# Analysis
# MTH301A

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Semester I, 2020-21

These are the lecture notes prepared for the course MTH301A[1]
I mostly referred in parts to the following texts:


- Carothers, N. L. Real analysis. Cambridge University Press, Cambridge, 2000. xiv+401 pp.
- Kumar, Ajit; Kumaresan, S. A basic course in real analysis. CRC Press, Boca Raton, FL, 2014. xx+302 pp.
- Kumaresan, S. Topology of metric spaces. Narosa Publishing House, New Delhi, 2005. xii+152 pp.
- M. Miklavčič, Applied Functional Analysis and Partial Differential Equations, World Scientific, Singapore, 1998.
- Pugh, Charles C. Real mathematical analysis. Undergraduate Texts in Mathematics. Springer, Cham, 2015. xi+478 pp.
- Simon, Barry Real analysis. With a 68 page companion booklet. A Comprehensive Course in Analysis, Part 1. American Mathematical Society, Providence, RI, 2015. xx+789 pp.
- Stein, Elias M.; Shakarchi, Rami Fourier analysis. An introduction. Princeton Lectures in Analysis, 1. Princeton University Press, Princeton, NJ, 2003. xvi+311 pp
- Stein, Elias M.; Shakarchi, Rami Real analysis. Measure theory, integration, and Hilbert spaces. Princeton Lectures in Analysis, 3. Princeton University Press, Princeton, NJ, 2005. xx+402 pp

# Syllabus

- $\mathbb{R}$, Completeness property. Countable and Uncountable
- Metric Spaces and Metric space topology
- Nested set theorem, Baire category theorem
- Compactness, Totally bounded, Characterizations of compactness, Finite intersection property, Continuous functions on compact sets, Uniform continuity
- Connectedness and Path connectedness, Continuous functions
- Riemann integration, Fundamental theorem of calculus. Set of measure zero, Cantor set, Integrable functions
- Convergence of sequence and series of functions: Pointwise and uniform convergence of functions, Series of functions, Power series, Dini's theorem, Ascoli's theorem
- Nowhere-differentiable Continuous functions, Weierstrass approximation theorem

# Course plan

- This course has been split into L + T + D.
- Lectures (3) and Tutorial (1) have been merged into 3 to 4 videos of time length 40-50 minutes every week. The links of these videos will be shared after the videos are released.[2]
- There will be a Discussion hour over zoom every Friday from 12.00-12.50 (afternoon).
- About assessment/Evaluation, we opt for
  - In-video questions for assignment submission (20 percent)
  - Assignments/homework (40 percent)
  - Online oral examination (40 percent)
- There will be mid-semester and end-semester examinations. These will be held during the prescribed examination period.
- For students with limited or no network access, the Institute will be making the course materials available as found feasible.

[2] The course will managed on https://hello.iitk.ac.in/

# Curiosity

**Questions.** Do you believe in real numbers ? Who are these creatures ? Are these real or imaginary ?

- B Assume the existence (be lazy be happy)
- A Prove the existence (hitting against the hard)

**Difficulty.** If you draw a point on a plane paper, and if see it by a magnifier, you find it to be disc-like! What/where is the point?

**Philosophy** Is there is a particle (point) of mass (measure) 0?

B $\downarrow$ A Understand $\mathbb{R}$ as a set containing all rational numbers, which satisfies (optimal) axioms. First two building blocks:

$\mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}$ and $\mathbb{Q} = \{m/n : m, n \in \mathbb{Z}, n \neq 0\}$

- Arithmetic operations of addition $+$ and multiplication $\cdot$
- Order structure $<$

# Axiomatic approach

Consider the set $\mathbb{R}$ of numbers which contains $\mathbb{Q}$ (and hence $\mathbb{Z}$) with arithmetic operations $+$ and $\cdot$ and order structure $<$ satisfying

- $x + y = y + x$ for all $x, y \in \mathbb{R}$
- $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbb{R}$
- $x + 0 = x$ for all $x \in \mathbb{R}$
- for every $x \in \mathbb{R}$, there exists a unique $y \in \mathbb{R}$ (denoted by $-x$) such that $x + y = 0$
- $x \cdot y = y \cdot x$ for all $x, y \in \mathbb{R}$
- $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all $x, y, z \in \mathbb{R}$
- $x \cdot 1 = x$ for all $x \in \mathbb{R}$
- for every nonzero $x \in \mathbb{R}$, there exists a unique $y \in \mathbb{R}$ (denoted by $x^{-1}$ or $1/x$) such that $x \cdot y = 1$
- $(x + y) \cdot z = x \cdot y + x \cdot z$ for all $x, y, z \in \mathbb{R}$
- $x < y$, $x = y$ or $x > y$ for all $x, y \in \mathbb{R}$ (exactly one possibility)

In the following, we understand that $x < y$ iff $y - x > 0$. Further, we assume that sum and product of positive numbers are positive.

