Chapter 3

Vector analysis

3.1 Triple products

3.1.1 Scalar triple product

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \tag{3.1}$$

Note that, scalar triple product represents volume of a parallelepiped, bounded by three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. The cross product is the area of a parallelogram, which is then multiplied by height to get the volume. Clearly, we can use any two sides as base, and the volume should not change. Using this, we can prove that $\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$ etc. Thus, it does not matter where we put the dot and cross product in a scalar triple product and often it is represented as $(\vec{A}\vec{B}\vec{C})$. However, it might pick a negative sign, like $\vec{A}\cdot(\vec{B}\times\vec{C}) = -\vec{A}\cdot(\vec{C}\times\vec{B})$ etc. It should be most convenient to figure out all possible combinations (with appropriate sign) from the determinant, because interchanging two rows introduces a negative sign.

3.1.2 Vector triple product

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B}) \tag{3.2}$$

It is not very difficult to realize that the above triple product is a linear combination of $\vec{B}$ and $\vec{C}$, i.e., a vector lying in the same plane, as the two vectors in parenthesis. The middle vector has a positive sign and coefficient of each vector is a dot product of the other two.

3.1.3 Exercise

1. Derive the formula for vector triple product, assuming $\vec{B}$ to be along x axis and $\vec{C}$ in the xy plane.

2. Let us change from rectangular to some general coordinate system (any three non-coplanar vectors, not perpendicular to each other). Derive the Jacobian, used in multiple integrals for changing variables.

3. Using reciprocal lattice vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$, find the direction perpendicular to the plane with Miller index (hkl). Also find the inter-planar spacing between (hkl) planes.

4. Mary L. Boas, chapter 6, section 3, problem 11-14.

3.2 First derivative of scalar and vector fields

3.2.1 Gradient and directional derivative

Let $\phi(x, y, z)$ be a scalar field. Gradient of ϕ (read as "grad ϕ" or "del ϕ") is defined as:

$$\boxed{\vec{\nabla}\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}}. \tag{3.3}$$

This is very useful for calculating directional derivative, given by

$$\boxed{\frac{d\phi}{du} = \vec{\nabla}\phi \cdot \hat{u}}, \tag{3.4}$$

where the unit vector is pointing in a direction along which the derivative is calculated.

3.2.2 Physical significance of gradient

Note that, rate of change of ϕ in some direction u is maximum, if $\hat{u}$ is in the direction of the $\vec{\nabla}\phi$ itself. Thus, *gradient is the direction along which the rate of change (inrease/decrease) of ϕ is maximum.*

3.2.3 Geometrical significance of gradient

Note that, the value of ϕ does not change along a contour line. Thus, if we draw a tangent at some point on the contour line, ϕ does not change along the line. On the other hand, we know that, gradient is the direction along which the rate of change of ϕ is maximum. Therefore, we conclude that *gradient is the direction normal to the surface at a given point.*

3.2.4 Divergence

$$\boxed{\mathrm{div}\vec{V} = \vec{\nabla}\cdot\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}.} \tag{3.5}$$

$$\boxed{\vec{\nabla}\cdot(\phi\vec{V}) = (\vec{\nabla}\phi)\cdot\vec{V} + \phi(\vec{\nabla}\cdot\vec{V}).} \tag{3.6}$$

Physical significance of divergence will be discussed later.

3.2.5 Curl

$$\boxed{\mathrm{curl}\vec{V} = \hat{i}\left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right) + \hat{j}\left(\frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}\right) + \hat{k}\left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right).} \tag{3.7}$$

$$\boxed{\vec{\nabla} \times (\phi \vec{V}) = (\vec{\nabla}\phi) \times \vec{V} + \phi(\vec{\nabla} \times \vec{V}).} \tag{3.8}$$

Physical significance of curl will be discussed later.

3.2.6 Exercise

1. Using Lagrange multiplier, find the maximum value of the directional derivative $d\phi/du$, subject to the constraint that $a^2+b^2+c^2=1$, where $\hat{u}=a\hat{i}+b\hat{j}+c\hat{k}$.

