Analysis
MTH301A

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These are the lecture notes prepared for the course MTH301A\(^1\)
I mostly referred in parts to the following texts:


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Syllabus

- \( \mathbb{R} \), Completeness property. Countable and Uncountable
- Metric Spaces and Metric space topology
- Nested set theorem, Baire category theorem
- Compactness, Totally bounded, Characterizations of compactness, Finite intersection property, Continuous functions on compact sets, Uniform continuity
- Connectedness and Path connectedness, Continuous functions
- Riemann integration, Fundamental theorem of calculus. Set of measure zero, Cantor set, Integrable functions
- Convergence of sequence and series of functions: Pointwise and uniform convergence of functions, Series of functions, Power series, Dini’s theorem, Ascoli’s theorem
- Nowhere-differentiable Continuous functions, Weierstrass approximation theorem
Course plan

- This course has been split into L + T + D.
- Lectures (3) and Tutorial (1) have been merged into 3 to 4 videos of time length 40-50 minutes every week. The links of these videos will be shared after the videos are released.²
- There will be a Discussion hour over zoom every Friday from 12.00-12.50 (afternoon).
- About assessment/Evaluation, we opt for
  - In-video questions for assignment submission (20 percent)
  - Assignments/homework (40 percent)
  - Online oral examination (40 percent)
- There will be mid-semester and end-semester examinations. These will be held during the prescribed examination period.
- For students with limited or no network access, the Institute will be making the course materials available as found feasible.

²The course will managed on https://hello.iitk.ac.in/
Curiosity

**Questions.** Do you believe in real numbers? Who are these creatures? Are these real or imaginary?

- B Assume the existence (be lazy be happy)
- A Prove the existence (hitting against the hard)

**Difficulty.** If you draw a point on a plane paper, and if see it by a magnifier, you find it to be disc-like! What/where is the point?

**Philosophy** Is there is a particle (point) of mass (measure) 0?

B ↓ A Understand $\mathbb{R}$ as a set containing all rational numbers, which satisfies (optimal) axioms. First two building blocks:

- $\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$
- Arithmetic operations of addition $+$ and multiplication $\cdot$
- Order structure $<$
Axiomatic approach

Consider the set $\mathbb{R}$ of numbers which contains $\mathbb{Q}$ (and hence $\mathbb{Z}$) with arithmetic operations $+$ and $\cdot$ and order structure $<$ satisfying

- $x + y = y + x$ for all $x, y \in \mathbb{R}$
- $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$
- $x + 0 = x$ for all $x \in \mathbb{R}$
- for every $x \in \mathbb{R}$, there exists a unique $y \in \mathbb{R}$ (denoted by $-x$) such that $x + y = 0$
- $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{R}$
- $x \cdot 1 = x$ for all $x \in \mathbb{R}$
- for every nonzero $x \in \mathbb{R}$, there exists a unique $y \in \mathbb{R}$ (denoted by $x^{-1}$ or $1/x$) such that $x \cdot y = 1$
- $(x + y) \cdot z = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$
- $x < y$, $x = y$ or $x > y$ for all $x, y \in \mathbb{R}$ (exactly one possibility)
In the following, we understand that $x < y$ iff $y - x > 0$. Further, we assume that sum and product of positive numbers are positive.

**Problem**

*Use the above axioms to verify the following:*

1. *If $x < y$ and $z \in \mathbb{R}$, then $x + z < y + z$.*
2. *If $x < y$ and $y < z$, then $x < z$.*
3. *If $x < y$ and $z \in \mathbb{R}$, then*
   
   $$\begin{cases} 
   x \cdot z < y \cdot z & \text{if } z > 0, \\
   y \cdot z < x \cdot z & \text{if } z < 0, \\
   x \cdot z = y \cdot z & \text{if } z = 0. 
   \end{cases}$$

4. *If $x < y$, then $-y < -x$.*
5. *Either $x^2 > 0$ or $x = 0$.*
6. *If $0 < x < y$, then $0 < 1/y < 1/x$."

**Hint.**

Since $z - x = z - y + y - x > 0$, 2 follows from 1.
Problem
Use only the last problem to verify the following:

1. Given two real numbers $x, z$ such that $x < z$, there exists $y \in \mathbb{R}$ such that $x < y < z$.
2. If $x \leq y + z$ for all $z > 0$, then $x \leq y$.
3. If $0 < x < y$, then for any positive integer $n$, $0 < x^n < y^n$.

Hint.
For 1, try $y = \frac{x+z}{2}$. For 2, prove by contradiction.

- $\mathbb{Q}$ satisfies all the axioms mentioned before the last problem.

Question. How do we differentiate $\mathbb{R}$ from $\mathbb{Q}$? Is there a property (of $\mathbb{R}$) not enjoyed by $\mathbb{Q}$?
Definition
Let \( \emptyset \neq A \subseteq \mathbb{R} \). We say that \( A \) is bounded from above if there is \( \alpha \in \mathbb{R} \) (an upper bound) such that \( a \leq \alpha \) for every \( a \in A \).

Problem
Show that any finite nonempty subset \( A \) of \( \mathbb{R} \) is bounded with an upper bound belonging to it. Give an infinite subset of \( \mathbb{R} \), where this fails.

Hint.
This can be proved by the induction on \( \text{card}(A) \). If \( \text{card}(A) = 1 \), then \( A = \{a\} \) for some \( a \in A \), and \( a \) is an upper bound. Suppose that the conclusion holds for all sets \( A \) with \( \text{card}(A) = n \), and let \( B \) be a subset of \( \mathbb{R} \) such that \( \text{card}(B) = n + 1 \). Since \( B \neq \emptyset \), there exists \( b \in B \). Apply the induction hypothesis to \( B \setminus \{b\} \) and use the law of trichotomy. For last part, take \( A = (0,1) \). None of the upper bounds of \( A \) belongs to \( A \).
Least upper bound

Definition
Let $\emptyset \neq A \subseteq \mathbb{R}$. We say that $\alpha$ is a least upper bound (lub) for $A$ if

- $\alpha$ is an upper bound for $A$, that is, $a \leq \alpha$ for all $a \in A$, and
- $\beta$ is an upper bound for $A$, then $\alpha \leq \beta$.

Remark. lub is unique. If a set contains its upper bound (for example, $(0, 1]$), then maximum of that set is its lub.

Problem
Show that $\alpha \in \mathbb{R}$ is lub for $A \subseteq \mathbb{R}$ iff $\alpha$ is an upper bound for $A$ and if $\beta < \alpha$, then there exists $a \in A$ such that $\beta < a \leq \alpha$.

Hint.
If $\beta < \alpha$, then $\beta$ is not an upper bound of $A$.

Example
1 is the lub for $A = (0, 1)$. Indeed, if $\alpha < 1$ is the lub, then $a := \frac{1+\alpha}{2} \in (\alpha, 1)$, so $\alpha$ is not an upper bound for $(0, 1)$.
Unboundedness of integers

It is tempting to conclude that $\mathbb{Z}$ is not bounded from above without a proof.

**Difficulty.** If $\mathbb{Z}$ is bounded above by $x \in \mathbb{R}$ how to ensure that there is an integer $n > x$ (this problem does not occur if the upper bound is an integer)?

**Solution.** If you can not prove an assertion, convert it into a property (with no disrespect for Archimedes)!

**Archimedean property.** Given $x, y \in \mathbb{R}$, with $x > 0$, there exists a positive integer $n$ such that $nx > y$.

**Interpretation.** Howsoever tiny may be your foot-step $x$, you can cover any finite distance $y$ (for example, the distance between earth and moon) in finitely many steps $n$. 
LUB property or completeness axiom

- Any nonempty subset of $\mathbb{R}$, which is bounded above in $\mathbb{R}$, has lub in $\mathbb{R}$.

This is the property which differentiates $\mathbb{R}$ from $\mathbb{Q}$:

**Example**

The subset $A = \{a \in \mathbb{Q} : a^2 < 2\}$ is nonempty and bounded above (1 $\in$ $A$ and 2 is an upper bound for $A$, but it does not have lub in $\mathbb{Q}$). To see the latter statement, assume that $\alpha$ is an lub of $A$. If $\alpha^2 < 2$, then for any positive integer $k$,

$$(\alpha + 1/k)^2 = \alpha^2 + 2\alpha/k + 1/k^2 \leq \alpha^2 + 5/k,$$

so that $(\alpha + 1/k)^2 < 2$ for sufficiently large $k$ (we need here the fact that $\mathbb{Z}$ is not bounded from above). This however implies that $\alpha + 1/k \in A$, contradicting the assumption that $\alpha$ is lub. Thus $\alpha^2 \not< 2$. Similarly, one can see that $\alpha^2 \not< 2$. Thus $\alpha^2 = 2$ or $\alpha \notin \mathbb{Q}$. 
LUB property and Archimedean property

**Theorem**

$LUB$ property $\implies$ Archimedean property.

**Proof.**

Assume the $LUB$ property, Suppose that the Archimedean property fails, that is, there exist $x, y$ with $x > 0$ such that $nx \leq y$ for every positive integer $n$. Thus $n \leq y/x$ for every (positive) integer $n$. Thus $\mathbb{Z}$ is bounded above and hence by $LUB$ property, it has lub, say, $\alpha$. Now $\alpha - 1$ is not an upper bound, so for some integer $N$, $\alpha - 1 < N$. However, this implies that $\alpha < N + 1$ and $N + 1$ is an integer, which contradicts the assumption that $\alpha$ is lub.

**Problem**

Describe $\cap_{n \geq 1} (0, 1/n)$ and $\cap_{n \geq 1} (n, \infty)$. 
Problem (Greatest Integer Function)

Assume the LUB property and let \( x \in \mathbb{R} \). Show that there exists a (unique) \( m \in \mathbb{Z} \) such that \( m \leq x < m + 1 \) (\( m \) is denoted by \([x]\)).

Hint.

Consider the nonempty set \( A = \{ k \in \mathbb{Z} : k \leq x \} \), which is bounded above. Let \( \alpha = \text{lub}(A) \). Since \( \alpha - 1 \) is not an upper bound, there exists \( m \in A \) such that \( \alpha - 1 < m \leq x \). Verify that \( x < m + 1 \).

Theorem

Assume the LUB property and let \( a, b \in \mathbb{R} \) be such that \( a < b \). Then there exists \( r \in \mathbb{Q} \) (resp. \( r \in \mathbb{R} \setminus \mathbb{Q} \)) such that \( a < r < b \).

Proof of Theorem.

By Archimedean property, we find an integer \( n \geq 1 \) such that \( n(b - a) > 1 \). By the last problem, there exists \( m \in \mathbb{Z} \) such that \( na < m \leq na + 1 \) (take \( m = [na] + 1 \)). Thus \( na < m < nb \) or \( r = m/n \). For the rest, apply this to \( a - \sqrt{2} \) and \( b - \sqrt{2} \).
Nested Interval Theorem

**Theorem**
Assume the LUB property and let $J_n = [a_n, b_n]$ be an interval in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all integers $n \geq 1$. Then $\cap_{n \geq 1} J_n \neq \emptyset$.

**Proof.**
Consider the set $A = \{ x \in \mathbb{R} : x = a_n $ for some $n \geq 1 \}$ and note that $A$ is nonempty and bounded above. Let $\alpha = \text{lub}(A)$. Since each $b_n$ is an upper bound for $A$ (since $J_{n+1} \subseteq J_n$ for all integers $n \geq 1$), we have $a_n \leq \alpha \leq b_n$, that is, $\alpha \in \cap_{n \geq 1} J_n$.

**Problem**
Assume the LUB (and GLB) property. Show that any bounded infinite subset $A$ of $\mathbb{R}$ has at least one accumulation point.

**Hint.**
Suppose $A \subseteq [a_1, b_1]$ with $a_1 = \text{glb}(A)$, $b_1 = \text{lub}(A)$. Let $[a_2, b_2]$ be one of the intervals from $[a_1, (a_1 + b_1)/2]$ and $[(a_1 + b_1)/2, b_1]$ which contains infinitely many points from $A$. Continue this.
We may understand now $\mathbb{R}$ as the ”complete ordered field” (the set $\mathbb{R}$ satisfying all the axioms mentioned earlier including the completeness axiom).

**Question.** Does a complete ordered field exist ? If yes, is it unique (up to isomorphism) ?

To address the issue of the existence of $\mathbb{R}$, we mention two approaches; one due to Dedekind (based on ”Dedekind cuts”) and other due to Cantor (based on the completion of $\mathbb{Q}$). We follow the second approach and for this, we need to know the convergence of sequences in $\mathbb{R}$.

Recall the definitions of Cauchy and convergent sequences in $\mathbb{R}$.

- Cauchy sequence is bounded and convergent sequence is a Cauchy sequence.
- Every Cauchy sequence with convergent subsequence is convergent.
Cauchy Completeness of $\mathbb{R}$

**Theorem**
_Assume the LUB property. Then $\mathbb{R}$ is Cauchy complete, that is, every Cauchy sequence in $\mathbb{R}$ is convergent._

**Proof 1.**
Let $A = \{x \in \mathbb{R} : \text{there exists } N \geq 1 \text{ such that } x < x_n \text{ for } n \geq N\}$ and let $\epsilon > 0$. Note that there exists $n_0 \geq 1$ (dependent on $\epsilon$) such that $x_{n_0} - \epsilon/2 \in A$ (since $\{x_n\}_{n \geq 1}$ is a Cauchy sequence). Thus $A$ is a nonempty set. Further, $A$ is bounded above by $x_{n_0} + \epsilon/2$ (verify by contradiction). Let $\alpha = \text{lub}(A)$. Since $x_{n_0} - \epsilon/2 \in A$ and $A$ is bounded above by $x_{n_0} + \epsilon/2$, $|x_{n_0} - \alpha| \leq \epsilon/2$. Thus for $n \geq n_0$,

$$|x_n - \alpha| \leq |x_n - x_{n_0}| + |x_{n_0} - \alpha| < \epsilon.$$ 

This completes the proof. □
It is possible to obtain the Cauchy completeness without using the LUB property.

**Proof II.**

Let \( A = \{ x \in \mathbb{R} : x = x_n \text{ for some } n \geq 1 \} \). Since \( A \) is bounded, \( A \subseteq [a_1, b_1] \) for some \( a_1, b_1 \in \mathbb{R} \). Let \([a_2, b_2]\) be one of the intervals from \([a_1, (a_1 + b_1)/2]\) and \([(a_1 + b_1)/2, b_1]\) which contains infinitely many points from \( A \). Continue this to obtain the intervals \([a_n, b_n]\), \( n \geq 1 \), containing infinitely many points of \( A \). By the nested interval theorem, \( \bigcap_{n \geq 1} [a_n, b_n] \) contains \( x \) for some \( x \in \mathbb{R} \).

Now for each \( n \geq 1 \), if we choose \( x_{k_n} \in [a_n, b_n] \) such that \( k_n \leq k_{n+1} \), we obtain a subsequence \( \{x_{k_n}\}_{n \geq 1} \) such that \( |x_{k_n} - x| \leq b_n - a_n = (b_1 - a_1)/2^n \to 0 \) as \( n \to \infty \). Since \( \{x_n\}_{n \geq 1} \) is a Cauchy sequence, it must be convergent now. \( \square \)

The proof above yields the following:

**Theorem (Bolzano-Weierstrass Theorem)**

Every bounded sequences admits a convergent subsequence.
Let \( R \) be the set of Cauchy sequences in \( \mathbb{Q} \). Let \( \{x_n\}_{n \geq 1}, \{y_n\}_{n \geq 1} \) belong to \( R \) and define the equivalence relation

\[
\{x_n\}_{n \geq 1} \sim \{y_n\}_{n \geq 1} \quad \text{if} \quad |x_n - y_n| \rightarrow 0.
\]

- Define the set \( \mathbb{R} \) as \( R/\sim \) (set of equivalence classes \( [\{x_n\}_{n \geq 1}] \)).
- Identify the element \( x \) in \( \mathbb{Q} \) with constant sequence \( \{x_n = x\}_{n \geq 1} \).
- \( \{x_n\}_{n \geq 1} < \{y_n\}_{n \geq 1} \) if \( \{y_n - x_n\}_{n \geq 1} \) eventually consists of positive numbers.
- It is a laborious work to verify that \( \mathbb{R} \) defines an ordered field which satisfies LUB property. Refer to the article \( \text{http://www.math.ucsd.edu/~tkemp/140A/Construction.of.R.pdf} \).^3

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^3Cauchy’s Construction of R, Todd Kemp
Theorem (Density of rationals)

If \( r = \{r_n\}_{n \geq 1} \in \mathbb{R} \) and any small (rational) number \( \epsilon \), then there exists a rational number \( q \) such that \( |r - q| < \epsilon \), that is, for some integer \( N \geq 1 \), \( |r_n - q| < \epsilon \) for all \( n \geq N \).

Proof.

Since \( \{r_n\}_{n \geq 1} \) is a Cauchy sequence, for some \( N \geq 1 \), \( |r_n - r_N| < \epsilon \) for all \( n \geq N \). Let \( q = r_N \in \mathbb{Q} \).

Let \( \mathbb{R} = \{[\{x_n\}_{n \geq 1}] : \{x_n\}_{n \geq 1} \in \mathbb{R}\} \) and define a metric on \( \mathbb{R} \) by

\[
d([\{x_n\}], [\{y_n\}]) = \lim_{n \to \infty} |x_n - y_n| \ (\text{an equivalence class})
\]

(the limit exists since \( \{x_n - y_n\}_{n \geq 1} \) is a Cauchy sequence and hence convergent). Then \( \mathbb{R} \) is a complete metric space, and if \( i : \mathbb{Q} \to \mathbb{R} \) is given by \( i(\{x\}) = [\{x\}] \) for every \( x \in \mathbb{Q} \), then \( i \) is injective.
Countable and Uncountable sets

**Definition**
A set $A$ is called **countably infinite** if there is a bijection $f : \mathbb{N} \rightarrow A$. If $A$ is finite or countably infinite, we say that $A$ is **countable**.

**Example**

- Any subset $A$ of $\mathbb{N}$ is countable (either $A$ is finite or $A = \{n_k\}_{k \geq 1}$; in the latter case, define $f(k) = n_k$)
- Any subset of a countable set is countable
- $\mathbb{Z}$ is countable (define $g : \mathbb{Z} \rightarrow \mathbb{N}$ by $g(n) = 2n$ if $n \geq 1$, and $g(n) = -2n + 1$ if $n \leq 0$; now let $f = g^{-1}$)
- $\mathbb{N} \times \mathbb{N}$ is countable (define $g(m, n) = 2^{m-1}(2n-1)$; let $f = g^{-1}$)
- Countable union of countable sets is countable (if $A = \{x_{m,n} : m \in \mathbb{N}, \ n \in \mathbb{N}\}$, then $g : \mathbb{N} \times \mathbb{N} \rightarrow A$ given by $f(m, n) = a_{m,n}$ is bijective)

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4 Cantor is a ‘corrupter of youth’. - L. Kronecker, as quoted by Schoenflies and requoted by Barry Simon
Problem
Verify the following:
1. If \( g : A \to B \) is bijective, show that \( A \) is countable iff so is \( B \).
2. If \( g : A \to B \) is injective, show that \( A \) is countable if so is \( B \).

Hint.
1. If \( h : \mathbb{N} \to A \) is bijective, then so is \( g \circ h : \mathbb{N} \to B \).
2. If \( h : \mathbb{N} \to g(B) \) is bijective, then so is \( g^{-1} \circ h : \mathbb{N} \to A \).

Problem
Show that \( \mathbb{Q} \) is countably infinite.

Hint.
Write \( \mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\} \) (union of positive/negative/zero rationals). To see that \( \mathbb{Q}_+ \) is countable, define \( f : \mathbb{Q}_+ \to \mathbb{N} \times \mathbb{N} \) by \( f(m/n) = (m, n) \), where \( m \) and \( n \) are coprime to each other. Note that \( f \) is injective and \( \mathbb{N} \times \mathbb{N} \) is countable.
Problem

For an index set $I$, let $\{J_\alpha\}_{\alpha \in I}$ be a collection of disjoint open intervals such that $\mathbb{R} = \bigcup_{\alpha \in I} J_\alpha$. Show that $I$ is countable.

Solution.

We know that each interval $J_\alpha$ contains a rational number $r_\alpha$. Define $g : I \to \mathbb{Q}$ by $g(\alpha) = r_\alpha$. If, for some $\alpha, \beta \in I$, $g(\alpha) = g(\beta)$, then $r_\alpha = r_\beta$, and hence $\alpha = \beta$ (if $\alpha \neq \beta$, then $J_\alpha \cap J_\beta = \emptyset$). This shows that $g$ is injective. Since $\mathbb{Q}$ is countable, by the problem on the previous slide, $I$ is also countable.

Problem

For an index set $I$, let $\{x_\alpha\}_{\alpha \in I}$ be a collection of mutually orthonormal vectors in an inner-product space $X$. If $X$ has a countable dense subset, then show that $I$ is countable.
Discontinuities of a monotone function

For a function $f : (0, 1) \to \mathbb{R}$ and $c \in (0, 1)$, let $f(c_-)$ and $f(c_+)$ denote the left and right hand side limits of $f$ at $c$, respectively.

**Theorem**

Let $f : (a, b) \to \mathbb{R}$ be monotone. Then the set $D$ of discontinuities of $f$ is countable.

**Proof.**

Without loss of generality, we may assume that $f$ is increasing. Note that $D = \{c \in (a, b) : I_c = (f(c_-), f(c_+)) \neq \emptyset\}$. For $c, d \in D$, let $t \in [a, b]$ be such that $c < t < d$. Then

$$f(c_+) = \text{glb}\{f(x) : x > c\} \leq f(t) \leq f(d_-) = \text{lub}\{f(y) : y < d\}.$$  

This shows that $I_c \cap I_d = \emptyset$. Since each interval $I_c$ contains a rational and rationals are countable, $D$ is countable.

**Problem**

Show that a monotone surjection $f : [a, b] \to [c, d]$ is continuous.
**Theorem**

There is no surjection \( f : \mathbb{N} \to [0, 1] \).

**Proof.**

Consider the decimal expansion (possibly more than one) of \( f(n) \):

\[
  f(n) = 0 \cdot a_{n1}a_{n2} \ldots a_{nn} \ldots, \quad n \geq 1
\]

(for the existence of decimal expansion, refer to Proposition 1.8 of [Real Analysis, Carothers, N. L.]). Let \( x = 0 \cdot a_1a_2 \ldots a_n \ldots \in \mathbb{R} \) where \( a_n \in \{1, \ldots, 8\} \setminus \{a_{nn}\} \). Note that \( x \neq f(n) \) for any \( n \geq 1 \), and hence \( x \notin f(\mathbb{N}) \).
Corollary (Cantor)

The interval $[0, 1]$ in $\mathbb{R}$ is not countable.

Problem

*Given two points in $\mathbb{R} \times \mathbb{R} \setminus \mathbb{Q} \times \mathbb{Q}$, show that there exists uncountably many paths (finite union of line segments) in $\mathbb{R} \times \mathbb{R} \setminus \mathbb{Q} \times \mathbb{Q}$, which join these two points.*

Hint.

Fix two points $A$ and $B$ in the plane and draw a square with one of the diagonals being the straight line $C$ joining these two points. Consider the diagonal $D$ orthogonal to $L$ and note that it is not countable. Now produce one path corresponding to each point on $D$ which joins $A$ and $B$. How many of these touch $\mathbb{Q} \times \mathbb{Q}$?
Example
Consider the power set \( P(\mathbb{N}) \) of all subsets of \( \mathbb{N} \). Let \( f : \mathbb{N} \to P(\mathbb{N}) \) be any function. We claim that \( f \) is not surjective. Indeed, define \( A \in P(\mathbb{N}) \) by the property that

\[
k \in A \iff k \notin f(k).
\]

Note that \( A \neq f(k) \) for any \( k \geq 1 \), and hence the claim stands verified. In particular, \( P(\mathbb{N}) \) is not countable.

Problem
Let \( A \) be a nonempty set. There is no map from \( A \) into the power set \( P(A) \) of \( A \), which is surjective.
Example

Let $C$ denote the Cantor set obtained by removing $2^{n-1}$ centrally situated disjoint open subintervals $U_{1,n}, \ldots, U_{2^n-1,n}$ of $[0,1]$ each of length $1/3^n$ at the $n$th stage, where $n = 1, 2, \ldots$. Specifically, if $U_{1,1} = (1/3, 2/3)$, $U_{2,1} = (1/9, 2/9)$, $U_{2,2} = (7/9, 8/9)$, $\ldots$, $U_{n,1} = (1/3^n, 2/3^n)$, $\ldots$, $U_{n,2^n-1} = (1 - 2/3^n, 1 - 1/3^n)$, then

$$C = \cap_{n \geq 1} C_n, \quad \text{where} \quad C_n = [0,1] \setminus \left( \bigcup_{k=1}^{n} \bigcup_{j=1}^{2^n-1} U_{k,j} \right).$$

Clearly, all end-points $1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots$ of $U_{j,k}$ belong to $\Delta$. Are there any other points in $C$?

**Answer** Yes, take $1/4$ (since $1/4 \notin U_{k,j}$ for all $j, k$)

**Question** What is the trick to generate more elements in $C$?
Example (continued ...)

- Any $x \in [0, 1]$ has two distinct representations of the form $\sum_{n=1}^{\infty} a_n/3^n$ with $a_n \in \{0, 1, 2\}$ if and only if $x$ belongs to the set $\{1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots\}$.

- Any $x \in C$ has unique representation $\sum_{n=1}^{\infty} a_n/3^n$, where $a_n \in \{0, 2\}$. (For instance, $1/3 = 0.1 = 0.0222\ldots$).

Let $\{0, 2\}^\mathbb{N}$ denote the collection of sequences in 0 and 2. Define the map $f : \{0, 2\}^\mathbb{N} \to C$ by $f(a_n) = \sum_{n=1}^{\infty} a_n/3^n$. By the above facts, $f$ is a bijection. The following shows that $C$ is not countable.

**Problem**

*There exists a bijection between $\{0, 2\}^\mathbb{N}$ and $P(\mathbb{N})$.***

**Hint.**

For a subset $A$ of $\mathbb{N}$, define a sequence $\{a_n\}_{n \geq 1}$ by

$$a_n = \begin{cases} 
2 & \text{if } n \in A, \\
0 & \text{if } n \notin A.
\end{cases}$$
Metric spaces

Definition
A metric \( d \) on a nonempty set \( X \) (refer as the metric space) is a map \( d : X \times X \to [0, \infty) \) that obeys

- (Symmetry) For all \( x, y \in X \), \( d(x, y) = d(y, x) \)
- (Triangle inequality) For all \( x, y, z \in X \),
  \[ d(x, z) \leq d(x, y) + d(y, z) \]
- (Zero property) For all \( x, y \in X \), \( d(x, y) = 0 \) iff \( x = y \).

Interpretation
- The distance of \( x \) from \( y \) is same as the distance of \( y \) from \( x \).
- If we consider the triangle \( \triangle \) with vertices \( x, y, z \) and lengths of sides being \( d(x, y), d(y, z) \) and \( d(x, z) \), then the length of any side is at most the sum of lengths of remaining two sides.
- The distance of a point from itself is 0, and if the distance between two points is 0, then these points must coincide.
Example

$\mathbb{R}$ is a metric space with metric $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. The symmetry and zero property follow immediately from the facts that $|x| = |-x|$ and $|x| = 0$ iff $x = 0$. To see the triangle inequality, note that since $x \leq |x|$, we have

$$|x - z|^2 = (x - y + y - z)^2 = (x - y)^2 + 2(x - y)(y - z) + (y - z)^2 \leq (|x - y| + |y - z|)^2.$$ 

Now take square roots on both sides.

Remark

- One may give an alternate proof of the triangle inequality which does not involve square root (consider the cases in which $x + y \geq 0$ and $x + y < 0$, and use the definition of $|\cdot|$).
- Note that $|x + y| = |x| + |y|$ iff either $x$ and $y$ are both nonnegative or both nonpositive (Exercise).
Young's Inequality

Theorem
Let \( p, q > 1 \) be conjugate exponents, that is, \( 1/p + 1/q = 1 \). For positive numbers \( a, b \), \( ab \leq \frac{a^p}{p} + \frac{b^q}{q} \). Equality holds iff \( a^p = b^q \).

Proof.
Given positive real numbers \( a \leq b \), consider

\[
D_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a, \ 0 \leq y \leq x^{p-1}\},
\]

\[
D_2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq b, \ 0 \leq x \leq y^{q-1}\}.
\]

We verify the following (see [page 43, Carothers] for a figure):

1. \( D_1 \cap D_2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a, \ y = x^{p-1}\} \). Indeed, since \((p - 1)(q - 1) = 1\), \( y = x^{p-1} \) iff \( x = y^{q-1} \).

2. \( R = \{(x, y) \in \mathbb{R} : 0 \leq x \leq a, \ 0 \leq y \leq b\} \subset D_1 \cup D_2 \).