### Problem

*Use the above axioms to verify the following:*

1. *If $x < y$ and $z \in \mathbb{R}$, then $x + z < y + z$.*
2. *If $x < y$ and $y < z$, then $x < z$.*
3. *If $x < y$ and $z \in \mathbb{R}$, then*

$$\begin{cases} x \cdot z < y \cdot z & \text{if } z > 0, \\ y \cdot z < x \cdot z & \text{if } z < 0, \\ x \cdot z = y \cdot z & \text{if } z = 0. \end{cases}$$

4. *If $x < y$, then $-y < -x$.*
5. *Either $x^2 > 0$ or $x = 0$.*
6. *If $0 < x < y$, then $0 < 1/y < 1/x$.*

### Hint.

Since $z - x = z - y + y - x > 0$, 2 follows from 1. $\square$

### Problem

*Use only the last problem to verify the following:*

1. *Given two real numbers* $x, z$ *such that* $x < z$, *there exists* $y \in \mathbb{R}$ *such that* $x < y < z$.
2. *If* $x \leqslant y + z$ *for all* $z > 0$, *then* $x \leqslant y$.
3. *If* $0 < x < y$, *then for any positive integer* $n$, $0 < x^n < y^n$.

### Hint.

For 1, try $y = \frac{x+z}{2}$. For 2, prove by contradiction. $\square$

- $\mathbb{Q}$ satisfies all the axioms mentioned before the last problem.

**Question.** How do we differentiate $\mathbb{R}$ from $\mathbb{Q}$? Is there a property (of $\mathbb{R}$) not enjoyed by $\mathbb{Q}$ ?

# Upper bounds

## Definition

Let $\emptyset \neq A \subseteq \mathbb{R}$. We say that $A$ is <u>bounded from above</u> if there is $\alpha \in \mathbb{R}$ (an <u>upper bound</u>) such that $a \leqslant \alpha$ for every $a \in A$.

## Problem

*Show that any finite nonempty subset $A$ of $\mathbb{R}$ is bounded with upper an bound belonging to it. Give an infinite subset of $\mathbb{R}$, where this fails.*

## Hint.

This can be proved by the induction on $\operatorname{card}(A)$. If $\operatorname{card}(A) = 1$, then $A = \{a\}$ for some $a \in A$, and $a$ is an upper bound. Suppose that the conclusion holds for all sets $A$ with $\operatorname{card}(A) = n$, and let $B$ be a subset of $\mathbb{R}$ such that $\operatorname{card}(B) = n + 1$. Since $B \neq \emptyset$, there exists $b \in B$. Apply the induction hypothesis to $B \setminus \{b\}$ and use the law of trichotomy. For last part, take $A = (0, 1)$. None of the upper bounds of $A$ belongs to $A$. $\square$

# Least upper bound

## Definition

Let $\emptyset \neq A \subseteq \mathbb{R}$. We say that $\alpha$ is a least upper bound (lub) for $A$ if

- $\alpha$ is an upper bound for $A$, that is, $a \leqslant \alpha$ for all $a \in A$, and
- $\beta$ is an upper bound for $A$, then $\alpha \leqslant \beta$.

**Remark.** lub is unique. If a set contains its upper bound (for example, $(0, 1]$), then maximum of that set is its lub.

## Problem

*Show that $\alpha \in \mathbb{R}$ is lub for $A \subseteq \mathbb{R}$ iff $\alpha$ is an upper bound for $A$ and if $\beta < \alpha$, then there exists $a \in A$ such that $\beta < a \leqslant \alpha$.*

## Hint.

If $\beta < \alpha$, then $\beta$ is not an upper bound of $A$. $\square$

## Example

1 is the lub for $A = (0, 1)$. Indeed, if $\alpha < 1$ is the lub, then $a := \frac{1+\alpha}{2} \in (\alpha, 1)$, so $\alpha$ is not an upper bound for $(0, 1)$.

# Unboundedness of integers

It is tempting to conclude that $\mathbb{Z}$ is not bounded from above without a proof.

**Difficulty.** If $\mathbb{Z}$ is bounded above by $x \in \mathbb{R}$ how to ensure that there is an integer $n > x$ (this problem does not occur if the upper bound is an integer) ?

**Solution.** If you can not prove an assertion, convert it into a property (with no disrespect for Archimedes)!

**Archimedean property.** Given $x, y \in \mathbb{R}$, with $x > 0$, there exists a positive integer $n$ such that $nx > y$.

**Interpretation.** Howsoever tiny may be your foot-step $x$, you can cover any finite distance $y$ (for example, the distance between earth and moon) in finitely many steps $n$.