Other coordinate systems

2. Define polar coordinate system. Write the transformation matrix from the cartesian to polar coordinate system. Note that, this is a 2D rotation matrix. Do you see why?
3. Write the gradient operator in polar, cylindrical and spherical coordinate system.
4. Take a function $f(r)=r$, where $r=\sqrt{x^2+y^2}$. Calculate $\vec{\nabla}f$ in polar, as well as cartesian coordinate system. Compare the answers and check whether you get the same answer or not.
5. Mary L. Boas, chapter 6, section 6, problem 18-20.

Equation of a line (normal to surface) and plane (tangent to surface)

6. Mary L. Boas, chapter 6, section 6, problem 6-9.

Finding the direction of heat flow

7. Mary L. Boas, chapter 6, section 6, problem 10-14. Note that, you should use a computer for better understanding.

3.3 Second derivative of scalar and vector fields

We can treat $\vec{\nabla}$ as a "vector" (you have to apply some common sense as well) and then get the following results.

3.3.1 Divergence of gradient or Laplacian

$$\boxed{\vec{\nabla} \cdot (\vec{\nabla}\phi) = (\vec{\nabla} \cdot \vec{\nabla})\phi = \nabla^2\phi.} \tag{3.9}$$

3.3.2 Laplacian of a vector field

$$\boxed{(\vec{\nabla} \cdot \vec{\nabla})\vec{V} = \nabla^2\vec{V}.} \tag{3.10}$$

3.3.3 Curl of gradient

$$\boxed{\vec{\nabla} \times \vec{\nabla}\phi = 0.} \tag{3.11}$$

Note that, this is just a consequence of $\frac{\partial^2\phi}{\partial x\partial y} = \frac{\partial^2\phi}{\partial y\partial x}$ etc. Using this, we can write a very important theorem. First note that, $\vec{\nabla}\phi$ is a vector field (say $\vec{U}$).

Theorem: if (curl $\vec{U}$)=0, then $\vec{U}$ must be the gradient of some scalar field ϕ.[1]

3.3.4 Divergence of curl

$$\boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{V}) = 0.} \tag{3.12}$$

Using this, we can write another very important theorem. Again, first we should note that, $\vec{\nabla} \times \vec{V}$ is a vector field (say $\vec{U}$).

Theorem: if (div $\vec{U}$)=0, then $\vec{U}$ must be the curl of some vector field $\vec{V}$.

3.3.5 Curl of curl

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - \nabla^2\vec{V}.} \tag{3.13}$$

3.3.6 Gradient of divergence

$$\boxed{\vec{\nabla}(\vec{\nabla} \cdot \vec{V}) = \hat{i}\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_y}{\partial x\partial y} + \frac{\partial^2 V_z}{\partial x\partial z}\right) + \hat{j}\left(\frac{\partial^2 V_x}{\partial x\partial y} + \frac{\partial^2 V_y}{\partial y^2} + \frac{\partial^2 V_z}{\partial y\partial z}\right) + \hat{k}\left(\frac{\partial^2 V_x}{\partial x\partial z} + \frac{\partial^2 V_y}{\partial y\partial z} + \frac{\partial^2 V_z}{\partial z^2}\right).} \tag{3.14}$$

3.3.7 Exercise

1. Starting from the gradient, divergence and curl (first derivatives), derive some second derivatives like: Laplacian, (curl grad), (div curl) and (grad div).

2. In order to memorize, we treated $\vec{\nabla}$ operator as a "vector" and it worked fine! Then, can we conclude that $(\vec{\nabla}\phi) \times (\vec{\nabla}\psi) = 0$?

3. What would be the expression for: $\vec{\nabla} \cdot (\vec{\nabla}\phi \times \vec{\nabla}\psi)$?

3.4 Line integrals

We know that, work done by a force is $dW = \vec{F} \cdot d\vec{s}$ and we have to calculate a *line integral* $\left[W = \int \vec{F} \cdot d\vec{s}\right]$ to get the total work done along certain path. The first thing to keep in mind while calculating a line integral is the fact that **there is only one independent variable along a curve.** Thus, first we have to express $\vec{F}(x, y, z)$ and $d\vec{s} = \hat{i}dx + \hat{j}dy + \hat{k}dz$ as functions of a single variable and then evaluate

[1] Later, we will find that it is related to Euler reciprocity relation and definition of an exact differential, which is also a path function in thermodynamics. A related concept is a conservative force field in classical mechanics and electrodynamics and work done is independent of path in a conservative force field.

the integral (of one variable) to find the total work done by the force to move an object from one point to other along a path.

