretation of ‘zero’ damping, henceforth, it is assumed that $\omega \in [0,2\sqrt{2}] \setminus S_c$, where $S_c = \{0,2,2\sqrt{2}\}$ [35].

Due to the dispersion relation (1.7), and (1.8), $\kappa_x$ and $\kappa_y$ are also complex numbers. Let, $\kappa$, the lattice wave number of incident lattice wave $u^i$, and, $\Theta \in (-\pi,\pi]$, the angle of incidence of $u^i$ be defined by the relations

$$\kappa_x = \kappa \cos \Theta, \kappa_y = \kappa \sin \Theta, \kappa = \kappa_1 + i \kappa_2, \kappa_1 \geq 0. \tag{1.9}$$

As a consequence of (1.8), $\kappa_2 > 0$ is assumed throughout this paper. The ‘zero’ damping is, alternatively, also interpreted as $\kappa_2 < 1$ (but non zero), and therefore, $\kappa$ in (1.9) is also expressed as $\kappa = \kappa_1 + i 0$. In view of this assumption, the Sommerfeld radiation condition is not used [35]. For the purpose of schematic, or numerical, illustrations, an exaggerated positive value of $\kappa_2$ (or $\omega_2$) is used to clarify some conceptual entities, such as branch cuts, integration contours, pole, etc. Dented contours and branch cuts are avoided in illustrations.

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1.3. Discrete Helmholtz Equation. The total displacement $u^i$ of an arbitrary particle in the lattice $\mathcal{S}$ is a sum of the incident wave displacement $u^i$ and the scattered wave displacement $u^s$ (which includes the reflected wave). Henceforth, the letter $u$ is used in place of $u^s$. The total displacement field $u$ satisfies the discrete Helmholtz equation

$$-\omega^2 u^i_{x,y} = \Delta u^i_{x,y}, \quad (x,y) \in \mathbb{Z}^2, \text{where } u^i_{x,y} = u^i_{x,y} + u^s_{x,y}, (x,y) \in \mathbb{Z}^2, \tag{1.10}$$

and certain other equations on the crack face $\Sigma$, analogous to “boundary conditions” in the continuum model [31], which are described below.

2. Wiener–Hopf Equations. In this section, Wiener–Hopf equation is derived in a manner analogous to [31]. Similar to the diffraction of an elastic SH wave by a semi-infinite mode III crack in a linear elastic continuum model [31] [14], some assumptions concerning the behaviour of $u$ are required before proceeding further. In a discrete model, the crack edge condition [19] is not relevant.

2.1. Conditions For Well-Posedness. The following discussion is a discrete analogue of a part of chapter II in [31]. The geometric part of $u$, the reflected wave in region (1) of Fig. 1.1 is expected to have the form

$$u^r_{x,y} := A \exp(-i \kappa \cos \Theta + i \kappa |y| \sin \Theta), \quad (x,y) \in \mathbb{Z}^2. \tag{2.1}$$

Let the “polar” coordinates $(R, \theta)$ (with $R > 0$, $\theta \in (-\pi,\pi]$) for $(x,y) \in \mathbb{Z}^2$, be specified by the relations

$$x = R \cos \theta, y + \frac{1}{2} = R \sin \theta. \tag{2.2}$$
Suppose the diffracted waves, defined by \( u^d := u - u' \), are expressed in polar coordinates \((R, \theta)\). When \( R \) tends to infinity, the lattice \( \mathcal{S} \) is divided into three regions as shown in Fig. 1.1: (1) – consists of a diffracted wave and a reflected wave; (2) – consists of a diffracted wave minus an incident wave; (3) – consists of only a diffracted wave. As \( R \) tends to infinity, since the diffracted wave \( u^d \) is regarded as produced by a point source at the ‘edge’ of the crack, it is expected \([29]\) to behave, for any fixed \( \theta \), as shown in Fig. 1.1: (1) – consists of a diffracted wave and a reflected wave; (2) –

\[
(\cos \theta, \sin \theta)
\]

where \( \kappa, \kappa_1, \kappa_2 > 0 \) and \( C_1 \) is a constant. From these statements, it is deduced that, for any fixed \( y \in \mathbb{Z} \),

\[
(a) \ |u_{x,y}| < C_2 \exp(\kappa_2 x \cos \Theta - \kappa_2 |y| \sin \Theta) \text{ for } -\infty < x < -|y| \cot \Theta, \\
(b) \ |u_{x,y}| < C_3 \exp(-\kappa_2 (x^2 + y^2)^{1/2}) \text{ for } -|y| \cot \Theta < x < +\infty.
\]

Based on the arguments above, analogous to those stated in \([31]\), for a fixed \( y \in \mathbb{Z} \), \( |u_{x,y}| < C_1 e^{-\kappa_2 x} \) as \( x \to +\infty \) and \( |u_{x,y}| < C_2 e^{\kappa_2 x \cos \Theta} \) as \( x \to -\infty \), where \( C_1, C_2 \) are positive constants. It follows that the series \( \sum_{x=0}^{+\infty} |u_{x,y} z^{-x}| \) (resp. \( \sum_{x=-\infty}^{-1} |u_{x,y} z^{-x}| \)) is absolutely convergent, provided that \( z \) satisfies the condition \( |z| > R_+ \) (resp. \( |z| < R_- \)) where

\[
R_+ = e^{-\kappa_2}, \quad R_- = e^{\kappa_2 \cos \Theta}.
\]

Note that \( R_+ < R_- \). Therefore, \( u_{y,+}(z) \) and \( u_{y,-}(z) \) are analytic at \( z \in \mathbb{C} \) such that \( |z| > R_+ \), \( |z| < R_- \), respectively, where

\[
u_{y,+}(z) = \sum_{x=0}^{+\infty} u_{x,y} z^{-x}, \quad u_{y,-}(z) = \sum_{x=-\infty}^{-1} u_{x,y} z^{-x}, \quad (2.5)
\]

for any fixed \( y \in \mathbb{Z} \). Putting both facts together it is concluded that \( u_{y}^F := u_{y,+} + u_{y,-} \) is analytic inside an annulus in \( \mathbb{C} \), defined by

\[
A_u := \{ z \in \mathbb{C} : R_L < |z| < R_- \}.
\]

See Appendix A for more details on the definitions and notation used in \([23]\).

2.2. Application of Discrete Fourier Transform. Based on above discussion, the discrete Fourier transform \( \mathcal{F} \) of the sequence \( \{u_{x,y}\}_{x \in \mathbb{Z}} \) is well defined for all \( y \in \mathbb{Z} \). Therefore, the discrete Helmholtz equation \([1.10]\) is expressed as

\[
(\mathcal{H}(z) + 2\mathcal{F}(z)) - (u_{y+1}^F(z) + u_{y-1}^F(z)) = 0, \quad z \in A_u,
\]

for all \( y \in \mathbb{Z} \) with \( y \neq -1 \). In \((2.7)\) the complex function \( \mathcal{H} \) is defined by

\[
\mathcal{H}(z) := 2 - z - z^{-1} - \omega^2, \quad z \in \mathbb{C}, \quad \text{and further define } \mathcal{R} := \mathcal{H} + 4.
\]

Note that both functions, \( \mathcal{H} \) and \( \mathcal{R} \), are analytic on \( A_u \). Using an elementary technique of solving a second order difference equation \([24]\), the general solution of \((2.7)\) is given by the expression

\[
u_{y}^F(z) = P(z)\lambda(z)^F + Q(z)\lambda(z)^{-F}, \quad z \in A_u,
\]

where \( P, Q \) are arbitrary analytic functions on \( A \). The annulus \( A \) is defined by

\[
A := A_u \cap A_L, \quad A_L := \{ z \in \mathbb{C} : R_L < |z| < R_L^{-1} \}, \quad R_L := \max\{|z_+|, |z_-|\},
\]

where \( \mathcal{F}(z) = e^{i\omega z} \).
where \( z_h \) and \( z_r \) are defined by (2.13). In (2.9), following [40], the function \( \lambda \) is defined by

\[
\lambda(z) := \frac{1 - L(z)}{1 + L(z)}, \quad z \in \mathbb{C} \setminus \mathcal{B},
\]