# LUB property or completeness axiom

• Any nonempty subset of $\mathbb{R}$, which is bounded above in $\mathbb{R}$, has lub in $\mathbb{R}$.

This is the property which differentiates $\mathbb{R}$ from $\mathbb{Q}$:

## Example

The subset $A = \{a \in \mathbb{Q} : a^2 < 2\}$ is nonempty and bounded above ($1 \in A$ and 2 is an upper bound for $A$, but it does not have lub in $\mathbb{Q}$. To see the latter statement, assume that $\alpha$ is an lub of $A$. If $\alpha^2 < 2$, then for any positive integer $k$,

$$(\alpha + 1/k)^2 = \alpha^2 + 2\alpha/k + 1/k^2 \leqslant \alpha^2 + 5/k,$$

so that $(\alpha + 1/k)^2 < 2$ for sufficiently large $k$ (we need here the fact that $\mathbb{Z}$ is not bounded from above). This however implies that $\alpha + 1/k \in A$, contradicting the assumption that $\alpha$ is lub. Thus $\alpha^2 \not< 2$. Similarly, one can see that $\alpha^2 \not> 2$. Thus $\alpha^2 = 2$ or $\alpha \notin \mathbb{Q}$.

# LUB property and Archimedean property

**Theorem**

*LUB property $\Longrightarrow$ Archimedean property.*

**Proof.**

Assume the LUB property, Suppose that the Archimedean property fails, that is, there exist $x, y$ with $x > 0$ such that $nx \leqslant y$ for every positive integer $n$. Thus $n \leqslant y/x$ for every (positive) integer $n$. Thus $\mathbb{Z}$ is bounded above and hence by LUB property, it has lub, say, $\alpha$. Now $\alpha - 1$ is not an upper bound, so for some integer $N$, $\alpha - 1 < N$. However, this implies that $\alpha < N + 1$ and $N + 1$ is an integer, which contradicts the assumption that $\alpha$ is lub. $\square$

**Problem**

*Describe* $\cap_{n\geqslant 1}(0, 1/n)$ *and* $\cap_{n\geqslant 1}(n, \infty)$.

# Density of rationals and irrationals in $\mathbb{R}$

## Problem (Greatest Integer Function)

*Assume the LUB property and let $x \in \mathbb{R}$. Show that there exists a (unique) $m \in \mathbb{Z}$ such that $m \leqslant x < m+1$ ($m$ is denoted by $[x]$).*

## Hint.

Consider the nonempty set $A = \{k \in \mathbb{Z} : k \leqslant x\}$, which is bounded above. Let $\alpha = \text{lub}(A)$. Since $\alpha - 1$ is not an upper bound, there exists $m \in A$ such that $\alpha - 1 < m \leqslant x$. Verify that $x < m+1$. $\square$

## Theorem

*Assume the LUB property and let $a, b \in \mathbb{R}$ be such that $a < b$. Then there exists $r \in \mathbb{Q}$ (resp. $r \in \mathbb{R}\backslash\mathbb{Q}$) such that $a < r < b$.*

## Proof of Theorem.

By Archimedean property, we find an integer $n \geqslant 1$ such that $n(b-a) > 1$. By the last problem, there exists $m \in \mathbb{Z}$ such that $na < m \leqslant na + 1$ (take $m = [na] + 1$). Thus $na < m < nb$ or $r = m/n$. For the rest, apply this to $a - \sqrt{2}$ and $b - \sqrt{2}$. $\square$

# Nested Interval Theorem

**Theorem**

*Assume the LUB property and let $J_n = [a_n, b_n]$ be an interval in $\mathbb{R}$ such that $J_{n+1} \subseteq J_n$ for all integers $n \geqslant 1$. Then $\cap_{n\geqslant 1} J_n \neq \emptyset$.*

**Proof.**

Consider the set $A = \{x \in \mathbb{R} : x = a_n \text{ for some } n \geqslant 1\}$ and note that $A$ is nonempty and bounded above. Let $\alpha = \text{lub}(A)$. Since each $b_n$ is an upper bound for $A$ (since $J_{n+1} \subseteq J_n$ for all integers $n \geqslant 1$), we have $a_n \leqslant \alpha \leqslant b_n$, that is, $\alpha \in \cap_{n\geqslant 1} J_n$. □

**Problem**

*Assume the LUB (and GLB) property. Show that any bounded infinite subset $A$ of $\mathbb{R}$ has at least one accumulation point.*

**Hint.**

Suppose $A \subseteq [a_1, b_1]$ with $a_1 = \text{glb}(A)$, $b_1 = \text{lub}(A)$. Let $[a_2, b_2]$ be one of the intervals from $[a_1, (a_1 + b_1)/2]$ and $[(a_1 + b_1)/2, b_1]$ which contains infinitely many points from $A$. Continue this. □

# Complete ordered field

We may understand now $\mathbb{R}$ as the "complete ordered field" (the set $\mathbb{R}$ satisfying all the axioms mentioned earlier including the completeness axiom).