Now, work required to move an object from one point to another may depend on the path (for example, because of energy dissipated due to friction). Such a field is known as a **non-conservative** force field. On the other hand, if the work required to move an object from one point to another is independent of the path taken, we call it a **conservative** force field.

Clearly, we can evaluate W along different path and find out whether the force field is conservative or not. Can we do this without evaluating the integral? The answer is yes and we have to think logically to recognize the following:

$$\boxed{\text{A vector field is conservative if } \vec{\nabla} \times \vec{F} = 0}.$$

A similar statement is:

$$\boxed{\text{A vector field is conservative if } \vec{F} = \vec{\nabla} W}.$$

The first two statements are correlated can can be stated as,

$$\boxed{\text{If } \vec{F} = \vec{\nabla} W, \text{ then curl } \vec{F} = 0}.$$

This is not entirely new, as we already know the reverse statement. In order to prove this, let us write the components of $\vec{F} = \vec{\nabla} W$: $F_x = \partial W / \partial x$, $F_y = \partial W / \partial y$ and $F_z = \partial W / \partial z$. Now, using the equality of second derivatives, i.e., $\partial^2 W / \partial x \partial y = \partial^2 W / \partial y \partial x$, we find that

$$\begin{aligned} \partial F_x / \partial y &= \partial F_y / \partial x, \\ \partial F_y / \partial z &= \partial F_z / \partial y, \\ \partial F_z / \partial x &= \partial F_x / \partial z. \end{aligned} \tag{3.15}$$

Is there a way to express the above set of equations in a compact form? We have to use the definition of curl and we can write a compact equation like $\vec{\nabla} \times \vec{F} = 0$.

Finally, we want to prove that work done is independent of the path for a conservative force field, i.e.,

$$\boxed{\int \vec{F} \cdot d\vec{s} \text{ is independent of path if } \vec{\nabla} \times \vec{F} = 0 \text{ or } \vec{F} = \vec{\nabla} W}.$$

Now, since $\vec{F} = \vec{\nabla} W$, we can write $\vec{F} \cdot d\vec{s} = \frac{\partial W}{\partial x} dx + \frac{\partial W}{\partial y} dy + \frac{\partial W}{\partial z} dz = dW$, where $d\vec{s} = \hat{i} dx + \hat{j} dy + \hat{k} dz$. Finally, the line integral

$$\int_A^B \vec{F} \cdot d\vec{s} = \int_A^B dW = W(B) - W(A), \tag{3.16}$$

is found to depend only on the value of W at the end points and independent of the path along which the integration is carried out. It is obvious that for a conservative force field, integral over a closed path:

$$\oint_C \vec{F} \cdot d\vec{s} = \oint_C dW = 0. \quad (3.17)$$

Next, we see two important applications of what we have learnt just now.

3.4.1 Exact Differential

In thermodynamics, we often see terms like exact and inexact differential. Let us understand what do they mean. Let dW be the infinitesimal difference between two adjacent values of W, i.e., $dW = W(x+dx, y+dy, z+dz) - W(x,y,z)$. Let us assume that we can express the differential (often termed as *total differential* and it is nothing but tangent approximation) as:

$$dW = \frac{\partial W}{\partial x}dx + \frac{\partial W}{\partial y}dy + \frac{\partial W}{\partial z}dz = F_x(x,y,z)dx + F_y(x,y,z)dy + F_z(x,y,z)dz. \quad (3.18)$$

This particular differential is an example of *exact differential*.[2] One can easily verify that Eq. 3.15 is satisfied for exact differential.[3] This is known as **Euler reciprocity relation** and it is simply based on the equality of the second derivative.