(2.11)

where

\[
L(z) := \frac{\delta(z)}{r(z)}, \quad \delta(z) := \sqrt{\mathcal{H}(z)}, \quad r(z) := \sqrt{\mathcal{R}(z)}, \quad z \in \mathbb{C} \setminus \mathcal{B},
\]

(2.12)

and \( \mathcal{B} \) denotes the union of branch cuts for \( \lambda \) (see below), borne out of the chosen branch (2.14) for \( \delta \) and \( r \) such that \(|\lambda(z)| \leq 1, z \in \mathbb{C} \setminus \mathcal{B}\). By definition of \( L \), its branch

points are the zeros of \( \mathcal{H} \) and \( \mathcal{R} \) and the branch cuts follow the pattern illustrated in Fig. 2.1(b). The zeros of \( \mathcal{H} \) are \( z_h \) and \( 1/z_h \) while the zeros of \( \mathcal{R} \) are \( z_r \) and \( 1/z_r \), where the complex numbers \( z_h \) and \( z_r \) are given by

\[
z_h := \frac{1}{2}(2 - \omega^2 \pm \omega \sqrt{\omega^2 - 4}), \quad z_r := \frac{1}{2}(6 - \omega^2 \pm \sqrt{\omega^4 - 12 \omega^2 + 32}).
\]

(2.13)

In (2.13) the \( \pm \) sign, in front of the square root, is chosen such that \( z_h, z_r \) lie inside the unit circle \( \mathbb{T} \) (note that \( \omega_2 > 0 \)). See Fig. 2.1(a) for an illustration of the zeros of \( \mathcal{H} \) and \( \mathcal{R} \) as \( \omega \) varies continuously as a parameter. Indeed, as discussed in [40], with \( \omega_2 > 0 \), for all \( z \in \mathbb{C} \setminus \mathcal{B} \), the condition

\[-\pi < \text{Arg} \mathcal{H}(z) < \pi, \text{Re} \delta(z) > 0, \text{Re} r(z) > 0, \text{sgn} \text{Im} \delta(z) = \text{sgn} \text{Im} r(z), \]

(2.14)

are sufficient to conclude that \(|r(z) - \delta(z)| < |r(z) + \delta(z)|, z \in \mathbb{C} \setminus \mathcal{B} \). Note that \( r \) and \( \delta \), as well as \( \lambda \), are analytic in \( \mathcal{A}_L \), and, therefore, also in \( \mathcal{A} \subset \mathcal{A}_L \). Using (2.13) it can be shown that and, as \( \mathcal{A} \) is non-empty, in fact \( \mathcal{A}_u \subset \mathcal{A}_L \) for \( \omega_2 \ll 1 \).

For the problem of a crack in unbounded lattice \( \mathfrak{S} \), in view of \( \kappa_2 > 0 \) it is expected that the scattered displacement field decays far away from the crack, for example see (2.3). So it is assumed that \( u^F(z) \to 0 \) when \( y \to \pm \infty \) and hence, using (2.9), the solution of (2.7) is expressed as

\[
u^F (z) = \begin{cases} u^F_0 (z) \lambda^\gamma (z) & (y \geq 0) \\ u^F_{-1} (z) \lambda^{-\gamma + 1} (z) & (y \leq -1) \end{cases}, \quad z \in \mathcal{A},
\]

(2.15)

where the function \( u^F_0 \) and \( u^F_{-1} \) are unknown functions. Using the definitions (2.6) and (2.10), it is clear that above ansatz, in particular, each of the unknown functions \( u^F_0 \) and \( u^F_{-1} \), is analytic in the annulus \( \mathcal{A} \).
\[ 2.3. \text{Discrete Wiener–Hopf Equation.} \] Let the internal force, in the ‘vertical’ bonds connecting the particles at \( y = 0 \) and \( y = -1 \), in front of the crack, be defined by

\[ v_x := \frac{1}{b^2}(u_{x,-1} - u_{x,0}), \quad x \geq 0, x \in \mathbb{Z}. \]  \hfill (2.16)

The force on particle at \((x, 0) \in \mathbb{Z}^2\) due to this bond is \(v_x\), while that at \((x, -1)\) due to the same bond is \(-u_x\). The external force at \((x, 0)\) due to the relative displacement of the particles at \(y = 0\) and \(y = -1\), induced by the incident wave \((1.6)\), be specified by

\[ v^i_x := \frac{1}{b^2}(u^i_{x,-1} - u^i_{x,0}), \quad x \in \mathbb{Z}. \]  \hfill (2.17)

The corresponding force on the particle at \((x, -1)\) is \(-v^i_x\). Taking into account the broken bonds between \(y = 0\) and \(y = -1\) for \(x < 0\), and the intact nature of all other bonds, the equation that must be satisfied by \(u\) at \(y = 0\) is found to be

\[-\omega^2 u_x,0 = -b^2 v^i_x H(-1 - x) + b^2 v_x H(x) + (u_{x+1,0} + u_{x-1,0} + u_{x,1} - 3u_{x,0}), \]  \hfill (2.18)

for all \(x \in \mathbb{Z}\). Similarly, at \(y = -1\), \(u\) satisfies

\[-\omega^2 u_{x,-1} = b^2 v^i_x H(-1 - x) - b^2 v_x H(x) + (u_{x+1,-1} + u_{x-1,-1} - 3u_{x,-1}), \]  \hfill (2.19)

for all \(x \in \mathbb{Z}\). Note that the incident wave automatically satisfies \((1.10)\) ahead of the crack and the consequent external forces on these particles at \(y = 0\) and \(y = -1\), do not appear at \(x \geq 0\).

By inspection of the equations \((2.18)\) and \((2.19)\), complementing \((1.10)\), for \(y = 0\) and \(y = -1\), respectively, it is assumed at the outset, that \(u\) is skew symmetric about the crack face \(\Sigma\), i.e.,

\[ u_{x,-y} = -u_{x-y} \quad \text{for } y \geq 0, x \in \mathbb{Z}. \]  \hfill (2.20)

In particular, \(u_{x,0} = -u_{x,-1}\). For a detailed reason behind \((2.20)\), see Appendix B. As a consequence of \((2.20)\), let \(u^F\) be defined by

\[ u^i = -u_{x-1}^F =: u^F. \]  \hfill (2.21)

in \((2.18)\). Using \((2.21)\), \((2.20)\), with \(v_+(z), v^i_+(z)\), as function of \(z \in \mathbb{C}\), defined as the discrete Fourier transforms of the sequences \(\{v_x H(x)\}_{x \in \mathbb{Z}}, \{v^i_x H(-1 - x)\}_{x \in \mathbb{Z}}\), respectively (see Eqs. \((2.16)\) and \((2.17)\)), it follows that

\[ v_+(z) = \frac{1}{b^2}((H(z) + 1)u^F(z) - u^F(z)) + v^i_+(z), \quad z \in \mathbb{A}. \]  \hfill (2.22)

Using \((2.8), (2.11), (2.15)\), and substituting \(u_+(z) = -b^2 v_+(z)/2\) (as a consequence of \((2.20)\)) into \((2.22)\), it follows that

\[ -\frac{1}{2}v_+(z) + \frac{1}{b^2}L(z)u_+(z) = -\frac{1}{2}(1 - L(z))v^i_+(z), \quad z \in \mathbb{A}, \]  \hfill (2.23)

where \(L\) is given by \((2.12)\). Indeed, \((2.23)\) relates the Fourier transform \(v_+\) of internal forces ahead of the crack and that of the crack opening displacement, i.e., \(u_+\), while
the right hand side depends on the input data prescribed on the crack boundary sites due to the time harmonic incident wave $u^i$. The function $L$, analytic in $A \subset \mathcal{A}_L$, is recognized as the Wiener–Hopf kernel of the discrete Wiener–Hopf equation (2.23).