Thus \( \text{Area}(R) = ab \leq \text{Area}(D_1 \cup D_2) \leq \text{Area}(D_1) + \text{Area}(D_2) = \frac{a^p}{p} + \frac{b^q}{q} \). Moreover, equality holds iff \( a^p = b^q \).
Hölder’s Inequality

Corollary

Let \( p, q > 1 \) be conjugate exponents, that is, \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^d = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbb{R}\} \). Then

\[
\left| \sum_{n=1}^{d} a_n b_n \right| \leq \left( \sum_{n=1}^{d} |a_n|^p \right)^{1/p} \left( \sum_{n=1}^{d} |b_n|^q \right)^{1/q}.
\]

Proof.

Let \( \|a\|_p := \left( \sum_{n=1}^{d} |a_n|^p \right)^{1/p} \) and \( \tilde{a}_n = |a_n|/\|a\|_p, \tilde{b}_n = |b_n|/\|b\|_q \), \( n = 1, \ldots, d \). By Young’s Inequality, \( |\tilde{a}_n| |\tilde{b}_n| \leq |\tilde{a}_n|^p/p + |\tilde{b}_n|^q/q \).

Thus

\[
\frac{1}{\|a\|_p \|b\|_q} \sum_{n=1}^{d} |a_n| |b_n| = \sum_{n=1}^{k} |\tilde{a}_n| |\tilde{b}_n|
\]

\[
\leq \sum_{n=1}^{d} |\tilde{a}_n|^p/p + \sum_{n=1}^{d} |\tilde{b}_n|^q/q = \frac{1}{p} + \frac{1}{q} = 1.
\]

\( \square \)
Example
For a positive integer $d \geq 1$, $\mathbb{R}^d = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbb{R}\}$ is a metric space with metric

$$d(x, y) = \sqrt{\sum_{j=1}^{d} |x_j - y_j|^2}, \quad x, y \in \mathbb{R}^d.$$ 

To see the triangle inequality, it suffices to check that

$$\|a + b\|_2 \leq \|a\|_2 + \|b\|_2, \quad a, b \in \mathbb{R}^d,$$

where $\|a\|_2 := \left(\sum_{n=1}^{d} |a_n|^2\right)^{1/2}$. To see this, note that by Hölder’s Inequality (with $p = 2 = q$),

$$\|a + b\|_2^2 = \sum_{j=1}^{d} |a_j|^2 + 2 \sum_{i,j=1}^{d} a_i b_j + \sum_{j=1}^{d} |b_j|^2$$

$$\leq \|a\|_2^2 + 2\|a\|_2\|b\|_2 + \|b\|_2^2 = (\|a\|_2 + \|b\|_2)^2.$$
Problem

For a positive integer \( d \geq 1 \) and \( p \geq 1 \), show that \( \mathbb{R}^d \) is a metric space with metric \( d_p \) given by

\[
d_p(x, y) = \left( \sum_{j=1}^{d} |x_j - y_j|^p \right)^{1/p}, \quad x, y \in \mathbb{R}^d.
\]

This result fails for \( 0 < p < 1 \).

Hint.

To see that \( \|a + b\|_p \leq \|a\|_p + \|b\|_p \), note that

\[
\sum_{j=1}^{d} |a_j + b_j|^p \leq \sum_{j=1}^{d} |a_j| |a_j + b_j|^{p-1} + \sum_{j=1}^{d} |b_j| |a_j + b_j|^{p-1},
\]

and apply Hölder’s inequality (two times). To see the second part, let \( d = 2 \), \( a = (1, 0) \), \( b = (0, 1) \) and \( p = 1/2 \).
Problem
For a positive integer \( d \geq 1 \), show that \( \mathbb{R}^d \) is a metric space with metric \( d_\infty \) given by \( d_\infty(x, y) = \max_{j=1}^{d} |x_j - y_j| \) for \( x, y \in \mathbb{R}^d \).

Problem
For \( p \geq 1 \) and for a sequence \( \{x_n\}_{n \geq 1} \) of real numbers, define
\[
\|x\|_p = \left( \sum_{j=1}^{\infty} |x_j|^p \right)^{1/p} \in [0, \infty].
\]
Verify the following:
\begin{enumerate}
\item \( \ell^p = \{\{x_n\}_{n \geq 1} : \|x\|_p < \infty\} \) is a vector space over \( \mathbb{R} \).
\item \( \ell^p \) is a metric space with metric \( d_p \) given by
\[
d_p(x, y) = \left( \sum_{j=1}^{\infty} |x_j - y_j|^p \right)^{1/p}, \quad x, y \in \ell^p.
\]
\end{enumerate}

Proof.
Both parts essentially follow from the triangle inequality. We already know the conclusion for truncated sequences \( \{a_n\}_{k=1}^{N} \) and \( \{b_n\}_{k=1}^{N} \) for every integer \( N \geq 1 \). Now let \( N \to \infty \). \( \square \)
Let \( a, b \in \mathbb{R} \) be such that \( a < b \).

**Lemma**

If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( \sup_{x \in [a, b]} |f(x)| < \infty \).

**Proof.**

Assume that \( \sup_{x \in [a, b]} |f(x)| = \infty \). Then, for any integer \( n \geq 1 \), there exists \( x_n \in [a, b] \) such that \( |f(x_n)| > n \). By Bolzano-Weierstrass theorem, \( \{x_n\}_{n \geq 1} \) admits a convergent subsequence \( \{x_{n_k}\}_{k \geq 1} \), and hence by (sequential) continuity of \( f \), \( \{f(x_{n_k})\}_{k \geq 1} \) is convergent. However, a convergent sequence is bounded, and on the other hand, \( |f(x_{n_k})| > n_k \). Not possible!  

**Example**

Consider the vector space \( C[a, b] \) of real-valued continuous functions defined on \([a, b]\). Then \( C[a, b] \) is a metric space with metric \( d_\infty \) given by

\[
d_\infty(f, g) = \sup_{x \in [a, b]} |f(x) - g(x)|. \quad f, g \in C[a, b].
\]
Let $X$ be a metric space with metric $d$ (denoted by $(X, d)$).

**Question** Given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, when can we say that these are equivalent/isomorphic metric spaces?

**Clue** We know that two sets $X$ and $Y$ are equivalent if there is a bijection between $X$ and $X'$. Similarly, two vector spaces $X$ and $Y$ are isomorphic if there is a bijective linear map between $X$ and $Y$.

**Guess** So, for two metric spaces to be equivalent, we need to impose some constraint on the given bijection that will take the metric structures into account (for instance, sequential continuity).

**Question** What do we mean by metric structures?

**Clue** The metric $d$ is the distance function, which measures distance between two points. One may look for a metric analog of a disc/ball in the metric space.
Metric balls

Let $(X, d)$ be a metric space. For $x_0 \in X$ and a real number $r > 0$, define the open ball and closed ball around $x_0$, respectively, by

$$B_r(x_0) = \{ x \in X : d(x, x_0) < r \}, \quad \overline{B_r(x_0)} = \{ x \in X : d(x, x_0) \leq r \}.$$

**Example (Discrete Geometry)**

Let $X$ be a nonempty set. For every $x, y \in X$, define the discrete metric $d_0$ by $d_0(x, x) = 0$ and $d_0(x, y) = 1$ if $x \neq y$. Then

$$B_r(x_0) = \begin{cases} \{x_0\} & \text{if } r \leq 1, \\ X & \text{otherwise} \end{cases}, \quad \overline{B_r(x_0)} = \begin{cases} \{x_0\} & \text{if } r < 1, \\ X & \text{otherwise} \end{cases}.$$

The above calculations may be applied to $X = \mathbb{R}$.

**Example (One dimensional geometry)**

Let $\mathbb{R}$ be the real line with $d_1(x, y) = |x - y|$, $x, y \in \mathbb{R}$. Then

$$B_r(x_0) = (x_0 - r, x_0 + r), \quad \overline{B_r(x_0)} = [x_0 - r, x_0 + r].$$
Example (Plane geometry)

Let $\mathbb{R}^2$ be the real plane with $d_2(x, y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. If $x_0 = (x_{01}, x_{02})$, then

$$B_r(x_0) = \{ x \in \mathbb{R}^2 : |x_1 - x_{01}|^2 + |x_2 - x_{02}|^2 < r^2 \}$$ (circular region),

$$\overline{B_r(x_0)} = \{ x \in \mathbb{R}^2 : |x_1 - x_{01}|^2 + |x_2 - x_{02}|^2 \leq r^2 \}.$$

What if $d_2$ is replaced by $d_\infty(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$?

$$B_r(x_0) = \{ x \in \mathbb{R}^2 : |x_1 - x_{01}| < r, |x_2 - x_{02}| < r \}$$ (square region),

$$\overline{B_r(x_0)} = \{ x \in \mathbb{R}^2 : |x_1 - x_{01}| \leq r, |x_2 - x_{02}| \leq r \}.$$

What if $d_2$ is replaced by $d_1(x, y) = |x_1 - y_1| + |x_2 - y_2|$? Why this metric is known as taxi-cab metric?

**Notation** Although the notations $B_r(x_0), \overline{B_r(x_0)}$ do not indicate dependence on the underlying metric, the dependence is evident.
Problem
Consider the metric space $C[0, 1]$ with the sup metric $d_\infty(f, g) = \sup_{x \in [0, 1]} |f(x) - g(x)|$, $f, g \in C[0, 1]$. Describe the open and closed balls $B_r(f), \overline{B_r(f)}$, where $f(x) = 0, x$ and $x^2$.

Hint.
Note that

$$B_r(f) = \left\{ g \in C[0, 1] : \sup_{x \in [0, 1]} |g(x) - f(x)| < r \right\}$$

$$= \left\{ g \in C[0, 1] : |g(x) - f(x)| < r \text{ for every } x \in [0, 1] \right\}$$

$$= \left\{ g \in C[0, 1] : g(x) \in (f(x) - r, f(x) + r) \text{ for every } x \in [0, 1] \right\}.$$  

Now draw the graphs of $f(x) - r$ and $f(x) + r$ in the above cases, and convince yourself that the ”tube” enclosed between these two graphs is the open ball $B_r(f)$. Draw the diagram.
In the following problem, we need the fact that every continuous function on $[0, 1]$ is Riemann integrable.

**Problem**

*Consider the metric space $C[0, 1]$ with the metric $d(f, g) = \int_{x \in [0, 1]} |f(x) - g(x)| \, dx$, $f, g \in C[0, 1]$.***

1. *Describe the open ball $B_1(0)$.***

2. *Give an example of $f$ belonging to $B_1(0)$ and also an example belonging to $C[0, 1] \setminus B_1(0)$.***

3. *Verify that $B_r(f) = \{ f + rg : g \in B_1(0) \}$.***

4. *Give an example of $g$ belonging to $B_r(f)$ and also an example belonging to $C[0, 1] \setminus B_r(f)$.***

**Hint.**

Note that $f \in B_1(0)$ if and only if the area under the graph of $|f(x)|$ is less than 1.
Problem

Let \((X, d)\) be a metric space and let \(A\) be a subset of \(X\). Verify the following:

1. A is a metric with metric \(d_A\) given by \(d_A(a, b) = d(a, b)\) for all \(a, b \in A\).

2. Describe the open ball in the metric space \((A, d_A)\).

3. If \((X, d) = (\mathbb{R}, d_1)\) (resp. \((\mathbb{R}^2, d_2)\)) and \(A = \mathbb{Z}\) (resp. \(\mathbb{Z} \times \mathbb{Z}\)), then describe \(d_A\) and the open ball in the metric space \((A, d_A)\).

Hint.

For the second part, verify that the open ball \(B^A_r(a)\) centred at \(a\) and of radius \(r\) in the \(d_A\) metric equals the intersection of \(A\) with the open ball \(B_r(a)\). For the last part, draw the lattice \(\mathbb{Z} \times \mathbb{Z}\) and examine its intersection with open balls.
Theorem

Let \((X, d)\) be a metric space and let \(x, y\) be distinct points in \(X\). Then there exist positive numbers \(r, s\) such that \(B_r(x) \cap B_s(y) = \emptyset\).

Proof.

Since \(x \neq y\), \(d(x, y) > 0\). Let \(r = d(x, y)/2 = s\). We must check that \(B_r(x) \cap B_s(y) = \emptyset\), or equivalently,

\[
B_r(x) \subseteq X \setminus B_s(y) = \{z \in X : d(z, y) \geq s\}.
\]

If \(z \in B_r(x)\), then by the symmetry and the triangle inequality,

\[
d(z, y) \geq d(x, y) - d(x, z) > d(x, y) - r = s,
\]

since \(r + s = d(x, y)\).

There exist ”topological spaces” without the Hausdorff property!
Open and closed sets

Definition
Let \((X, d)\) be a metric space and let \(A\) be a subset of \(X\). We say that \(A\) is open in \(X\) if either \(A\) is an empty set or if for every \(a \in A\), there exists \(\epsilon > 0\) such that \(B_\epsilon(a) \subseteq A\). We say that \(A\) is closed if the complement of \(A\) in \(X\) is open.

- Both the empty set \(\emptyset\) and \(X\) are open, and hence closed.
- The open ball \(B_r(x_0)\) is open. To see this, let \(a \in B_r(x_0)\).
  Thus \(d(a, x_0) = r - \delta\) for some \(\delta > 0\). We must find an \(\epsilon > 0\) such that \(B_\epsilon(a) \subseteq A = B_r(x_0)\). For any \(b \in B_\epsilon(a)\), by the triangle inequality,
  \[
d(b, x_0) \leq d(b, a) + d(a, x_0) \leq \epsilon + r - \delta,
  \]
  which is less than \(r\) provided \(\epsilon < \delta\).
Problem
Show that the closed ball \( \overline{B_r(x_0)} \) in a metric space \((X, d)\) is closed.

Hint.
Note that \( X \setminus \overline{B_r(x_0)} = \{x \in X : d(x, x_0) > r\} \). Let \( a \in X \setminus \overline{B_r(x_0)} \) and write \( \delta = d(x, x_0) - r > 0 \). Use the triangle inequality to find conditions on \( \epsilon > 0 \) (in terms of \( \delta \)), so that \( B_\epsilon(a) \subseteq A = X \setminus \overline{B_r(x_0)} \).

The closed ball could be an open set in some metric spaces!

Example (Discrete topology)
Consider \( \mathbb{R} \) endowed with the discrete metric \( d_0 \) given by \( d_0(x, x) = 0 \) and \( d_0(x, y) = 1 \) if \( x \neq y \in \mathbb{R} \). Since single-tons are open balls (\( \{x_0\} = B_1(x_0) \)), any subset of \( \mathbb{R} \) is open!

This can not happen in \( \mathbb{R}^d \) endowed with the metric \( d_p \), \( 1 \leq p \leq \infty \) (to be seen later).
Problem

Answer the following:

1. Which of the sets are open/closed in $\mathbb{R}$ with metric $d_1$
   \[ x, y \in \mathbb{R} \] ?
   
   1.1 (0, 1), [0, 1), (0, 1], [0, 1], $\mathbb{Z}$, $\mathbb{Q}$

2. Which of the sets are open/closed in $\mathbb{R}^2$ with metric $d_2$ ?
   \( (0, 1) \times (0, 1), [0, 1) \times [0, 1), (0, 1] \times [0, 1), [0, 1] \times [0, 1], \mathbb{Z} \times \mathbb{Z}, \mathbb{Q} \times \mathbb{Q}, \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \{(x, y) \in \mathbb{R}^2 : xy \neq 0\} \)

Justify your answer.

Hint.
The answers in 1 are open, not open and not closed, not open and not closed, closed, closed, not open and not closed (in order).
The answers in 2 are open, not open and not closed, not open and not closed, closed, closed, not open and not closed, closed, open (in order).
Let $B[0,1]$ denote the metric space of bounded functions $f : [0,1] \to \mathbb{R}$ with the metric $d_\infty$ given by

$$d_\infty(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|, \quad f, g \in B[0,1].$$

**Problem**

*Whether the subset $C[0,1]$ of $B[0,1]$ is open in $B[0,1]$? Justify your answer.*

**Hint.**

No. To see this, given any $\epsilon > 0$, find $f \in B[0,1] \setminus C[0,1]$ such that $f \in B_\epsilon(0)$ ($\epsilon$-neighbourhood of the 0 function), that is, a bounded function $f$ discontinuous at least at a point in $[0,1]$ such that $\|f\|_\infty < \epsilon$.

**Problem**

*Let $(X, d)$ be a metric space. For $a \in X$ and $r > 0$, show that $S_r(a) = \{x \in X : d(x, a) = r\}$ is closed in $X$.***
Let \((X, d)\) be a metric space and let \(\mathcal{O}\) be the collection of all open subsets of \(X\).

- \(\mathcal{O}\) is nonempty: \(\emptyset\) and \(X\) belong to \(\mathcal{O}\)
- \(\mathcal{O}\) is closed under arbitrary (finite or infinite) union: If \(\{U_\alpha\}\) \(\subseteq \mathcal{O}\), then \(\bigcup_\alpha U_\alpha\) is open (if \(a \in \bigcup_\alpha U_\alpha\), then \(a \in U_\alpha\) for some \(\alpha\), and hence there is \(r > 0\) such that \(B_r(a) \subseteq U_\alpha \subseteq \bigcup_\alpha U_\alpha\))
- \(\mathcal{O}\) need not be closed under countable intersection: Consider \(X = \mathbb{R}\) with usual metric \(d_1(x, y) = |x - y|, x, y \in \mathbb{R}\). Then for \(U_n = (-1/n, 1/n) \in \mathcal{O}\), \(\bigcap_{n \geq 1} U_n = \{0\} \notin \mathcal{O}\)
- \(\mathcal{O}\) is closed under finite intersection: If \(U_1, \ldots, U_k \in \mathcal{O}\), then \(\bigcap_{j=1}^k U_j \in \mathcal{O}\) (if \(a \in \bigcap_{j=1}^k U_j\), then for some \(r_j > 0\), \(B_{r_j}(a) \subseteq U_j\), and hence \(B_r(a) \subseteq \bigcap_{j=1}^k U_j\) with \(r = \min\{r_1, \ldots, r_k\}\))

**Theorem**

*For any metric space \((X, d)\), the collection \(\mathcal{O}\) (known as topology of \(X\)) of open subsets of \(X\) contains empty set and \(X\). Further, \(\mathcal{O}\) is closed under arbitrary union and finite intersection.*
Problem

Let $U$ be an open set in $\mathbb{R}$ and let $x \in U$. Show that there exists largest open interval containing $x$ and contained in $U$.

Hint.

Let $a_x = -\infty$ if $\{a \in \mathbb{R} : a < x, (a, x) \subseteq U\}$ is not bounded below, and $b_x = \infty$ if $\{b \in \mathbb{R} : b > x, (x, b) \subseteq U\}$ is not bounded above. Otherwise, set

$$a_x = \text{glb}\{a \in \mathbb{R} : a < x, (a, x) \subseteq U\},$$
$$b_x = \text{lub}\{b \in \mathbb{R} : b > x, (x, b) \subseteq U\},$$

and consider the open interval $I_x = (a_x, b_x)$. Then $x \in I_x$ (otherwise either $x \leq a_x$ or $x \geq b_x$). We claim that the interval $I_x$ is contained in $U$. If $I_x \nsubseteq U$, then there is $y \in I_x \setminus U$, and hence either $(x, y)$ or $(y, x)$ intersects the complement of $U$, and hence $b_x \leq y$ or $a_x \geq y$. That’s the contradiction since $y \in I_x$. 

\[\square\]
Theorem
Any open set $U$ in $\mathbb{R}$ can be expressed as the disjoint union of countably many open intervals.

Proof.
Let $I_x$ denote the largest open interval containing $x$ and contained in $U$. Then $U = \bigcup_{x \in U} I_x$. If $x, y \in U$, then either $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Since rationals are countable and distinct intervals contains distinct rationals, the collection $\{I_x\}_{x \in U}$ contains at most countably many disjoint intervals whose union is $U$. \hfill \Box$

The role of an open interval in $\mathbb{R}$ may be replaced by open rectangles $(a, b) \times (c, d)$ in $\mathbb{R}^2$.

Question Is it possible to express an open subset of $\mathbb{R}^2$ as disjoint union of countably many open rectangles?
Example

Consider the open unit disc in $\mathbb{R}^2$:

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$ 

We claim that $D$ can not be written as disjoint union of countably many open rectangles. Indeed, if $D = \bigcup_{n \geq 1} R_n$ is union of open rectangles $R_n = (a_n, b_n) \times (c_n, d_n) \subseteq D$, then at least some point of the form $(a_n, y)$, $c_n < y < d_n$ will lie in $D$, and hence $(a_n, y)$ must lie in some open rectangle $R_m$, $m \neq n$. In that case, $R_n \cap R_m \neq \emptyset$.

- Major difference between $\mathbb{R}$ and $\mathbb{R}^2$ is the “natural” order structure (and LUB property).
Let $(X, d)$ be a metric space.

• Given the closed ball $B_r(a) = \{ x \in X : d(x, a) \leq r \}$ in $X$, we can distinguish the points lying in $B_r(a) = \{ x \in X : d(x, a) < r \}$ from the remaining points $\{ x \in X : d(x, a) = r \}$.

**Question** Can we do the same for an arbitrary set?

**Definition**
Let $A$ be a subset of $X$ and let $a \in A$. We say that $a$ is an interior point of $A$ if there exists $r > 0$ such that $B_r(a) \subseteq A$.

Let $A^\circ = \{ a \in A : a$ is an interior point of $A \}$.

• Note that $A$ is open in $X$ if and only if $A^\circ = A$. 
Let \((X, d)\) be a metric space and let \(a \in X\). We know that any point in \(B_r(a)\) is an interior point of

\[
\overline{B_r(a)} = \{x \in X : d(x, a) \leq r\}.
\]

**Example**

In general, the interior of \(\overline{B_r(a)}\) need not be \(B_r(a)\). Indeed, if \(x \in \overline{B_r(a)} \setminus B_r(a)\), then \(x\) could be an interior point of \(\overline{B_r(a)}\). To see this, let \(X = \mathbb{R}\) with discrete metric \(d_0\) and \(a = 0, r = 1\). Then \(B_1(0) = \{0\}\) and the interior of \(\overline{B_1(0)} = \mathbb{R}\) is \(\mathbb{R}\).

This does not happen in \(X = \mathbb{R}^d\) endowed with any metric \(d_p\), \(p \geq 1\), as the following example illustrates.
Example

In \((\mathbb{R}^d, d_p)\) the interior of \(\overline{B_r(a)}\) is \(B_r(a)\). We have seen that the \(B_r(a)\) is contained in the interior of \(\overline{B_r(a)}\). Let \(b \in \overline{B_r(a)}\) be such that \(d_p(b, a) = r\). We claim that for any \(s > 0\), the ball \(B_s(b)\) intersects \(\mathbb{R}^d \setminus \overline{B_r(a)}\).

The line joining \(a\) and \(b\) is given by \(a + t(b - a), \ t \geq 0\). Note that

\[
d_p(a + t(b - a), b) = \left( \sum_{j=1}^{d} |1 - t|^p |a_j - b_j|^p \right)^{1/p},
\]

which can be made less than \(s\) if \(|1 - t|\) is very small. Thus \(a + t(b - a)\) belongs to \(B_s(b)\) if \(|1 - t|\) is very small. Also, if \(t > 1\),

\[
d_p(a + t(b - a), a) = \left( \sum_{j=1}^{d} t^p |a_j - b_j|^p \right)^p > d_p(b, a) = r.
\]
Example

Consider the metric space \((\mathbb{Q}, d_1)\), where \(d_1(x, y) = |x - y|\), \(x, y \in \mathbb{Q}\). Let \(\zeta \in \mathbb{R} \setminus \mathbb{Q}\) (e.g. \(\zeta = \sqrt{2}\)) and let

\[ A = \{ x \in \mathbb{Q} : x < \zeta \}. \]

We claim that \(A\) is open and closed in \(\mathbb{Q}\).

- \(A\) is open since \(A = (-\infty, \zeta) \cap \mathbb{Q}\) and \((-\infty, \zeta)\) is open in \(\mathbb{R}\).
- \(A\) is closed since \(\mathbb{Q} \setminus A = \{ x \in \mathbb{Q} : x \geq \zeta \} = \{ x \in \mathbb{Q} : x > \zeta \}\), which is again open since it is equal to \(\mathbb{Q} \cap (\zeta, \infty)\).

The phenomenon above can not occur in \(\mathbb{R}\)! This means that there are no proper open and closed subsets of \(\mathbb{R}\) (to be seen later).

Problem

*Find a proper subset of \(\mathbb{Q} \times \mathbb{Q}\) with the metric \(d_2(x, y) = \|x - y\|_2\) \((x, y \in \mathbb{Q} \times \mathbb{Q})\), which is open and closed.*
Limit point of a set

Let \((X, d)\) be a metric space.

**Definition**
Let \(A \subseteq X\) and let \(a \in X\). We say that \(a\) is a

- limit point\(^5\) of \(A\) if for every \(r > 0\), \(B_r(a) \cap A \neq \emptyset\).
- cluster point of \(A\) if for every \(r > 0\), \(B_r(a) \cap (A \setminus \{a\}) \neq \emptyset\).

- A limit point need not belong to \(A\) (0 is a limit point of \((0, 1)\))
- \(a\) is not a limit point iff there exists some \(r > 0\) such that \(B_r(a) \cap A = \emptyset\) (2 is not a limit point of \((0, 1)\) for \(B_1(2) \cap (0, 1) = \emptyset\))
- An interior point is a limit point (this follows from definition)
- A cluster point of \(A\) is a limit point of \(A\), but not conversely (e.g. \((\mathbb{R}, d_0), 0\) is a limit point of \(A = \{0\}\) but not a cluster point)

---

\(^5\)Some authors define limit point as the point for which \(B_r(a) \cap (A \setminus \{a\}) \neq \emptyset\)
Example
Let $X = \mathbb{R}$ and let $d_1(x, y) = |x - y|$, $x, y \in \mathbb{R}$. Let $\overline{A}$ denote the set of limit points and let $\text{cl}(A)$ be the set of cluster points of $A$:

- If $A = [a, b)$, then $\overline{A} = [a, b] = \text{cl}(A)$
- If $A = (a, b] \cup \{a + b\}$, then $\overline{A} = [a, b] \cup \{a + b\}$, $\text{cl}(A) = [a, b]$
- If $A = \{1/n : n \in \mathbb{N}\}$, then $\overline{A} = A \cup \{0\}$, $\text{cl}(A) = \{0\}$ (apply the Archimedean property)
- If $A = \mathbb{Z}$, then $\overline{A} = \mathbb{Z}$, $\text{cl}(A) = \emptyset$

Problem
Let $(X, d) = (\mathbb{R}, d_1)$. Find all limit points and cluster points of $\mathbb{Q}$.

Hint.
Since rationals are dense in $\mathbb{R}$, any open interval contains infinitely many rationals. So every real number is a cluster point of $\mathbb{R}$.  \[ \square \]
Closure of a set

Theorem

Let \((X, d)\) be a metric space and let \(A \subseteq X\). Then the set \(\overline{A}\) of limit points of \(A\) is closed in \(X\).

Proof.

We must check that \(X \setminus \overline{A}\) is open. To see this, let \(x \in X \setminus \overline{A}\). Thus, there exists \(r > 0\) such that \(B_r(x) \cap A = \emptyset\) or \(B_r(x) \subseteq X \setminus A\).

We claim that \(B_r(x) \subseteq X \setminus \overline{A}\). Otherwise, there exists \(y \in B_r(x) \cap \overline{A}\), and hence \(B_s(y) \cap A \neq \emptyset\) for every \(s > 0\). However, for small \(s > 0\), \(B_s(y) \subseteq B_r(x)\) (since \(B_r(x)\) is open), and hence \(B_r(x)\) intersects with \(A\), which is a contradiction.

Thus \(B_r(x) \subseteq X \setminus \overline{A}\) and hence \(X \setminus \overline{A}\) is open or \(\overline{A}\) is closed. \(\square\)

- \(\overline{A}\) is known as the closure of \(A\).
Problem

Let \((X, d)\) be a metric space and let \(A \subseteq X\). Then \(\overline{A}\) is the smallest closed set containing \(A\), that is, if \(L \subseteq X\) is a closed set such that \(A \subseteq L \subseteq \overline{A}\), then \(L = \overline{A}\). Conclude that \(A\) is closed iff \(\overline{A} = A\).

Solution.

Let \(x \in X \setminus L\). Since \(L\) is closed, there exists \(r > 0\) such that \(B_r(x) \cap X \setminus L \subseteq X \setminus A\), or \(B_r(x) \cap A = \emptyset\), so \(x \in X \setminus \overline{A}\). Thus \(X \setminus L \subseteq X \setminus \overline{A}\) or \(\overline{A} \subseteq L\), and hence \(L = \overline{A}\).

Problem

Let \((X, d)\) be a metric space and let \(A \subseteq X\). Then \(A^\circ\) is the largest open set contained in \(A\), that is, if \(O \subseteq X\) is an open set such that \(A^\circ \subseteq O \subseteq A\), then \(O = A^\circ\).

Hint.

Show that \(O\) is contained in \(A^\circ\).

- For any set \(A\) of \(X\), \(A^\circ \subseteq A \subseteq \overline{A}\) (strict inclusion may hold).
Convergence in metric spaces

By a sequence \( \{x_n\}_{n \geq 1} \) in a metric space \( X \), we understand a function from \( \mathbb{N} \) into \( X \), which maps \( n \) to \( x_n \).