State and path functions in thermodynamics:

Do you see a connection between conservative force fields and exact differentials? Note that, Eq. 3.15 implies that $\vec{\nabla} \times \vec{F} = 0$. Thus, we can also state that: $dW = \vec{F} \cdot d\vec{s}$ is exact differential if $\vec{\nabla} \times \vec{F} = 0$. The last statement is true for a conservative force field, for which work done is independent of path. Thus, we conclude that, **line integrals of exact differentials are path independent**, such that Eq. 3.16 and Eq. 3.17 are valid. In thermodynamics, exact differentials are related to *state functions*, while inexact differentials are related to *path functions*.

One should note the connection between an exact differential and a conservative vector field. If I give you a conservative vector field $\vec{F}$, you can find an exact differential $dW = \vec{F} \cdot d\vec{s}$. On the other hand, if I give you an exact differential, $dW = Xdx + Ydy + Zdz$, then you can define a conservative vector field like $\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$. You will find problems related to differentials in thermodynamics, while problems related to force fields are important in mechanics or electrodynamics.

3.4.2 Scalar potential for a conservative force field

In mechanics, we define a scalar potential for a conservative force field. Note that, $\vec{F} = \vec{\nabla} W$ implies that W is the work done by the force $\vec{F}$. For example, if we lift

[2] Every differential $dw = P(x,y,z)dx + Q(x,y,z)dy + R(x,y,z)dz$ need not be an exact differential. If the differetial is an exact differential, then only we can write it as $dw = \frac{\partial w}{\partial x}dx + \frac{\partial w}{\partial y}dy + \frac{\partial w}{\partial z}dz$. See problem set for examples.

[3] This is the test to check whether a given differential is an exact differential or an inexact differential.

a mass, the work done against the gravitational force is $W = -mgh$. But we are increasing the potential energy by $\phi = +mgh$ and we conclude that $W = -\phi$. Thus, we can write

$$\boxed{\vec{F} = -\vec{\nabla}\phi.} \tag{3.19}$$

3.4.3 Exercise

Evaluation of line integrals

1. Mary L. Boas, chapter 6, section 8, problem 1-7.

Given a conservative force field, find the scalar potential

2. Mary L. Boas, chapter 6, section 8, problem 8-15.

Given the differential, determine the function

3. Test if $dz = \frac{1}{x^2}dx - \frac{y}{x^3}dy$ is exact or inexact differential. If it is exact, find $z(x, y)$.
4. Test if $dz = (2x + y)dx + (x + y)dy$ is exact or not. If exact, then find $z(x, y)$.
5. Given $dP = \frac{RT}{V-b}dT + \left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right] dV$, find out the function $P(T, V)$.

3.5 Green's theorem in plane

Line integral around a *closed path* is equal to the double integral over the area A enclosed by the *path*.

$$\oint_c [P(x, y)dx + Q(x, y)dy] = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy \tag{3.20}$$

The line integral should be evaluated in the counterclockwise direction around the boundary of A. Consult the textbook for detailed proof. There is a simple way of memorizing the formula. If we consider the differential in the L.H.S. to be an exact differential, then the line integral should be zero. Now, the condition for exact differential is: $\partial Q/\partial x = \partial P/\partial y$ and these two are present in the R.H.S. in such a combination that the integral is going to be zero if the differential in the L.H.S. is an exact differential.

3.5.1 Exercise

1. Using Green's theorem, prove that:

$$\int\int_A \vec{\nabla} \cdot \vec{V} dxdy = \oint_{\partial A} \vec{V} \cdot \hat{n} ds \tag{3.21}$$

Note that, this can be generalized to write the divergence theorem.

2. Using Green's theorem, prove that:

$$\int\int_A (\vec{\nabla} \times \vec{V}) \cdot \hat{k} dx dy = \oint_{\partial A} \vec{V} \cdot d\vec{r} \tag{3.22}$$

   Note that, this can be generalized to write the Stokes' theorem.