Also (2.23) can be written in terms of $u_+$ and $u_-$ only, and the resulting equation is

$$u_+(z) + L(z)u_-(z) = -\frac{1}{2}b^2(1 - L(z))v^i_1(z), \quad z \in A. \quad (2.24)$$

2.4. Wiener–Hopf Factorization and the Solution. As a major step in any application of the Wiener–Hopf technique [31, 44], the factorization of the kernel function (2.12) is required. The multiplicative factorization of $L$ is [31, 40]

$$L(z) = L_+(z)L_-(z), \quad z \in \mathcal{A}_L, \quad (2.25)$$

where the factors $L_\pm$ are given by the Cauchy projectors [30, 31, 40]. Indeed,

$$L_\pm(z) = \exp(\pm \frac{1}{2\pi i} \int_{\gamma_z} \log L(\zeta) d\zeta), \quad z \in \mathbb{C} \text{ such that } |z| \gtrless R_L^{\pm1}, \quad (2.26)$$

where $\gamma_z$ is any rectifiable, closed, counterclockwise contour that lies in the annulus of analyticity for $L$, that is $A_L$ defined by (2.10) (see Fig. 2.1b)). In (2.26), it has been implicitly assumed that $L_\pm(z) = L_\pm(z^{-1})$, which makes the representation unique. In the factorization of $L$, described by (2.25) and (2.26), the function $L_+$ (resp. $L_-$) is analytic, in fact it has neither poles nor zeros, in the exterior (resp. interior) of a disk centered at 0 in $\mathbb{C}$ with radius $R_L$ (resp. $R_L^{-1}$). This means that $1/L_+$ (resp. $1/L_-$) is analytic in the same region as $L_+$ (resp. $L_-$).

Using (2.8) in (2.12), the explicit expression [10, 41, 40] for $L_\pm$ is given by

$$L_+(z) = L_-(z^{-1}) = C_z \sqrt{\frac{1 - z_k z^{-1}}{1 - z_k z}}, \quad z \in \mathbb{C} \text{ such that } |z| > R_L, \quad (2.27)$$

with $C_z = (z_k/z_k^2)^{1/4} \in \mathbb{C}$. It follows from (2.17), (2.24), and (2.25) that

$$\frac{u_+(z)}{L_+(z)} + \frac{u_-(z)}{L_-(z)} = C(z), \quad z \in A, \quad (2.28a)$$

where $C(z) = \frac{1}{2}(z_k^{-1}L_+(z) - L_-(z))\Lambda(1 - e^{i\kappa z})\delta_{D-}(zz_p^{-1}), \quad z \in A, \quad (2.28b)$

$$z_p := e^{-i\kappa z}, \quad \text{and } \delta_{D-}(z) := \sum_{n=-\infty}^{-1} z^{-n}, \quad z \in \mathbb{C} \text{ such that } |z| < 1. \quad (2.28c)$$

Clearly, $\delta_{D-}$ is analytic inside the unit disk in $\mathbb{C}$. Note that the only singularity of $\delta_{D-}(zz_p^{-1})$ is a simple pole at $z = z_p$, which lies outside the annulus $A$ (2.10).

Therefore, an additive factorization, $C = C_+(z) + C_-(z)$, is constructed by elementary means [31] with

$$C_\pm(z) = \pm \frac{1}{2} A(1 - e^{i\kappa z})\left(\frac{1}{L_+(z)} - \frac{1}{L_{\pm}^{\pm1}(z)}\right)\delta_{D-}(zz_p^{-1}), \quad z \in A. \quad (2.29)$$

$C_+(z)$ and $C_-(z)$ are analytic at $z \in \mathbb{C}$ with $|z| > \max\{R_+, R_L\}$, $|z| < \min\{R_-, R_L^{-1}\}$, respectively ($R_\pm$ are given by (2.24)). On substitution of (2.29) in (2.28a), after
rearrangement, let
\[ J(z) := \frac{u_+(z)}{L_+(z)} - C_+(z) = -L_-(z)u_-(z) + C_-(z), \quad z \in A. \] (2.30)

The function \( J(z) \) is analytic at \( z \in \mathbb{C} \) with \(|z| > \max\{R_+, R_L\}\) and also at \( z \in \mathbb{C} \) with \(|z| < \min\{R_-, R_L^{-1}\}\), i.e., in the whole of the complex plane \( \mathbb{C} \), since the two regions overlap in the annulus \( A \). Using (2.27), (2.5), and (2.29), as \( z \to 0 \), \( L_-(z) \to C_1 \), \( u_-(z) \to 0 \), and \( C_-(z) \to 0 \), while as \( z \to \infty \), \( L_+(z) \to C_2 \), \( u_+(z) \to C_3 \) and \( C_+(z) \to C_4 \), hence, it follows that that \( J(z) \) is bounded on the complex plane and tends to zero as \( z \) tends to 0. As a consequence of the Liouville’s theorem, \( J(z) \) must be identically zero.

By the reasoning established above, the solution of the discrete Wiener–Hopf equation (2.28a), in the form of a discrete Fourier transform, is written as
\[ u^F(z) = \frac{1}{2\pi i} A C_0 \int_{\mathcal{C}} \frac{\mathcal{X}(z)}{z - z_p} \lambda^y z^x dz, \quad (x, y) \in \mathbb{Z}^2, y \geq 0, \] (2.33)

where \( \mathcal{C} \) is a rectifiable, closed, counterclockwise contour in the annulus \( \mathcal{A} \). Along with \( u_{x,y} = -u_{-x,-y-1} \), \((x, y) \in \mathbb{Z}^2, y \geq 0 \), (2.33) provides the complete solution of the diffraction problem in integral form.

3. Far Field Approximation for \( \omega > 1 \). The ‘zero’ damping case, with interpretation stated earlier, is considered, and
\[ \xi_\omega = \begin{cases} -\pi & \text{for } \omega \in (0, 2), \\ 0 & \text{for } \omega \in (2, 2\sqrt{2}) \end{cases}, \quad \xi_f = \begin{cases} +\pi & \text{for } \omega \in (0, 2), \\ 2\pi & \text{for } \omega \in (2, 2\sqrt{2}) \end{cases}. \] (3.1)

For \( \omega \in (0, 2) \), the pair \( z_\omega^{+1} \) is close to the unit circle \( \mathcal{T} \) in \( \mathbb{C} \), and is written as \( e^{\pm i\xi_\omega + \phi(\xi_\omega)} \) with \( \xi_\omega \in (0, 2\pi) \). At the same time, the pair \( z_\omega^{-1} \) is real and is expressed as \( e^{\pm i\xi_\omega} \) (\( \text{Im} \xi_\omega > 0 \)). On the other hand, for \( \omega \in (2, 2\sqrt{2}) \), the pair \( z_\omega^{+1} \) of complex zeros of \( \mathcal{H} \) also lies close to the unit circle \( \mathcal{T} \) in \( \mathbb{C} \), and is written as \( e^{\pm i(\xi_\omega + \phi(\xi_\omega))} \) with \( \xi_\omega \in (0, \pi) \), while the pair of reciprocally related zeros \( z_\omega^{-1} \), that is \( e^{\pm i\xi_\omega} \) of \( \mathcal{H} \) is real (\( \text{Im} \xi_\omega > 0 \)).