**Question.** Does a complete ordered field exist ? If yes, is it unique (up to isomorphism) ?

To address the issue of the existence of $\mathbb{R}$, we mention two approaches; one due to Dedekind (based on "Dedekind cuts") and other due to Cantor (based on the completion of $\mathbb{Q}$). We follow the second approach and for this, we need to know the convergence of sequences in $\mathbb{R}$.

Recall the definitions of Cauchy and convergent sequences in $\mathbb{R}$.

- Cauchy sequence is bounded and convergent sequence is a Cauchy sequence.
- Every Cauchy sequence with convergent subsequence is convergent.

# Cauchy Completeness of $\mathbb{R}$

**Theorem**

*Assume the LUB property. Then $\mathbb{R}$ is Cauchy complete, that is, every Cauchy sequence in $\mathbb{R}$ is convergent.*

**Proof I.**

Let $A = \{x \in \mathbb{R} : \text{there exists } N \geqslant 1 \text{ such that } x < x_n \text{ for } n \geqslant N\}$ and let $\epsilon > 0$. Note that there exists $n_0 \geqslant 1$ (dependent on $\epsilon$) such that $x_{n_0} - \epsilon/2 \in A$ (since $\{x_n\}_{n \geqslant 1}$ is a Cauchy sequence). Thus $A$ is a nonempty set. Further, $A$ is bounded above by $x_{n_0} + \epsilon/2$ (verify by contradiction). Let $\alpha = \text{lub}(A)$. Since $x_{n_0} - \epsilon/2 \in A$ and $A$ is bounded above by $x_{n_0} + \epsilon/2$, $|x_{n_0} - \alpha| \leqslant \epsilon/2$. Thus for $n \geqslant n_0$,

$$|x_n - \alpha| \leqslant |x_n - x_{n_0}| + |x_{n_0} - \alpha| < \epsilon.$$

This completes the proof. $\square$

It is possible to obtain the Cauchy completeness without using the LUB property.

Proof II.

Let $A = \{x \in \mathbb{R} : x = x_n \text{ for some n } \geqslant 1\}$. Since $A$ is bounded, $A \subseteq [a_1, b_1]$ for some $a_1, b_1 \in \mathbb{R}$ Let $[a_2, b_2]$ be one of the intervals from $[a_1, (a_1 + b_1)/2]$ and $[(a_1 + b_1)/2, b_1]$ which contains infinitely many points from $A$. Continue this to obtain the intervals $[a_n, b_n]$, $n \geqslant 1$, containing infinitely many points of $A$. By the nested interval theorem, $\cap_{n \geqslant 1}[a_n, b_n]$ contains $x$ for some $x \in \mathbb{R}$. Now for each $n \geqslant 1$, if we choose $x_{k_n} \in [a_n, b_n]$ such that $k_n \leqslant k_{n+1}$, we obtain a subsequence $\{x_{k_n}\}_{n \geqslant 1}$ such that $|x_{k_n} - x| \leqslant b_n - a_n = (b_1 - a_1)/2^n \to 0$ as $n \to \infty$. Since $\{x_n\}_{n \geqslant 1}$ is a Cauchy sequence, it must be convergent now. □

The proof above yields the following:

Theorem (Bolzano-Weierstrass Theorem)

*Every bounded sequences admits a convergent subsequence.*

# Cauchy's Construction of $\mathbb{R}$ (Incomplete)

Let $R$ be the set of Cauchy sequences in $\mathbb{Q}$. Let $\{x_n\}_{n\geqslant 1}, \{y_n\}_{n\geqslant 1}$ belong to $R$ and define the equivalence relation

$$\{x_n\}_{n\geqslant 1} \sim \{y_n\}_{n\geqslant 1} \text{ if } |x_n - y_n| \to 0.$$

• Define the set $\mathbb{R}$ as $R/\sim$ (set of equivalence classes $[\{x_n\}_{n\geqslant 1}]$).

• Identify the element $x$ in $\mathbb{Q}$ with constant sequence $\{x_n = x\}_{n\geqslant 1}$

• $\{x_n\}_{n\geqslant 1} < \{y_n\}_{n\geqslant 1}$ if $\{y_n - x_n\}_{n\geqslant 1}$ eventually consists of positive numbers.

• It is a laborious work to verify that $\mathbb{R}$ defines an ordered field which satisfies LUB property. Refer to the article `http://www.math.ucsd.edu/~tkemp/140A/Construction.of.R.pdf`) [3].