3. Mary L. Boas, chapter 6, section 9, problems 2-12.

3.6 Divergence and divergence theorem

3.6.1 Physical significance of divergence

Let us develop our understanding of divergence using mass flux $\vec{J}$ [velocity $\times$ density]. Note that, whatever we discuss is true for any flux and in general, for any vector field. We already have defined divergence as:

$$\boxed{\text{div}\vec{J} = \vec{\nabla} \cdot \vec{J} = \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}.} \tag{3.23}$$

Divergence represents *net outflow per unit volume*. See class notes/text book for a simple proof for a cubic volume element.

3.6.2 Equation of continuity

By *net outflow*, we mean *outgoing minus incoming* and in general they are not equal, such that the **equation of continuity** is:

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0. \tag{3.24}$$

In steady state,

$$\vec{\nabla} \cdot \vec{J} = 0. \tag{3.25}$$

Note that, the above equations are correct if there exist no source and sink. Otherwise, we have to add a term ψ to take into account the source minus sink part,

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = \psi. \tag{3.26}$$

3.6.3 Divergence theorem: volume and surface integral

As shown in Fig. 3.1, imagine that water is flowing through the cylinder. Now, amount of water crossing the area A' in time t is $(vt)A'\rho$ and this is equal to the amount crossing the area A in time t. Since $A' = A\cos\theta$, we can write the following:

$$(\rho)(vt)(A') = (\rho v)A't = (\rho v)(A\cos\theta)t = (\rho v\cos\theta)At = (\vec{J} \cdot \hat{n})At. \tag{3.27}$$

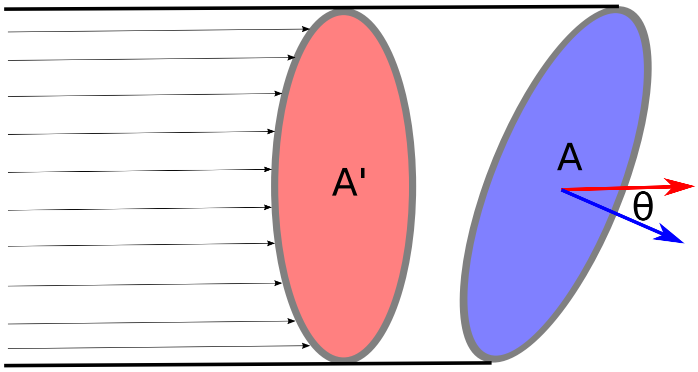

Figure 3.1: Amount of water crossing through area A' is same as amount of water crossing through the area A.

Note that, θ is the angle between the direction of $\vec{v}$ and $\hat{n}$ (unit normal to the surface A). Thus, net amount of water crossing per unit area and per unit time is given by $\boxed{\vec{J}\cdot\hat{n}}$. Now, we can take some area element da on any surface enclosing some volume (for example surface of a sphere) and unit normal $\hat{n}$ to the surface. Thus, mass of water flowing out of the area is given by $(\vec{J}\cdot\hat{n})da$ and the total outflow from the volume enclosed by the surface is

$$\int\int(\vec{J}\cdot\hat{n})da = \int\int\vec{J}\cdot d\vec{a}. \tag{3.28}$$

We already know that divergence is net outflow per unit volume. We can easily argue that (consult textbook or class notes): *net outflow from the volume enclosed by the surface must be equal to the the net outflow from a surface enclosing the volume*, which leads to the **divergence theorem**:

$$\boxed{\int\int\int(\vec{\nabla}\cdot\vec{J})dV = \int\int(\vec{J}\cdot\hat{n})da}. \tag{3.29}$$

Note that, the L.H.S. is a triple integral over the entire volume enclosed by the surface A and R.H.S. is a double integral over the entire surface enclosing the volume V.

3.6.4 Exercise

1. Given that, $\vec{B} = \vec{\nabla}\times\vec{A}$, using the divergence theorem, prove that $\oint\vec{B}\cdot\hat{n}da$ over any closed surface is zero. Can you justify this in terms of simple arguments.