**Definition**

Let \((X, d)\) be a metric space and let \( \{x_n\}_{n \geq 1} \) be a sequence in \( X \). We say that \( \{x_n\}_{n \geq 1} \) is a convergent sequence in \( X \) if there exists \( x \in X \) (limit of \( \{x_n\}_{n \geq 1} \)) such that

\[
d(x_n, x) \to 0 \quad \text{as} \quad n \to \infty.
\]

**Remark**

- The limit \( x \) of a convergent sequence is unique: If \( d(x_n, x) \to 0 \) and \( d(x_n, y) \to 0 \), then

\[
0 \leq d(x, y) \leq d(x, x_n) + d(x_n, y)
= d(x_n, x) + d(x_n, y) \to 0 \quad \text{as} \quad n \to \infty.
\]
Theorem
Let \((X,d)\) be a metric space, let \(A\) be a subset of \(X\) and let \(x \in X\). Then \(x\) is a limit point of \(A\) if and only if there exists a sequence \(\{a_n\}_{n \geq 1} \subseteq A\) such that \(d(a_n,x) \to 0\) as \(n \to \infty\).

Proof.
Let \(x\) be a limit point of \(A\).

- If \(x \in A\), then take the constant sequence \(\{a_n = x\}_{n \geq 1}\).

- Suppose that \(x \notin A\). Then \(x\) is a cluster point. Thus for every \(r > 0\), \(B_r(x) \cap (A \setminus \{x\}) \neq \emptyset\). By induction, after letting \(r = 1/n\), we obtain \(a_n \in B_{1/n}(x) \cap (A \setminus \{x\})\), and since \(d(a_n,x) < 1/n\), we have \(d(a_n,x) \to 0\) as \(n \to \infty\).

Conversely, assume that there exists a sequence \(\{a_n\}_{n \geq 1} \subseteq A\) such that \(d(a_n,x) \to 0\) as \(n \to \infty\). Thus for every \(r > 0\), there exists \(N \geq 1\) such that \(d(a_n,x) < r\) for every \(n \geq N\). Thus \(a_N \in B_r(x) \cap A\), and hence \(x\) is a limit point of \(A\).
Cauchy sequences and complete metric spaces

Definition
Let \((X, d)\) be a metric space and let \(\{x_n\}_{n \geq 1}\) be a sequence in \(X\). We say that \(\{x_n\}_{n \geq 1}\) is a Cauchy sequence in \(X\) if \(d(x_m, x_n) \to 0\) as \(m, n \to \infty\). A metric space is said to be complete if every Cauchy sequence is convergent.

Remark
• Every convergent sequence is Cauchy: 
\[
0 \leq d(x_m, x_n) \
\leq d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) \to 0 \text{ as } m, n \to \infty
\]

Example
The metric space \((\mathbb{R}, d_1)\) is complete.

Problem
Which of the following are complete metric spaces: \(([a, b], d_1)\), \(((a, b), d_1)\), \(([a, b), d_1)\), where \(d_1\) is the absolute metric.
Theorem
Let \((X, d)\) be a complete metric space. Then every closed subset \(A\) of \(X\) is complete when endowed with the (relative) metric

\[
d^A(x, y) = d(x, y), \quad x, y \in A.
\]

Proof.
Let \(\{a_n\}_{n \geq 1}\) be a Cauchy sequence in \((A, d^A)\). Clearly, \(\{a_n\}_{n \geq 1}\) is a Cauchy sequence in \((X, d)\). However, \(X\) is complete, so that \(\{a_n\}_{n \geq 1}\) converges to some \(a \in X\). It follows that \(a\) is a limit point of \(A\). Since \(A\) is closed, \(a \in A\), and it follows that \(\{a_n\}_{n \geq 1}\) is convergent in \(A\).

Corollary
Any closed subset of \((\mathbb{R}, d_1)\) is complete
Example

Consider the metric space \((\mathbb{Q}, d_1)\), where \(d_1(x, y) = |x - y|\), \(x, y \in \mathbb{Q}\). Let \(x_1 \in \mathbb{Q}\) such that \(\sqrt{2} - 1 < x_1 < \sqrt{2}\) (by density of rationals). Next choose \(x_2 \in \mathbb{Q}\) such that \(\sqrt{2} - 1/2 < x_2 < \sqrt{2}\), and hence by induction, for every positive integer \(n\), there exists \(x_n \in \mathbb{Q}\) such that \(\sqrt{2} - 1/n < x_n < \sqrt{2}\). Thus, as \(m, n \to \infty\),

\[
d_1(x_m, x_n) = |x_m - x_n| \leq |x_m - \sqrt{2}| + |\sqrt{2} - x_n| \leq 1/m + 1/n \to 0.
\]

However, \(\{x_n\}_{n \geq 1}\) is not convergent in \(\mathbb{Q}\) (since the limit being unique is necessarily \(\sqrt{2}\) and \(\sqrt{2} \notin \mathbb{Q}\)).

• \((\mathbb{Q}, d_1)\) is not complete.

Problem

*Show that \((\mathbb{R} \setminus \mathbb{Q}, d_1)\) is not complete.*
Problem
What are the Cauchy sequences in the discrete metric space $(X, d_0)$?

Solution.
Let $\{x_n\}_{n \geq 1}$ be a Cauchy sequence in $(X, d_0)$. Thus for $\epsilon = 1$, there exists $N \geq 1$ such that
\[ d_0(x_n, x_m) < 1 \text{ for every } m, n \geq N. \]

However, for the metric $d_0$, the distance between two points is either 0 or 1. It follows that $x_n = x_m$ for every $m, n \geq N$.

Problem
Which of the following are complete metric spaces: $([a, b], d_0)$, $(\langle a, b \rangle, d_0)$, $([a, b), d_0)$, where $d_0$ is the discrete metric.
Theorem
Let \((X, d)\) be a metric space. Then a Cauchy sequence \(\{x_n\}_{n \geq 1}\) is convergent if and only if it has a convergent subsequence.

Proof.
If \(\{x_n\}_{n \geq 1}\) is convergent, then clearly any subsequence of \(\{x_n\}_{n \geq 1}\) is convergent (follows from the definition).

Let \(\{x_{n_k}\}_{k \geq 1}\) be a convergent subsequence of \(\{x_n\}_{n \geq 1}\) and let \(x \in X\) be its limit. Thus given \(\epsilon > 0\), there exists \(N \geq 1\) such that \(d(x_{n_k}, x) < \epsilon/2\) for every \(k \geq N\). Also, for some \(N' \geq N\), \(d(x_n, x_m) \leq \epsilon/2\) for all \(m, n \geq N'\). Thus for any \(n \geq N'\),

\[
d(x_n, x) \leq d(x_n, x_{n_{N'}}) + d(x_{n_{N'}}, x) < \epsilon
\]

(since \(n_{N'} \geq N' \geq N\)).
Bounded sets and diameter

Let \((X, d)\) be a metric space and let \(\emptyset \neq A \subseteq X\). We say that \(A\) is bounded if there exists \(r > 0\) and \(x \in X\) such that \(A \subseteq B_r(x)\).

- Every Cauchy sequence \(\{x_n\}_{n \geq 1}\) is bounded:
  
  Indeed, for \(\epsilon = 1\), there exists \(N \geq 1\) such that \(d(x_n, x_N) < 1\) for all \(n \geq N\). Thus \(\{x_n\}_{n \geq 1} \subseteq B_r(x)\), where \(x = x_N\) and \(r = \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\} + 1\).

Consider \(S = \{d(x, y) : x, y \in A\}\). The diameter of \(A\) is defined as

\[
\text{diam}(A) = \begin{cases} 
\text{lub } S & \text{if } S \text{ is bounded above}, \\
\infty & \text{otherwise}.
\end{cases}
\]

Problem

Let \((X, d)\) be a metric space and let \(\emptyset \neq A \subseteq X\). Show that \(A\) is bounded if and only if \(\text{diam}(A) < \infty\).

Hint.

\(\text{diam}(B_r(x)) = 2r\), and if \(A \subseteq B\), then \(\text{diam}(A) \leq \text{diam}(B)\). \(\square\)
Normed linear spaces

Definition
A normed linear space $X$ (denoted by $(X, \| \cdot \|)$) is a vector space over $\mathbb{R}$ with a function assigning $\|x\| \in \mathbb{R}$ to every $x \in X$ such that for every $x, y, z \in X$ and $\alpha \in \mathbb{R}$,

N1 (Non-negativity) $\|x\| \geq 0$.

N2 (Positive Definiteness) $\|x\| = 0$ if and only if $x = 0$.

N3 (Dilation) $\|\alpha x\| = |\alpha|\|x\|$.

N4 (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$.

Remark Every normed linear space is a metric space with metric $d(x, y) = \|x - y\|$, $x, y \in X$. This metric has the additional property that $d(\alpha x, 0) = |\alpha|d(x, 0)$ for every $\alpha \in \mathbb{R}$ and $x \in X$. 
Example

We contend that $\mathbb{R}^d$ with the norm $\| \cdot \|_p$ is complete, where

$$\| x \|_p = \left( \sum_{j=1}^{d} |x_j|^p \right)^{1/p}, \quad x \in \mathbb{R}^d.$$  

To see this, let $\{x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)})\}_{n \geq 1}$ be a Cauchy sequence in $\mathbb{R}^d$. Thus for $k = 1, \ldots, d$,

$$|x_{k}^{(m)} - x_{k}^{(n)}| \leq \left( \sum_{j=1}^{d} |x_{j}^{(m)} - x_{j}^{(n)}|^p \right)^{1/p} = \|x^{(m)} - x^{(n)}\|_p \to 0,$$

that is, $\{x_{k}^{(n)}\}_{n \geq 1}$ is a Cauchy sequence in $\mathbb{R}$. Since $(\mathbb{R}, d_1)$ is complete, there exists $x_k \in \mathbb{R}$ such that $|x_{k}^{(n)} - x_{k}| \to 0$ as $n \to \infty$. Clearly, $\|x^{(n)} - x\|_p \to 0$ as $n \to \infty$, where $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$.

Problem

Show that $\mathbb{R}^d$ with the norm $\| \cdot \|_\infty$ is complete.
Problem
Let \( \{x^{(n)} = \{(x_1^{(n)}, x_2^{(n)}, \ldots)\} \}_{n \geq 1} \) be a Cauchy sequence in \( l^p \), \( 1 \leq p < \infty \). For \( \epsilon > 0 \), verify the following:

1. \( \{x_i^{(n)}\}_{n \geq 1} \subseteq \mathbb{R} \) converges to some \( x_i \in \mathbb{R} \) for every \( i \geq 1 \).
2. For \( k \geq 1 \), there exists \( n_0 \geq 1 \) (independent of \( k \)) such that
   \[
   \sum_{i=1}^{k} |x_i^{(n)} - x_i|^p \leq \epsilon \text{ for all } n \geq n_0.
   \]
3. For \( k \geq 1 \), \( \sum_{i=1}^{k} |x_i|^p \leq (\epsilon + \|x^{(n_0)}\|_p)^p \).
4. The normed linear space \( l^p \) is complete.

Hint.
For part 1, argue as in the last example. For part 2, use the definition of Cauchy sequence and part 1. For part 3, use the triangle inequality. By parts 2-3, \( x \in l^p \) and \( \|x^{(n)} - x\|_p \to 0. \)
Problem

Let \( p \) be such that \( p \geq 1 \). For \( f \in C[0,1] \), define

\[
\| f \|_p = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p}.
\]

Show that \((C[0,1], \| \cdot \|_p)\) is an incomplete normed linear space.

Hint.

To see triangle inequality, apply Hölder’s equality (see the argument on Page 35). To see that \( C[0,1] \) is not complete, let

\[
f(x) = \begin{cases} 
1 & \text{if } x \in [0,1/2), \\
0 & \text{if } x \in [1/2,1],
\end{cases}
\]

\[
f_n(x) = \begin{cases} 
1 & \text{if } x \in [0,1/2 - 1/n), \\
(-n/2)x + (n/2 + 1)/2 & \text{if } x \in [1/2 - 1/n, 1/2 + 1/n], \\
0 & \text{if } x \in [1/2 + 1/n,1].
\end{cases}
\]

Verify that \( \{f_n\}_{n \geq 3} \) converges to \( f \notin C[0,1] \).
Dense sets and separable metric spaces

Definition
Let $(X, d)$ be a metric space and let $A \subseteq X$. We say that $A$ is dense in $X$ if every open ball in $X$ intersects $A$, that is, for every $x \in X$ and every $r > 0$, $B_r(x) \cap A \neq \emptyset$.

Remark $A$ is dense in $X$ if and only if every $x \in X$ is a limit point of $A$, or equivalently, if and only if $\overline{A} = X$.

Example
- $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $(\mathbb{R}, d_1)$
- $\mathbb{Z}$ is not dense in $(\mathbb{Q}, d_1)$
- $\mathbb{Q}^d$ is a countable dense in $(\mathbb{R}^d, d_p)$

A metric space is separable if it has a countable dense subset.
Theorem
Let $1 \leq p < \infty$. Then $l^p$ is separable.

Proof.
For $k \geq 1$, let $A_k = \{(r_1, \ldots, r_k, 0, \ldots, ) : r_k \in \mathbb{Q}\}$.

- $A_k$ is countable: There is a bijection between $A_k$ and $\mathbb{Q}^k$
- $A = \bigcup_{k=1}^{\infty} A_k$ is countable: Countable union of countable sets is countable

We claim that $A$ is dense in $l^p$: Let $r > 0$, $x = (x_1, x_2, \ldots, ) \in l^p$. Then $\sum_{k=1}^{\infty} |x_k|^p < \infty$, and hence for some integer $N \geq 2$,

$$(\sum_{k=N}^{\infty} |x_k|^p)^{1/p} < r/2.$$  

For $j = 1, \ldots, N - 1$, let $r_j \in \mathbb{Q}$ be such that $|x_j - r_j| < \frac{r}{2(N-1)^{1/p}}$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then for $y = (r_1, \ldots, r_{N-1}, 0, \ldots, ) \in A$,

$$
\|x - y\|_p \leq \left( \sum_{j=1}^{N-1} |x_j - r_j|^p \right)^{1/p} + \left( \sum_{k=N}^{\infty} |x_k|^p \right)^{1/p} < r/2 + r/2 = r.
$$

Thus for every $x \in l^p$ and $r > 0$, $y \in B_r(x) \cap A$. \qed
Problem

Show that the set $l^\infty$ of bounded sequences is complete normed linear space with norm $\|x\|_\infty = \sup_{j \geq 1} |x_j|$, $x \in l^\infty$.

Example

We claim that $l^\infty$ is not separable. Consider the uncountable set

$$\{0, 1\}^\mathbb{N} = \{x \in l^\infty : x_n = 0 \text{ or } 1\}.$$ 

If $x, y \in \{0, 1\}^\mathbb{N}$, then $\|x - y\|_\infty = 1$ if $x \neq y$. Thus

$$\bigcup_{x \in \{0,1\}^\mathbb{N}} B_{1/2}(x) \subseteq l^\infty \text{ (disjoint union).}$$

Now if $l^\infty$ has a countable dense set, then each $B_{1/2}(x)$ would contain at least one element from this set. However, there are uncountably disjoint balls of the form $B_{1/2}(x)$, and hence $l^\infty$ can not admit a countable dense subset.

Question  Whether $(C[0,1], d_\infty)$ is separable ?
Definition
Let \((X, d)\) be a metric space and \(A \subseteq X\). A point \(x \in X\) is said to be a **boundary point** of \(A\) in \(X\) if for every \(r > 0\),

\[
B_r(x) \cap A \neq \emptyset \quad \text{and} \quad B_r(x) \cap (X \setminus A) \neq \emptyset.
\]

The **boundary** of \(A\) in \(X\) (denoted by \(\partial A\)) is the set of boundary points of \(A\) in \(X\).

**Remark** Every boundary point is a limit point, that is, \(\partial A \subseteq \overline{A}\).

**Example**
- If \((X, d) = (\mathbb{R}, d_1)\), then for \(a, b \in \mathbb{R}\) such that \(a < b\),

  \[
  \partial[a, b] = \partial(a, b) = \partial[a, b] = \partial(a, b) = \{a, b\}.
  \]

- If \((X, d) = (\mathbb{R}^d, d_p)\), then \(\partial B_r(x) = \{y \in \mathbb{R}^d : d_p(x, y) = r\}\).
Theorem
Let $(X, d)$ be a metric space and $\emptyset \neq A \subseteq X$. Then $\partial A = \overline{A} \setminus A^o$.

Proof.
Clearly, $x \in \overline{A} \setminus A^o$ if and only if $x \in \overline{A}$ and $x \notin A^o$. Note that
- $x \in \overline{A}$ if and only if for every $r > 0$, $B_r(x) \cap A \neq \emptyset$.
- $x \notin A^o$ if and only if for every $r > 0$, $B_r(x) \cap (X \setminus A) \neq \emptyset$.
Thus $x \in \overline{A} \setminus A^o$ if and only if for every $r > 0$, $B_r(x) \cap A \neq \emptyset$ and $B_r(x) \cap (X \setminus A) \neq \emptyset$, that is, $x \in \partial A$. 

Problem
Find the boundary of $A = \{(x, y) \in \mathbb{R}^2 : x > 0, \ y > 0\}$.

Hint.
Since $A$ is open, $A^o = A$. Also, $\overline{A} = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0\}$. It follows that $\partial A = \{(x, y) \in \overline{A} : xy = 0\}$. 

- The boundary of $\mathbb{Q}$ is equal to $\mathbb{R}$. 

Continuity

**Definition**
Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Let $f : X \rightarrow Y$ be a function and let $a \in X$. We say that $f$ is

1. **continuous at $a$** if for every $\epsilon > 0$, there exists $\delta > 0$ such that
   \[
   \left( x \in X, \quad d(x, a) < \delta \implies \rho(f(x), f(a)) < \epsilon \right).
   \]
2. **s-continuous at $a$** (or sequential continuous at $a$) if
   \[
   \left( d(x_n, a) \rightarrow 0 \text{ as } n \rightarrow \infty \implies \rho(f(x_n), f(a)) \rightarrow 0 \text{ as } n \rightarrow \infty \right).
   \]

We say that $f$ is continuous (resp s-continuous) if it is continuous (resp s-continuous) at every $a \in X$.

**Remark**
- $f$ is continuous at $a$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B^d_\delta(a)) \subseteq B^\rho_\epsilon(f(a))$.
- $f$ is s-continuous at $a$ if and only if $f$ maps convergent sequences to convergent sequences.
Problem
Let \((X, d)\) and \((Y, \rho)\) be two metric spaces. Let \(f : X \to Y\) be a function and let \(a \in X\). Then \(f\) is continuous at \(a\) if and only if \(f\) is s-continuous at \(a\).

Hint.
\(\Rightarrow\) follows from the definition: Given \(\epsilon > 0\), there exists \(\delta > 0\) such that C1 holds. Now find an integer \(N \geq 1\) such that \(d(x_n, a) < \delta\), so by continuity, \(\rho(f(x_n), f(a)) < \epsilon\).

For \(\Leftarrow\), argue by contradiction: Suppose that C2 holds but C1 fails to hold. So, for some \(\epsilon > 0\), for \(\delta = 1/n, n \geq 1\), find \(x_n \in X\) such that \(d(x_n, a) < 1/n\) but \(\rho(f(x_n), f(a)) \geq \epsilon\). Then \(d(x_n, a) \to 0\) but \(\rho(f(x_n), f(a)) \not\to 0\), a contradiction.

Problem
Let \((X, d), (Y, \rho)\) and \((Z, \eta)\) be metric spaces. If \(f : X \to Y\) and \(g : Y \to Z\) are continuous, then so is \(g \circ f : X \to Z\).
Example
Let \( p : \mathbb{R} \to \mathbb{R} \) be a polynomial, that is, for some \( a_0, \ldots, a_k \in \mathbb{R} \),
\[
p(x) = a_0 + a_1 x + \cdots + a_k x^k, \quad x \in \mathbb{R}.
\]
Notice that its not easy to verify the continuity of \( p \) right from its definition (given \( \epsilon \), how to find \( \delta \) ?). However, \( s \)-continuity of \( p \) follows immediately from the facts that sum and product of convergent sequences is convergent:
\[
x_n \to x, y_n \to y \Rightarrow x_n + y_n \to x + y, \; x_n^m \to x^m, \; m \geq 1.
\]
Thus if \( x_n \to x \), then \( p(x_n) \to p(x) \) as \( n \to \infty \). So \( p \) is continuous.

Theorem
Let \((X, d)\) be a metric space. Then the set \( C(X) \) of continuous functions \( f : X \to \mathbb{R} \) forms an algebra, that is, \( C(X) \) is a vector space over \( \mathbb{R} \) equipped with the binary operation \((f, g) \to fg\) from \( C(X) \times C(X) \) to \( C(X) \), which is a binary form.

Proof.
The set of convergent sequences in \( \mathbb{R} \) forms an algebra. \( \square \)
Let \((X, d)\) be a metric space. Let \(A\) be a non-empty subset of \(X\), and for \(x \in X\), let

\[
d(x, A) = \text{glb } S_x, \text{ where } S_x = \{d(x, a) : a \in A\}
\]

(since \(d(x, a) \geq 0\), \(S_x\) is bounded from below. Also, since \(A\) is nonempty, so is \(S_x\). Hence \(\text{glb} \) of \(S\) exists).

**Problem**

*Show that \(f : X \rightarrow [0, \infty)\) given by \(f(x) = d(x, A)\) is continuous.*

**Solution.**

To see this, for \(x, y \in X\) and \(a \in A\), note that

\[
f(x) = \text{glb } S_x \leq d(x, a) \leq d(x, y) + d(y, a)
\]

\[
f(x) - d(x, y) \leq d(y, a) \text{ for every } a \in A \implies f(x) - d(x, y) \leq \text{glb } S_y = f(y) \text{ or } f(x) - f(y) \leq d(x, y).\]

Changing roles of \(x\) and \(y\), we obtain \(|f(x) - f(y)| \leq d(x, y)\). Let \(\delta = \epsilon\). \(\square\)
Theorem (Urysohn’s Lemma)

Let \((X, d)\) be a metric space. Given closed non-empty disjoint subsets \(A\) and \(B\) of \(X\), there exists a continuous function \(f : X → [0, 1]\) such that \(f|_A = 0\) and \(f|_B = 1\).

Proof.
For \(C ⊆ X\) closed and \(x ∈ X\), let \(d(x, C) = \inf\{d(x, a) : a ∈ C\}\).

- \(d(x, C)\) is a continuous function of \(x\).
- \(d(x, C) = 0\) if and only if \(x ∈ C\).
- \(d(x, A) + d(x, B) > 0\) for every \(x ∈ X\).

Define \(f : X → [0, 1]\) by

\[
f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad x ∈ X.
\]

Clearly, \(f\) is a continuous function. Note that \(f(a) = 0\) and \(f(b) = 1\) for every \(a ∈ A\) and \(b ∈ B\).
A characterization

Theorem

Let $(X, d)$ and $(Y, ρ)$ be two metric spaces. Let $f : X \rightarrow Y$ be a function. Then the following statements are equivalent:

(a) $f$ is continuous.

(b) For every open subset $O$ of $Y$, $f^{-1}(O) = \{ a \in X : f(a) \in O \}$ is an open subset of $X$.

Proof.

(a) ⇒ (b). Let $O \subseteq Y$ be open and $a \in f^{-1}(O)$. Thus $f(a) \in O$. Since $O$ is open, there exists $\epsilon > 0$ such that $B_\epsilon^ρ(f(a)) \subseteq O$. By the continuity, there is $\delta > 0$ such that $f(B_\delta^d(a)) \subseteq B_\epsilon^ρ(f(a)) \subseteq O$. Thus $B_\delta^d(a) \subseteq f^{-1}(O)$, which shows that $f^{-1}(O)$ is open.

(b) ⇒ (a). Let $a \in X$ and $\epsilon > 0$. Since $O = B_\epsilon^ρ(f(a))$ is an open subset of $Y$, $a \in f^{-1}(B_\epsilon^ρ(f(a)))$ is an open subset of $X$. Hence there exists $\delta > 0$ such that $B_\delta^d(a) \subseteq f^{-1}(B_\epsilon^ρ(f(a)))$, and hence $f$ is continuous.
Problem
Consider the vector space $M_n(\mathbb{R})$ (over $\mathbb{R}$) of $n \times n$ matrices $A = (a_{i,j})_{1 \leq i,j \leq n}$ with real entries. For $A \in M_n(\mathbb{R})$, set

$\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$. Verify the following:

1. $\| \cdot \|$ defines a norm on $M_n(\mathbb{R})$.
2. The determinant $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous.
3. The set $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices is open in $M_n(\mathbb{R})$.

Hint.
For 1, the triangle inequality follows from the triangle inequality of real numbers. For 2, note that determinant is a polynomial in $n^2$ variables, and hence $\det$ is s-continuous. For 3, note that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ and $\mathbb{R} \setminus \{0\}$ is open.

Remark Can you see that $GL_n(\mathbb{R})$ has a proper clopen subset?
Equivalent norms

We say that two norms \( \| \cdot \| \) and \( \| \cdot \|' \) on a normed linear space \( X \) are *equivalent* if there exists \( m, M > 0 \) such that

\[
m\|x\| \leq \|x\|' \leq M\|x\|, \quad x \in X.
\]

**Problem**

*Verify that the equivalence of norms is an equivalence relation.*

*Conclude that the all norms \( \| \cdot \|_p, p \geq 1, \) on \( \mathbb{R}^n \) are equivalent.*

**Hint.**

The first part is a routine verification. To see the second part, by transitivity of norms, it suffices to check that \( \| \cdot \|_p \) is equivalent to \( \| \cdot \|_\infty \). This follows from \( \|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty \). This in turn follows from \( |x_j| \leq \sum_{j=1}^n |x_j|^p \leq n\left(\max_{j=1}^n |x_j|\right)^p, \) \( j = 1, \ldots, d. \) \( \square \)
Problem

Let $X$ be a normed linear space with two norms $\| \cdot \|$ and $\| \cdot \|'$. Then the norms $\| \cdot \|$ and $\| \cdot \|'$ are equivalent if and only if the identity map $I$ from $(X, \| \cdot \|)$ onto $(X, \| \cdot \|')$ is continuous at 0 with inverse being continuous at 0.

Hint.

For $\Rightarrow$, show that the identity map and its inverse are s-continuous. For $\Leftarrow$, note that the continuity of $I : (X, \| \cdot \|) \rightarrow (X, \| \cdot \|')$ at 0 implies that for $\epsilon = 1$, there exists $\delta > 0$ such that

$$\| x \| < \delta \Rightarrow \| x \|' < 1.$$ 

For $0 \neq x \in X$, let $y = \frac{\delta}{2} \frac{x}{\| x \|}$. Thus $\| y \| = \frac{\delta}{2}$, and hence $\| y \|' < 1$, that is, $\frac{\delta}{2} \frac{\| x \|'}{\| x \|} < 1$ or $\| x \|' \leq \frac{2}{\delta} \| x \|$ for every $x \in X$. Similarly, one may get the other inequality.

Problem

If two norms $\| \cdot \|$ and $\| \cdot \|'$ on $X$ are equivalent, show that $(X, \| \cdot \|)$ is complete if and only if $(X, \| \cdot \|')$ is complete.
**Question** What are all norms on $\mathbb{R}^d$ (up to equivalence)?

Let us analyze the case of $d = 1$:

**Problem**
Describe all norms on $\mathbb{R}$.

**Solution.**
Let $\| \cdot \|$ be any norm on $\mathbb{R}$. For any $x \in \mathbb{R}$, note that by the dilation property,

$$\|x\| = \|x \cdot 1\| = |x| \|1\|.$$

This means that any norm $\| \cdot \|$ is of the form $\alpha | \cdot |$, $\alpha > 0$. \hfill \square

- Note that all norms on $\mathbb{R}$ are equivalent. Indeed, if $\| \cdot \|$ and $\| \cdot \|^\prime$ are two norms on $\mathbb{R}$, then for $m = \frac{\|1\|^\prime}{\|1\|} = M$, we have

$$m\|x\| = \|x\|^\prime = M\|x\|, \quad x \in \mathbb{R}.$$

What about the case of dimension $d > 1$?
Theorem

Consider \((\mathbb{R}^d, \| \cdot \|_\infty)\). Let \(S = \{x \in \mathbb{R}^d : \|x\|_\infty = 1\}\) and let \(f : S \to \mathbb{R}\) be a continuous function. Then there exists \(a \in S\) such that \(f(x) \geq f(a)\) for every \(x \in S\).

Proof.

Let \(A = \{f(x) : x \in S\}\) and let \(\alpha = \inf A\). Thus there exists a sequence \(\{x^{(n)}\}_{n \geq 1} \subseteq S\) such that \(f(x^{(n)}) \to \alpha\) as \(n \to \infty\). Now if \(x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)})\), then \(\{x_1^{(n)}\}_{n \geq 1}\), being a bounded sequence, has a convergent subsequence (by the Bolzano-Weierstrass Theorem), say, \(\{x_1^{(n_j)}\}_{j \geq 1}\). Since \(\{x_2^{(n_j)}\}_{j \geq 1}\) is bounded, it has a convergent subsequence, and continuing this, we obtain a convergent subsequence of \(x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)})\) converging to say \(a\) in \(\| \cdot \|_\infty\). It follows that \(f(a) = \alpha\) and \(f(x) \geq \inf A = f(a)\) for every \(x \in S\). Since \(S\) is closed, \(a \in S\). \(\square\)
Norms on $\mathbb{R}^d$

**Theorem**

*All norms on $\mathbb{R}^d$ are equivalent.*

**Proof.**

Let $\| \cdot \|$ be an arbitrary norm on $\mathbb{R}^d$.