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[3] Cauchy's Construction of R, Todd Kemp

### Theorem (Density of rationals)

*If $r = \{r_n\}_{n \geqslant 1} \in \mathbb{R}$ and any small (rational) number $\epsilon$, then there exists a rational number $q$ such that $|r - q| < \epsilon$, that is, for some integer $N \geqslant 1$, $|r_n - q| < \epsilon$ for all $n \geqslant N$.*

### Proof.

Since $\{r_n\}_{n \geqslant 1}$ is a Cauchy sequence, for some $N \geqslant 1$, $|r_n - r_N| < \epsilon$ for all $n \geqslant N$. Let $q = r_N \in \mathbb{Q}$. $\square$

Let $\mathbb{R} = \{[\{x_n\}_{n \geqslant 1}] : \{x_n\}_{n \geqslant 1} \in R\}$ and define a metric on $\mathbb{R}$ by

$$d([\{x_n\}], [\{y_n\}]) = \lim_{n \to \infty} |x_n - y_n| \text{ (an equivalence class)}$$

(the limit exists since $\{x_n - y_n\}_{n \geqslant 1}$ is a Cauchy sequence and hence convergent). Then $\mathbb{R}$ is a complete metric space, and if $i : \mathbb{Q} \to \mathbb{R}$ is given by $i(\{x\}) = [\{x\}]$ for every $x \in \mathbb{Q}$, then $i$ is injective.

# Countable and Uncountable sets [4]

### Definition

A set $A$ is called <u>countably infinite</u> if there is a bijection $f : \mathbb{N} \to A$. If $A$ is finite or countably infinite, we say that $A$ is <u>countable</u>.

### Example

• Any subset $A$ of $\mathbb{N}$ is countable (either $A$ is finite or $A = \{n_k\}_{k \geqslant 1}$; in the latter case, define $f(k) = n_k$)
• Any subset of a countable set is countable
• $\mathbb{Z}$ is countable (define $g : \mathbb{Z} \to \mathbb{N}$ by $g(n) = 2n$ if $n \geqslant 1$, and $g(n) = -2n + 1$ if $n \leqslant 0$; now let $f = g^{-1}$)
• $\mathbb{N} \times \mathbb{N}$ is countable (define $g(m, n) = 2^{m-1}(2n - 1)$; let $f = g^{-1}$)
• Countable union of countable sets is countable (if $A = \{x_{m,n} : m \in \mathbb{N},\ n \in \mathbb{N}\}$, then $g : \mathbb{N} \times \mathbb{N} \to A$ given by $f(m, n) = a_{m,n}$ is bijective)

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[4] Cantor is a 'corrupter of youth'. - L. Kronecker, as quoted by Schoenflies and requoted by Barry Simon

## Problem

*Verify the following:*

1. *If $g : A \to B$ is bijective, show that $A$ is countable iff so is $B$.*
2. *If $g : A \to B$ is injective, show that $A$ is countable if so is $B$.*

## Hint.

1. If $h : \mathbb{N} \to A$ is bijective, then so is $g \circ h : \mathbb{N} \to B$.
2. If $h : \mathbb{N} \to g(B)$ is bijective, then so is $g^{-1} \circ h : \mathbb{N} \to A$. $\square$

## Problem

*Show that $\mathbb{Q}$ is countably infinite.*

## Hint.

Write $\mathbb{Q} = \mathbb{Q}_+ \cup \mathbb{Q}_- \cup \{0\}$ (union of positive/negative/zero rationals). To see that $\mathbb{Q}_+$ is countable, define $f : \mathbb{Q}_+ \to \mathbb{N} \times \mathbb{N}$ by $f(m/n) = (m, n)$, where $m$ and $n$ are coprime to each other. Note that $f$ is injective and $\mathbb{N} \times \mathbb{N}$ is countable. $\square$

## Problem

*For an index set $I$, let $\{J_\alpha\}_{\alpha \in I}$ be a collection of disjoint open intervals such that $\mathbb{R} = \cup_{\alpha \in I} J_\alpha$. Show that $I$ is countable.*

## Solution.

We know that each interval $J_\alpha$ contains a rational number $r_\alpha$. Define $g : I \to \mathbb{Q}$ by $g(\alpha) = r_\alpha$. If, for some $\alpha, \beta \in I$, $g(\alpha) = g(\beta)$, then $r_\alpha = r_\beta$, and hence $\alpha = \beta$ (if $\alpha \neq \beta$, then $J_\alpha \cap J_\beta = \emptyset$). This shows that $g$ is injective. Since $\mathbb{Q}$ is countable, by the problem on the previous slide, $I$ is also countable. $\square$

## Problem

*For an index set $I$, let $\{(x_\alpha\}_{\alpha \in I}$ be a collection of mutually orthonormal vectors in an inner-product space. $X$. If $X$ has a countable dense subset, then show that $I$ is countable.*

# Discontinuities of a monotone function

For a function $f : (0,1) \to \mathbb{R}$ and $c \in (0,1)$, let $f(c_-)$ and $f(c_+)$ denote the left and right hand side limits of $f$ at $c$, respectively.