2. Mary L. Boas, chapter-6, section-10, problem 1-10 and 15-16.

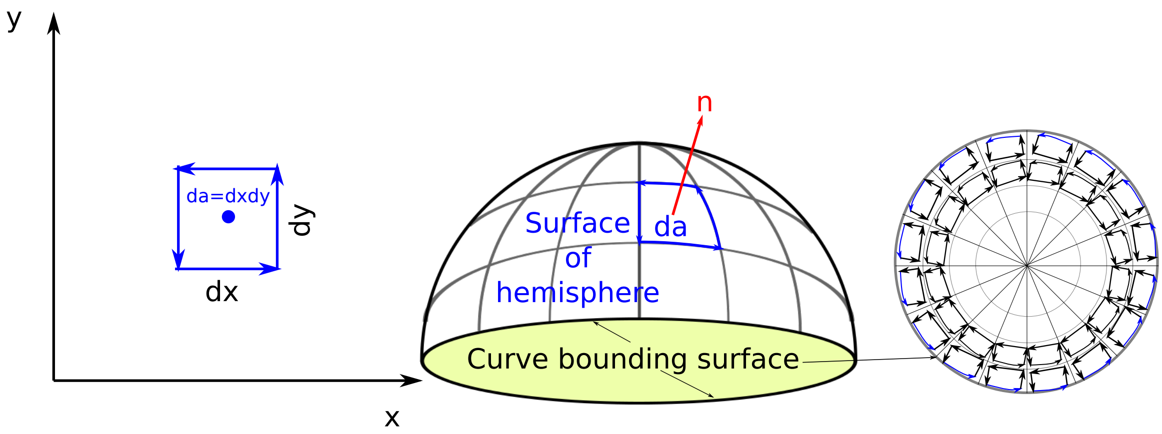

Figure 3.2: (left) Line integral over a closed path in xy plane, such that the normal is pointing towards the $\hat{k}$ at the given point. (center) We generalize the area element and take it to be on the surface of a hemisphere. (right) Flat view of the hemisphere.

3.7 Curl and Stokes' theorem

3.7.1 Physical significance of curl

Again we consider fluid flow and let $\vec{v}$ be the vector field representing the velocity. Now, $\vec{\nabla} \times \vec{v}$ represents the *angular velocity* of the fluid in the neighborhood of a given point. If $\vec{\nabla} \times \vec{v} = 0$ in some region, then the flow is said to be *irrotational* in that region. Interestingly, this is the same mathematical condition for a force field $\vec{F}$ (which is a vector field) to be *conservative*.

3.7.2 Stoke's theorem: surface and line integral

Let $\vec{V}$ be a vector field. For example, $\vec{V}$ can be a force field $\vec{F}$ or it can be $\vec{V} = \vec{v}\rho$ for fluid flow. We want to evaluate the line integral $\oint_c \vec{V} \cdot d\vec{r}$ over the closed path shown in Fig. 3.2. We have already evaluated such integrals (for example, work done $\oint_c \vec{F} \cdot d\vec{r}$ over a closed path). Let the vector field be $\vec{V} = V_x\hat{i} + V_y\hat{j} = P\hat{i} + Q\hat{j}$. You can easily verify that,

$$Pdx + Qdy = \vec{V} \cdot d\vec{r}, \tag{3.30}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} = (\vec{\nabla} \times \vec{V}) \cdot \hat{k}.$$

Now, using Greens theorem $\oint_c (Pdx + Qdy) = \int\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$, we can write that,

$$\oint_c \vec{V} \cdot d\vec{r} = \int\int (\vec{\nabla} \times \vec{V}) \cdot \hat{k} dxdy. \tag{3.31}$$

Now, there is nothing special about xy plane and we can easily generalize the above to write:

$$\oint_c \vec{V} \cdot d\vec{r} = \int\int (\vec{\nabla} \times \vec{V}) \cdot \hat{n} da, \tag{3.32}$$