Let \( \xi_\phi := \kappa_2 \). Using the mapping \( z = e^{-i\xi} \) and polar coordinates \((R, \theta)\) for \((x, y) \in \mathbb{Z}^2 \), (2.22) is written as
\[ u_{x,y} = -\frac{1}{2\pi} A C_0 \int_{\mathcal{C}_x} \mathcal{K}(e^{-i\xi}) e^{-i\xi_\phi} \eta(\xi) e^{iR_\phi(\xi)} e^{i(\xi - \xi_0)} - d\xi, \quad (x, y) \in \mathbb{Z}^2, y \geq 0, \] (3.2)

where \( \mathcal{C}_x \) is a contour, traversed from \( \xi_\phi \) to \( \xi_f \), in the strip
\[ S = \{ \xi \in \mathbb{C} : \xi_1 \in [\xi_\phi, \xi_f], -\kappa_2 < \xi_2 < \kappa_2 \cos \Theta \}, \] (3.3)
the phase function, \( \phi \), and the complex function \( \eta \) are given by

\[
\phi(\xi) = \eta(\xi) \sin \theta - \xi \cos \theta, \quad \eta(\xi) = -i \log(\lambda(e^{-i \xi})), \quad \xi \in \mathcal{S}.
\] (3.4)

Note that \( \eta \) is well defined on \( \mathcal{S} \) because \( |\lambda(z)| < 1 \) for all \( z \) in the cut plane \( \mathbb{C} \setminus \mathcal{B} \) (Fig. 2.1(b)). See Figs. 3.2(a), 3.2(c) for a schematic illustration of \( \mathcal{C}_\xi \). Note that the strip \( \mathcal{S} \) has vanishing thickness in this 'zero' damping case, and \( \mathcal{S} \) does not intersect with the real axis in \( \mathbb{C} \) for admissible \( \Theta \in (\pi/2, \pi] \).

The essence of an integral such as that appearing in (3.2), so called **diffraction integral** (marked by the presence of a pole and stationary point that may collide), is captured by reduction to one of its contour of steepest descents [6, 14, 31]. All the integrals (marked by the presence of a pole and stationary point that may collide), is

\[
\int_{\text{contour}} \ldots
\]

on \( \mathcal{C} \) has vanishing thickness in this 'zero' damping case, and \( \mathcal{S} \) does not intersect with the real axis in \( \mathbb{C} \) for admissible \( \Theta \in (\pi/2, \pi] \).

The existence and uniqueness of saddle point follow directly by using the convexity of level sets of dispersion relation (1.7) [35], but the arguments given below are based on elementary calculus, hence included in this paper. The only relevant pole, in context of \( \mathcal{C}_\xi \), is located at \( \xi = \xi_P \) for all \( \omega \in (0, 2\omega) \), or for all \( \omega \in (2, 2\sqrt{2}) \) and admissible \( \Theta \in [0, \pi/2] \), and at \( \xi_P + 2\pi \) for all \( \omega \in (2\sqrt{2}, 3) \) admissible \( \Theta \in (\pi/2, \pi) \). Note that the diffraction integral in (3.2) is evaluated along the contour \( \mathcal{C}_\xi \) that passes ‘below’ the pole at \( \xi = \xi_P \) (see Figs. 3.2(a), 3.2(c)).

### 3.1. Saddle point

The function \( \phi \) (3.4) possesses a saddle point at \( \xi = \xi_S \) on \( \mathcal{C}_\xi \) if \( [14, 6] \)

\[
\phi'((\xi_S) = \eta'(\xi_S) \sin \theta - \cos \theta = 0,
\]

\[
\phi''((\xi_S) = \eta''((\xi_S) \sin \theta \neq 0.
\]

(3.5a)

(3.5b)

Let \( \omega \) denote the expression \( 2 - \frac{1}{\ell^2} \). Since \( \text{Im} \omega < 0 \) and \( \eta(\xi) > 0 \) (as \( |\lambda| < 1 \)), the function \( \eta \) in (3.4) is, equivalently, expressed as \( \eta(\xi) = \cos^{-1}(\omega - \cos \xi), \xi \in \mathcal{C}_\xi \), where \( \cos^{-1} \) is defined according to the branch cut mentioned in the notation. Therefore, (3.5a) is written as

\[
\frac{\sin \theta}{\sqrt{1 - (\omega - \cos \xi_S)^2}} = \cot \theta.
\] (3.6)

Simplifying further, (3.6) yields \( \cos \xi_S = \frac{1}{2}(\omega + \tau(\theta)) \), where \( \tau \) is a function defined by

\[
\tau(\theta) = \sin^{-1}\left(\sqrt{\frac{\omega^2 \sin^2 \theta + 4 \cos^2 \theta}{2}}\right), \quad \theta \in [0, \pi].
\] (3.7)

The issue of \( \pm \) sign in (3.7) is settled below (see (3.9)). Consider the case when \( \theta \in [0, \pi/2) \). The definition (3.7) of \( \tau \) implies that it is a solution of the equation \( (\tau^2 + \omega^2 - 4)\chi - 2\omega \tau = 0 \), where \( \chi = \cos 2 \theta \) for fixed \( \theta \). Since \( d\chi/d\tau = -2\omega(\tau^2 + 4 - \omega^2)/(\tau^2 + \omega^2 - 4)^2 \) for \( |\tau| \neq \sqrt{4 - \omega^2} \) and \( |\tau| \leq 2 \) for \( \omega \in (0, 2\sqrt{2}) \), it follows that \( \text{sgn } d\chi/d\tau = \text{sgn } \omega \). In other words, \( \chi \) is monotone with respect to \( \tau \) for fixed \( \omega \) such that \( |\omega| \leq 2 \). For \( \theta \in [0, \pi/2) \), \( \chi(\theta) = \cos 2 \theta \) is a monotone function of \( \theta \), hence, it is follows that \( \tau(\theta) \) is a monotone function of \( \theta \) as well. Clearly, the same conclusion holds when \( \theta \in (\pi/2, \pi] \) since the last argument depends on the monotonicity of \( \chi \). These arguments establish the following Lemma.

**Lemma 3.1.** For \( \omega \in (0, 2\sqrt{2}) \), \( \tau \), specified by (3.7), is a monotone function of \( \theta \) for all \( \theta \in [0, \pi/2) \) and \( \theta \in (\pi/2, \pi] \).
Using the definition (3.7) of $\tau$, $\lim_{\theta \to \pi/4} \frac{1}{2}(\varpi + \tau(\theta))$ exists if and only if $+ \tau$ is chosen in the expression of $\tau$ for $\varpi < 0$, and $-\tau$ otherwise. Also

$$\lim_{\theta \to \omega} \frac{1}{2}(\varpi + \tau(\theta)) = \lim_{\theta \to -\pi} \frac{1}{2}(\varpi + \tau(\theta)) = \varpi \pm 1, \quad \lim_{\theta \to -\pi/2} \frac{1}{2}(\varpi + \tau(\theta)) = \mp 1, \quad (3.8)$$

where $\pm, \mp$ signs concur with $\pm$ in the definition (3.7). Note that $\omega \in (0, 2)$ implies $\varpi \in (0, 2)$, while $\omega \in (2, 2\sqrt{2}) \Rightarrow \varpi \in (-2, 0)$. By Lemma 3.1, $\cos^{-1} \frac{1}{2}(\varpi + \tau(\theta))$ is well defined only when $\pm$ sign is chosen in (3.7) in such a way that

$$\tau(\theta) = \sec 2\theta(\varpi - \operatorname{sgn}(\varpi)\sqrt{\varpi^2 \sin^2 2\theta + 4\cos^2 2\theta}), \quad \theta \in [0, \pi]. \quad (3.9)$$

It follows that, $\cos^{-1} \frac{1}{2}(\varpi + \tau(\theta))$ is a monotone function of $\theta$ on $[0, \pi/2]$ as well as on $(\pi/2, \pi]$. Motivated by $\cos \xi_S = \frac{1}{2}(\varpi + \tau(\theta))$, let $\Xi$ be defined according to

$$\Xi(\theta) = \cos^{-1} \frac{1}{2}(\varpi + \tau(\theta)), \quad \theta \in [0, \pi], \quad (3.10)$$

where $\tau$ is given by (3.9). The next two Lemmas follow immediately after the statements made above.