## Theorem

*Let $f : (a,b) \to \mathbb{R}$ be monotone. Then the set $D$ of discontinuities of $f$ is countable.*

## Proof.

Without loss of generality, we may assume that $f$ is increasing. Note that $D = \{c \in (a,b) : I_c = (f(c_-), f(c_+)) \neq \emptyset\}$. For $c, d \in D$, let $t \in [a,b]$ be such that $c < t < d$. Then

$$f(c_+) = \operatorname{glb}\{f(x) : x > c\} \leqslant f(t) \leqslant f(d_-) = \operatorname{lub}\{f(y) : y < d\}.$$

This shows that $I_c \cap I_d = \emptyset$. Since each interval $I_c$ contains a rational and rationals are countable, $D$ is countable. $\square$

## Problem

*Show that a monotone surjection $f : [a,b] \to [c,d]$ is continuous.*

# Cantor's diagonalization theorem

**Theorem**

*There is no surjection $f : \mathbb{N} \to [0, 1]$.*

**Proof.**

Consider the decimal expansion (possibly more than one) of $f(n)$:

$$f(n) = 0 \cdot a_{n1} a_{n2} \dots a_{nn} \dots, \quad n \geqslant 1$$

(for the existence of decimal expansion, refer to Proposition 1.8 of [Real Analysis, Carothers, N. L.]). Let $x = 0 \cdot a_1 a_2 \dots a_n \dots \in \mathbb{R}$ where $a_n \in \{1, \dots, 8\} \backslash \{a_{nn}\}$. Note that $x \neq f(n)$ for any $n \geqslant 1$, and hence $x \notin f(\mathbb{N})$. $\square$

### Corollary (Cantor)

*The interval* $[0,1]$ *in* $\mathbb{R}$ *is not countable.*

### Problem

*Given two points in* $\mathbb{R}\times\mathbb{R}\setminus\mathbb{Q}\times\mathbb{Q}$*, show that there exists uncountably many paths* (*finite union of line segments*) *in* $\mathbb{R}\times\mathbb{R}\setminus\mathbb{Q}\times\mathbb{Q}$*, which join these two points.*

### Hint.

Fix two points $A$ and $B$ in the plane and draw a square with one of the diagonals being the straight line $C$ joining these two points. Consider the diagonal $D$ orthogonal to $L$ and note that it is not countable. Now produce one path corresponding to each point on $D$ which joins $A$ and $B$. How many of these touch $\mathbb{Q}\times\mathbb{Q}$ ? $\square$

# Power set of $\mathbb{N}$

### Example

Consider the power set $P(\mathbb{N})$ of all subsets of $\mathbb{N}$. Let $f : \mathbb{N} \to P(\mathbb{N})$ be any function. We claim that $f$ is not surjective. Indeed, define $A \in P(\mathbb{N})$ by the property that

$$k \in A \Longleftrightarrow k \notin f(k).$$

Note that $A \neq f(k)$ for any $k \geqslant 1$, and hence the claim stands verified. In particular, $P(\mathbb{N})$ is not countable.

### Problem

*Let $A$ be a nonempty set. There is no map from $A$ into the power set $P(A)$ of $A$, which is surjective.*

# Cantor set

## Example

Let $C$ denote the Cantor set obtained by removing $2^{n-1}$ centrally situated disjoint open subintervals $U_{1,n}, \cdots, U_{2^{n-1},n}$ of $[0,1]$ each of length $1/3^n$ at the $n$th stage, where $n = 1, 2, \cdots$. Specifically, if $U_{1,1} = (1/3, 2/3), U_{2,1} = (1/9, 2/9), U_{2,2} = (7/9, 8/9), \ldots, U_{n,1} = (1/3^n, 2/3^n), \ldots, U_{n,2^{n-1}} = (1 - 2/3^n, 1 - 1/3^n)$, then

$$C = \cap_{n \geqslant 1} C_n, \quad \text{where } C_n = [0,1] \setminus \big( \cup_{k=1}^{n} \cup_{j=1}^{2^{n-1}} U_{k,j} \big).$$

Clearly, all end-points $1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots$ of $U_{j,k}$ belong to $\Delta$. Are there any other points in $C$ ?

**Answer** Yes, take $1/4$ (since $1/4 \notin U_{k,j}$ for all $j, k$)

**Question** What is the trick to generate more elements in $C$ ?

## Example (continued ...)

- Any $x \in [0,1]$ has two distinct representations of the form $\sum_{n=1}^{\infty} a_n/3^n$ with $a_n \in \{0,1,2\}$ if and only if $x$ belongs to the set $\{1/3, 2/3, 1/9, 2/9, 7/9, 8/9, \ldots\}$.
- Any $x \in C$ has unique representation $\sum_{n=1}^{\infty} a_n/3^n$, where $a_n \in \{0,2\}$. (For instance, $1/3 = 0.1 = 0.0222\ldots$).