(a) Let $e_1, \cdots, e_d$ denote the standard basis of $\mathbb{R}^d$. By the triangle inequality, $\| x \| \leq \left( \sum_{i=1}^{d} \| e_i \| \right) \| x \|_\infty$.

(b) By (a), the function $f : (\mathbb{R}^d, \| \cdot \|_\infty) \to \mathbb{R}$ given by $f(x) = \| x \|$ is (sequential) continuous.

(c) $f$ attains its minimum on $S = \{ x \in \mathbb{R}^d : \| x \|_\infty = 1 \}$. Thus there exists $a \in S$ such that $\| x \| \geq \| a \| > 0$ for every $x \in S$.

(d) By (a) and (c), the norm $\| \cdot \|$ is equivalent to $\| \cdot \|_\infty$.

The desired conclusion now follows from the transitivity of the equivalence of norms. □
One can produce inequivalent norms on infinite-dimensional spaces.

**Example**

Let $\mathbb{R}[x]$ denote the vector space over $\mathbb{R}$ of polynomials $p(x) = \sum_{n=0}^{k} a_n x^n$ in $x$. For $c := \{c_n\}_{n=0}^{\infty}$, define

$$\|p\|_c := \sum_{n=0}^{k} |c_n||a_n|.$$ 

- $\|\cdot\|_c$ defines a norm on $\mathbb{R}[x]$ if $c_n \neq 0$ for every $n \geq 0$. This is a routine verification.
- $\|\cdot\|_c$ and $\|\cdot\|_d$ are not equivalent if $c_n = 1/(n+1)$, $d_n = n + 1$. This is far from being obvious.
Example (Example continued …)

If possible, then assume that there exists $m, M > 0$ such that

$$m \| p \|_c \leq \| p \|_d \leq M \| p \|_c, \quad p \in \mathbb{R}[x].$$

Thus, for any $p \in \mathbb{R}[x],

$$\| p \|_d = \sum_{n=0}^{k} (n + 1) |a_n| < 1 \text{ whenever } \| p \|_c = \sum_{n=0}^{k} \frac{|a_n|}{n + 1} < \frac{1}{M}.$$

Choose an integer $k$ large enough so that $\frac{k}{M} > 2$. Letting $a_0 = 0$ and $a_n = \frac{1}{2(n+1)M} \ (n = 1, \ldots, k)$, we get

$$\sum_{n=0}^{k} \frac{|a_n|}{n + 1} = \frac{1}{2M} \sum_{n=1}^{k} \frac{1}{(n + 1)^2} < \frac{1}{M}.$$

However, $\sum_{n=0}^{k} (n + 1)|a_n| = \frac{k}{2M} > 1$, which is a contradiction.
Theorem
Let \((X, d)\) and \((Y, \rho)\) be two metric spaces. Let \(f : X \to Y\) be a function. Then the following statements are equivalent:

(a) \(f\) is continuous.
(b) For every closed subset \(U\) of \(Y\), \(f^{-1}(U) = \{a \in X : f(a) \in U\}\) is a closed subset of \(X\).

Proof.
Recall that \(f^{-1}(Y \setminus U) = X \setminus f^{-1}(U)\). Now apply the fact that \(f\) is continuous iff for every open subset \(O\) of \(Y\), \(f^{-1}(O)\) is an open subset of \(X\).

Example
Let \(X\) be a metric space endowed with the discrete metric \(d_0\). Then any \(f : X \to Y\) is continuous. Indeed, since every subset of \(X\) is closed, by the previous theorem, \(f\) is continuous. In particular, if \(\mathbb{Z}\) carries the (relative) metric induced from \((\mathbb{R}, d_1)\), then any function \(f : \mathbb{Z} \to Y\) is continuous.
Problem

Consider the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices $A = (a_{i,j})_{1\leq i,j\leq n}$ with real entries. Define $d : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ by $d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|$. Verify the following:

1. For every positive integer $k$, the map $p_k : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $p_k(A) = A^k$ is continuous.

2. If $\mathcal{N}_n(\mathbb{R}) = \{ A \in M_n(\mathbb{R}) : \text{there is } k \geq 1 \text{ such that } A^k = 0 \}$, then $\mathcal{N}_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.

Hint.

To see that $p_k$ is s-continuous, note that the entries of $A^k$ are polynomials in the entries of $A$ (please verify this for $k = 2$ and $n = 2$). To see the part 2, note that

$$\mathcal{N}_n(\mathbb{R}) = \bigcup_{k \geq 1} \ker p_k = \bigcup_{k=1}^{n} \ker p_k$$

(justify the second equality). Since kernel of a continuous map is closed, $\mathcal{N}_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$. \qed
Homeomorphisms

**Definition**
Let \((X, d)\) and \((Y, \rho)\) be two metric spaces.

- A function \(f : X \to Y\) is a **homeomorphism** if \(f\) is continuous, one-one, and onto with a continuous inverse.
- We say that \(X\) and \(Y\) are **homeomorphic** if there exists a homeomorphism \(f : X \to Y\). In this case, we say that \(X \cong Y\).

**Remark** Let \(f : X \to Y\) be a homeomorphism.

1. \(f^{-1}\) is also a homeomorphism.
2. If \(U\) is open in \(X\) then \(f(U)\) is open in \(Y\).
3. If \(A\) is a subset of \(X\) then \(A\) and \(f(A)\) are homeomorphic.
4. Composition of homeomorphisms is again a homeomorphism. In particular, if \(X\) is homeomorphic to \(Y\), and \(Y\) is homeomorphic to \(Z\) then \(X\) is homeomorphic to \(Z\).

Note that \(\cong\) is an equivalence relation.
Example

Consider the subsets $A$ of the metric space $(\mathbb{R}, d_1)$ as the metric spaces with relative metric $d_1^A(x, y) = |x - y|$, $x, y \in A$.

- $(0, 1) \not\cong (0, 1]$: If $f : (0, 1) \to (0, 1]$ is a homeomorphism, then $f(c) = 1$ for some $c \in (0, 1)$, and hence it follows that $f^{-1}$ maps $(0, 1)$ onto $(0, c) \cup (c, 1)$, which is not possible in view of the intermediate value property.

Problem

Show that the interval $(a, b) \subseteq \mathbb{R}$ is homeomorphic to any other interval $(c, d) \subseteq \mathbb{R}$.

Hint.

Try $\alpha(t - b) + \beta(t - a)$ for appropriate scalars $\alpha$ and $\beta$.

Problem

Show that $e^{-x}$ is a homeomorphism from $(0, \infty)$ onto $(0, 1)$. 
Problem
Which of spaces $X$ and $Y$ are homeomorphic:

(1) $X = \mathbb{R}$ and $Y = [0, 1)$
(2) $X = \mathbb{R}$ and $Y = [0, 1]$
(3) $X = [1, \infty)$ and $Y = (0, 1]$
(4) $X = (-1, 0)$ and $Y = (-\infty, -1)$
(5) $X = \mathbb{R}$ and $Y = (0, 1)$
(6) $X = \mathbb{Q}$ and $Y = \mathbb{Z}$

Hint. For (1), (2), use intermediate value property (No). For (3), (4), check that $1/x$ is the desired homeomorphism (Yes). For (5), write $X = (-\infty, 1) \cup [1, \infty)$, and note that by (3), $[1, \infty) \cong (0, 1]$. Also, by (4), $(-\infty, 1) \cong (-\infty, -1) \cong (-1, 0)$ (No). For (6), choose an open ball centred at an integer (which contains finitely many elements), and analyze its image in $\mathbb{Q}$ (No).

---

6 The answer to (5) is Yes. Please try tan (thanks to Sudip and Satyam for catching a careless assertion!)
Theorem
\( \mathbb{R}^n \) is homeomorphic to \( \mathbb{R} \) iff \( n = 1 \).

Proof.
Suppose that for \( n > 1 \). Thus there is a continuous bijection \( f : \mathbb{R}^n \to \mathbb{R} \), and hence \( g = f|_X \) is a homeomorphism from \( X := \mathbb{R}^n \setminus \{0\} \) onto \( Y := \mathbb{R} \setminus \{y_0\} \) for some \( y_0 \in \mathbb{R} \). Choose \( y_1, y_2 \in Y \) such that \( y_1 < y_0 < y_2 \) and let \( x_1, x_2 \in X \) be such that \( f(x_1) = y_1 \) and \( f(x_2) = y_2 \). Let \( L \) denote the line segment connecting \( x_1 \) and \( x_2 \).

- If \( L \) does not pass through \( 0 \) then let \( \gamma(t) = (1 - t)x_1 + tx_2 \).
- If \( L \) passes through \( 0 \) then choose any point \( x_3 \not\in L \) and let

\[
\gamma(t) = \begin{cases} 
(1 - 2t)x_1 + 2tx_3 & \text{if } 0 \leq t \leq 1/2, \\
2(1 - t)x_3 + (2t - 1)x_2 & \text{if } 1/2 \leq t \leq 1.
\end{cases}
\]

Thus \( \gamma : [0, 1] \to X \) is a continuous function (since \( \gamma(1/2) = x_3 \)) such that \( \gamma(0) = x_1 \) and \( \gamma(1) = x_2 \).
Proof continued ...
We consider the continuous function $h : [0, 1] \rightarrow [y_1, y_2]$ by

$$h(t) = g(\gamma(t)), \quad t \in [0, 1].$$

By the intermediate value property, there exists $t_0 \in [0, 1]$ such that $h(t_0) = y_0$, that is, $g(\gamma(t_0)) = y_0$. This implies that $y_0$ belongs to the image of $g = f|_X$, which is a contradiction.

Remark It is highly non-trivial fact that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $m = n$ (beyond the scope of this course).

Problem
Is an open annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$ homeomorphic to the unit circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$? Justify your answer.

Hint.
Argue as in the last theorem.
**Objective** To find when two metric spaces are homeomorphic, one needs to look for ”invariants” which are preserved under homeomorphisms (e.g, existence of a proper clopen set).

**Problem**
*Show that the function* \( g : (0, 1) \rightarrow \mathbb{R} \) *given below is continuous on irrationals and discontinuous on rationals:*

\[
g(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0, 1) \text{ and } x = \frac{p}{q} \text{ in reduced form} \\ 0 & \text{otherwise.} \end{cases}
\]

**Question** Does there exist a function \( g : (0, 1) \rightarrow \mathbb{R} \) which is continuous on rationals and discontinuous on irrationals?

Answer: No

A solution was first provided by Vito Volterra.
G-delta sets

Definition
A set is a $G_\delta$ set if it is countable intersection of open sets.

Example
The irrationals $\mathbb{R} \setminus \mathbb{Q}$ form a $G_\delta$ set for $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r \in \mathbb{Q}} \mathbb{R} \setminus \{r\}$.

Problem
The rationals do not form a $G_\delta$ set.

Solution.
Suppose that $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ for open sets $U_n$. Since $\mathbb{Q} \subseteq U_n$, each $U_n$ is dense in $\mathbb{R}$. Now note that $\emptyset = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ is countable intersection of open dense sets $U_n$, $n \geq 1$ and $\mathbb{R} \setminus \{r\}$, $r \in \mathbb{Q}$. This contradicts BCT (to be proven later).

Theorem (Baire Category Theorem)
A countable intersection of open dense sets in $\mathbb{R}$ is dense in $\mathbb{R}$. 

A Theorem of Volterra Vito

Theorem
Let $U$ be an open subset of $\mathbb{R}$ and let $f : U \rightarrow \mathbb{R}$ be a function defined on $U$. Then $A = \{a \in U : f \text{ is continuous at } a\}$ is a $G_\delta$ set.

Proof.
For positive integer $n$, consider the set $A_n$ given by

$$\{x_0 \in U : \exists \delta > 0 \text{ such that } |f(x) - f(y)| < 1/n, \ x, y \in (x_0 - \delta, x_0 + \delta)\}.$$  

Note that $A_n$ is open and $A = \bigcap_{n=1}^{\infty} A_n$. □

Since rationals do not form a $G_\delta$ set, we obtain the following:

Corollary (Volterra Vito)

There is no function $g : (0, 1) \rightarrow \mathbb{R}$ which is continuous on rationals and discontinuous on irrationals.
**Baire category theorem**

**Theorem (Baire Category Theorem)**

Let \((X, d)\) be a complete metric space and let \(\{U_n\}_{n\geq 1}\) be a sequence of open dense subsets of \(X\). Then \(\bigcap_{n\geq 1} U_n\) is dense in \(X\).

**Proof.**

Let \(x \in X\) and \(r > 0\). We claim that \(B_r(x) \cap (\bigcap_{n\geq 1} U_n) \neq \emptyset\).

- Since \(U_1\) is dense in \(X\), there exists \(x_1 \in B_r(x) \cap U_1\). Since \(U_1\) is open, for some \(0 < r_1 < 1\), \(B_{r_1}(x_1) \subseteq B_r(x) \cap U_1\).

- Since \(U_2\) is dense in \(X\), there exists \(x_2 \in B_{r_1}(x_1) \cap U_2\). Since \(U_2\) is open, for some \(0 < r_2 < 1/2\), \(B_{r_2}(x_2) \subseteq B_{r_1}(x_1) \cap U_2\).

By induction on \(n \geq 1\), there exist \(\{x_n\}_{n \geq 1} \subseteq X\) and \(0 < r_n < 1/n\) such that \(B_{r_{n+1}}(x_{n+1}) \subseteq B_{r_n}(x_n) \cap U_{n+1}\) for every \(n \geq 1\). Thus \(\{x_n\}_{n \geq 1}\) is a Cauchy sequence (since \(d(x_n, x_m) < 1/n\) for every \(m \geq n\)), and hence converges to some \(x_0 \in X\) (since \(X\) is complete). Note that \(x_0 \in B_r(x) \cap (\bigcap_{n\geq 1} U_n)\).
Problem
Let \( X \) be a complete metric space. If \( X \) is a union of closed sets \( A_1, A_2, \ldots \), then show that there exists \( N \geq 1 \) such that \( A_N \) has non-empty interior.

Hint.
Assume that \( X = \bigcup_{n \geq 1} A_n \) and that \( A_n^\circ = \emptyset \) for every integer \( n \geq 1 \). Then \( \emptyset = \bigcap_{n \geq 1} (X \setminus A_n) \), not possible (since \( X \setminus A_n = X \)).

Remark \( \mathbb{R}^2 \) can not be written as the countable union of lines. Also, \( \mathbb{R}^3 \) can not be written as the countable union of planes.

Problem
Let \( X \) be a complete metric space. If \( f : X \rightarrow X \) is surjective, then show that there exists an integer \( n \geq 1 \) and \( x_0 \in X \) such that \( f(B_n(x_0)) \) has nonempty interior.

Hint.
Apply the last problem to appropriate closed subsets \( A_n \) of \( X \).
Theorem

Let $X$ be a complete normed linear space containing an infinite linearly independent sequence $\{x_n\}_{n \geq 1}$ in $X$. Then the linear span of $\{x_n\}_{n \geq 1}$ is a proper subspace of $X$.

Proof.

For $m \geq 1$, let $Y_m := \text{linspan}\{x_1, \cdots, x_m\}$.

- $Y_m$ is a proper closed subspace of $X$, and hence $Y_m$ has empty interior (if $B_r(x) \subseteq Y_m$, then for any nonzero $y \in X$, $x + \frac{r}{2} \frac{y}{\|y\|} \in Y_m$ and hence $y \in Y_m$).

- The complement of $Y_m$ is open and dense. By the Baire Category Theorem, the intersection $\bigcap_{m \geq 1}(X \setminus Y_m)$ is dense in $X$. If $\text{linspan} \{x_n\}_{n \geq 1} = X$, then any element in $\bigcap_{m \geq 1}(X \setminus Y_m)$ (which exists) belongs to $Y_N$ for large $N$, and hence it belongs to $Y_N$ and its complement $X \setminus Y_N$ simultaneously, a contradiction. \qed
Problem

If \((X, d)\) is a complete metric space such that every \(x \in X\) is a cluster point of \(X \setminus \{x\}\), then show that \(X\) is not countable.

Solution.

If possible, assume that \(X\) is countable. So one can enumerate \(X\) as a sequence \(\{x_n\}_{n \geq 1}\). Note that for every integer \(n \geq 1\), \(X \setminus \{x_n\}\) is open and dense in \(X\). By the Baire Category Theorem, 
\[ \bigcap_{n=1}^{\infty} (X \setminus \{x_n\}) \text{ is dense in } X. \]
However, 
\[ \bigcap_{n=1}^{\infty} (X \setminus \{x_n\}) = \emptyset, \]
and hence \(X\) must be uncountable.

\((\mathbb{Z}, d_0)\) is a complete metric space even if \(\mathbb{Z}\) is countable (\(\{n\}\) is open for every \(n \in \mathbb{Z}\)). Also, the above problem is applicable to \((\mathbb{R}, d_1)\) providing indirect verification of uncountability of \(\mathbb{R}\).

Theorem (Nested Set Theorem)

Let \(X\) be a complete metric space and let \(\{J_n\}_{n \geq 1}\) be a decreasing sequence of nonempty closed sets in \(X\) such that \(\text{diam}(J_n) \to 0\) as \(n \to \infty\). Then \(\bigcap_{n \geq 1} J_n\) contains exactly one point.
Uniform continuity

Definition
Let \((X, d)\) and \((Y, \rho)\) be two metric spaces. Let \(f : X \to Y\) be a function. We say that \(f\) is uniformly continuous if for every \(\epsilon > 0\), there exists \(\delta > 0\) such that

\[
\left( x, y \in X, \quad d(x, y) < \delta \implies \rho(f(x), f(y)) < \epsilon \right).
\]

- Every uniformly continuous function is continuous at every point in \(X\). The major difference is that unlike the definition of continuity at a point, single \(\delta\) works for all points in \(X\).
- \(f(x) = 1/x, \ x \in (0, 1)\) is continuous but not uniformly continuous. Indeed, if \(m, n \geq 1\) are integers such that \(m > n > \delta/2\), then \(|1/n - 1/m| < \delta\) & \(|f(1/n) - f(1/m)| \geq 1\).
- For any subset \(A\) of a normed linear space \(X\), the function \(f(x) = \text{dist}(x, A), \ x \in X\) is uniformly continuous. Indeed, since \(|f(x) - f(y)| \leq \|x - y\|, \ x, y \in X\), one may take \(\delta = \epsilon\).
Problem
Let $(\mathbb{R}, d_1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then

1. $f$ maps bounded sets to bounded sets.
2. $f$ maps Cauchy sequences to Cauchy sequences.

Hint.
To see 1, let $A$ be a bounded subset of $\mathbb{R}$. Thus $A \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < b$. We must check that $f([a, b])$ is bounded. Note that $[a, b]$ can be covered by finitely many open intervals of length less than $\delta$ (the one obtained from the definition of uniform continuity with $\epsilon = 1$). Part 2 follows from definition.

Problem
Let $X, Y$ be normed linear spaces and let $T : X \to Y$ be such that is, $T(x + \alpha y) = T(x) + \alpha T(y)$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$. Show that $T$ is continuous at 0 iff $T$ is uniformly continuous.

Hint.
For $0 \neq x \in X$, let $x' = \frac{\delta}{2 \|x\|} x$. Now use the continuity at 0.
Theorem
Let \( a, b \in \mathbb{R} \) be such that \( a < b \). If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous.

Proof.
Let \( \epsilon > 0 \) and let \( x \in [a, b] \). Then, by the continuity of \( f \), there exists \( \delta_x \) such that

\[
\left( y \in [a, b], \ |x - y| < \delta_x \implies |f(x) - f(y)| < \epsilon/2 \right).
\]

However, we want single \( \delta \), which will work for every \( x \in [a, b] \). Note that \( [a, b] \subseteq \bigcup_{x \in [a,b]} (x - \delta_x/2, x + \delta_x/2) \). It turns out that there are finitely many \( x_1, \ldots, x_k \in [a, b] \) such that

\[
[a, b] \subseteq \bigcup_{n=1}^k (x_n - \delta_{x_n}/2, x_n + \delta_{x_n}/2)
\]

(to be seen later). Let \( \delta = \min_{n=1}^k \delta_{x_n}/2 \). If \( |x - y| < \delta \), then \( x \in (x_n - \delta_{x_n}/2, x_n + \delta_{x_n}/2) \) for some \( n \), and hence \( |y - x_n| < \delta_{x_n} \).

It follows that \( |f(x) - f(y)| < \epsilon \). \( \square \)
Definition
Let \((X, d)\) be a metric space. We say that \(X\) is compact if for any collection \(\{U_\alpha : \alpha \in I\}\) of open subsets of \(X\) such that \(X = \bigcup_{\alpha \in I} U_\alpha\), there exists finitely many indices \(\{\alpha_1, \ldots, \alpha_k\}\) in \(I\) such that \(X = \bigcup_{j=1}^k U_{\alpha_j}\).

- \(\{U_\alpha : \alpha \in I\}\): open cover of \(X\)
- \(\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}\): open subcover of \(X\)

Remark If \(f : X \to Y\) is a continuous surjection and \(X\) is compact, then \(Y\) is also compact. Indeed, if \(\{V_\alpha : \alpha \in I\}\) is an open cover of \(Y\), then \(\{f^{-1}(V_\alpha) : \alpha \in I\}\) is an open cover of \(X\), and since \(X\) is compact, \(\{f^{-1}(V_\alpha) : \alpha \in I\}\) admits a finite subcover \(\{f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_k})\}\) of \(X\). It follows that \(Y = \bigcup_{j=1}^k V_{\alpha_j}\).
Problem

Show that none of the following metric spaces \((X, d)\) is compact.

1. \(X = (0, 1] \), \(d = \text{relative metric induced by } d_1\)
2. \(X = \mathbb{Z} \), \(d = d_0\)
3. \(X = \mathbb{R}^d \), \(d = d_p\)
4. \(X = \text{normed linear space}, d(x, y) = \|x - y\|\)

Solution.

For 1, consider the open cover \(\{(1/n, 1] : n \geq 2\}\) of \((0, 1]\). For 2, consider the open cover \(\{\{n\} : n \in \mathbb{Z}\}\). Part 3 follows from 4, and for 4, consider the open cover \(\{B_n(0) : n \geq 1\}\). Verify that none of above open covers admits a finite subcover.

Problem

Let \(X\) be an arbitrary set endowed with a discrete metric \(d_0\). Then \(X\) is compact if and only if \(

\square

\)

\)
Example

We claim that the finite interval $[a, b]$ with relative metric $d_1$ is compact. If this is false, then there exists an open cover $\mathcal{U} = \{ U_\alpha : \alpha \in I \}$ of $[a, b]$, which has no finite subcover.

- Note that $\mathcal{U}$ is also an open cover for $[a, (a + b)/2]$ and $[(a + b)/2, b]$, and $\mathcal{U}$ does not admits finite subcover for at least one of subintervals, say, $[a_1, b_1]$. Clearly, $b_1 - a_1 = \frac{b-a}{2}$.

- Note that $\mathcal{U}$ is also an open cover for $[a_1, (a_1 + b_1)/2]$ and $[(a_1 + b_1)/2, b_1]$, and $\mathcal{U}$ does not admits finite subcover for at least one of subintervals, say, $[a_2, b_2]$. Clearly, $b_2 - a_2 = \frac{b-a}{4}$.

- By induction, for every integer $n \geq 1$, $\mathcal{U}$ does not admits finite subcover for $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ and $b_n - a_n = \frac{b-a}{2^n}$.

- By the nested interval theorem, there exists $c \in \bigcap_{k=1}^\infty [a_k, b_k]$. Since $\mathcal{U}$ is an open cover for $[a_1, b_1]$, $c \in U_\alpha$ for some $\alpha \in I$.

- $c$ is an interior point of $U_\alpha$, $a_k \uparrow c$ and $b_k \downarrow c$, for large $k$, $[a_k, b_k] \subseteq U_\alpha$, that is, $[a_k, b_k]$ admits a finite subcover. $\Rightarrow \Leftarrow$
Theorem (Heine-Borel Theorem)
Any closed and bounded subset $A$ of $(\mathbb{R}, d_1)$ is compact.

Proof.
We claim that any closed subset $A$ of a compact metric space $(X, d)$ is compact. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of $A$.

- $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of $X$.
- There exists a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\} \cup X \setminus A$ of $X$ (since $X$ is compact).
- $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is an open subcover of $A$.

Since $A$ is bounded, $A \subseteq [a, b]$ (finite interval). Since $[a, b]$ is compact and $A$ is a closed subset of $[a, b]$, $A$ is also compact.

- Every closed subset of a compact metric space is compact.

Question Does there exist a closed and bounded set which is not compact?
Question Let \((X, d)\) be a metric space and \(Y \subseteq X\) endowed with the relative metric \(d^Y(x, y) = d(x, y), x, y \in X\). If \(K \subseteq Y\), then whether or not

\((K, d)\) compact \iff \((K, d^Y)\) compact?

- If \(\{U_\alpha : \alpha \in I\}\) is an open cover of \(K\) in \(X\) then \(\{U_\alpha \cap Y : \alpha \in I\}\) is an open cover of \(K\) in \(Y\).

- If \(\{V_\alpha : \alpha \in I\}\) is an open cover of \(K\) in \(Y\), then for every \(\alpha \in I\), \(V_\alpha = U_\alpha \cap K\) for some open set \(U_\alpha\) in \(X\). Note that \(\{U_\alpha : \alpha \in I\}\) is an open cover of \(K\) in \(X\).

Conclude that \((K, d)\) is compact \iff \((K, d^Y)\) is compact.
Theorem

A compact subset $K$ of a metric space $(X, d)$ is closed & bounded.

Proof.

To see that $K$ is closed, it suffices to check that $X \setminus K$ is open.

• Let $x \in X \setminus K$ and let $r_y = d(x, y)/2 > 0$ for every $y \in K$.

• $\{B_{r_y}(y) : y \in K\}$ is an open cover of $K$. So it admits a finite subcover $\{B_{r_{y_j}}(y_j) : j = 1, \ldots, N\}$ (since $K$ is compact).

• If $r = \min\{r_{y_1}, \ldots, r_{y_N}\}$, then $B_r(x) \subseteq X \setminus K$. Indeed, if $y \in K \cap B_r(x)$, then $y \in B_{r_{y_j}}(y_j) \cap B_r(x)$ for some $j$, and hence $d(y, y_j) < d(x, y_j)/2$, $d(x, y) < r \leq d(x, y_j)/2$, $\implies \iff$.

Thus $x$ is an interior point of $X \setminus K$, and hence $K$ is closed.

To see that $K$ is bounded, let $x \in X$. Consider the open cover $\{B_r(x) \cap K : r > 0\}$ of $K$. Since $K$ is compact, there exists $r > 0$ such that $K \subseteq B_r(x)$, and hence it is bounded. □
Example
Consider \( X = \{ r \in \mathbb{Q} : 1 < r^2 < 2 \} \) with \( d \) being the relative metric induced by \( d_1 \) is not compact. Then
\[
X = \{ r \in \mathbb{Q} : 1 < |r| < \sqrt{2} \} = \left( (1, \sqrt{2}) \cap \mathbb{Q} \right) \cup \left( (-\sqrt{2}, -1) \cap \mathbb{Q} \right),
\]
which is not closed in \((\mathbb{Q}, d_1)\).

Problem
Assume that \( n \geq 2 \). Show that \( M_n(\mathbb{R}) \setminus GL_n(\mathbb{R}) \) is not compact as a metric space endowed with the (relative) metric \( d : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R} \) by \( d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}| \).

Hint.
\( M_n(\mathbb{R}) \setminus GL_n(\mathbb{R}) \) is closed in \( M_n(\mathbb{R}) \) but not bounded. Indeed, there are diagonal matrices \( B \) in \( M_n(\mathbb{R}) \setminus GL_n(\mathbb{R}) \) for which \( d(0, B) \) is of arbitrarily large magnitude provided \( n > 1 \).
Problem

Consider \( \mathbb{C}^2 \) endowed with the metric

\[ d((z, w), (z', w')) = (|z - z'|^2 + |w - w'|^2)^{1/2}, \quad (z, w), (z', w') \in \mathbb{C}^2. \]

Show that \( \{(z, w) \in \mathbb{C}^2 : z^2 + w^2 = 1\} \) is not compact.

Hint.

Find \( w \in \mathbb{C} \) such that \( w^2 = 1 + n^2 \). Thus, for \( \iota = \sqrt{-1} \), we have \( (\iota n, \sqrt{1 + n^2}) \in \{(z, w) \in \mathbb{C}^2 : z^2 + w^2 = 1\} \).

Question What if \( z^2 + w^2 - 1 \) is replaced by any nonconstant polynomial in \( z, w \)?

The answer is yes (this can be deduced from the Fundamental theorem of algebra).
Theorem

If \((X, d)\) is compact, then every sequence in \(X\) has a convergent subsequence.

Proof.

We claim that an infinite subset \(A\) of \(X\) has a cluster point in \(X\).