**Lemma 3.2.** $\Xi$ is a monotone function of $\theta$ for $\theta \in [0, \pi/2]$ and $\theta \in (\pi/2, \pi]$.

**Lemma 3.3.** (a) For $\omega \in (0, 2)$, when $\theta \to 0^+$ or $\theta \to \pi^-$, $\Xi(\theta) \to \Xi(\theta)$ in the same range for $\omega$, when $\theta \to \frac{\pi}{2} \pm \Xi(\theta) \to 0$.

(b) For $\omega \in (2, 2\sqrt{2})$, when $\theta \to 0^+$ or $\theta \to \pi^-$, $\Xi(\theta) \to \Xi(\theta)$ in the same range for $\omega$, when $\theta \to \frac{\pi}{2} \pm \Xi(\theta) \to \pm \pi$.

All possible candidates for the saddle point $\xi_S$ on $C_\xi$, are therefore, given by $\pm \Xi(\theta)$ for $\omega \in (0, 2)$ and $+\Xi(\theta), 2\pi - \Xi(\theta)$ for $\omega \in (2, 2\sqrt{2})$. Assuming $\theta \in (0, \pi/2)$, from (3.6) that describes $\xi_S$, indeed, it follows that $\xi_S(\theta) = (-\pi, 0)$ for $\omega \in (0, 2)$ and $\xi_S(\theta) = (\pi, 2\pi)$ for $\omega \in (2, 2\sqrt{2})$, and, putting together with first sentence in this paragraph, this implies $\xi_S = -\Xi(\theta)$ for $\omega \in (0, 2)$, and $2\pi - \Xi(\theta)$ for $\omega \in (2, 2\sqrt{2})$. On the other hand, for $\theta \in (\pi/2, \pi)$, it is necessary that $\xi_S(\theta) \in (0, \pi)$, so that $\xi_S = +\Xi(\theta)$ for these values of $\theta$ and $\omega \in (0, 2\sqrt{2})$.

By (3.4), (3.5a), assuming $\cot \theta \neq 0$, one finds that

$$\eta''(\xi_S)(\varpi + \sin \xi_S) = -(1 - \tan^2 \theta) \cos \xi_S + \varpi. \quad (3.11)$$

Since $\cos \xi_S$ lies between $-1$ and $1$ and is a monotone function of $\theta$ for $\theta \in [0, \pi/2)$ and $\theta \in (\pi/2, \pi]$, to verify (3.11) it is sufficient to check whether the right side of (3.11) possesses different values at $\theta = 0$ versus $\theta = \pi/2$, and also, at $\theta = \pi/2$ versus $\theta = \pi$. By evaluating the corresponding values as limits, using the expression for $\xi_S$ specified in the previous paragraph, it is concluded that $\eta''(\xi_S)$ does not vanish at any value of $\theta \in [0, \pi]$. Finally, putting together the gist of several facts stated so far, the following statement holds.

**Theorem 3.4.** For $\phi$ defined by (3.4), there exists a unique saddle point $\xi_S \in C_\xi$, satisfying both conditions in (3.5). $\xi_S$ is, explicitly, given by

$$\xi_S = \begin{cases} -\Xi(\theta) + 2\pi H(\varpi - 2) & \text{for } \theta \in (0, \pi/2), \\ +\Xi(\theta) & \text{for } \theta \in (\pi/2, \pi), \end{cases} \quad (3.12)$$

where $\Xi$ is defined by (3.10) and (3.9). $\xi_S$ depends monotonically on $\theta$ in $(0, \pi/2)$ and $(\pi/2, \pi)$. $\xi_S$ lies in $[-\xi_S, \xi_S]$ for $\omega \in (0, 2)$ and in $[\xi_S, 2\pi - \xi_S]$ for $\omega \in (2, 2\sqrt{2})$. Moreover, the limiting values of $\xi_S$ at $\theta \in \{0, \pi/2, \pi\}$ are given by Lemma 3.3.
For a range of values of $\omega$, are shown in Fig. 3.1 (top). Note that $|\xi_0|$ is less than $\xi_\delta$ for $\omega \in (0, 2)$ (left plot) while $\xi_0$ lies between $\xi_\alpha$ and $\xi_\delta$, for $\omega \in (2, 2\sqrt{2})$ (right plot). The symmetry between both plots is a result of the peculiar form of dispersion relation (1.7). Fig. 3.1 (bottom) also illustrates $\xi_0$ and $\xi_\delta$ curves and the symmetry is obvious (see [35]). This illustration concludes the discussion of saddle point. Below is the presentation of asymptotic approximation of the scattered displacement in far field due to diffraction by the semi-infinite crack.

### 3.2. Approximation for $\xi_0$ and $\omega \in (0, 2)$

When $\omega \in (0, 2)$, as discussed in the paragraph following equation (3.1), $\xi_\delta \in (0, \pi)$. Therefore, using the transformation $\xi = \xi_\delta \cos \theta$, integral in (3.2) is reduced to an integral over a contour $C_\alpha$ in the $\alpha$-plane (see Fig. 3.2(b)), allowing the expression (3.2) to be written as

$$ u_{x,y} = \frac{\lambda}{\sqrt{2\pi}} \int_{C_\alpha} \mathcal{D}(\alpha) e^{i \xi_\delta R \psi(\alpha)} d\alpha, \quad (x, y) \in \mathbb{Z}^2; y \geq 0, \quad (3.13) $$

where $\mathcal{D}(\alpha)$, so called diffraction coefficient [13, 18], is defined as

$$ \mathcal{D}(\alpha) = C_0 \frac{e^{-i \frac{\pi}{4}}}{\sqrt{2\pi}} \mathcal{A}(e^{-i \xi_\delta \cos \alpha}) \frac{e^{-i \frac{1}{2} \eta_0 \cos \alpha} - e^{-i \xi_\delta \cos \alpha}}{e^{-i \xi_\delta \cos \alpha} - e^{-i \xi_\delta \cos \alpha}}, \quad (3.14) $$

and the phase function, $\psi$, is given by

$$ \psi(\alpha) = \xi_0^{-1} \phi(\xi_\delta \cos \alpha), \quad (3.15) $$

using $\phi$ defined by (3.4). The end points of the contour $C_\alpha$ are given by $\alpha_i$ and $\alpha_f$ in $\mathbb{C}$, satisfying the relations $\xi_\delta \cos \alpha_i = -\pi, \xi_\delta \cos \alpha_f = +\pi$, with $\alpha_i = \pi - ia_i, \alpha_f = \pi + ia_f$. In the following sections, $\mathcal{D}(\alpha)$ and $\psi(\alpha)$ will be discussed further.
0 + ia_f, a_i ≥ 0, a_f ≥ 0. After some calculations it turns out that a_i = a_f. In Fig. 3.2(b), the contour \( C_\alpha \) is shown, along with the steepest descent contour \( C_{\alpha,S} \) in the complex plane \( \mathbb{C} \) for the variable \( \alpha \), conveniently called \( \alpha \)-plane. The contour \( C_{\alpha,S} \) intersects \( C_\alpha \) at \( \alpha = \alpha_S \). Using (3.15), and by definition [13] of the saddle point \( \alpha_S \), \( \Psi'(\alpha_S) = 0 \) and \( \Psi''(\alpha_S) \neq 0 \). By definition of \( \cos^{-1} \) stated in the notation, the properties of \( \alpha_S \) follow from those of \( \xi_S \) according to Theorem 3.4.