Let $\{0,2\}^{\mathbb{N}}$ denote the collection of sequences in 0 and 2. Define the map $f : \{0,2\}^{\mathbb{N}} \to C$ by $f(a_n) = \sum_{n=1}^{\infty} a_n/3^n$. By the above facts, $f$ is a bijection. The following shows that $C$ is not countable.

## Problem

*There exists a bijection between $\{0,2\}^{\mathbb{N}}$ and $P(\mathbb{N})$.*

## Hint.

For a subset $A$ of $\mathbb{N}$, define a sequence $\{a_n\}_{n\geqslant 1}$ by

$$a_n = \begin{cases} 2 & \text{if } n \in A, \\ 0 & \text{if } n \notin A. \end{cases}$$

# Metric spaces

## Definition

A metric $d$ on a nonempty set $X$ (refer as the metric space) is a map $d : X \times X \to [0, \infty)$ that obeys

- (Symmetry) For all $x, y \in X$, $d(x, y) = d(y, x)$
- (Triangle inequality) For all $x, y, z \in X$, $d(x, z) \leqslant d(x, y) + d(y, z)$
- (Zero property) For all $x, y \in X$, $d(x, y) = 0$ iff $x = y$.

**Interpretation**

- The distance of $x$ from $y$ is same as the distance of $y$ from $x$.
- If we consider the triangle $\triangle$ with vertices $x, y, z$ and lenghts of sides being $d(x, y), d(y, z)$ and $d(x, z)$, then the length of any side is at most the sum of lengths of remaining two sides.
- The distance of a point from itself is 0, and if the distance between two points is 0, then these points must coincide.

# Real line as a metric space

## Example

$\mathbb{R}$ is a metric space with metric $d(x, y) = |x - y|$ for $x, y \in \mathbb{R}$. The symmetry and zero property follow immediately from the facts that $|x| = |-x|$ and $|x| = 0$ iff $x = 0$. To see the triangle inequality, note that since $x \leqslant |x|$, we have
$|x - z|^2 = (x - y + y - z)^2 = (x - y)^2 + 2(x - y)(y - z) + (y - z)^2$
$\leqslant (|x - y| + |y - z|)^2$. Now take square roots on both sides.

**Remark**

- One may give an alternate proof of the triangle inequality which does not involve square root (consider the cases in which $x + y \geqslant 0$ and $x + y < 0$, and use the definition of $|\cdot|$).
- Note that $|x + y| = |x| + |y|$ iff either $x$ and $y$ are both nonnegative or both nonpositive (Exercise).

# Young's Inequality

### Theorem

*Let $p, q > 1$ be conjugate exponents, that is, $1/p + 1/q = 1$. For positive numbers $a, b$, $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$. Equality holds iff $a^p = b^q$.*

### Proof.

Given positive real numbers $a \leq b$, consider

$$D_1 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a,\ 0 \leq y \leq x^{p-1}\},$$

$$D_2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq b,\ 0 \leq x \leq y^{q-1}\}.$$

We verify the following (see [page 43, Carothers] for a figure):

1. $D_1 \cap D_2 = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq a\ , y = x^{p-1}\}$. Indeed, since $(p-1)(q-1) = 1$, $y = x^{p-1}$ iff $x = y^{q-1}$.
2. $R = \{(x, y) \in \mathbb{R} : 0 \leq x \leq a,\ 0 \leq y \leq b\} \subseteq D_1 \cup D_2$.

Thus $\text{Area}(R) = ab \leq \text{Area}(D_1 \cup D_2) \leqslant \text{Area}(D_1) + \text{Area}(D_2) = a^p/p + b^q/q$. Moreover, equality holds iff $a^p = b^q$. □

# Hölder's Inequality

## Corollary

*Let* $p, q > 1$ *be conjugate exponents, that is,* $\frac{1}{p} + \frac{1}{q} = 1$. *Let* $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d) \in \mathbb{R}^{\mathrm{d}} = \{(x_1, \ldots, x_d) : x_1, \ldots, x_d \in \mathbb{R}\}$. *Then*

$$\left|\sum_{n=1}^{d} a_n b_n\right| \leq \left(\sum_{n=1}^{d} |a_n|^p\right)^{1/p} \left(\sum_{n=1}^{d} |b_n|^q\right)^{1/q}.$$

## Proof.

Let $\|a\|_p := \left(\sum_{n=1}^{d} |a_n|^p\right)^{1/p}$ and $\tilde{a}_n = |a_n|/\|a\|_p$, $\tilde{b}_n = |b_n|/\|b\|_q$, $n = 1, \ldots, d$. By Young's Inequality, $|\tilde{a}_n||\tilde{b}_n| \leq |\tilde{a}_n|^p/p + |\tilde{b}_n|^q/q$.