• Suppose \(A\) has no cluster point in \(X\), that is, for every \(x \in X\), there exists \(r_x > 0\) such that \(B_{r_x}(x) \cap A \subseteq \{x\}\).

• \(\{B_{r_x}(x) : x \in X\}\) is an open cover for \(X\), and hence \(X\) has finite subcover \(\{B_{r_{x_j}}(x_j) : j = 1, \ldots, N\}\).

• \(A = \bigcup_{j=1}^{N}(B_{r_{x_j}}(x_j) \cap A) \subseteq \{x_1, \ldots, x_N\}\) is a finite set.

Applying the above fact to \(A = \{x_n\}_{n \geq 1}\), we conclude that \(A\) has a cluster point in \(X\). However, for every cluster point, there exists a sequence in \(A\) converging to this cluster point (see slide 62). \(\square\)
Let $Y$ be a proper, closed subspace of a normed linear space $X$.

- Choose $x_1 \in X \setminus Y$, and note that $d(x_1, Y) > 0$.
- There exists $x_0 \in Y$ such that $\|x_1 - x_0\| < 2d(x_1, Y)$.
- Note that $\|x_1 - x_0\|y + x_0$ belongs to $Y$ for any $y \in Y$.
- If $x = \frac{x_1 - x_0}{\|x_1 - x_0\|}$ (unit vector), then for any $y \in Y$,
  \[
  \|x - y\| = \frac{\|x_1 - x_0 + \|x_1 - x_0\|y\|}{\|x_1 - x_0\|} \geq \frac{1}{2}.
  \]

Thus there exists a unit vector $x \in X$ such that $d(x, Y) \geq 1/2$.

**Theorem**

*If $X$ is infinite dimensional, then unit sphere in $X$ is not compact.*

**Proof.**

- Let $x_1 \in X$ be a unit vector in $X$ and let $X_1 := \text{span}\{x_1\}$.
- There exists $x_2 \in X$ such that $\|x_2\| = 1$ and $d(x_2, X_1) \geq 1/2$.
- Note that $x_2 \notin X_1$. Let $X_2 := \text{span}\{x_1, x_2\}$.
- There exists $x_3 \in X_2$ such that $\|x_3\| = 1$ and $d(x_3, X_2) \geq 1/2$.

Continuing this, we get $\{x_n\}_{n \geq 1}$ with no cgt subsequence. \qed
Definition
A metric space $X$ is sequentially compact if every sequence in $X$ has a subsequence convergent in $X$.

- A compact metric space is sequentially compact (slide 117).
- If $X$ is sequentially compact and $f : X \to Y$ is a continuous surjection, then $Y$ is sequentially compact (if $\{y_n = f(x_n)\}_{n \geq 1}$ is given, then $\{x_n\}_{n \geq 1}$ has a convergent subsequence, and hence $\{f(x_n)\}_{n \geq 1}$ has a convergent subsequence).
- Every sequentially compact metric space is complete. Indeed, if $\{x_n\}_{n \geq 1}$ is a Cauchy sequence, then since $\{x_n\}_{n \geq 1}$ has a convergent subsequence, $\{x_n\}_{n \geq 1}$ is convergent (slide 67).

- A complete metric space is not necessarily sequentially compact. For example, $\{n\}_{n \in \mathbb{N}}$ in $(\mathbb{R}, d_1)$ has no convergent subsequence.

**Question** What are all complete metric spaces which are also sequentially compact?
Problem

Consider the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices $A = (a_{i,j})_{1 \leq i,j \leq n}$ with real entries. For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (seen as a column vector), note that $AX \in \mathbb{R}^n$. Define

$$\|A\| = \sup_{X \in \mathbb{R}^n, \|X\|_2 = 1} \|AX\|_2, \quad A \in M_n(\mathbb{R}).$$

Verify the following:

1. $(M_n(\mathbb{R}), \| \cdot \|)$ is a complete normed linear space.
2. $ISO = \{ A \in M_n(\mathbb{R}) : \|AX\|_2 = \|X\|_2 \text{ for every } X \in \mathbb{R}^n \}$ is sequentially compact.

Hint.

Since $\| \cdot \|_2$ is a norm on $\mathbb{R}^n$, $\| \cdot \|$ defines a norm on $M_n(\mathbb{R}^n)$. To see that $M_n(\mathbb{R})$ is complete, note that $f : M_n(\mathbb{R}) \to (\mathbb{R}^{n^2}, \| \cdot \|_2)$ given by $f(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn})$ is a linear homeomorphism (since $\|A_m - A\| \to 0$ if and only if $(i,j)$th entry of $A_m$ converges to $(i,j)$th entry of $A$ for every $1 \leq i,j \leq n$). To see 2, note that $ISO$ is closed and bounded in $M_n(\mathbb{R})$. □
Theorem

Let \((X, d)\) be a metric space and let \(A \subseteq X\) be sequentially compact. Then, for every \(\epsilon > 0\), there exist finitely many points \(x_1, \ldots, x_N \in X\) such that \(A \subseteq \bigcup_{j=1}^{N} B_\epsilon(x_j)\).

Proof.

Let \(\epsilon > 0\) and let \(x_1 \in A\).

- If \(A \subseteq B_\epsilon(x_1)\), then we are done.
- Otherwise, there exists \(x_2 \in A \setminus B_\epsilon(x_1)\). Thus \(d(x_1, x_2) \geq \epsilon\).
- If \(A \subseteq B_\epsilon(x_1) \cup B_\epsilon(x_2)\), then we are done.
- Otherwise, there exists \(x_3 \in A \setminus (B_\epsilon(x_1) \cup B_\epsilon(x_2))\). Thus \(d(x_1, x_3) \geq \epsilon\) and \(d(x_2, x_3) \geq \epsilon\).

Continuing this, we either have \(A \subseteq \bigcup_{j=1}^{N} B_\epsilon(x_j)\) for some integer \(N \geq 1\) or there exists a sequence \(\{x_n\}_{n \geq 1}\) such that \(d(x_n, x_m) \geq \epsilon\) for all positive integers \(m \neq n\). The latter case does not arise since \(A\) is sequentially compact. \(\square\)
Totally bounded spaces

**Definition**

A metric space $X$ is said to be **totally bounded** if for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_N \in X$ such that

$$X \subseteq \bigcup_{j=1}^{N} B_\epsilon(x_j).$$

- Unlike compactness and sequential compactness, total boundedness is not preserved under homeomorphism. To see this, note that $(0, 1)$ is totally bounded. Indeed, if choose an integer $N \geq 1$ such that $N\epsilon > 1$, then

$$(0, 1) \subseteq (-\epsilon, \epsilon) \cup (\epsilon/2, 3\epsilon/2) \cup (\epsilon, 3\epsilon) \cup \cdots \cup (N\epsilon, (N + 2)\epsilon).$$

On the other hand, $\mathbb{R}$ is homeomorphic to $(0, 1)$, but it is not totally bounded.
Compact $\rightarrow$ Sequentially compact $\rightarrow$ Complete and totally bounded

(see slides 117, 119, 121)

**Question** Whether Sequentially Compact $\Rightarrow$ Compact ?

**Question** Whether Complete and Totally bounded $\Rightarrow$ Compact ?

- This will give complete the diagram!
Problem

Every totally bounded set is bounded.

Hint.
Note that union of finitely many open balls is contained in a single open ball of sufficiently large radius.

Problem

Show that a subset of a totally bounded metric space is again totally bounded.

Solution.
Suppose $X$ is totally bounded and $A \subseteq X$. Then, for every $\epsilon > 0$, there exist $x_1, \ldots, x_N \in X$ such that $A \subseteq \bigcup_{j=1}^N B_{\epsilon/2}(x_j)$. For each $j$, choose $a_j \in A \cap B_{\epsilon/2}(x_j)$, and note that by the triangle inequality, $A \subseteq \bigcup_{j=1}^N B_{\epsilon}(a_j)$. 
Problem
Show that union of finitely many totally bounded sets is totally bounded.

Solution.
This follows from the definition.

Problem
Show that every totally bounded metric space is separable.

Solution.
For every integer $k \geq 1$, there exist $x_1^{(k)}, \ldots, x_{N_k}^{(k)} \in X$ such that $X \subseteq \bigcup_{j=1}^{N_k} B_{1/k}(x_j^{(k)})$. Clearly, $Y = \bigcup_{k \geq 1} \{x_1^{(k)}, \ldots, x_{N_k}^{(k)}\}$ is countable. Verify that $Y$ is dense in $X$, that is, every ball of radius $\epsilon$ in $X$ intersects $Y$ non-trivially.
Theorem
Let \((X, d)\) be a metric space. Let \(\{x_n\}_{n \geq 1}\) be a sequence in \(X\) and let \(A = \{ x \in X : x = x_n \text{ for some } n \geq 1 \}\). Then

1. If \(\{x_n\}_{n \geq 1}\) is Cauchy, then \(A\) is totally bounded.
2. If \(A\) is totally bounded, then \(\{x_n\}_{n \geq 1}\) has a Cauchy subsequence.

Proof.
To see (1), let \(\epsilon > 0\). Since \(\{x_n\}_{n \geq 1}\) is Cauchy, there exists \(N \geq 1\) such that \(d(x_n, x_m) < \epsilon\) for all \(m, n \geq N\). Thus

\[
A = \{ x_1 \} \cup \cdots \cup \{ x_{N-1} \} \cup \{ x_n : n \geq N \} \subseteq B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_{N-1}) \cup B_\epsilon(x_N).
\]

To see (2), we may assume that \(A\) is infinite (otherwise, \(\{x_n\}_{n \geq 1}\) is eventually constant, and hence it has a convergent sequence).

- Cover \(A\) by finitely many balls of radius 1, and at least one of these balls, say, \(A_1\) contains infinitely many points in \(A\).
- Cover \(A_1\) by finitely many balls of radius \(1/2\), and at least one of these balls, say, \(A_2\) contains infinitely many points in \(A\).
Proof continued ...
Continuing this, we obtain a decreasing sequence $\{A_n\}_{n \geq 1}$ of balls $A_n$ of radius $1/n$, where each $A_n$ contains infinitely many points of $A$. Choose now a sequence $\{x_{n_k}\}_{k \geq 1}$ such that $x_{n_k} \in A_k$ and check that $\{x_{n_k}\}_{k \geq 1}$ is a Cauchy sequence. 

Corollary

Let $(X, d)$ be a metric space. Then $X$ is totally bounded and complete if and only if it is sequentially compact.

Proof.

We have already seen $\impliedby$ (see slides 119 and 121). To see $\implies$, let $\{x_n\}_{n \geq 1}$ be given and and let

$$A = \{x \in X : x = x_n \text{ for some } n \geq 1\}.$$ 

Since $X$ is totally bounded, so is $A$ (see slide 124). By the last theorem, $\{x_n\}_{n \geq 1}$ has a Cauchy subsequence, say, $\{x_{n_k}\}_{k \geq 1}$. However, $X$ is complete, so that $\{x_{n_k}\}_{k \geq 1}$ is convergent in $X$, and hence $X$ is sequentially compact. 

Example
Let $H$ denote the space of all sequences in $[0, 1]$. Thus

$$H = \{ x = \{x_n\}_{n \geq 1} : 0 \leq x_n \leq 1 \text{ for every } n \geq 1 \}.$$ 

One may think of $H$ as the infinite product $\prod_{n=1}^{\infty} [0, 1]$. Define $d : H \times H \rightarrow [0, \infty)$ by

$$d(x, y) = \sup_{n \geq 1} \left( \frac{|x_n - y_n|}{2^n} \right), \quad x, y \in H.$$

- $d$ is a metric. Indeed, $0 \leq d(x, y) \leq 1/2$, $d(x, y) = 0 \iff x = y$, and $d(x, z) \leq d(x, y) + d(y, z)$ for every $x, y, z \in H$.

- $d(x^{(n)}, x) \rightarrow 0$ as $n \rightarrow \infty \iff |x_k^{(n)} - x_k| \rightarrow 0$ as $n \rightarrow \infty$ for every $k \geq 1$. The part $\Rightarrow$ is clear. To see $\Leftarrow$, choose $N \geq 2$ such that $2^{-N} \leq \epsilon/4$, and choose $N_0 \geq 1$ such that $|x_k^{(n)} - x_k| < \epsilon/2$ for all $k = 1, \ldots, N - 1$ and for all $n \geq N_0$. 

Example (Continued ...)

- Let $r > 0$ and $x \in H$. Let $N \geq 2$ be such that $2^N r > 2$. Then

$$B_r(x) = \{ y \in H : d(x, y) < r \} = \{ y \in H : \sup_{n \geq 1} \left( \frac{|x_n - y_n|}{2^n} \right) < r \}$$

$$\subseteq \{ y \in H : |x_n - y_n| < 2^n r \text{ for every } n \geq 1 \}$$

$$= \{ y \in H : |x_n - y_n| < 2^n r \text{ for } n = 1, \ldots, N - 1 \}.$$  

$$\implies B_r(x) = \{ y \in H : |x_n - y_n| < 2^n r \text{ for } n = 1, \ldots, N - 1 \}.$$  

Thus the open ball $B_r(x)$ in $H$ is of the form

$$B_{2r}(x_1) \times \cdots \times B_{2^{N-1}r}(x_{N-1}) \times [0,1] \times \cdots .$$

We claim that $H$ is sequentially compact. It suffices to check that $H$ is complete and totally bounded.
Example (Continued ...)

- The Hilbert cube $H$ is totally bounded.

Let $\epsilon > 0$. Find an integer $N \geq 1$ such that $2^N \epsilon > 2$. Then, by the discussion on the previous slide, the ball $B_\epsilon(x)$ is of the form

$$B_{2\epsilon}(x_1) \times \cdots \times B_{2^{N-1}\epsilon}(x_{N-1}) \times [0, 1] \times \cdots.$$ 

Since $[0, 1]$ is totally bounded, for every $j = 1, \ldots, N - 1$, it can be covered by finitely many balls (intervals) of the form $B_{2^j\epsilon}(x_{ij})$ (with finitely choices of $i = 1, \ldots, N_j$). If one takes the product

$$B_{2\epsilon}(x_{i_11}) \times \cdots \times B_{2^{N-1}\epsilon}(x_{i_{N-1}1}) \times [0, 1] \times \cdots,$$

then $H$ can be covered by finitely many balls of radius $\epsilon$. 
Example (Continued ...)

- The Hilbert cube $H$ is complete.

To see that $H$ is complete, let $\{x^{(n)}\}_{n \geq 1}$ be Cauchy in $H$.

- Note that $d(x^{(n)}, x^{(m)}) \to 0$ as $m, n \to \infty$ if and only if $|x_k^{(n)} - x_k^{(m)}| \to 0$ as $n \to \infty$ for every $k \geq 1$.

- Since $\mathbb{R}$ is complete, $\{x_k^{(n)}\}_{n \geq 1}$ converges to some $x_k$ in $\mathbb{R}$ for every $k \geq 1$.

- Note that $x = \{x_k\}_{k \geq 1}$ belongs to $H$. It follows from the discussion on slide 128 that $d(x^{(n)}, x) \to 0$ as $n \to \infty$.

Since complete and totally bounded metric space is sequentially compact, the discussion above shows that the Hilbert cube $H$ is indeed sequentially compact.
Problem

Show that \( \{ x = \{ x_n \}_{n \geq 1} : -1 \leq x_n \leq 1 \text{ for every } n \geq 1 \} \) as a subset of \( l^\infty \) is not compact.

Problem

Give an example of infinite sequentially compact subset of \( l^\infty \).

Solution.

Let \( K = \{ x = \{ x_n \}_{n \geq 1} : 0 \leq x_n \leq 1/2^n \text{ for every } n \geq 1 \} \) and note that \( f : H \to l^\infty \) given by

\[
     f(\{x_1, x_2, x_3, \ldots, \}) = \{x_1/2, x_2/4, x_3/8 \ldots, \}
\]

satisfies \( \|f(x) - f(y)\|_{\infty} = d(x, y) \). Thus \( f \) is continuous, and since \( H \) is sequentially compact, so is \( f(H) \). However, \( f(H) = K \) is a subset of \( l^\infty \).

The above example provides an infinite compact subset of \( l^\infty \).
Theorem

A sequentially compact metric space is compact.

Proof.

Let \( \{U_\alpha : \alpha \in I\} \) be an open cover of \( X \). For \( x \in X \), let \( r_x = \sup \{ r \in \mathbb{R} : B_r(x) \subseteq U_\alpha \text{ for some } \alpha \} \) (\(< \infty \) since \( X \) is bounded).

- For \( \epsilon = \inf \{ r_x : x \in X \} \), there is \( \{x_n\}_{n \geq 1} \) such that \( r_{x_n} \to \epsilon \).
- By sequential compactness, \( \{x_{n_k}\}_{k \geq 1} \) converges to \( x \in X \).
- \( x \in U_\alpha \) for some \( \alpha \), and hence \( B_r(x) \subseteq U_\alpha \) for some \( r > 0 \).
- If \( d(x_{n_k}, x) < r/2 \) for all large \( k \), then \( r_{x_{n_k}} > r/2 \). Thus \( \epsilon > 0 \).
- Let \( x_1 \in X \). Either \( X = B_{\epsilon/2}(x_1) \subseteq U_\alpha \) for some \( \alpha \) or there exists \( x_2 \in X \setminus B_{\epsilon/2}(x_1) \).
  - Either \( X = B_{\epsilon/2}(x_1) \cup B_{\epsilon/2}(x_2) \subseteq U_\alpha \cup U_\beta \) for some \( \alpha, \beta \) or there exists \( x_3 \in X \setminus (B_{\epsilon/2}(x_1) \cup B_{\epsilon/2}(x_2)) \).

One may continue this. However, this process ends in finitely many steps (otherwise we will get a sequence without a convergent subsequence contradicting sequential compactness). \( \square \)
Problem
Let \((X, d), (Y, \rho)\) be two metric spaces. For \((x_j, y_j) \in X \times Y\) define

\[D((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), \rho(y_1, y_2)\},\]

Verify:

1. \(D\) defines a metric on \(X \times Y\).
2. If \(X\) and \(Y\) are compact, then so is \(X \times Y\).

Hint.
For 2, it suffices to check that \(X \times Y\) is sequentially compact.

- Let \(\{(x_n, y_n)\}_{n \geq 1}\) be a sequence in \(X \times Y\), Then \(\{x_n\}_{n \geq 1} \subseteq X\) and \(\{y_n\}_{n \geq 1} \subseteq Y\).
- Since \(X\) is sequential compact, \(\{x_{n_k}\}_{k \geq 1}\) is convergent.
- Now \(\{y_{n_k}\}_{k \geq 1} \subseteq Y\), so by the sequential compactness of \(Y\), \(\{y_{n_{k_l}}\}_{l \geq 1}\) is convergent.

Thus \(\{x_{n_{k_l}}\}_{l \geq 1}\) and \(\{y_{n_{k_l}}\}_{l \geq 1}\) are convergent. Then \(\{(x_{n_{k_l}}, y_{n_{k_l}})\}_{l \geq 1}\) is convergent in \(X \times Y\). So \(X \times Y\) is sequentially compact.

Question Can you say that \(X\) and \(Y\) are compact if so is \(X \times Y\) ?
One may define sum of two subsets $A$ and $B$ of $\mathbb{R}^n$ as

$$A + B = \{a + b : a \in A, b \in B\}.$$ 

Problem

Let $A$ be closed, $B$ be compact in $\mathbb{R}^n$. Show that $A + B$ is closed.

Solution.

Let $\{x_n\}_{n \geq 1}$ be a sequence in $A + B$ such that $x_n \to x$ in $\mathbb{R}^n$.

- Thus $x_n = a_n + b_n$ for some $\{a_n\}_{n \geq 1} \subseteq A$ and $\{b_n\}_{n \geq 1} \subseteq B$.
- $\{b_{n_k}\}_{k \geq 1}$ converges to $b \in B$ ($B$ is sequentially compact).
- $\{x_{n_k}\}_{k \geq 1}$ converges to $x$, and $\{a_{n_k}\}_{k \geq 1}$ converges to $a \in A$ (difference of cgt sequences is cgt and $A$ is closed).

Thus $x = a + b$ for $a \in A$ and $b \in B$. Hence $A + B$ contains all its limit points showing that it is closed.

Problem

Let $A$ and $B$ be compact in $\mathbb{R}^n$. Show that $A + B$ is compact.

Problem

Give an example of two closed sets whose sum is not closed.
Corollary (Lebesgue covering lemma)

Let \((X, d)\) be a compact metric space. Let \(\{U_\alpha : \alpha \in I\}\) be an open cover of \(X\). Then there is a \(\delta > 0\) such that if \(A \subseteq X\) with diameter \(\text{diam}(A) < \delta\), then there is \(\alpha \in I\) such that \(A \subseteq U_\alpha\).

Proof.

Suppose there is no \(\delta > 0\) with the above property.

- For a positive integer \(n\), letting \(\delta = 1/n\), there exists \(A_n \subseteq X\) with diameter \(\text{diam}(A_n) < 1/n\), then \(A_n \not\subseteq U_\alpha\) for every \(\alpha \in I\).
- Choose any \(x_n \in A_n\) and note that \(\{x_n\}_{n \geq 1}\) has a subsequence \(\{x_{n_k}\}_{k \geq 1}\) converging to some \(x \in X\) (by compactness of \(X\)).
- \(x \in U_\alpha\) for some \(\alpha \in I\) (since \(\{U_\alpha : \alpha \in I\}\): open cover of \(X\)).
- Since \(U_\alpha\) is open, \(B_r(x) \subseteq U_\alpha\) for some \(r > 0\).
- Find \(k \geq 1\) such that \(n_k r > 2\) and \(x_{n_k} \in B_{r/2}(x)\).
- Now if \(a \in A_{n_k}\), then \(d(x, a) \leq d(x, x_{n_k}) + d(x_{n_k}, a) < r\).

This shows that \(A_{n_k}\) is contained in \(B_r(x) \subseteq U_\alpha\), \(\Rightarrow \Leftarrow\).
Lebesgue number

Definition
Let \( U = \{ U_\alpha : \alpha \in I \} \) be an open cover of \((X, d)\). A real number \( \delta > 0 \) is said to be \textit{Lebesgue number} for \( U \) if \( A \subseteq X \) with diameter \( \text{diam}(A) < \delta \), then there is \( \alpha \in I \) such that \( A \subseteq U_\alpha \).

Example
Consider the metric space \([0, 1]\) with relative metric induced by \( d_1 \). Let \( \delta > 0 \) be given. We claim that the Lebesgue number for the open covering \( U = \{ [0, \delta) \} \cup \{ (1/n, 1] : n \geq 1 \} \) of \([0, 1]\) equals \( \delta \).

- If \( N \) is a positive integer such that \( 1/N < \delta \) (which exists by AP), then \( \{ [0, \delta), (1/N, 1] \} \) is a finite open subcover of \([0, 1]\).
- Let \( A \) be such that \( \text{diam}(A) < \delta \). Then either \( A \subseteq [0, \delta) \) or \( A \cap [\delta, 1] \neq \emptyset \). Let \( a \in A \) be such that \( a \geq \delta \).
- If \( A \notin (1/n, 1] \) for any \( n \geq 1 \), (so \( a > \delta \)) then there is \( a_M \in A \) such that \( a_M \leq 1/M < a - \delta < a \). Thus \( \text{diam}(A) > \delta \).  

\[\[\text{7}\] I thought for a moment that there is a glitch but this calculation is fine.
Continuous functions on compact metric spaces

**Theorem**

*If* \( f \) *is a continuous function from a compact metric space* \((X, d)\) *to any other metric space* \((Y, \rho)\), *then* \( f(X) \) *is bounded.*

**Proof.**

Fix \( y_0 \in Y \) and write \( Y = \bigcup_{n \geq 1} B^\rho_n(y_0) \).

- \( f^{-1}(B^\rho_n(y_0)) \) is open for every \( n \geq 1 \) (since \( f \) is continuous).
- \( X = \bigcup_{n \geq 1} f^{-1}(B^\rho_n(y_0)) \).
- Since \( X \) is compact, there exist positive integers \( n_1, \ldots, n_k \) such that \( X = \bigcup_{j=1}^k f^{-1}(B^\rho_{n_j}(y_0)) \).

Let \( N = \max\{n_1, \ldots, n_k\} \) and note that \( X = f^{-1}(B^\rho_N(y_0)) \). Thus \( f(X) \subseteq B^\rho_N(y_0) \) is bounded.

- Let \( f : X \to \mathbb{R} \) be a continuous function. Then \( \sup_{x \in X} f(x) \) and \( \inf_{x \in X} f(x) \) exist. Can we find \( x_0 \) and \( x_1 \) in \( X \) such that \( \sup_{x \in X} f(x) = f(x_0) \) and \( \inf_{x \in X} f(x) = f(x_1) \) ?
Suppose that there is no $x_0 \in X$ such that $M = f(x_0)$, where $M = \sup_{x \in X} f(x)$. Thus $f(x') < \sup_{x \in X} f(x)$ for every $x' \in X$.

- $\{x' \in X : f(x') < M - 1/n\}$ is an open subset of $X$.
- $X = \bigcup_{n \geq 1} \{x' \in X : f(x') < M - 1/n\}$.
- There exist positive integers $n_1, \ldots, n_k$ such that

$$X = \bigcup_{j=1}^{k} \{x' \in X : f(x') < M - 1/n_j\}.$$ 

- $X = \{x' \in X : f(x') < M - 1/N\}$ with $N = \max\{n_1, \ldots, n_k\}$.

This would imply that $f(x) < M - 1/N$ for every $x \in X$, and hence $M < M - 1/N \Rightarrow \Leftarrow$

- Similarly, one can see that there exists $x_1 \in X$ such that $\inf_{x \in X} f(x) = f(x_1)$ (Exercise).

**Problem**

*Let $X$ be a compact metric space. Then there exists no continuous map from $X$ onto $(0, 1)$.*
Theorem (Criterion for homeomorphism)

Let $X, Y$ be metric spaces and let $f : X \rightarrow Y$ be a bijective continuous map. If $X$ is compact, then $f$ is a homeomorphism.

Proof.
To check that $f^{-1}$ is continuous, let $A$ be a closed subset of $X$.

- $A$ is compact (being closed subset of a compact space $X$).
- $f(A)$ is compact (continuous image of a compact space).
- $f(A)$ is closed (compact subset of a metric space is closed).

Thus continuous image of a closed set is closed. Hence $f^{-1}$ is continuous.

Problem

Let $X$ be a normed linear space with two norms $\| \cdot \|$ and $\| \cdot \|'$. Let $Y$ be a compact subset of $X$. If there exists $M > 0$ such that $\|x\| \leq M\|x\|'$ for every $x \in Y$, then show that $(Y, \| \cdot \|)$ and $(Y, \| \cdot \|')$ are homeomorphic.
Theorem
Any continuous function $f$ from a compact metric space $(X, d)$ to any other metric space $(Y, \rho)$ is uniformly continuous.

Proof.
Since $f$ is continuous at $x$, given $\epsilon > 0$, there exists $\delta_x > 0$ such that $\rho(f(x), f(x')) < \epsilon/2$ whenever $x' \in X$ and $d(x, x') < \delta_x$.

- $\{B_{\delta_x}(x) : x \in X\}$ is an open cover of $X$.
- There is a $\delta > 0$ (Lebesgue number) such that if $A \subseteq X$ with diameter $\text{diam}(A) < \delta$, then there is $x \in X$ such that $A \subseteq B_{\delta_x}(x)$ (by Lebesgue covering lemma).
- If $x_1, x_2 \in X$ are such that $d(x_1, x_2) < \delta$, then apply the above to $A = \{x_1, x_2\}$, we get $x \in X$ such that $\{x_1, x_2\} \subseteq B_{\delta_x}(x)$.

Thus $d(x_1, x) < \delta_x$ and $d(x_2, x) < \delta_x$. Hence, by continuity at $x$, $\rho(f(x_1), f(x_2)) \leq \rho(f(x_1), f(x)) + \rho(f(x_2), f(x)) < \epsilon$. \qed
Example
Let $f : (0, 1) \to \mathbb{R}$ be a bounded, continuous function, which is monotonically increasing.

- $\inf_{x \in (0,1)} f(x)$ and $\sup_{x \in [0,1)} f(x)$ exist (since $f$ is bounded).
- Define $g : [0, 1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} 
  f(x) & \text{if } x \in (0, 1), \\
  \inf_{x \in (0,1)} f(x) & \text{if } x = 0, \\
  \sup_{x \in [0,1)} f(x) & \text{if } x = 1.
\end{cases}$$

Note that $g : [0, 1] \to \mathbb{R}$ is a continuous function (since $f$ is increasing), and hence it is uniformly continuous.

- One can not drop the assumption of boundedness in the above example (e.g. $f(x) = 1/x, \ x \in (0, 1)$).
Problem

Given \((X, d), (Y, \rho)\), such that \(X\) is compact, verify:

1. \(\sup_{x \in X} \rho(f(x), y) < \infty\) for any \(y \in Y\).
2. Consider \(C(X, Y) = \{ f : X \to Y : f \text{ is continuous} \}\). Define

\[
D(f, g) = \sup_{x \in X} \rho(f(x), g(x)), \quad f, g \in C(X, Y).
\]

Then \((C(X, Y), D)\) is a metric space.

Hint.