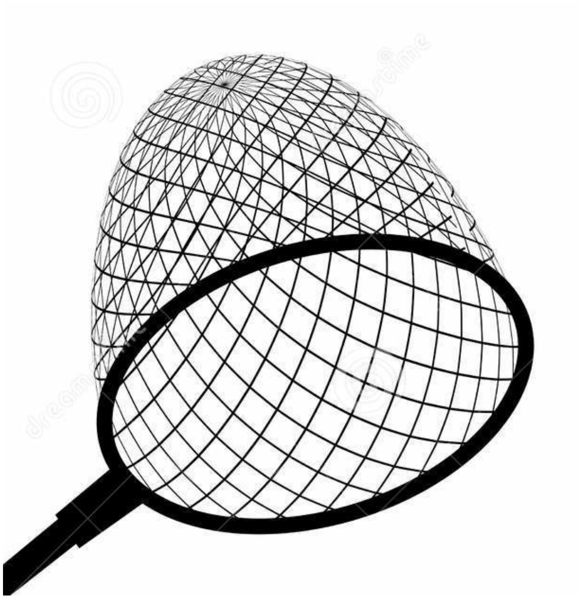

Figure 3.3: In a fishing net, the net forms the open surface and the rim (made of metal or plastic) is the curve bounding the open surface.

where $\hat{n}$ is normal to the area element da and c is the curve surrounding the area element da (see Fig. 3.2).

Now, imagine a surface which is not flat, for example, surface of a hemisphere. We can divide the entire surface of the hemisphere in small area elements da and add all the terms obtained from the above equation. As shown in Fig. 3.2, all the interior line integals cancel each other, because along the border, two adjacent integrals are in opposite direction. However, line integrals around the curve bounding the hemisphere (the outermost circle) do not cancel each other. Thus, surface integral over entire surface of the hemisphere is equal to the line integral around the curve (the circle) bounding the hemisphere. This is the statement of *Stokes' theorem, which relates an integral over an open surface to a line integral around a curve bounding the surface:*

$$\boxed{\oint_{\text{surface boundary}} \vec{V} \cdot d\vec{r} = \int\int_{\text{surface}} (\vec{\nabla} \times \vec{V}) \cdot \hat{n} da}. \tag{3.33}$$

Let us think of a small fishing net, as shown in Fig. 3.3. The net forms the *open surface*, while the rim is the *curve bounding the open surface*. Note that, we can deform the net easily, but the rim does not change. Now, let us think of a net of the shape of a hemisphere. We can deform the net to any other shape, keeping the rim unchanged. If we do this, whatever we agrued to get Eq. 3.33, still remains valid. Let us further assume that the net is made of a stretchable material, which looks like a fishing net when stretched, but converts to something like a badminton racket when unstretched. Accoding to our logic, integral over the surface should be the same for the stretched and unstretched net. Thus, we conclude that, what matters is *the curve bounding the sufrace, not the surface itself.*

This further implies that, all we need is to calculate a surface integral over a flat surface (similar to the badminton racket), instead of a curved surface (similar to the fishing net). The result is going to be same as long as the rim (curve bounding the net) remains same. For example, if we take the bounding curve to be a circle, it does not matter whether we have a perfect hemisphere or deformed hemisphere on top of the circle. We need not even try to calculate the surface integral over a deformed (or perfect) hemisphere. All we need to calculate is a surface integral over the circle (bounding the "hemisphere" of whatever shape).

3.7.3 Vector potential

A vector field is *solenoidal* if $\vec{\nabla} \cdot \vec{V} = 0$. Using the fact that $div(curl)$ and $curl(grad)$ is zero, we can write,

$$\vec{V} = \vec{\nabla} \times \vec{A} + \vec{\nabla} u, \tag{3.34}$$

where $\vec{A}$ is a vector field (vector potential) and u is a scalar field.

3.7.4 Exercise

1. Let us verify the fact that integral over a hemisphere is same as integral over a circle bounding the hemisphere. Assume $\vec{V} = 4y\hat{i} + x\hat{j} + 2z\hat{k}$.
   (a) Find $\int\int(\vec{\nabla} \times \vec{V}) \cdot \hat{n} da$ over the hemisphere $x^2 + y^2 + z^2 = a^2, z \geq 0$.
   (b) Verify that the result will be same if we evaluate the integral over the circle bounding the hemisphere.

   Apply Stokes' theorem to evaluate the integrals

2. Mary L. Boas, chapter-8, section-11, problems 1-15.

   Find vector potential, given the vector field

3. Mary L. Boas, chapter-8, section-11, problems 18-22.