The contour \( C_\alpha \) starts at \( \alpha_1 \) and ends on \( \alpha_f \), as illustrated in Fig. 3.2(b), and can be deformed into \( C_{\alpha,S} \), with/without contribution of the pole at \( \alpha_p \) if \( \alpha_S \leq \alpha_p \). Since the approximation, discussed below, is based on only the dominant asymptotic term, the complications due to a possible change of Riemann sheet [13] do not arise.

Assuming \( \alpha_S \neq \alpha_P \), by (formally) deforming the contour \( C_\alpha \) to \( C_{\alpha,S} \) (see Fig. 3.2(b)), it is found that

\[
\begin{align*}
\mathbf{u}_{x,y} &= \mathbf{u}_{x,y}|_S \quad \mathbf{u}_{x,y}|_P,
\end{align*}
\]

where \( \mathbf{u}_{x,y}|_S \), dominated by the contribution of the saddle point located at \( \alpha_S \) as \( \xi_\delta R \to +\infty \), is given by

\[
\mathbf{u}_{x,y}|_S = \frac{e^{i\frac{\pi}{4}}}{\xi_\delta} \int_{\alpha_1}^{\alpha_f} \mathcal{D}(\alpha) e^{i\xi_\delta R} \Psi(\alpha) d\alpha \quad \text{along } C_{\alpha,S},
\]

and \( \mathbf{u}_{x,y}|_P \), the contribution of the pole located at \( \alpha_P \), is given by

\[
\mathbf{u}_{x,y}|_P = \sqrt{2\pi} A \xi_\delta \mathcal{C}_P H(\alpha_P - \alpha_S) e^{i\frac{\pi}{4}} \xi_\delta e^{i\xi_\delta R} \Psi(\alpha_P),
\]

with \( \mathcal{C}_P = \lim_{\alpha \to \alpha_P} \mathcal{D}(\alpha)(\alpha - \alpha_P) \). Simplifying \( \mathbf{u}_{x,y}|_P \) further, using \( e^{i\pi|z|} = e^{i\kappa z} \) as \( |\lambda| < 1 \), the expression for \( \mathbf{u}_{x,y}|_P \) becomes

\[
\mathbf{u}_{x,y}|_P = \frac{1}{2(1 - e^{i\kappa y})}(\frac{1}{\mathcal{L}(z_P) - y}) H(\xi_S - \xi_P) \mathbf{u}_{x,y}^t, \quad y \geq 0,
\]

where \( \mathbf{u}_{x,y}^t = A e^{-i \kappa x + i \kappa y} |y| \) with \( y \in \mathbb{Z} \). Using expression for \( \mathcal{L} \), it can be shown that the scalar coefficient appearing in front of \( u' \) in (3.19) is \( \lambda(z_P) \), which leads to the expected form of solution in the ‘shadow’ region using the skew symmetry (2.20).

As \( \xi_\delta R \to +\infty \), the asymptotic approximation of the expression \( \mathbf{u}_{x,y}|_S \) is found by the application of method of stationary phase [12] [14] so that \( \mathbf{u}_{x,y}|_S \sim \mathbf{u}_{x,y}|_S; \text{interior} + \)
The intact lattice is same as (1.5) with $b$ replaced by $\varepsilon$

$$\varepsilon_S^k x = \cos \theta \xi \sin \phi$$

where $(\pi, \alpha, \xi)$ also defined by expressions similar to (3.14) and (3.15), respectively. By periodicity $D\omega \xi$ of the contour by Theorem 3.4 on saddle point, and $\alpha$ of the pole to the asymptotic approximation occurs when the criterion $2\alpha_\xi \approx 2\alpha_{\xi P}$ as the

$$\omega \xi$$

of the contour $C\eta$ in the $\alpha$-plane (see Fig. 3.2(c)). Then $u_{x,y}$ is given by an expression analogous to (3.13). The diffraction coefficient $D$ and phase function $\psi$ are also defined by expressions similar to (3.14) and (3.15), respectively. By periodicity of $\xi_P$, $\alpha_P$ is associated with $\xi_P + 2\pi$ for admissible $\Theta \in (\pi/2, \pi)$. The end points of the contour $C\eta$ are given by $\alpha_i$ and $\alpha_f$, satisfying the relations $(\pi - \xi_i) \cos \alpha_i = \pi, (\pi - \xi_f) \cos \alpha_f = -\pi$, with $\alpha_i = -\alpha_i, \alpha_f = 2\pi + \alpha_f, a_i$ and $a_f$ are positive real numbers. When $\omega \in (2, 2\sqrt{2})$, there exists a unique point $\xi_S = (\xi_i, \xi_f)$ by Theorem 3.4 on saddle point, and $\alpha_S$ satisfies $\xi_S = (\pi - \xi_i) \cos \alpha_S$. Virtually everything follows the same way as earlier case where $\omega \in (0, 2\sqrt{2})$. For small $\epsilon$, one can see that $\xi_S \ll \xi_P$ as $\epsilon \to 0$. Let $\xi_S \ll \xi_P < \xi_i$ in (3.19). While for admissible $\Theta \in (\pi/2, \pi)$ the same happens when the $\epsilon \to 0$ for $2\pi + \xi_P < \xi_S$ in (3.19) is satisfied.

4. Low Frequency Approximation. Let $L$ denote a macroscopic length scale in the diffraction problem. For instance, $L$ may be described the ‘size’ of a specimen used in diffraction experiment. The ratio $b/L$ is denoted by parameter $\epsilon$. As $L$ increases (relative to $b$), the ‘macroscopic’ behavior is captured by the continuum limit $[1] [9]$, that is $\epsilon \to 0$. Due to the nature of problem, the limit $\epsilon \to 0$ is also same as the low frequency approximation, $\omega \ll 1$. To avoid notational complications, let $L$ denote unit length and $b$ be replaced by $\epsilon^3$. Let $M = \rho b^3 = \rho \epsilon^3, K = \mu b = \mu \epsilon$, and

$$u(x, y, \tilde{t}) = u_{x,y}(\tilde{t})$$

with $x = x \epsilon, y = (y + \frac{1}{2}) \epsilon$, (4.1) where $(x, y)$ are the macroscopic coordinates for $(x, y)$. As $\epsilon \to 0$, from (1.2), dropping the $O(\epsilon)$ terms and using $\tilde{t} = \epsilon t$, the equation of motion in the continuum limit is obtained as $\frac{\partial^2}{\partial \tau^2} \epsilon_0 (x, y, \tilde{t}) = \Delta u_0 (x, y, \tilde{t})$. The subscript 0 on $u_0$ and decoration $\epsilon$ on $\tilde{t}$ is further ignored, and the classical wave equation in two dimensional space results. Reverting back to the discrete model for square lattice $\xi$, the equation of motion of the intact lattice is same as (1.5) with $b$ replaced by $\epsilon$. In terms of the macroscopic coordinates, the incident lattice wave (1.6) is expressed as

$$u^\dagger(x, y, \tilde{t}) = A e^{i k_x x + k_y y} e^{-i (k_x x + k_y y) - i \omega \tilde{t}}$$

with $k_x = \kappa_x / \epsilon, k_y = \kappa_y / \epsilon$. (4.2)

It is natural to call $(k_x, k_y)$ as the macroscopic wave vector of $u^\dagger$. In the continuum limit $\epsilon \to 0$, $\omega^2 = \epsilon^2 \omega^2 \approx \epsilon^2 (k_x^2 + k_y^2) = \kappa_x^2 + \kappa_y^2$. Indeed, $\omega^2 = \kappa_x^2 + \kappa_y^2$ is recognized
as the macroscopic dispersion relation \[ \omega \] \cite{1}. In the same way as \( \kappa \) is determined by \eqref{1.9}, its continuous analogue, \( k \), is defined by \( k := \epsilon^{-1} \kappa \), so that \( k \) is interpreted as the macroscopic wave number of incident wave. Note that in the continuum limit, \( \omega \approx \kappa \) (or \( \omega = k \)), however, by nature of the discrete model for large \( \epsilon \), the dependence of \( \omega \) on \( (\kappa_x, \kappa_y) \) (or \( \kappa \) and \( \Theta \)) and \( \omega \) on \( (k_x, k_y) \) (or \( k \) and \( \Theta \)) is significantly different \[ \omega \].