For a positive integer \(k\), define \(U_k = \{ x \in X : \rho(f(x), y) < k \}\). Verify that \(\{ U_k : k \geq 1 \}\) is an open cover of \(X\). Now (1) follows from the compactness of \(X\).

By (1) and the triangle inequality, \(D(f, g) < \infty\) for every \(f, g \in C(X, Y)\). Since \(\rho\) is a metric, so is \(D\).
**Theorem**

Let \((X, d), (Y, \rho)\) be metric spaces and \(X\) be compact. If \(Y\) is complete, then \(C(X, Y)\) is complete.

**Proof.**

Let \(\{f_n\}_{n \geq 1}\) be a Cauchy sequence in \(C(X, Y)\), that is, given \(\epsilon > 0\), there exists \(N \geq 1\) such that \(D(f_n, f_m) < \epsilon / 3\) for every \(m, n \geq N\).

- For every \(x \in X\), \(\{f_n(x)\}_{n \geq 1}\) is a Cauchy sequence in \(Y\).
- For every \(x \in X\), \(\{f_n(x)\}_{n \geq 1}\) converges to some \(f(x) \in Y\).
- \(\rho(f_n(x), f_m(x)) < \frac{\epsilon}{3}, m, n \geq N \Rightarrow \rho(f(x), f_m(x)) \leq \frac{\epsilon}{3}, m \geq N\)

To see that \(f \in C(X, Y)\), note that for any \(x, x' \in X\),

\[
\rho(f(x), f(x')) \leq \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x')) + \rho(f_N(x'), f(x')),
\]

which is \(\leq 2\epsilon / 3 + \rho(f_N(x), f_N(x'))\). However, \(f_N\) is continuous and \(X\) is compact, so that for some \(\delta > 0\),

\[
\rho(f_N(x), f_N(x')) < \epsilon / 3 \text{ whenever } d(x, x') < \delta.
\]

Thus \(\rho(f(x), f(x')) < \epsilon\) whenever \(d(x, x') < \delta\). \(\square\)
**Corollary**

*Let $(X, d)$ be a compact metric space and consider*

\[ C(X) = \{ f : X \to \mathbb{R} : f \text{ is continuous} \}. \]

*If $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $f \in C(X)$, then $C(X)$ is a complete normed linear space with $\| \cdot \|_\infty$.\*

The rest of the notes is devoted to the study of this metric space.

- What are compact subsets of $C(X)$? (Arzela-Ascoli Theorem)
- Whether $C(X)$ is separable? (Weierstrass Theorem)
- Is there any proper clopen subset of $C(X)$?

Answer to the last question is No (Why?)
Theorem
Let \((X, d)\) be a compact metric space and let \(F\) be a totally bounded subset of \(C(X)\). Then for each \(\epsilon > 0\), there exists \(\delta > 0\) such that \(\sup_{f \in F} |f(x) - f(y)| < \epsilon\) whenever \(x, y \in X\) and \(d(x, y) < \delta\).

Proof.
Let \(\epsilon > 0\). Then

- there exist \(f_1, \ldots, f_N \in F\) such that \(F \subseteq \bigcup_{j=1}^{N} B_{\epsilon/3}(f_j)\).
- there exists \(\delta_j > 0\) such that \(|f_j(x) - f_j(y)| < \epsilon/3\) whenever \(d(x, y) < \delta_j\) (since each \(f_j\) is uniformly continuous; slide 141).

If \(\delta = \min\{\delta_1, \ldots, \delta_N\}\) and \(d(x, y) < \delta\), then any \(f \in F\) lies in some ball \(B_{\epsilon}(f_j)\), and hence

\[
|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < \epsilon.
\]
Equicontinuity

**Definition**
Let \((X, d)\) be a compact metric space and let \(F\) be a subset of \(C(X)\). We say that \(F\) is equicontinuous if for each \(\epsilon > 0\), there exists \(\delta > 0\) such that \(\sup_{f \in F} |f(x) - f(y)| < \epsilon\) whenever \(x, y \in X\) and \(d(x, y) < \delta\).

- Every compact subset of \(C(X)\) is equicontinuous (since compact metric space is totally bounded; see slides 121 and 133).

**Question** What are all compact subsets of \(C(X)\)?

- A compact subset of \(C(X)\) is closed and bounded (see slide 114).

**Question** Are equicontinuous, closed and bounded subsets of \(C(X)\) all compact subsets of \(C(X)\)?
Theorem (Generalized Arzela-Ascoli Theorem)

Suppose that \((X, d)\) is a compact metric space and let \(\{f_n\}_{n \geq 1}\) be a sequence in \(C(X)\) such that

(A) \(\sup_{n \geq 1} |f_n(x)| < \infty\) for every \(x \in X\),

(B) for each \(\epsilon > 0\) and \(x \in X\), there exists \(\delta_x > 0\) such that
\[
\sup_{n \geq 1} |f_n(x) - f_n(y)| < \epsilon
\]
whenever \(y \in X\) and \(d(x, y) < \delta_x\).

Then there exist a subsequence \(\{f_{n_k}\}_{k \geq 1}\) of \(\{f_n\}_{n \geq 1}\) and \(f \in C(X)\) such that \(\|f_{n_k} - f\|_\infty \to 0\) as \(k \to \infty\).

Remark Let \(\mathcal{F}\) be a subset of \(C(X)\).

- If \(\mathcal{F}\) is bounded, then any sequence \(\{f_n\}_{n \geq 1}\) in \(\mathcal{F}\) satisfies (A).
- Condition (B) is the "equicontinuity" of \(\{f_n\}_{n \geq 1}\) at \(x\).
- If all sequences in \(\mathcal{F}\) satisfy (A) and (B), then \(\mathcal{F}\) is compact.

Corollary (Arzela-Ascoli Theorem)

A subset \(\mathcal{F}\) of \(C(X)\) is (sequentially) compact if and only if it is closed, bounded and equicontinuous.
Proof of Generalized Arzela-Ascoli Theorem.

The proof is divided into two steps:

**Step 1**

- Let \( \{x_n\}_{n \geq 1} \) be a countable dense subset of \( X \) (a compact metric space is separable; see slide 125).
- There exists a convergent subsequence \( \{f_{n_k}(x_1)\}_{k \geq 1} \) of the bounded sequence \( \{f_n(x_1)\}_{n \geq 1} \) (use (A) and H-B Theorem).
- There exists a convergent subsequence \( \{f_{n_{k_l}}(x_2)\}_{l \geq 1} \) of the bounded sequence \( \{f_{n_k}(x_2)\}_{k \geq 1} \) (use (A) and H-B Theorem).
- \( \{f_{n_{k_l}}(x_1)\}_{l \geq 1} \) is convergent (subsequence of a convergent sequence is convergent).

Continuing this, we obtain a subsequence \( \{f_{n_k}\}_{k \geq 1} \) such that \( \{f_{n_k}(x_j)\}_{k \geq 1} \) is convergent for every \( j \geq 1 \) (form the set \( \{n_k\}_{k \geq 1} \) by picking up first element in first subsequence, second element in second subsequence and so on).
Proof of Generalized Arzela-Ascoli Theorem continued ...

Step 2 Let \( \delta_x \) be as given in (B).

- There exist \( a_1, \ldots, a_m \in X \) such that \( X = \bigcup_{k=1}^{m} B_{\delta_{a_k}}(a_k) \) (since \( X = \bigcup_{a \in X} B_{\delta_a}(a) \) and \( X \) is compact)
- Choose \( y_k \in B_{\delta_{a_k}}(a_k) \cap \{x_n\}_{n \geq 1} \) and let \( N \geq 1 \) be such that \( |f_{n_i}(y_k) - f_{n_j}(y_k)| < \epsilon \) for \( i, j \geq N \) and \( k = 1, \ldots, m \) (Step 1)
- If \( x \in X \) then \( x \in B_{\delta_{a_k}}(a_k) \) for some \( k = 1, \ldots, m \).

Note that for \( i, j \geq N \), (use (B) four times)

\[
|f_{n_i}(x) - f_{n_j}(x)| \leq |f_{n_i}(x) - f_{n_i}(a_k)| + |f_{n_i}(a_k) - f_{n_i}(y_k)| + |f_{n_i}(y_k) - f_{n_j}(y_k)| \\
+ |f_{n_j}(y_k) - f_{n_j}(a_k)| + |f_{n_j}(a_k) - f_{n_j}(x)| \leq 5 \epsilon.
\]

Thus the sequence \( \{f_{n_j}\}_{j \geq 1} \) is Cauchy in \( C(X) \). Since \( C(X) \) is complete, \( \{f_{n_j}\}_{j \geq 1} \) is convergent.
Problem

Let \((X, d)\) be a compact metric space and let \(F : X \times X \to \mathbb{R}\) be a continuous function. For \(y \in X\), define \(f_y(x) = F(x, y)\). Show that the family \(\{f_y\}_{y \in X}\) is bounded and equicontinuous.

Solution.

Note that \(\|f_y\|_\infty = \sup_{x \in X} |F(x, y)| \leq \|F\|_\infty < \infty\) (since \(F\) is continuous and \(X \times X\) is compact; see slide 134). Thus \(\sup_{y \in X} \|f_y\|_\infty < \infty\), and hence \(\{f_y\}_{y \in X}\) is bounded.

Since \(F\) is continuous on a compact metric space, \(F\) is uniformly continuous. Thus, for given \(\epsilon > 0\), there exists \(\delta > 0\) such that \(|f_y(x) - f_{y'}(x')| < \epsilon\) whenever \(x, x', y, y' \in X\), \(d(x, x') < \delta\) and \(d(y, y') < \delta\). Let \(y' = y\).

- By (Generalized) Arzela-Ascoli Theorem, any sequence in \(\{f_y\}_{y \in X}\) has a subsequence convergent in \(C(X)\).
Definition
Let \((X, d)\) be a metric space. We say that \(X\) is connected if the only subsets of \(X\), which are both open and closed are \(\emptyset\) and \(X\).

Example
We check that \(\mathbb{R}\) is connected. Write \(\mathbb{R} = A \sqcup B\) (disjoint union) for nonempty, open sets \(A\) and \(B\). Let \(a \in A\), \(b \in B\) with \(a < b\).

- \([a, b] = A_0 \sqcup B_0\), where \(A_0 = A \cap [a, b]\) and \(B_0 = B \cap [a, b]\).
- Let \(c = \sup A_0 \in [a, b]\). Thus either \(c \in A_0\) or \(c \in B_0\).
- If \(c \in A_0\) then either \(c = a\) or \(a < c < b\) (since \(A\) is closed).
- Since \(A\) is open, there is \(r > 0\) such that \([c, c + r) \subseteq A\).

This contradicts that \(c = \sup A_0\). Similarly, prove that \(c \notin B_0\).

Problem
Show that for any \(a \in \mathbb{R}\), \(\mathbb{R} \setminus \{a\}\) is not connected.
Example
Any normed linear space $X$ over $\mathbb{R}$ is connected. To see this, assume that $X = A \sqcup B$ such that $A$ and $B$ are open subsets of $X$. We claim that one of the $A$ and $B$ is empty. If not, then for $a \in A$ and $b \in B$, consider $f : \mathbb{R} \to X$ given by

$$f(t) = (1 - t)a + tb, \quad t \in \mathbb{R}.$$ 

Then $f(0) = a$ and $f(1) = b$. Since $f$ is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open subsets of $\mathbb{R}$. Since these are proper and disjoint, we arrive at the conclusion that $\mathbb{R}$ is not connected $\Rightarrow \Leftarrow$

Example
The space $\mathbb{Q}$ is not connected. In fact, for any irrational $x \in \mathbb{R}$, the set $[-x, x] \cap \mathbb{Q} = (-x, x) \cap \mathbb{Q}$ is both open and closed in $\mathbb{Q}$. 
Theorem
\[
\text{Let } f : X \rightarrow Y \text{ be continuous. If } X \text{ is connected, then so is } f(X) .
\]

Proof.
Suppose \( f(X) \) has a nonempty open and closed subset, say, \( U \). Then \( f^{-1}(U) \) is an open and closed subset of \( X \). However, \( X \) is connected, so \( f^{-1}(U) = \emptyset \) or \( f^{-1}(U) = X \). Since \( f^{-1}(U) \neq \emptyset \), we conclude that \( f^{-1}(U) = X \), and hence \( U = f(X) \). □

- A path connecting points \( x, y \) is a continuous function \( f : [a, b] \rightarrow X \) such that \( f(a) = x \) and \( f(b) = y \). Since \([a, b]\) is connected, so is \( f([a, b]) \).

Problem
Show that union of line segments \( L_1, \ldots, L_m \) in a normed linear space is connected if \( L_1 \cap L_2 \neq \emptyset \), \( L_2 \cap L_3 \neq \emptyset \), \ldots, \( L_m \cap L_1 \neq \emptyset \).
Example
Let us see that $\mathbb{R}^n \setminus \{0\}$ is connected. Suppose there exists a proper set $U$ which is both open and closed in $\mathbb{R}^n \setminus \{0\}$.

- Let $x \in U$ and $y \in (\mathbb{R}^n \setminus \{0\}) \setminus U$.
- $x$ and $y$ can be connected by union $L$ of line segments.
- $L$ contains open and closed proper subset $L \cap U$.

However, $L$ being connected, $L \cap U = L$ or $L \cap U = \emptyset \Rightarrow \Leftarrow$

Example
The unit sphere $S^n = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ in $\mathbb{R}^n$ is connected.
Indeed, $\mathbb{R}^n \setminus \{0\}$ is connected and $g : \mathbb{R}^n \setminus \{0\} \to S^n$ given by

$$g(x) = x/\|x\|_2, \quad x \in \mathbb{R}^n \setminus \{0\}$$

is a continuous surjection.
Theorem
Let \((X, d)\) be a metric space and let \(A \subseteq X\). If \(A\) is connected and \(B\) is a set such that \(A \subseteq B \subseteq \overline{A}\), then \(B\) is also connected.

Proof.
Let \(B = C \sqcup D\) for open sets \(C\) and \(D\).

- \(A = (A \cap C) \sqcup (A \cap D)\) for open sets \(A \cap C\) and \(A \cap D\).
- Since \(A\) is connected, either \(A \cap C = \emptyset\) or \(A \subseteq C\).
- Thus either \(A \subseteq C\) or \(A \subseteq D\).
- Since \(C\) and \(D\) are closed in \(B\), we obtain \(\overline{A} \cap B \subseteq C\) or \(\overline{A} \cap B \subseteq D\).

It follows that either \(C = B\) or \(D = B\). However, in this case, either \(C = \emptyset\) or \(D = \emptyset\), which shows that \(B\) is connected.

Corollary
The closure of a connected set is connected.
Example (Topologist’s sine curve)
Consider the continuous function $s : (0, 1] \rightarrow [0, 1]$ given by

$$s(x) = \sin(1/x), \quad x \in (0, 1].$$

Let $S$ denote the graph of $s$ given by

$$S = \{(x, s(x)) \in \mathbb{R}^2 : x \in (0, 1]\}.$$

Note that $S = f((0, 1])$, where $f(x) = (x, s(x))$ is a continuous function. Thus $S$ is connected, and hence $\overline{S}$ is also connected.

The metric space $\overline{S}$ (with relative metric induced by $d_2$) is commonly known as the topologist’s sine curve.
Path-connected metric spaces

**Definition**
Let $(X, d)$ be a metric space. We say that $X$ is **path-connected** if for any two points $x, y \in X$, there exists a continuous function $f : [0, 1] \rightarrow X$ such that $f(0) = x$ and $f(1) = y$.

**Remark** If $U$ is path-connected then for open sets $V, W$ of $U$ such that $U = V \sqcup W$ and for any path $f$ in $U$, the range of $f$ being the continuous image of the connected set $[0, 1]$ is connected, and hence lies entirely either in $V$ or $W$. This shows that one of $V, W$ must be empty, that is, $U$ is connected.

**Problem**
*Show that the complement of any countable subset $C$ in $\mathbb{R}^2$ is path-connected.*

**Hint.**
Given $p, q \in \mathbb{R}^2 \setminus C$, consider the uncountable set $\mathcal{F}$ given by $\{f : f$ is a path connecting $p$ and $q\}$.
Example (Comb space)
Let $K$ denote the set $\{1/n : n \geq 1\}$ and consider

$$E = ([0, 1] \times \{0\}) \cup (K \times [0, 1]) \subseteq \mathbb{R}^2.$$ 

- $E$ is path-connected (see diagram).

The comb space $C$ is defined to be the space $E \cup (\{0\} \times [0, 1])$.
- $C$ is also connected (since $E$ is connected and $\overline{E} = C$)

The deleted comb space $C_0$ is defined as $E \cup \{(0, 1)\}$.
- $C_0$ is connected (since $E \subseteq C_0 \subseteq \overline{E}$)
- $C_0$ is not path-connected (since there is no path which connects the points $p = (0, 1)$ and $q = (1, 0)$)

Theorem

The deleted comb space is not path-connected.

Proof.
Suppose, contrary to this, that there is a path $\gamma : [0, 1] \to C_0$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

- $\gamma^{-1}({p})$ is a closed subset of $[0, 1]$. and hence it is compact. Let $t_0 \in [0, 1]$ be its maximum.
- Consider the projection $P_1(x, y) = x$ of $\mathbb{R}^2$ onto the $X$-axis.
- Let $\{t_n\}_{n \geq 1} \subseteq (t_0, 1]$ be a sequence converging to $t_0$.

If, for every $n \geq 1$, there exists $t_0 < s_n < t_n$ such that $\gamma(s_n) = (x_n, 0)$ for some $x_n \in [0, 1] \setminus K$, then $\{s_n\}_{n \geq 1}$ converges to $t_0$, by the continuity, $(x_n, 0) = \gamma(s_n) \to \gamma(t_0) = p = (0, 1)$.

- There exists $t_1 \in (t_0, 1]$ such that $(P_1 \circ \gamma)(t_0, t_1) \subseteq K$.
- $(P_1 \circ \gamma)(t_0, t_1)$ is a connected subset of $K$ containing 1 (wlog)

Thus $(P_1 \circ \gamma)(t_0, t_1) = \{1\}$. By continuity, $(P_1 \circ \gamma)[t_0, t_1) = \{1\}$, which is impossible since $(P_1 \circ \gamma)(t_0) = 0$. \qed
**Theorem**

Let \( f : X \rightarrow Y \) be a continuous surjection. If \( X \) is path-connected, then so is \( Y \).

**Proof.**

Let \( y_0, y_1 \in Y \). Let \( x_0, x_1 \in X \) be such that \( f(x_0) = y_0 \) and \( f(x_1) = y_1 \). If \( \gamma \) is a path connecting \( x_0 \) and \( x_1 \), then \( f \circ \gamma \) is a path connecting \( y_0 \) and \( y_1 \).

If \( p \) is a polynomial in the real variables \( x_1, \ldots, x_n \) and \( Z(p) = \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^d : p(x) = 0 \} \), then \( \mathbb{R}^n \setminus Z(p) \) is not necessarily path-connected.

- If \( p(x, y) = x^2 + y^2 - 1 \), then \( Z(p) \) is equal to
  \[
  \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.
  \]

  Clearly, \( \mathbb{R}^2 \setminus Z(p) \) is not connected.
Corollary

If $p$ is a polynomial in the complex variables $z_1, \ldots, z_n$ and $Z(p) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : p(z) = 0\}$, then $\mathbb{C}^n \setminus Z(p)$ is path-connected.

Proof.

Let $z, w \in \mathbb{C}^n \setminus Z(p)$. Consider the straight-line path

$$
\gamma(t) = (1 - t)z + tw, \quad t \in \mathbb{C}.
$$

- $Z = \{t \in \mathbb{C} : \gamma(t) \in Z(p)\}$ is the set of zeros of $p \circ \gamma$
- $Z$ is a finite subset of $\mathbb{C}$ ($p \circ \gamma$ is a polynomial in one variable)
- $\mathbb{C} \setminus Z$ is path-connected (see slide 158)
- $\gamma$ maps $\mathbb{C} \setminus Z$ continuously into $\mathbb{C}^n \setminus Z(p)$

In particular, $z$ and $w$ belong to the path-connected subset $\gamma(\mathbb{C} \setminus Z)$ of $\mathbb{C}^n \setminus Z(p)$. 

\qed
Example
The general linear group $GL_n(\mathbb{C})$ of all invertible $n \times n$ matrices with complex entries is path-connected.

- Define $f : GL_n(\mathbb{C}) \rightarrow \mathbb{C}^{n^2}$ by
  
  $$f(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn}), \quad A \in GL_n(\mathbb{C}).$$

- $f$ is a (linear) homeomorphism
- $f$ maps $GL_n(\mathbb{C})$ onto $\mathbb{C}^{n^2} \setminus Z(\text{det})$, where $\text{det}$ is the complex polynomial in the variables $a_{i,j}, 1 \leq i, j \leq n$, which sends $f(A)$ to the determinant of $A$.

Thus $GL_n(\mathbb{C})$ is path-connected (since so is $\mathbb{C}^{n^2} \setminus Z(\text{det})$).

Problem

Show that $GL_n(\mathbb{R})$ is not connected.
Problem
For an open subset $U$ of $\mathbb{R}^n$, show that $U$ is connected if and only if $U$ is path-connected.

Solution.
We have already seen that path-connected space is connected. To see the converse, consider for any $p \in U$, the set

$$S = \{x \in U : \text{there is a path connecting } p \text{ and } x\}.$$  

We claim that $S$ is the whole of $U$.

- $S$ is nonempty (since $p \in U$)
- $S$ is open (if $x \in S$ and $B_r(x) \subseteq U$ for some $r > 0$, then any $y \in B_r(x)$ can be connected to $x$ and $x$ can be connected to $p$, so $y$ can be connected to $p \Rightarrow B_r(x) \subseteq S$)
- $S$ is closed (let $\{x_n\}_{n \geq 1}$ be a sequence in $S$ converging to $x$, connect $x$ and $x_N$ (for large $N$) and $x_N$ to $p$)

Since $U$ is connected, $S = U$. \qed
Example

For $n \geq 1$, consider the function $f_n(x) = x^n$ for $x \in [0, 1]$. Note that $\{f_n\}_{n \geq 1}$ converges pointwise to $f$, where $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$. Thus the pointwise limit of a sequence of continuous functions is not necessarily continuous.

Problem

For $m \geq 1$, consider the function $f_m(x) = \lim_{n \to \infty} (\cos(m!\pi x))^n$ for $x \in \mathbb{R}$. Verify the following:

(1) $\{f_m\}_{m \geq 1}$ converges pointwise to $f$, where $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, and $f(x) = 1$ for $x \in \mathbb{Q}$.

(2) $f$ is discontinuous everywhere.

Hint.

If $x = p/q \in (0, 1)$, $q \neq 0$ then $\lim_{n \to \infty} (\cos(m!\pi x))^n = 1$ for every $m \geq q$, and hence $f(p/q) = 1$. If $x \notin \mathbb{Q}$, then $\cos(m!\pi x) < 1$, and hence $\lim_{n \to \infty} (\cos(m!\pi x))^n = 0$ and $f(x) = 0$. □
For a metric space \((X, d)\), consider the vector space problem of all bounded functions \(f : X \to \mathbb{R}\).

- \(B(X)\) is a normed linear space with norm \(\|f\|_\infty = \sup_{x \in X} |f(x)|\)

**Theorem**

\(B(X)\) is a complete normed linear space.

**Proof.**

Let \(\{f_n\}_{n \geq 1}\) be a Cauchy sequence in \(B(X)\).

- \(\sup_{n \geq 1} \|f_n\|_\infty < \infty\) (every Cauchy sequence is bounded)
- For any \(x \in X\), \(\{f_n(x)\}_{n \geq 1}\) is Cauchy. Indeed,

\[
|f_m(x) - f_n(x)| \leq \|f_m - f_n\|_\infty, \quad m, n \geq 1
\]

- Define \(f(x) = \lim_{n \to \infty} f_n(x), x \in X\)
- \(f \in B(X)\) (since \(\|f\|_\infty \leq \|f_m - f_n\|_\infty + \|f_n\|_\infty\))

Given \(\epsilon > 0\), there exists \(N \geq 1\) such that \(\|f_m - f_n\| < \epsilon\) for all \(m, n \geq N\). Thus for every \(x \in X\), \(|f_m(x) - f_n(x)| \leq \epsilon\) for every \(m, n \geq N\). Now let \(n \to \infty\) and take supremum over \(X\). \(\square\)
Problem
Let \((X, d)\) be a metric space and fix \(z \in X\). For every \(x \in X\), define \(f_x(y) = d(x, y) - d(y, z)\). Verify the following:

1. For each \(x \in X\), \(f_x \in B(X)\).
2. \(F : X \to B(X)\) by \(F(x) = f_x\) satisfies \(\|F(x)\|_\infty \leq d(x, z)\).
3. For every \(x, y \in X\), \(\|F(x) - F(y)\|_\infty = d(x, y)\).

Conclude that \(X\) is homeomorphic to \((F(X), \|\cdot\|_\infty)\).

Solution.
(1) follows from \(|f_x(y)| \leq d(x, z)\), while (2) follows from (1). To see (3), note that for any \(x, y \in X\), by the triangle inequality,

\[
\|F(x) - F(y)\|_\infty = \sup_{w \in X} |d(x, w) - d(y, w)| \leq d(x, y).
\]

Since \(|f_x(x) - f_y(x)| = | - d(x, z) - d(y, x) + d(x, z)| = d(x, y)\), (3) follows.

• The subspace \(\overline{F(X)}\) of \(B(X)\) is said to be the completion of \(X\).  

Uniform convergence

Let \((X, d)\) be a metric space. For \(n \geq 1\), let \(f_n, f : X \to \mathbb{R}\) be such that \(f_n - f \in B(X)\). The sequence \(\{f_n\}_{n \geq 1}\) converges uniformly to \(f\) if

\[
\|f_n - f\|_\infty = \sup_{x \in X} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.
\]

**Theorem**

Let \((X, d)\) be a compact metric space. Let \(\{f_n\}_{n \geq 1}\) be a sequence of continuous functions on \(X\). If \(\{f_n\}_{n \geq 1}\) converges uniformly to \(f\) on \(X\), then \(f\) is continuous.

**Proof.**

This follows from the fact that \(C(X)\) is complete metric space. \(\Box\)

- In general, pointwise convergence \(\not\Rightarrow \) uniform convergence
Theorem

Let \( \{f_n\}_{n \geq 1} \) be a sequence of continuous functions. If \( \{f_n\}_{n \geq 1} \) converges uniformly to \( f \) on \([a, b]\) then

\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.
\]

Proof.

Note that

\[
\left| \int_a^b f_n(x) \, dx - \int_a^b f(x) \, dx \right| = \left| \int_a^b (f_n(x) - f(x)) \, dx \right| \leq \int_a^b |f_n(x) - f(x)| \, dx \leq \|f_n - f\|_{\infty} (b - a).
\]

Problem

For \( n \geq 1 \), consider the function \( f_n(x) = nx(1 - x^2)^n \) for \( x \in [0, 1] \).

Verify that \( \{f_n\}_{n \geq 1} \) converges pointwise to \( f \), where \( f(x) = 0 \) for all \( x \in [0, 1] \), but \( \{f_n\}_{n \geq 1} \) does not converge uniformly to \( f \).

Hint.

For second part, use \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx \neq \int_0^1 f(x) \, dx \).
Dini’s Theorem

Sometimes pointwise convergence $\Rightarrow$ uniform convergence.

**Theorem**

Let $(X, d)$ be a compact metric space. Let $\{f_n\}_{n \geq 1}$ be a sequence in $C(X)$ converging pointwise to a continuous function $f$. If $\{f_n(x)\}_{n \geq 1}$ is decreasing for all $x \in X$, then $\{f_n\}_{n \geq 1}$ converges uniformly to $f$.

**Proof.**

Let $g_n = f_n - f \geq 0$. For $\epsilon > 0$, let $K_n = \{x \in X : g_n(x) \geq \epsilon\}$.

- $K_n$ is compact and $K_{n+1} \subseteq K_n$ (since $K_n$ is closed, $g_n \geq g_{n+1}$)
- If each $K_n \neq \emptyset$, then so is finitely many sets from $\{K_n\}_{n \geq 1}$
- If each $K_n \neq \emptyset$, then by the finite intersection property (see [Assignment 7, Exercise 2]), $\cap_{n=1}^{\infty} K_n \neq \emptyset$

However, if $x \in X$, then since $g_n(x) \to 0$, $x \notin K_n$ for sufficiently large $n$. Hence $K_N$ is empty for some $N$, that is, $0 \leq g_n(x) < \epsilon$ for every $x \in X$ and for every $n \geq N$. 

□
Example
Define a sequence \( \{p_n\}_{n \geq 0} \) of polynomials by \( p_0(x) = 0 \), and
\[
p_{n+1}(x) = p_n(x) + (x^2 - p_n(x)^2)/2, \quad n \geq 0.
\]
A routine calculation shows that
\[
|x| - P_{n+1}(x) = (|x| - P_n(x))(1 - (|x| + P_n(x))/2), \quad n \geq 0.
\]
One may now verify inductively that
\[
0 \leq p_n(x) \leq p_{n+1}(x) \leq |x|, \quad x \in [-1, 1], \quad n \geq 0.
\]
In particular, \( \{p_n(x)\}_{n \geq 0} \) converges pointwise to \( |x| \). Now apply Dini’s Theorem to \( f_n = -p_n, \quad n \geq 0 \) to conclude that \( \{p_n\}_{n \geq 0} \) converges uniformly to \( f(x) = |x| \) on \( [-1, 1] \).
Problem
Show that the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } x \geq 0. \end{cases}$$

Show that for any $\alpha > 0$, $g$ can be uniformly approximated by polynomials on $[-\alpha, \alpha]$.

**Hint.**
Note that $g(x) = \frac{1}{2}(x + |x|)$.

Problem
Let $\{p_n\}_{n \geq 1}$ be a sequence of polynomials of degree $d_n$. Suppose that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$ for some $f \in C([a, b])$. If $f$ is not a polynomial, then show that $d_n \to \infty$ as $n \to \infty$.

**Hint.**
The subspace of polynomials of degree less than or equal to $d$ is finite-dimensional, and hence it is closed in $C([a, b])$. 


Theorem
• \( \overline{P} = \{ f \in B(X) : \exists \{ p_n \}_{n \geq 1} \text{ such that } \| p_n - f \|_\infty \to 0 \} \subseteq C(X) \).
• There are compact metric spaces \( X \) with \( \overline{P} \not\subseteq C(X) \).

The first part follows from the completeness of \( C(X) \).

Example (Failure of polynomial approximation)
Let \( X = \{ z \in \mathbb{C} : |z| = 1 \} \) with \( d(z, w) = |z - w|, \ z, w \in \mathbb{C} \).
• \( X \) is a compact metric space (\( X \subseteq \mathbb{C} \) is closed and bounded)
• Consider the continuous function \( f(z) = \bar{z} \) (\( \mathbb{C} \)-conjugate)
• If \( \gamma(t) = e^{it}, \ 0 \leq t \leq 2\pi \), then \( \int_0^{2\pi} p(\gamma(t))\gamma'(t)dt = 0 \) for any polynomial \( p \) and \( \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = 1 \)
• If \( \{ p_n \}_{n \geq 1} \) is a sequence of complex polynomials \( p_n \) in \( z \) such that \( \| p_n - f \|_\infty \to 0 \), then
\[
\int_0^{2\pi} p_n(\gamma(t))\gamma'(t)dt \to \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \text{ as } n \to \infty \quad \Rightarrow \Leftarrow
\]
Thus \( f \in C(X) \setminus \overline{P} \).
Lemma

Let \((X, d)\) be a compact metric space. The following are true:

1. \(\overline{P}\) is a complete normed linear space.
2. Given \(f \in C(X)\), if, for every \(\epsilon > 0\), there exists \(g_\epsilon \in \overline{P}\) such that \(\|f - g_\epsilon\|_\infty < \epsilon\), then \(f \in \overline{P}\).

Proof.

(1): Clearly, linear combination of functions approximated uniformly by polynomials are again in \(\overline{P}\). Since \(\overline{P}\) is a closed subset of \(C(X)\) and \(C(X)\) is complete, \(\overline{P}\) is complete.

(2): Every ball \(B_\epsilon(f)\) intersects \(\overline{P}\), and hence \(f\) is a limit point of \(\overline{P}\). Thus \(f \in \overline{P}\). \(\square\)

Theorem (Weierstrass’ approximation theorem)

Any \(f \in C([a, b])\) can be approximated uniformly by polynomials.
Lebesgue’s Proof of Weierstrass’ Theorem*\(^8\).

Since \( f \) is uniformly continuous on \([a, b]\), there exists an integer \( N \geq 1 \) such that \(|f(x) - f(y)| < \epsilon\) whenever \(|x - y| < 1/N\). For \( x_i := a + (b - a)(i/N) \) \((i = 0, \cdots, N)\), consider the function \( h(x) \) with graph equal to a polygon (see diagram) of vertices at

\[(a, f(a)), (x_1, f(x_1)), \cdots, (x_{n-1}, f(x_{n-1})), (b, f(b)).\]

\(\bullet\) \(\|f - h\|_\infty < \epsilon\). Indeed, if \( x \in (x_i, x_{i+1}) \) then

\[f(x) - h(x) = f(x) - ((1 - t)f(x_i) + tf(x_{i+1})) = (1 - t)(f(x) - f(x_i)) + t(f(x) - f(x_{i+1})) \Rightarrow |f(x) - h(x)| < \epsilon\]

\(\bullet\) If \( h(x) = f(a) + \sum_{i=0}^{N-1} c_i g(x - x_i) \) \((x \in [a, b])\) for some scalars \( c_0, \cdots, c_{N-1} \) and \( g(x) = \frac{1}{2}(x + |x|) \)

Since \( g \) can be approximated uniformly by polynomials (see slide 172), \( h \in \overline{P} \), and hence \( f \in \overline{P} \) (see the lemma on last slide). \(\square\)

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\(^8\) J. Burkill, Lectures on Approximation by Polynomials, Lecture Notes, TIFR, Bombay, 1959
Problem
Use Weierstrass’ Theorem to show that $C([a, b])$ is separable.

Hint.
Polynomials with rational coefficients are countable and dense.

Problem
Let $f \in C[a, b]$ be such that $\int_a^b t^n f(t) \, dt = 0$ for all non-negative integers $n$. Show that $f(t) = 0$ for every $t \in [a, b]$.

Solution.
Note that $\int_a^b p(t) f(t) \, dt = 0$ for any polynomial $p$. If $\{p_n\}$ is a sequence converging uniformly to $f$, then $\int_a^b f(t)^2 \, dt = 0$, and hence $f = 0$. 
Problem

Let \( f : [a, b] \to \mathbb{R} \) be a continuous function. Show that there exists a sequence \( \{p_n\}_{n \geq 1} \) of polynomials such that

\[
\int_a^b |p_n(x) - f(x)|^p \, dx \to 0 \text{ as } n \to \infty.
\]

Recall that \( \chi_A \) is 1 on \( A \) and 0 outside \( A \).

Problem

For \( 0 \leq x < y \leq 1 \), consider \( \chi_{[x,y]} : [0, 1] \to \mathbb{R} \). Show that there exists a sequence \( \{p_n\} \) of polynomials such that

\[
\int_0^1 |p_n(x) - \chi_{[x,y]}(x)| \, dx \to 0 \text{ as } n \to \infty.
\]

Conclude that finite linear combination of indicator functions of subintervals of \([0, 1]\) can be approximated by polynomials.

Hint.

Approximate \( \chi_{[x,y]} \) by continuous functions.
Problem
Let $f : [a, b] \to \mathbb{R}$ be a continuously differentiable function. Show that there exists a sequence $\{r_n\}_{n \geq 1}$ of polynomials such that

$$
\|r_n - f\|_\infty \to 0 \text{ and } \|r'_n - f'\|_\infty \to 0 \text{ as } n \to \infty.
$$

Conclude that $C^1[a, b]$ (the space of continuously differentiable functions on $[a, b]$) is a separable normed linear space with norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$.

Solution.
Let $g(x) = f(x) - f(a)$ and note that $g' = f'$. Find a sequence $\{q_n\}_{n \geq 1}$ of polynomials such that $\|q_n - g'\|_\infty \to 0$. Set $p_n(x) := \int_a^x q_n(t)dt$. Note that $p'_n = q_n$, and hence $\|p'_n - g'\|_\infty \to 0$. Also, $|p_n(x) - g(x)| = \left| \int_a^x q_n(t)dt - \int_a^x g'(t)dt \right| \leq (b - a)\|q_n - g'\|_\infty$.

Let $r_n(x) := p_n(x) + f(a)$. \qed
We have seen that the pointwise limit of sequence of functions can be discontinuous at every point (see slide 165).

**Question** Can the pointwise limit of sequence of continuous functions be discontinuous at every point?

**Answer** No

A subset $A$ of a metric space $(X, d)$ is said to be **nowhere dense** if the interior of the closure of $A$ is empty, that is, $(\overline{A})^\circ = \emptyset$.

**Theorem (Baire-Osgood Theorem)**

If $f : [a, b] \rightarrow \mathbb{R}$ is a pointwise limit of a sequence of continuous functions on $[a, b]$, then the set $D(f)$ of discontinuities of $f$ is a countable union of closed nowhere dense sets.

For a proof of this theorem, we need some preliminaries.
Oscillation

Let \( f : [a, b] \rightarrow \mathbb{R} \) and let \( I(c, r) = (c - r, c + r) \) for \( c \in [a, b] \) and \( r > 0 \). Define the oscillation of \( f \) on \( I(c, r) \) by

\[
\text{osc}(f, c, r) = \sup_{x,y \in [a, b] \cap I(c, r)} |f(x) - f(y)|.
\]

- \( \text{osc}(f, c, r) \) exists if \( f \) is bounded
- \( \text{osc}(f, c, r) \geq 0 \) and \( \text{osc}(f, c, r) \) is decreasing in \( r \)

Define the oscillation of \( f \) at \( c \) by

\[
\text{osc}(f, c) = \begin{cases} 
\lim_{r \to 0} \text{osc}(f, c, r) & \text{if } f \text{ is bounded near } c \\
\infty & \text{otherwise}.
\end{cases}
\]

Problem

Show that \( f \) is continuous at \( c \) if and only if \( \text{osc}(f, c) = 0 \).

Hint.

The delta in the definition of continuity plays here the role of \( r \).
For $\epsilon > 0$, consider the set

$$A_\epsilon = \{ c \in [a, b] : \text{osc}(f, c) \geq \epsilon \}.$$ 

- The set $D(f)$ of discontinuities of $f$ is equal to $\bigcup_{\epsilon > 0} A_\epsilon$.

**Lemma**

*For every $\epsilon > 0$, $A_\epsilon$ is a compact subset of $[a, b]$.***

**Proof.**

Let $\{c_n\}_{n \geq 1}$ be a sequence in $A_\epsilon$ converging to $c \in [a, b]$.

- If $c \notin A_\epsilon$, then $\delta = \epsilon - \text{osc}(f, c) > 0$ and hence $\text{osc}(f, c, r) < \epsilon - \delta/2$ for some $r$

- If $|c_n - c| < r/2$, then $I(c_n, r/2) \subseteq I(c, r)$, and hence $\text{osc}(f, c_n, r/2) = \sup_{x,y \in [a, b] \cap I(c_n, r/2)} |f(x) - f(y)| < \epsilon$ (for $c \notin A_\epsilon$)

Since $\text{osc}(f, c_n) \leq \text{osc}(f, c_n, r/2)$, $\text{osc}(f, c_n) < \epsilon \iff (\text{for } c_n \in A_\epsilon)$

Thus $A_\epsilon$ is closed and hence it is compact. \qed
Proof of Baire-Osgood Theorem.

Note that \( D(f) = \bigcup_{n \geq 1} A_{1/n} \). Since \( A_\epsilon \) is closed for every \( \epsilon > 0 \), it suffices to check that \( A_\epsilon^\circ = \emptyset \).

Claim For any closed interval \( J \subseteq [a, b] \), \( J \not\subseteq A_\epsilon \)

- \( E_n = \bigcap_{i,j \geq n} \{ x \in [a, b] : |f_i(x) - f_j(x)| \leq \epsilon/5 \} \) is closed (since \( f_i - f_j \) is continuous for every \( i, j \geq n \))
- \( \bigcup_{n \geq 1} E_n = [a, b] \) (since \( \lim_{n \to \infty} f_n(x) = f(x), x \in [a, b] \))
- \( J = \bigcup_{n=1}^{\infty} (E_n \cap J) \)
- There exists \( N \geq 1 \) such that \( E_N \cap J \) has nonempty interior (since \( J \) is complete, BCT is applicable, see slide 103)

Thus there exists an open interval \( K \) contained in \( E_N \cap J \).

SubClaim: \( K \subseteq [a, b] \setminus A_\epsilon \) (\( \Rightarrow J \not\subseteq A_\epsilon \), and Claim follows)

- \( |f_i(x) - f_j(x)| \leq \epsilon/5 \) for all \( x \in K \) and \( i, j \geq N \) (since \( K \subseteq E_N \))
- \( |f_N(x) - f(x)| \leq \epsilon/5 \) for all \( x \in K \) (let \( i = N, f_j(x) \to f(x) \))
Proof of Baire-Osgood Theorem continued ...

- For $x_0 \in K$, there exists $\delta > 0$ such that $|f_N(x) - f_N(x_0)| < \epsilon/5$ whenever $x \in K$ and $|x - x_0| < \delta$ (by the continuity of $f_N$ at $x_0$)
- $|f(x) - f_N(x_0)| \leq 2\epsilon/5$ for every $x \in K$ such that $|x - x_0| < \delta$
- $|f(x) - f(y)| \leq 4\epsilon/5$ for every $x, y \in K$ such that $|x - y| < \delta$

This shows that $\text{osc}(f, x_0, \delta) \leq 4\epsilon/5$, and hence $\text{osc}(f, x_0) < \epsilon$.

Thus the claim $K \subseteq [a, b] \setminus A_\epsilon$ stands verified.

Corollary

If $f : [a, b] \to \mathbb{R}$ is a pointwise limit of a sequence of continuous functions on $[a, b]$, then the set of continuities of $f$ (that is, $[a, b] \setminus D(f)$) is a dense subset of $[a, b]$.

Proof.

Since $[a, b] \setminus D(f) = \cap_{n \geq 1}([a, b] \setminus A_{1/n})$ and each $[a, b] \setminus A_{1/n}$ is dense in $[a, b]$, BCT yields the desired conclusion.
Example
Consider the function $f : [a, b] \rightarrow \mathbb{R}$ such that $f = 1$ on rationals and $f = 0$ on irrationals. Note that $D(f) = [a, b]$. It follows from the Baire-Osgood Theorem that $f$ can not be a pointwise limit of a sequence of continuous functions.

Problem
If $f : [a, b] \rightarrow \mathbb{R}$ is continuous and $f : (a, b) \rightarrow \mathbb{R}$ is differentiable, then show that the set of continuities of $f'$ is dense in $(a, b)$.

Solution.
For any $x \in (a, b)$, note that $f'(x) = \lim_{n \to \infty} \frac{f(x+1/n) - f(x)}{1/n}$. Thus $f'$ is a pointwise limit of continuous functions. By Baire-Osgood Theorem, the set of continuities of $f'$ is dense in any closed interval $[c, d]$ contained in $(a, b)$. Hence it is dense in $(a, b)$. □
Convergence of series of functions

Let \((X, d)\) be a metric space. Let \(\{f_n\}_{n \geq 1}\) be a sequence of bounded functions and let \(f : X \to \mathbb{R}\) be a bounded function. We say that the series \(\sum_{n=1}^{\infty} f_n\) converges uniformly to \(f\) if the sequence \(\{\sum_{n=1}^{k} f_n\}_{k \geq 1}\) converges uniformly to \(f\), that is,

\[
\| \sum_{n=1}^{k} f_n - f \|_\infty = \sup_{x \in X} | \sum_{n=1}^{k} f_n(x) - f(x) | \to 0 \text{ as } k \to \infty.
\]

In this case, we write \(f = \sum_{n=1}^{\infty} f_n\).

**Theorem**

Let \((X, d)\) be compact. Let \(\{f_n\}_{n \geq 1}\) be a sequence of functions in \(C(X)\). If \(\sum_{n=1}^{\infty} f_n\) converges uniformly to \(f\), then \(f \in C(X)\).

**Proof.**

If each \(f_n\) is continuous, then so is \(\sum_{n=1}^{k} f_n\). Now use the fact that uniform limit of continuous functions is continuous. \(\square\)
Problem

Let \((X, d)\) be a compact metric space and let \(\{f_n\}_{n \geq 1}\) be a sequence in \(C(X)\). Assume that \(\sum_{n \geq 1} f_n(x)\) converges pointwise to some \(f \in C(X)\) for every \(x \in X\). If \(f_n(x) \geq 0\) for every \(x \in X\) and every \(n \geq 1\), then \(\sum_{n \geq 1} f_n\) is uniformly convergent.

Hint.

Use Dini’s Theorem.

Problem

Let \((X, d)\) be a compact metric space and let \(A\) be a closed subset of \(X\). Assume that \(X \setminus A = \bigsqcup_{n=1}^\infty X_n\) (disjoint union), where each \(X_n\) is a clopen set in \(X\). Let \(f \in C(X)\) be such that \(f(a) = 0\) for every \(a \in A\). Show that \(\sum_{n=1}^\infty f \chi_{X_n}\) converges uniformly to \(f\).

Hint.

Note that each \(\chi_{X_n}\) is continuous. Now use the last problem.
Weierstrass M-test

Theorem

Let \((X, d)\) be a metric space and let \(\{f_n\}_{n \geq 1}\) be a sequence of bounded functions \(f_n : X \to \mathbb{R}\). If \(\sum_{n \geq 1} \|f_n\|_\infty < \infty\) (convergence in \(\mathbb{R}\)), then \(\sum_{n \geq 1} f_n\) is uniformly convergent. Moreover,

\[
\| \sum_{n \geq 1} f_n \|_\infty \leq \sum_{n \geq 1} \| f_n \|_\infty .
\]

Proof.

For \(n \geq 1\), let \(g_n = \sum_{k=1}^{n} f_n\) and \(g_0 = 0\). For a positive integers \(m < n\), note that by triangle inequality,

\[
\|g_n - g_m\|_\infty = \| \sum_{k=m+1}^{n} f_k \|_\infty \leq \sum_{k=m+1}^{n} \| f_k \|_\infty ,
\]

and hence \(\{g_n\}_{n \geq 1}\) is Cauchy in \(B(X)\). Since \(B(X)\) is complete (see slide 166), \(\{g_n\}_{n \geq 1}\) is uniformly convergent. Let \(m = 0\), \(n \to \infty\) in (1) to get the remaining part. \(\square\)
Definition
A **power series** is an expansion of the form

\[ \sum_{n=0}^{\infty} a_n x^n, \text{ where } a_n \in \mathbb{R}. \]

\( \sum_{n=0}^{\infty} a_n x^n \) converges absolutely if \( \sum_{n=0}^{\infty} |a_n||x|^n < \infty \).

**Definition (Domain of Convergence)**

\( D = \{ w \in \mathbb{R} : \sum_{n=0}^{\infty} |a_n||w|^n < \infty \} \)

Note that

- \( w_0 \in D \implies \pm w_0 \in D \) for any \( \theta \in \mathbb{R} \)
- \( w_0 \in D \implies w \in D \) for any \( w \in \mathbb{R} \) with \( |w| \leq |w_0| \)

Conclude that \( D \) is either \( \mathbb{R} \), \((-R, R)\) or \([-R, R]\) for some \( R \geq 0 \).
**Definition**

The radius of convergence (for short, RoC) of $\sum_{n=0}^{\infty} a_n x^n$ is defined as

$$R = \sup\{|x| : \sum_{n=0}^{\infty} |a_n||x|^n < \infty\}.$$ 

**Remark** The series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any $0 < r < R$.

**Example (Geometric series)**

Consider the series $\sum_{n=0}^{\infty} x^n$, $x \in \mathbb{R}$.

- This series converges absolutely to $\frac{1}{1-x}$ for any $|x| < 1$. Indeed,
  $$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}, \quad x^{n+1} \to 0 \text{ as } n \to \infty$$

- Thus $R \geq |x|$ for every $|x| < 1$, and hence $R \geq 1$
- $R = 1$ (since $\sum_{n=0}^{\infty} x^n$ diverges at $x = 1$)
Theorem (Cauchy-Hadamard Formula)

The RoC of $\sum_{n=0}^{\infty} a_n x^n$ is given by

$$R = \frac{1}{\lim \sup |a_n|^{1/n}},$$

where we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

Proof.

Assume $R < \infty$. If $r > R$, then $\lim \sup |a_n|^{1/n} > 1/r$, and hence $\lim_{k \to \infty} \sup_{n \geq k} |a_n|^{1/n} > 1/r$. Thus there exists a subsequence $\{n_k\}_{k \geq 1}$ such that $|a_{n_k}|^{1/n_k} > 1/r$, that is, $r^{n_k} |a_{n_k}| \to 0$, and hence $\sum_{n=0}^{\infty} a_n r^n$ is divergent. Thus $\text{RoC} \not> R = \lim \sup |a_n|^{1/n}$.

If $r < R$, then $|a_n| r^n < 1$ for all integers $n \geq N$. Thus, for $|x| < r$,

$$\sum_{n=N}^{\infty} |a_n| |x|^n \leq \sum_{n=N}^{\infty} \left( \frac{|x|}{r} \right)^n \leq \frac{1}{1 - |x|/r} < \infty.$$

Thus $\text{RoC} \geq r$ for any $r < R$ or $\text{RoC} \geq R = \lim \sup |a_n|^{1/n}$.
Examples

\[ \sum_{n=0}^{k} a_n x^n, \quad a_n = 0 \text{ for } n > k, \quad R = \infty \]

\[ \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad a_n = \frac{1}{n!}, \quad R = \infty \]

\[ \sum_{n=0}^{\infty} x^n, \quad a_n = 1, \quad R = 1 \]

\[ \sum_{n=0}^{\infty} n! x^n, \quad a_n = n!, \quad R = 0 \]

The coefficients of a power series may not be given by a single formula.

Example

Consider the power series \( \sum_{n=0}^{\infty} x^{n^2} \). Then

\[ a_k = 1 \text{ if } k = n^2, \text{ and } 0 \text{ otherwise.} \]

Clearly, \( \lim \sup |a_n|^{1/n} = 1 \), and hence \( R = 1 \).
Sometimes RoC can be computed without knowing the coefficients explicitly.

**Example**

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_n$ is number of divisors of $n^{1111}$. Note that

$$1 \leq a_n \leq n^{1111}.$$

Note that $1 \leq \limsup |a_n|^{1/n} \leq \limsup (n^{1111})^{1/n} = 1$, and hence the RoC of $\sum_{n=0}^{\infty} a_n x^n$ equals 1.

**Theorem**

*If the RoC of $\sum_{n=0}^{\infty} a_n x^n$ is $R$ then the RoC of the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also $R$.***

**Proof.**

Since $\lim_{n \to \infty} n^{1/n} = 1$, $R = \frac{1}{\limsup |n a_n|^{1/n}} = \frac{1}{\limsup |a_n|^{1/n}}$. \(\square\)
Theorem
If \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) is a power series of radius \( R > 0 \), then \( f \) is infinitely differentiable with \( f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \).

Proof.
Let \( x_0 \in (-R, R) \), \( h \in \mathbb{R} \), \( r > 0 \) with \( \max\{|x_0|, |x_0 + h|\} < r < R \).

- \( S_k(x) = \sum_{n=0}^{k} a_n x^n \), \( E_k(x) = \sum_{n=k+1}^{\infty} a_n x^n \)
- \( \frac{f(x_0+h) - f(x_0)}{h} - g(x_0) = A + (S_k'(x_0) - g(x_0)) + B \), where

\[
A = \left( \frac{S_k(x_0 + h) - S_k(x_0)}{h} - S_k'(x_0) \right), \quad B = \left( \frac{E_k(x_0 + h) - E_k(x_0)}{h} \right)
\]

- \( |B| \leq \sum_{n=k+1}^{\infty} |a_n| \left| \frac{(x_0 + h)^n - x_0^n}{h} \right| \leq \sum_{n=k+1}^{\infty} |a_n| n r^{n-1} \)

Since \( f' \) is a power series, \( f \) is infinitely differentiable on \((-R, R)\).
A nowhere differentiable continuous function

- Any continuous function on \([0, 1]\) is a uniform limit of infinitely differentiable functions (Weierstrass’ Theorem).

It is quite striking that uniform limit of infinitely real differentiable functions could be nowhere differentiable.

- Let \(\phi(x) = |x|\) for \(x \in [-1, 1]\), which is extended periodically (with period 2) to \(\mathbb{R}\) by setting \(\phi(x + 2) = \phi(x), x \in \mathbb{R}\)
- \(\phi\) is a continuous function such that \(0 \leq \phi(x) \leq 1, x \in \mathbb{R}\)
- Plot graphs of \(\phi(x), \phi(4x)\) (period 1/2), \(\phi(16x)\) (period 1/8)
- Define the function \(f : \mathbb{R} \rightarrow \mathbb{R}\) by \(f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)\)

Note that \(f\) is continuous. Indeed,

\[
\left| f(x) - \sum_{n=0}^{k} \left(\frac{3}{4}\right)^n \phi(4^n x) \right| \leq \sum_{n=k+1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x) \leq \sum_{n=k+1}^{\infty} \left(\frac{3}{4}\right)^n \rightarrow 0.
\]

Since \(\phi\) is continuous, so is \(f\).
Theorem

For any \( x \in \mathbb{R} \), there exists \( \{\delta_m\}_{m \geq 1} \) converging to \( 0 \) such that

\[
| (f(x + \delta_m) - f(x))/\delta_m | \to \infty \quad \text{as} \quad m \to \infty
\]

In particular, \( f \) is not differentiable at any point in \( \mathbb{R} \).

Proof.

For an integer \( m \geq 1 \), set \( \delta_m = \pm \frac{1}{2}4^{-m} \), where the sign is so chosen that no integer lies between \( 4^m x \) and \( 4^m(x + \delta_m) \). Define

\[
\gamma_n = (\phi(4^n(x + \delta_m)) - \phi(4^n x))/\delta_m.
\]

- If \( n > m \), then \( \phi(4^n(x + \delta_m)) = \phi(4^n x \pm 4^{n-m}/2) = \phi(4^n x) \), and hence \( \gamma_n = 0 \).
- \( |\gamma_m| = 4^m \) (since \( 4^n(x + \delta_m) \) and \( 4^n x \) are both \( \geq 0 \) or \( < 0 \))
- When \( 0 \leq n < m \),

\[
|\gamma_n| = \frac{||4^n(x + \delta_m)| - |4^n x||}{|\delta_m|} \leq \frac{|4^n \delta_m|}{|\delta_m|} = 4^n
\]

Now we complete the argument.
Proof continued ...

Since $\gamma_n = 0$ for $n > m$, $|\gamma_m| = 4^m$ and $|\gamma_n| \leq 4^m$, $0 \leq n < m$,

\[
|(f(x + \delta_m) - f(x))/\delta_m| = \left| \sum_{n=0}^{\infty} \left( \frac{3}{4} \right)^n \gamma_n \right| = \left| \sum_{n=0}^{m} \left( \frac{3}{4} \right)^n \gamma_n \right|
\]

\[
= \left| \sum_{n=0}^{m-1} \left( \frac{3}{4} \right)^n \gamma_n \pm 3^m \right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1),
\]

where we used $\sum_{n=0}^{m-1} 3^n = (3^m - 1)/2$.

Thus $(f(x + \delta_m) - f(x))/\delta_m$ blows up to $\infty$ as $m \to \infty$.

Geometrically, a continuous nowhere differentiable function has a continuous graph with "corner" at every point!

- The set of nowhere differentiable continuous functions turns out to be dense in $C[0, 1]$ (this may be deduced from BCT).
Riemann integral

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function.

- $P = \{x_0 = a, \ldots, x_n = b\}$ (partition)
- $I_i = [x_i, x_{i+1}]$, $[a, b] = \bigcup_{i=0}^{n-1} I_i$, $\ell(I_i)$ (length of $I_i$)
- $\|P\| = \max_{0 \leq i \leq n-1} \ell(I_i)$ (width of $P$)
- $m_i = \inf_{x \in I_i} f(x)$, $M_i = \sup_{x \in I_i} f(x)$
- $L(P, f) = \sum_{i=0}^{n-1} m_i \ell(I_i)$ (lower Riemann sum)
- $U(P, f) = \sum_{i=0}^{n-1} M_i \ell(I_i)$ (upper Riemann sum)
- $\int_a^b f(x) \, dx = \sup_P L(P, f)$ (lower Riemann integral)
- $\int_a^b f(x) \, dx = \inf_P U(P, f)$ (upper Riemann integral)
**Definition**
A bounded function \( f : I \to \mathbb{R} \) is Riemann integrable (or Darboux integrable) if \( \int f(x) \, dx = \int^\prime f(x) \, dx \), say \( \int_I f(x) \, dx \).

**Interpretation** Area of the region \( R \) enclosed by the lines \( x = a \), \( x = b \), and the curve \( y = f(x) \geq 0 \).

**Example**
For \( x_0 \in I \), consider the function \( f : I \to \mathbb{R} \) defined by

\[
  f(x) = \begin{cases} 
    1 & \text{if } x \neq x_0, \\ 
    0 & \text{if } x = x_0. 
  \end{cases}
\]

For \( \epsilon > 0 \), let \( P \) be a partition with \( \|P\| = \max_{0 \leq i \leq n-1} \ell(I_i) < \epsilon \).

- If \( x_0 \in I_{i_0} \) is an interior point, then \( U(P, f) - L(P, f) = \ell(I_{i_0}) < \epsilon \)
- If \( x_0 \in I_{i_0} \) is an end point, then \( U(P, f) - L(P, f) < 2\epsilon \)
- \( L(P, f) \leq \int f(x) \, dx \leq \int^\prime f(x) \, dx \leq U(P, f) \) (Exercise)

Conclude that \( f \) is Riemann integrable.
Theorem

If \( f : [a, b] \to \mathbb{R} \) is a bounded function, then \( f \) is Riemann integrable if and only if for every \( \epsilon > 0 \), there exists a partition \( P \) of \([a, b]\) such that \( U(P, f) - L(P, f) < \epsilon \).

Proof.