The continuum limit of the discrete Helmholtz equation is the classical two-dimensional Helmholtz equation. The continuum model has a well defined solution \cite{12,31}, though an additional restriction specifying the singular behavior at the crack tip (though the assumption of bounded \( \epsilon \)) is required for uniqueness \cite{19,31}. It is of independent interest to verify that \[ \omega \] asymptotically approaches the continuum solution as \( \epsilon \rightarrow 0 \) without the prior assumption of singular behavior at the crack tip (though the assumption of a bounded solution is implicit in the adopted discrete formulation).

Similar to the change of variables and definitions, used in far field approximation for \( \omega_2 \ll 1 \) and \( \omega_1 \in (0,2) \) to simplify \eqref{3.2}, let \( \xi = \epsilon^{-1} \xi_\delta \cos \alpha, \psi(\alpha) = \epsilon^{-1} \phi(\xi \delta \cos \alpha) \), where \( \phi \) is given by \eqref{3.4}. In the limit, as \( \epsilon \rightarrow 0 \), the two zeros of \( k \) approach \( \epsilon \pm i \xi_\delta \), with \( \xi_\delta = c \omega \). But note that \( \epsilon \omega = \epsilon k \), so \( \xi = k \cos \alpha, \psi'(\alpha) = \epsilon^{-1} \phi(\epsilon k \cos \alpha) \). Similar to the case of \( k \), the two zeros of \( r \) approach \( \xi_\delta \), with \( \xi_\delta = ik_r, k_x, k_y \approx 1.76 \). After a long calculation, using above substitutions, it is found that

\[
\quad u(x, y) \approx -2 \pi \sqrt{\frac{k}{2}} \sin \frac{1}{2} \Theta \int_{\gamma_{\xi, \Theta}} \int \frac{1}{\sqrt{k-\xi}} e^{i(-x - y + \sqrt{\xi^2 + k^2})} d\xi,
\]

as \( \epsilon \rightarrow 0 \), where \( \gamma_{\xi, \Theta} \) coincides with the contour present in the traditional continuum solution (in integral form) \cite{31}. Indeed, the expression \eqref{4.3} coincides with the solution of the continuum problem \cite{31,18}.

5. Numerical Results.

5.1. Contribution of Pole. The criteria for non-zero contribution of pole, present in \( \omega_2 \ll 1 \) and \( \omega_1 \in (0,2) \) and its dependence for \( \omega \in (2,2\sqrt{2}) \), can be expressed in terms of \( \theta \) and \( \Theta \) using the fact that \( \delta \) varies monotonically with \( \theta \) (Theorem \[ \Theta \]). Fig. \[ 5.1 \] gives a numerical illustration, for a range of values of \( \omega \) as indicated, of this criterion in \( \theta - \Theta \) plane where the coalescence of pole and saddle point of diffraction integral \eqref{3.2}, that is \( \xi_P = \xi_S \), is depicted as solid curves. As \( \omega \) increases from 0 to 2 (left plot in Fig. \[ 5.1 \]) the range of \( \theta - \Theta \) relation and its deviation from the \( \theta + \Theta = \pi \) (corresponding to the continuum model \cite{31}) increases and quite significantly when \( \omega \) approaches 2. At \( \omega = 2 \) (right plot in Fig. \[ 5.1 \]) the \( \theta - \Theta \) relation does not have any resemblance to that for the continuum model \cite{31}. Note that the criterion does not change with \( \omega \in (0,2) \) for \( \Theta \in \{ \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \pi \} \) (and their negative counterparts), as well as with \( \omega \in (2,2\sqrt{2}) \) for \( \Theta \in \{ \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \pi \} \) (and their negative counterparts). These sets of “special” incidence angles of the lattice wave are, naturally, associated with symmetries of the square lattice \( \Theta \).

5.2. Far field approximation. Numerical solution of the diffraction problem \eqref{1.10} and \eqref{2.18} for \( \omega \in (0,2) \), together with \( \omega \rightarrow 2 \) \cite{21}, is obtained on \( (2N_{\text{grid}} + 1) \times (2N_{\text{grid}} + 1) \) grid using a method outlined in Appendix D. In Fig. \[ 5.2 \] the numerical solution is compared with the asymptotic approximation \eqref{3.2}. The modulus and argument of displacement of every particle located on a (fixed) discrete rectangle (shown at the bottom of Fig. \[ 5.2 \]) is plotted, with relevant details in the figure caption. The “jump”
equal to $\pi$ in the argument, with a symmetry in the modulus, at the midpoint exhibits the skew symmetry of solution about the crack. For $\omega = 2\pi$ (lowest plot in Fig. 5.2), this is not apparent but only because of the presence of ‘oscillation’ with a small wave length. The places where disparity between analytical and numerical solution occurs are: (a) boundary of shadow/illuminated region with diffraction dominated region, as the asymptotic approximation fails there (owing to the proximity of pole and saddle, that is $\xi_P \approx \xi_S$), (b) $\theta \approx 0$, where the disparity disappears when a larger value of ‘grid size’ $N_{\text{grid}}$ is used.

### 5.3. Low frequency approximation

The asymptotic approximation (3.2) in far field is compared with the exact solution of the continuum model on a semi-circle in the upper half plane as shown in Fig. 5.3. The solution in lower half plane is not shown as the scattered displacement is ‘negative’ with respect to the upper half plane. Since the asymptotic approximation fails at the boundary of illuminated region with diffraction dominated region (owing to the proximity of pole and saddle, that is $\xi_P \approx \xi_S$), there is a visible disparity between the discrete solution and the continuum solution at the boundary between shaded and unshaded portion (see zoomed out portion in the inset). The amplitude plots (top) for the small values of $\omega$, of discrete solution, fall almost on top of one another and, therefore, are not distinguishable. Both plots (top & bottom), for the listed values of $\omega$, show an increased departure from the continuum solution (thick black curve) in the diffraction dominated region (unshaded region), as $\omega$ increases. Note that the first peak in illuminous region shifts away from the expected ray of reflection, as the frequency $\omega$ increases (significantly for $\omega \gtrsim 2$). This happens, partly, due to the deviation, from the continuum model, in criterion for coalescence of saddle and pole of the diffraction integral, as shown graphically in Fig. 5.1. The choice $R_\infty = 50/\kappa$ also contributes to the shift (since $\kappa$ does not capture the discreteness induced anisotropic effects for high $\omega$).
6. Concluding Remarks. In this paper, an analysis of a discrete analogue of Sommerfeld diffraction by a semi-infinite crack is presented following Noble’s approach [31]. The exact solution is obtained by the discrete Wiener–Hopf method. An asymptotic approximation of the exact solution, in far field away from the region corresponding to divergent behavior associated with proximity of pole and saddle, co-
Fig. 5.3. $|u|$ and Arg $u$ according to asymptotic approximation of \[ (3.2) \] for the particles lying close to a continuous counter-clockwise semi-circular arc with radius $R_\infty = 50/\kappa$ (i.e., radius depending on $\omega$), for $\omega \in \{0.1, 0.2, \ldots, 1.8, 1.9, 1.95\}$ shown as dashed curves, with dash length proportional to $\omega$. The pointed arrows indicated direction of decreasing $\omega$ for exact continuum solution \[ (3.2) \] with $k R_\infty = 50$ is shown as undashed black curve. The light gray region represents the illuminous region according to continuum model. Inset shows an enlarged portion of the amplitude field as indicated. The purpose of gap in dashed curves for top plot, at the boundary between shaded and unshaded portion, is to avoid unnecessary clutter. $A = 1$ in all plots.

incides with numerical solution. Low frequency approximation of the exact solution turns out to be the same as exact solution for the continuum model (in integral form).