The sufficiency part \( \Leftarrow \) follows from

\[
L(P, f) \leq \int f(x)dx \leq \int f(x)dx \leq U(P, f).
\]

To see the necessity part \( \Rightarrow \), find partitions \( P \) and \( Q \) of \([a, b]\) such that \( U(P, f) - \int f(x)dx < \epsilon/2 \) and \( \int f(x)dx - L(Q, f) < \epsilon/2 \).

Let \( R \) be a partition obtained from \( P \cup Q \), and note that

\[
L(Q, f) \leq L(R, f) \leq U(R, f) \leq U(P, f).
\]

Then \( U(R, f) - L(R, f) \leq U(P, f) - L(Q, f) = U(P, f) - \int f(x)dx + \int f(x)dx - L(Q, f) < \epsilon. \)
Corollary

Every continuous function \( f : I \rightarrow \mathbb{R} \) is Riemann integrable.

Proof.

Given \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( |f(x) - f(x')| < \epsilon \) whenever \( x \in I, |x - x'| < \delta \). Thus for partition \( P \) with \( \|P\| < \delta \),

\[
U(P, f) - L(P, f) \leq \sum_{i=0}^{m-1} |M_i - m_i|\ell(I_i).
\]

However, \( M_i = f(a_i) \) and \( m_i = f(b_i) \) for some \( a_i, b_i \in I_i \). Thus

\[
U(P, f) - L(P, f) = \sum_{i=0}^{m-1} |f(a_i) - f(b_i)|\ell(I_i) \leq \epsilon\ell(I).
\]

Hence \( f \) is Riemann integrable.

Problem

Show that the indicator function \( \chi_{[0,1/2]} : [0, 1] \rightarrow \mathbb{R} \) (which is 1 on \([0, 1/2]\) and 0 outside) is Riemann integrable.
Problem

Show that a bounded function, which is continuous except at a finite subset $F$ of $[a, b]$, is Riemann integrable.

Hint.

Consider first the case in which $F = \{x_0\}$. Assuming $f \neq 0$, let $I_\epsilon = (x_0 - \frac{\epsilon}{4\|f\|_\infty}, x_0 + \frac{\epsilon}{4\|f\|_\infty})$. Apply the last corollary to $I = [a, b] \setminus I_\epsilon$ to find a partition $P$ of $I$ such that $U(P, f) - L(P, f) < \epsilon/2$. Let $\tilde{P}$ be the partition formed by taking union of $P$ and the interval $\tilde{I}_\epsilon$. Check that $U(P, f) - L(P, f) < \epsilon$. Extend this argument to any finite set $F$.

Question Does there exist a Riemann integrable function, which is discontinuous at countably infinite points?

Answer There are plenty of such functions!
Theorem

Every bounded monotone function \( f : [a, b] \to \mathbb{R} \) is integrable.

Proof.

Let \( f \) be increasing, \( P \) be a partition with \( \ell(I_i) = \frac{b-a}{m} \) for all \( i \).

- \( U(P, f) = \sum_{i=0}^{m-1} M_i \ell(I_i) = \sum_{i=0}^{m-1} f(x_{i+1}) \ell(I_i) \) and
- \( L(P, f) = \sum_{i=0}^{m-1} m_i \ell(I_i) = \sum_{i=0}^{m-1} f(x_i) \ell(I_i) \)

\( U(P, f) - L(P, f) = (f(x_m) - f(x_0)) \frac{b-a}{m} = \frac{f(b) - f(a)}{b-a} \frac{1}{m} \to 0. \)

Problem

Consider the Zeno’s staircase function \( Z : [0, 1] \to \mathbb{R} \) given by

\[
Z(x) = \begin{cases} 
\frac{1}{2} & \text{if } 0 \leq x < \frac{1}{2}, \\
\frac{3}{4} & \text{if } \frac{1}{2} \leq x < \frac{3}{4}, \\
\frac{7}{8} & \text{if } \frac{3}{4} \leq x < \frac{7}{8}, \\
\vdots & \\
\frac{2^{k-1}}{2^k} & \text{if } \frac{2^{k-1}-1}{2^{k-1}} \leq x < \frac{2^{k-1}}{2^k}, \quad k \geq 1.
\end{cases}
\]

Then \( Z \) is integrable with countably infinite discontinuites.
Sets of measure 0

**Question** Does there exist a Riemann integrable function with uncountably many discontinuities?

**Question** Does there exist a Riemann integrable function with discontinuities containing an open interval?

**Definition**

Let $E$ be a subset of $\mathbb{R}$. We say that $E$ is a set of measure 0 if for every $\epsilon > 0$, there exists a countable family of open intervals $\{I_k\}_{k \geq 1}$ such that $E \subseteq \bigcup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) < \epsilon$.

- In case $E$ is compact, there exists a finite family of open intervals $\{I_k\}_{k=1}^{N}$ satisfying the conditions above.
- If $A$ is of measure 0 and $B \subseteq A$, then $B$ is of measure 0.

**Problem**

*Show that the Cantor set is of measure 0.*

**Hint.**

$C = \bigcap_{n \geq 1} C_n$ (see slide 28) and $\ell(C_n) \to 0$ as $n \to \infty$.  \[\square\]
Problem

Show that a countable union of sets of measure 0 is of measure 0.

Solution.

Let $E_1, E_2, \ldots$, be countably many sets of measure 0. Let $\epsilon > 0$.

- Let $\{I_k^{(1)}\}_{k \geq 1}$ be a sequence of open intervals such that $E_1 \subseteq \bigcup_{k=1}^{\infty} I_k^{(1)}$ and $\sum_{k=1}^{\infty} \ell(I_k^{(1)}) < \epsilon/2$.
- Let $\{I_k^{(2)}\}_{k \geq 1}$ be a sequence of open intervals such that $E_2 \subseteq \bigcup_{k=1}^{\infty} I_k^{(2)}$ and $\sum_{k=1}^{\infty} \ell(I_k^{(2)}) < \epsilon/4$.

Continue like this to get for every $j \geq 1$, $\{I_k^{(j)}\}_{k \geq 1}$ such that $E_j \subseteq \bigcup_{k=1}^{\infty} I_k^{(j)}$ and $\sum_{k=1}^{\infty} \ell(I_k^{(j)}) < \epsilon/2^j$. Thus $\bigcup_{j=1}^{\infty} E_j \subseteq \bigcup_{j,k=1}^{\infty} I_k^{(j)}$ (countable union) and $\sum_{j,k=1}^{\infty} \ell(I_k^{(j)}) < \sum_{j=1}^{\infty} \epsilon/2^j = \epsilon$.

Problem

Show that any countable subset of $\mathbb{R}$ is of measure 0.

Proof.

Any finite set (and hence single-ton) is of measure 0.
Let \( f : [a, b] \to \mathbb{R} \) be bounded with \( M = \sup_{x \in [a, b]} |f(x)| < \infty \).

- \( A_\varepsilon = \{ c \in [a, b] : \text{osc}(f, c) \geq \varepsilon \} \) is compact (see slide 181)

**Lemma**

*If \( A_\varepsilon \) is a set of measure 0, then there exists a partition \( P \) of \([a, b]\) such that \( U(P, f) - L(P, f) < (2M + b - a)\varepsilon \).*

**Proof.**

Since \( A_\varepsilon \) is a compact set of measure 0, there exists open intervals \( \{I_k\}_{k=1}^N \) such that \( A_\varepsilon \subseteq \bigcup_{k=1}^N I_k \) and \( \sum_{k=1}^N \ell(I_k) < \varepsilon \).

- \( K = [a, b] \setminus (\bigcup_{k=1}^N I_k) \) is a compact subset of \([a, b]\)

For any \( c \in K \), there exists an interval, open in \( J_c \subseteq [a, b] \setminus A_\varepsilon \) such that \( c \in J_c \implies \sup_{x, y \in J_c} |f(x) - f(y)| \leq \varepsilon \)

- There exists \( J_1, \ldots, J_{N'} \) such that \( K \subseteq \bigcup_{j=1}^{N'} J_j \) (\( K \) is compact)

If \( P : \) partition formed by end-points of \( I_1, \ldots, I_N, J_1, \ldots, J_{N'} \), then

\[
U(P, f) - L(P, f) \leq 2M \sum_{j=1}^N \ell(I_j) + \varepsilon(b - a) < (2M + b - a)\varepsilon.
\]

\( \Box \)
• $A_\varepsilon$ is a subset of $D(f)$ (the set of discontinuities of $f$)
• If $D(f)$ is of measure 0, then so is $A_\varepsilon$ for every $\varepsilon > 0$
• If $D(f)$ is of measure 0, then $f \in R[a, b]$ (apply Lemma)

**Theorem (Lebesgue’s criterion for Riemann integrability)**

*If $f : [a, b] \to \mathbb{R}$ is bounded, then $f \in R[a, b]$ if and only if $D(f)$ is of measure 0.*

**Proof.**

To see $(D(f)$ is of measure 0 $\iff f \in R[a, b])$, let $f \in R[a, b]$.

• Countable union of sets of measure zero is of measure 0 (see slide 204) and $D(f) = \bigcup_{n \geq 1} A_{1/n}$  
  **Claim** Each $A_{1/n}$ is of measure 0

• Let $\varepsilon > 0$ and choose a partition $P = \{x_0, \ldots, x_N\}$ such that $U(P, f) - L(P, f) < \varepsilon/n$ (since $f$ is Riemann integrable)

• $I_j = (x_{j-1}, x_j)$ and $I_{j_1}, \ldots, I_{j_{N'}}$ be sets intersecting with $A_{1/n}$

• $\sup_{x \in I_{j_k}} f(x) - \inf_{x \in I_{j_k}} f(x) \geq \sup_{x \in I_{j_k} \cap A_{1/n}} f(x) - \inf_{x \in I_{j_k} \cap A_{1/n}} f(x) \geq 1/n$

Finally, $\frac{1}{n} \sum_{k=1}^{N'} \ell(I_{j_k}) \leq U(P, f) - L(P, f) < \frac{\varepsilon}{n}$. Choose $I'_{j_k} \supseteq I_{j_k}$ so that $\sum_{k=1}^{N'} \ell(I'_{j_k}) < 2\varepsilon$ and $A_{1/n} \subseteq \bigcup_{k=1}^{N'} I'_{j_k}$.  

$\square$
Example
Consider the function $g : [0, 1] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0, 1) \text{ and } x = \frac{p}{q} \text{ in reduced form} \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in R[0, 1]$ since $D(g) = \mathbb{Q} \cap (0, 1]$ is of measure 0.

Problem
Show by an example that composition of Riemann integrable functions need not be Riemann integrable.

Hint.
Take $f = \chi_{(0,1]}$ and $g$ as above.

Problem
Let $f : [a, b] \to [c, d]$ be Riemann integrable. If $\phi : [c, d] \to \mathbb{R}$ is continuous, then $\phi \circ f$ is Riemann integrable.

Hint.
$D(\phi \circ f) \subseteq D(f)$ is of measure 0.
**Problem**

Show that sum and product of Riemann integrable functions is Riemann integrable.

**Hint.**

\[ D(f + g) \subseteq D(f) \cup D(g) \] and \[ D(fg) \subseteq D(f) \cup D(g). \]

**Theorem**

The set \( R[a, b] \) of Riemann integrable functions \( f : [a, b] \to \mathbb{R} \) is a vector space over \( \mathbb{R} \). Moreover, for every \( f, g \in R[a, b] \) and \( \alpha \in \mathbb{R} \),

\[ \int_{a}^{b} (f(x) + \alpha g(x))dx = \int_{a}^{b} f(x)dx + \alpha \int_{a}^{b} g(x)dx. \]

For \( f \in R[a, b] \), define \( \|f\|_1 = \int_{a}^{b} |f(x)|dx. \)

**Question** Is \( \| \cdot \|_1 \) a norm on \( R[a, b] \)?

- Clearly, \( \|f\|_1 \geq 0 \), \( \|\alpha f\|_1 = |\alpha|\|f\|_1 \) and \( \|f + g\|_1 \leq \|f\|_1 + \|g\|_1 \) for every \( f, g \in R[a, b] \) and \( \alpha \in \mathbb{R} \)

- If \( \|f\|_1 = 0 \) for \( f \in R[a, b] \), then \( f \) need not be 0

- \( \chi_{[0,1/2]} - \chi_{[0,1/2)} \) is Riemann integrable, \( \|\chi_{[0,1/2]} - \chi_{[0,1/2)}\|_1 = 0 \) but \( \chi_{[0,1/2]} \neq \chi_{[0,1/2)} \)
Discontinuities of positive measure

Example

Let \( \hat{C} \) denote the Cantor-like set obtained by removing \( 2^{k-1} \) centrally situated open subintervals \( I_{1k}, \cdots, I_{2^{k-1}k} \) of \( I = [0, 1] \) each of length \( 1/4^k \) at the \( k \)th stage, where \( k = 1, 2, \cdots \).

- \( \ell(\hat{C}) = 1 - \sum_{k=1}^{\infty} \frac{2^{k-1}}{4^k} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = 1/2 \)

- Let \( F_k : I \rightarrow I \) be a continuous function such that \( F_k = 1 \) on \( I \setminus \bigcup_{i=1}^{2^{k-1}} I_{ik} \) and \( F_k = 0 \) at the mid-points of \( I_{1k}, \cdots, I_{2^{k-1}k} \)

- If \( f_n = \prod_{i=1}^{n} F_i \), then \( f_{n+1}(x) \leq f_n(x) \) for every \( x \in I \), \( n \geq 1 \)

- Let \( f : I \rightarrow I \) be the pointwise limit of \( \{f_n\}_{n \geq 1} \)

- For \( x \in \hat{C} \), there exists a sequence \( \{x_n\}_{n \geq 1} \) converging to \( x \) such that \( f(x_n) = 0 \)

- \( f \) is discontinuous on \( \hat{C} \) (since \( f(x) = 1 \) for every \( x \in \hat{C} \))

Conclude that \( f \) is not Riemann integrable.

- It turns out that \( \|f_n - f\|_1 \rightarrow 0 \) as \( n \rightarrow \infty \)!
Problem
For \( f, g \in R[ a, b ] \), define \( f \sim g \) if \( f = g \) outside a set of measure \( 0 \). Verify the following:

(1) \( \sim \) defines an equivalence relation.

(2) If \([ f ]\) denotes the equivalence relation containing \( f \) and \( \|[ f ]\|_1 = \int_a^b | f ( x ) | \, dx \), then \( \|[ f ]\|_1 = 0 \) if and only if \( f \sim 0 \).

(3) \( \mathcal{R} = \{ [ f ] : f \in R[ a, b ] \} \) is a normed linear space endowed with the norm \( \| \cdot \|_1 \).

(4) \( (\mathcal{R}, \| \cdot \|_1) \) is incomplete.

Problem
Consider the intervals \( l_1 = [0, 1], l_2 = [0, 1/2], l_3 = [1/2, 1], l_4 = [0, 1/4], l_5 = [1/4, 1/2], l_6 = [1/2, 3/4], l_7 = [3/4, 1] \) and so on. For \( f_n = \chi_{l_n} \) and \( f = 0 \), show that \( f_n(x) \not\rightarrow f(x) \) for any \( x \in [0, 1] \).

Hint.
Any \( x \in l_1 \) lies in infinitely many \( l_n \) & infinitely many \([0, 1] \setminus l_n\). □

\( \int_{[0,1]} |f_n(x) - f(x)| \, dx = \ell(l_n) \to 0 \) as \( n \to \infty \).
Problem (Integrable function with uncountable discontinuities)

Consider the indicator function $\chi_C$ of the Cantor set, that is,

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \in [0, 1] \setminus C. \end{cases}$$

Show that $\chi_C$ is discontinuous precisely at every $x \in C$. Conclude that $\chi_C \in R[0, 1]$.

**Hint.**

Note that $C$ is closed and nowhere dense, and hence $[0, 1] \setminus C$ is dense. Thus for every $x \in C$, there exists a sequence of points $x_n \in [0, 1] \setminus C$ such that $x_n \to x$. Clearly, $\chi_C(x_n) = 0$ does not converge to $\chi_C(x) = 1$, and hence $C \subseteq D(\chi_C)$.

Let $x \in [0, 1] \setminus C$. Then, some neighborhood of $x$ does not intersect $C$ (otherwise, $x$ is a limit point of $C$), and hence $\chi_C$ is sequentially continuous at $x$. Thus $D(\chi_C) = C$. Finally, since $C$ is of measure 0, by Lebesgue’s criterion, $\chi_C \in R[0, 1]$. □
Theorem
The uniform limit $f$ of a sequence of Riemann integrable functions $f_n$ is Riemann integrable and $\lim_{n \to \infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$.

Proof.
Let $\{f_n\}_{n \geq 1} \subseteq R[a, b], f : [a, b] \to \mathbb{R}$ be such that $\|f_n - f\|_\infty \to 0$.

- $f$ is bounded (since $\|f\|_\infty \leq \|f_n - f\|_\infty + \sup_{n \geq 1} \|f_n\|_\infty$)
- Given $\epsilon > 0$, there exists $N \geq 1$ such that for every $n \geq N$ and for every $x \in [a, b]$, $f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon$
- For partitions $P, Q$ of $[a, b]$, $L(f_n - \epsilon, P) \leq L(f, P) \leq \int_a^b f(x)\,dx \leq \int_a^b f(x)\,dx \leq U(Q, f) \leq U(f_n + \epsilon, Q)$

Now take supremum over LHS and infinimum over RHS to get

$$\int_a^b (f_n(x) - \epsilon)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b (f_n(x) + \epsilon)\,dx.$$  

Thus $0 \leq \int_a^b f(x)\,dx - \int_a^b f(x)\,dx \leq 2\epsilon$, or $f \in R[a, b]$ and

$$\left| \int_a^b f_n(x)\,dx - \int_a^b f(x)\,dx \right| \leq (b - a)\epsilon \text{ for every } n \geq N.$$  

$\square$
• A power series can be integrated termwise

**Problem**

Let \( f(x) = \sum_{n=0}^{\infty} a_n x^n \) be a power series of radius of convergence \( R > 0 \). Show that for any \(-R < a < b < R\),

\[
\int_a^b f(x) \, dx = \sum_{n=0}^{\infty} a_n \left( \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \right).
\]

**Hint.**

The partial sum of power series is continuous and it converges to \( f(x) \) uniformly on \([a, b]\). Now apply last problem.

• One may use the last problem to compute \( \int_a^b \frac{1}{x} \, dx \) for every \( 1 < a < b < 2 \). Indeed, \( \frac{1}{x} = \sum_{n=0}^{\infty} (1 - x)^n \) converges uniformly on \( |1 - x| \leq 1 - \epsilon \) for every \( \epsilon > 0 \), and hence on \([a, b]\). Thus

\[
\int_a^b \frac{1}{x} \, dx = \sum_{n=0}^{\infty} \left( \frac{(1 - b)^{n+1}}{n+1} - \frac{(1 - a)^{n+1}}{n+1} \right) = \log b - \log a.
\]
Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $x_0 \in [a, b] \setminus D(f)$.

- Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

- Let $J$ be an interval of length $\ell(J) < \delta/2$ that contains $x_0$.

- $f(x_0) - \frac{1}{\ell(J)} \int_J f(x) \, dx = \frac{1}{\ell(J)} \int_J (f(x_0) - f(x)) \, dx$.

- $\left| f(x_0) - \frac{1}{\ell(J)} \int_J f(x) \, dx \right| \leq \frac{1}{\ell(J)} \int_J |f(x_0) - f(x)| \, dx < \epsilon$ (see 9).

This proves the following:

**Theorem (Fundamental Theorem of Calculus-I)**

For every $f \in R[a, b]$ and every $x_0 \in [a, b] \setminus D(f)$,

$$
\lim_{\ell(J) \to 0} \frac{1}{\ell(J)} \int_J f(x) \, dx = f(x_0).
$$

**Remark** The quantity $\frac{1}{\ell(J)} \int_J f(x) \, dx$ is the “average of $f$ over $J$”.

---

9We used $|\int_a^b g(x) \, dx| \leq \int_a^b |g(x)| \, dx$, $g \in R[a, b]$ (Exercise)
Corollary

Let $f \in R[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by $F(x_0) = \int_a^{x_0} f(x)dx$, $x_0 \in [a, b]$. Then

(1) $F$ is continuous at every point in $[a, b]$,

(2) $F$ is differentiable at every $x_0 \in [a, b] \setminus D(f)$ & $F'(x_0) = f(x_0)$.

Proof.

The first part follows from the estimate

$$|F(x) - F(x_0)| = \left| \int_x^{x_0} f(t)dt \right| \leq \left( \sup_{t \in [a, b]} |f(t)| \right) |x - x_0|, \quad x \in [a, b].$$

To see the second part, let $J = [x_0, x_0 + h]$ in FTC-I to obtain

$$\frac{F(x_0 + h) - F(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0 + h} f(x)dx \to f(x_0) \text{ as } h \to 0.$$

This also shows that $F'(x_0) = f(x_0)$. \qed
Corollary

Let \( f \in R[a, b] \). Define \( F : [a, b] \to \mathbb{R} \) by \( F(x_0) = \int_a^{x_0} f(x) \, dx \), \( x_0 \in [a, b] \). If \( D(f) \) denotes the set of points of discontinuities of \( f \) and \( \text{Diff}(F) \) denotes the set of points at which \( F \) is differentiable, then \( [a, b] \setminus D(f) \subseteq \text{Diff}(F) \).

Problem

Give an example of a continuous function on \([0, 1]\), which is differentiable at irrationals in \([0, 1]\) (we are not asking for a function differentiable precisely at irrationals in \([0, 1]\)).

Hint.

Use the last corollary.

- There exist functions differentiable precisely at rationals

Definition
Given a partition \( P = \{ a = x_0 < x_1 \cdots < x_n = b \} \) of \([a, b]\) and \( g : [a, b] \to \mathbb{R} \), the variation of \( g \) over \( P \) is given by

\[
V(g, P) = \sum_{j=1}^{n} |g(x_j) - g(x_{j-1})|.
\]

• \( g \) is of bounded variation on \([a, b]\) if the total variation \( V^b_a g = \sup_P V(g, P) \) of \( g \) over \([a, b]\) is finite.

Theorem
Let \( f \in R[a, b] \). Define \( F : [a, b] \to \mathbb{R} \) by \( F(x_0) = \int_a^{x_0} f(x) \, dx \), \( x_0 \in [a, b] \). Then \( F \) is of bounded variation.

Proof.
Since \( V(F, P) = \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \leq \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |f(x)| \, dx \), \( V^b_a F \) is at most \((b - a) \sup_{x \in [a, b]} |f(x)|\). \( \square \)
Theorem (Fundamental Theorem of Calculus-II)

Let $f \in R[a, b]$. If there is a continuous function $F : [a, b] \to \mathbb{R}$ differentiable on $(a, b)$ such that $F'(x) = f(x)$, $x \in (a, b)$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

Proof.

Let $\epsilon > 0$ and $P = \{a = x_0 < x_1 \cdots < x_n = b\}$ be a partition of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.

- For $i = 1, \ldots, n$, there exists $c_i \in [x_i, x_{i-1}]$ such that

  $$F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$$

  (Mean Value Theorem)

- $$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} (F(x_i) - F(x_{i-1})) = F(b) - F(a)$$

- $$L(P, f) \leq \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) \leq U(P, f)$$

  ($m_i \leq f(c_i) \leq M_i$)

- $$L(P, f) \leq \int_{a}^{b} f(x)dx \leq U(P, f)$$

Thus $\int_{a}^{b} f(x)dx$ and $F(b) - F(a)$ lie in interval $[L(U, f), U(P, f)]$ of length less than $\epsilon$. Hence

$$|F(b) - F(a) - \int_{a}^{b} f(x)dx| < \epsilon.$$
The Fundamental Theorem of Calculus-II can be used to compute integrals provided derivatives are known.

- \[ \int_0^x t^n dt = \frac{x^{n+1}}{n+1} \] (since the derivative of \( F(x) = \frac{x^{n+1}}{n+1} \) equals \( x^n \))
- \[ \int_1^x \frac{1}{t} dt = \log x \] (since \( (\log x)' = \frac{1}{x} \))

**Definition**

We say that a function \( G : [a, b] \to \mathbb{R} \) is an anti-derivative of \( g \in R[a, b] \) if \( G \) is differentiable on \([a, b]\) and \( G'(x) = g(x) \) at every \( x \in [a, b] \).

**Example**

The jump function \( g = \chi_{(0,1]} \) does not have an anti-derivative \( G \). Indeed, if there exists a differentiable function \( G \) such that \( G' = g \), then by the Fundamental Theorem of Calculus-II,

\[
1 - x = \int_0^1 g(x) dx = G(1) - G(x), \quad 0 < x < 1.
\]

This shows that \( G(x) = G(1) - 1 + x \) for every \( x \in (0, 1] \), and hence by continuity, \( G(x) = G(1) - 1 + x \) for every \( x \in [0, 1] \).

However, \( G'(0) = 1 \neq g(0) \Rightarrow \Leftarrow \)
Corollary (Integration by parts)

Suppose that $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable in $[a, b]$, and $f', g'$ are integrable on $[a, b]$. Then

$$\int_{a}^{b} f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_{a}^{b} f'(x)g(x)dx.$$ 

Proof.
Define $F(x) = f(x)g(x)$, $x \in [0, 1]$. Then $F$ is continuous on $[a, b]$, $F$ is differentiable and

$$F'(x) = f'(x)g(x) + f(x)g'(x), \quad x \in (a, b).$$

By the Fundamental Theorem of Calculus-II,

$$\int_{a}^{b} \left( f'(x)g(x) + f(x)g'(x) \right) dx = F(b) - F(a) = f(b)g(b) - f(a)g(a).$$

This completes the proof.
• There exists \( f \in R[a, b] \) such that \( F' = f \) outside a set of measure 0, yet, \( \int_a^b f(x)dx \neq F(b) - F(a) \).

**Example (Cantor Function)**

Let \( C \) denote the Cantor set obtained by removing \( 2^{n-1} \) centrally situated disjoint open subintervals \( U_{1,n}, \ldots, U_{2^{n-1},n} \) of \([0, 1]\) each of length \( 1/3^n \) at the \( n \)th stage, where \( n = 1, 2, \ldots \). Thus

\[
C = \cap_{n \geq 1} C_n, \quad \text{where} \quad C_n = [0, 1] \setminus (\cup_{k=1}^n \cup_{j=1}^{2^{n-1}} U_{k,j}).
\]

• \( F_1 : [0, 1] \to \mathbb{R} \) a continuous increasing function so that \( F_1(0) = 0, F_1 = 1/2 \) on \([1/3, 2/3]\), \( F_1(1) = 1 \), \( F_1 \) linear on \( C_1 \)

• \( F_2 : [0, 1] \to \mathbb{R} \) a continuous increasing function so that \( F_2(0) = 0, F_2 = 1/4 \) on \([1/9, 2/9]\), \( F_2 = 1/2 \) on \([1/3, 2/3]\), \( F_2 = 3/4 \) on \([7/9, 8/9]\), \( F_1(1) = 1 \), \( F_2 \) linear on \( C_2 \)

Continuing this, we obtain a sequence of continuous increasing functions \( \{F_n\}_{n \geq 1} \) such that \( |F_{n+1}(x) - F_n(x)| \leq 2^{-n-1} \).
Example (Example continued ...)

Check that \( \{ F_n \}_{n \geq 1} \) is Cauchy in \( C[0, 1] \). Indeed, for \( m > n \),

\[
|F_m(x) - F_n(x)| \leq \sum_{j=n+1}^{m} |F_j(x) - F_{j-1}(x)| \leq \sum_{j=n+1}^{m} 2^{-j},
\]

and hence \( \{ F_n \}_{n \geq 1} \) converges uniformly to \( F \in C[0, 1] \).

- We refer to \( F \) as Cantor function or devil’s staircase function.
  - \( F \) is increasing (since so is \( F_n \) for every \( n \geq 1 \))
  - \( F' = 0 \) on \([0, 1] \setminus C\) (since \( F \) is constant on each interval of the complement of \( C \))
  - \( F' = 0 \) outside a set of measure 0 (since \( C \) is of measure 0)

This shows that \( \int_{0}^{1} F'(x)dx = 0 \) (why?) and \( F(1) - F(0) = 1 \), so

\[
\int_{0}^{1} F'(x)dx \neq F(1) - F(0).
\]

- \( F \) is non-constant, yet, \( F' = 0 \) outside a set of measure 0!
Let us summarize the discussion above:

- If \( f \in R[a, b] \), then \( F(x) = \int_a^x f(x)dx \) is differentiable outside the set \( D(f) \) of measure 0.
- In general, \( \int_a^b F'(x)dx \neq F(b) - F(a) \).

This raises the following question:

**Question** What conditions on a function \( F \) on \([a, b]\) guarantee that \( F'(x) \) exists (outside a set of measure 0), that this function is integrable, and that moreover \( \int_a^b F'(x)dx = F(b) - F(a) \)?

This problem is closely related to the "averaging problem":

**Question** What are conditions on a function \( f \) on \([a, b]\) for which

\[
\lim_{\ell(J) \to 0} \frac{1}{\ell(J)} \int_J f(x)dx = f(x_0)
\]

holds for \( x_0 \in [a, b] \) outside a set of measure 0?

To answer these questions, one needs to venture into the theory of Lebesgue integration!