At frequency, roughly, in the middle of the pass band, the effects due to discreteness and anisotropy begin to dominate.

The calculation of the near-tip field for diffraction on square lattice by a semi-

infinite crack is presented in a companion paper \[37\]. A rigorous operator-theoretic

analysis for the problem of diffraction due to finite and semi-infinite crack is also

discussed therein \[37\]. Some natural generalizations of the problem shall appear elsewhere,

for e.g., diffraction of surface wave by a semi-infinite rigid constraint on square

\[36, 38\], triangular, honeycomb lattices; diffraction by interface crack; diffraction on

square lattice with ‘vertical’ and ‘horizontal’ bonds with unequal moduli; etc.

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Appendix A. Discrete Fourier Transform. The (complex) discrete Fourier

transform of a sequence $\{a_m\}_{m \in \mathbb{Z}}$ in $\mathbb{C}$ is defined \[20, 7\] as

\[ a_F(z) = a_+(z) + a_-(z), \quad \text{where} \quad a_+(z) = \sum_{m=0}^{+\infty} a_m z^{-m}, \quad a_-(z) = \sum_{m=-\infty}^{-1} a_m z^{-m}. \quad (A.1) \]
It is usually assumed that \( a_+ \) (resp. \( a_- \)) is analytic at \( z \in \mathbb{C} \) such that \(|z| > r_+\) (resp. \(|z| < r_-\)). In general, the discrete Fourier transform which is analytic outside (inside) a finite disc \( \{ z \in \mathbb{C} : |z| > r_+\} \) (\( \{ z \in \mathbb{C} : |z| < r_-\}\)), is denoted by ‘+’ (‘-’). Note that \( a_+ \) (\( a_- \)) may have singular points in the interior (exterior) of such disk. The corresponding sequence obtained by inversion of such discrete transform has support \( m \geq 0 \) \( (m < 0) \). When it is assumed that \( r_+ < r_-\), the transform \( a^F \) is an analytic function on the complex annulus \( \{ r_+ < |z| < r_-\} \). In such cases, the sequence \( a_m \) is obtained by means of the inverse transform

\[
a_m = \frac{1}{2\pi i} \oint_{\mathcal{C}_z} a^F(z) z^{m-1} dz, \quad m \in \mathbb{Z},
\]

where \( \mathcal{C}_z \) denotes any rectifiable, closed, counter clockwise contour in the annulus \( \{ z \in \mathbb{C} : r_+ < |z| < r_-\} \), which is the intersection of regions of analyticity for \( a_{\pm} \). Also, \( a_0 = \lim_{z \to \infty} a_+(z), \) \( a_1 = \lim_{z \to \infty}(a_+(z) - a_0)z, \) \( a_{-m} = \lim_{z \to \infty} a_-(z) z^{-m}, \) \( a_{-m-1} = \lim_{z \to \infty} a_-(z) z^{-m-1}, \) and in general,

\[
a_m = \lim_{z \to \infty} (a_+(z) - \sum_{j=0}^{m-1} a_j z^j) z^m, \quad a_{-m} = \lim_{z \to \infty} (a_-(z) - \sum_{j=-m+1}^{-1} a_j z^j) z^{-m-1}
\]

for \( m = 2, 3, \ldots \).

**Appendix B. Skew Symmetry.** Adding equations \( (2.18) \) and \( (2.19) \), with \( w_x = u_{x,0} + u_{x,-1} \), it follows that

\[
-\omega^2 w_x = w_{x+1} + w_{x-1} + u_{x,1} + u_{x,-2} \quad x \in \mathbb{Z}.
\]

Using the discrete Fourier transform of the sequence \( \{ u_{x,0}\}_{x \in \mathbb{Z}} \) and equations \( (2.15), (2.18) \), it follows that \( \frac{1}{2} k(z) (\hat{k}(z) + r(z)) w^F(z) = 0 \), \( \forall z \). But \( \hat{k}(\hat{k} + r) \) does not vanish anywhere on \( \mathbb{A} \), hence, it follows that \( w^F \) is zero. This implies \( u_0^F = -u_{-1}^F \). In the context of \( (2.15) \), it follows that \( w_x = -w_{x-1} \), \( x \geq 0 \), i.e., \( (2.20) \) holds.

**Appendix C. Boundary Stationary Phase Approximation.** Consider the case \( \omega \in (0, 2] \) only, since the other \( \omega \in (2, 2\sqrt{2}] \) is analogous. Since the contour \( \mathcal{C}_\alpha \) contains finite endpoints, the contributions from the end points \( \alpha_i \) and \( \alpha_f \) [18] are

\[
u_{x,y}^{\text{S;boundary}} = \frac{1}{i \xi_4 R} \frac{e^{i\xi_4 R}}{2\pi} \frac{C_0}{\psi'(\alpha_f) - \psi'(\alpha_i)} \frac{D(\alpha_f)}{\psi'(\alpha_f)} e^{i\xi_4 R} R \psi(\alpha_f) - \frac{D(\alpha_i)}{\psi'(\alpha_i)} e^{i\xi_4 R} R \psi(\alpha_i),
\]

as \( \xi_4 R \to +\infty \). Using the values of \( \alpha_0 \) and \( \alpha_f \), and the definitions of relevant functions stated in the main text, it can be shown that \( \eta(\pm \pi) = \eta(-\pi), \) \( D(\alpha_f) = D(\alpha_i), \) \( \psi(\alpha_f) = \psi(\alpha_i) - 2 \xi_4^{-1} \pi \cos \theta, \psi'(\alpha_f) = \psi'(\alpha_i) \), so that, after some algebraic simplifications,

\[
u_{x,y}^{\text{S;boundary}} = \frac{A C_0}{R} \frac{\mathcal{A}(-1) e^{-i\xi_4 R} R \eta(\eta(\pi) \sin \theta)}{(-\eta(\pi) \sin \theta + \cos \theta)} \sin(R \pi \cos \theta) e^{-i\xi_4 R} + \frac{1}{2}.
\]

**Appendix D. Numerical Scheme.** Since the equations \( (1.10), (2.18), \) and \( (2.19) \) are algebraic, the numerical solution on a \( (2N_{\text{grid}} + 1) \times (2N_{\text{grid}} + 1) \) grid \( \Omega \) (with crack tip located at the center of grid) is straightforward and the only outstanding issue is which boundary conditions to employ on the outer edges of \( \Omega \). To construct a
workable scheme, a variant of perfectly matched layers (PML) is adopted for simulation of an ‘infinite’ domain. Introducing

\[ \sigma_{x,y} = \frac{1}{N_{grid}} H(|x| - N_{pml})N_{pml} - |x|| + \frac{1}{N_{grid}} H(|y| - N_{pml})N_{pml} - |y||, \]

the discrete Helmholtz equation in intact lattice is replaced by

\[ \Delta u_{x,y} + \omega^2 (1 - \frac{\sigma_{x,y}}{i\omega})^2 u_{x,y} = 0, (x,y) \in \Omega \setminus \Sigma. \]  

(D.1)

Using \( N_{pml} \) specified in Fig. 5.2, this scheme is used to obtain the plots.

REFERENCES


[38] ——, Near Tip Field For Diffraction On Square Lattice by Rigid Constraint, 2014 (accepted by ZAMP).