$$\text{Thus} \quad \frac{1}{\|a\|_p \|b\|_q} \sum_{n=1}^{d} |a_n||b_n| = \sum_{n=1}^{k} |\tilde{a}_n||\tilde{b}_n|$$

$$\leq \sum_{n=1}^{d} |\tilde{a}_n|^p/p + \sum_{n=1}^{d} |\tilde{b}_n|^q/q = \frac{1}{p} + \frac{1}{q} = 1. \qquad \square$$

## Example

For a positive integer $d \geqslant 1$, $\mathbb{R}^{\mathrm{d}} = \{(x_1, \dots, x_d) : x_1, \dots, x_d \in \mathbb{R}\}$ is a metric space with metric

$$d(x,y) = \sqrt{\sum_{j=1}^{d} |x_j - y_j|^2}, \quad x, y \in \mathbb{R}^{\mathrm{d}}.$$

To see the triangle inequality, it suffices to check that

$$\|a+b\|_2 \leqslant \|a\|_2 + \|b\|_2, \quad a, b \in \mathbb{R}^{\mathrm{d}},$$

where $\|a\|_2 := \left(\sum_{n=1}^{d} |a_n|^2\right)^{1/2}$. To see this, note that by Hölder's Inequality (with $p = 2 = q$),

$$\|a+b\|_2^2 = \sum_{j=1}^{d} |a_j|^2 + 2\sum_{i,j=1}^{d} a_j b_j + \sum_{j=1}^{d} |b_j|^2$$

$$\leqslant \|a\|_2^2 + 2\|a\|_2\|b\|_2 + \|b\|_2^2 = (\|a\|_2 + \|b\|_2)^2.$$

## Problem

*For a positive integer $d \geqslant 1$ and $p \geqslant 1$, show that $\mathbb{R}^{\mathrm{d}}$ is a metric space with metric $d_p$ given by*

$$d_p(x, y) = \Big( \sum_{j=1}^{d} |x_j - y_j|^p \Big)^{1/p}, \quad x, y \in \mathbb{R}^{\mathrm{d}}.$$

*This result fails for $0 < p < 1$.*

## Hint.

To see that $\|a + b\|_p \leqslant \|a\|_p + \|b\|_p$, note that

$$\sum_{j=1}^{d} |a_j + b_j|^p \leqslant \sum_{j=1}^{d} |a_j||a_j + b_j|^{p-1} + \sum_{j=1}^{d} |b_j||a_j + b_j|^{p-1},$$

and apply Hölder's inequality (two times). To see the second part, let $d = 2$, $a = (1, 0)$, $b = (0, 1)$ and $p = 1/2$. □

## Problem

*For a positive integer $d \geqslant 1$, show that $\mathbb{R}^{\mathrm{d}}$ is a metric space with metric $d_\infty$ given by $d_\infty(x, y) = \max_{j=1}^{d} |x_j - y_j|$ for $x, y \in \mathbb{R}^{\mathrm{d}}$.*

## Problem

*For $p \geqslant 1$ and for a sequence $\{x_n\}_{n\geqslant 1}$ of real numbers, define $\|x\|_p = \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{1/p} \in [0, \infty]$. Verify the following:*

1. $\ell^p = \{\{x_n\}_{n\geqslant 1} : \|x\|_p < \infty\}$ *is a vector space over* $\mathbb{R}$.
2. $\ell^p$ *is a metric space with metric* $d_p$ *given by*

$$d_p(x, y) = \Big(\sum_{j=1}^{\infty} |x_j - y_j|^p\Big)^{1/p}, \quad x, y \in \ell^p.$$

## Proof.

*Both parts essentially follow from the triangle inequality. We already know the conclusion for truncated sequences $\{a_n\}_{k=1}^{N}$ and $\{b_n\}_{k=1}^{N}$ for every integer $N \geqslant 1$. Now let $N \to \infty$.* □

Let $a, b \in \mathbb{R}$ be such that $a < b$.

### Lemma

*If $f : [a, b] \to \mathbb{R}$ is continuous, then* $\sup_{x \in [a,b]} |f(x)| < \infty$.

### Proof.

Assume that $\sup_{x \in [a,b]} |f(x)| = \infty$. Then, for any integer $n \geqslant 1$, there exists $x_n \in [a, b]$ such that $|f(x_n)| > n$. By Bolzano-Weierstrass theorem, $\{x_n\}_{n \geqslant 1}$ admits a convergent subsequence $\{x_{n_k}\}_{k \geqslant 1}$, and hence by (sequential) continuity of $f$, $\{f(x_{n_k}\}_{k \geqslant 1}$ is convergent. However, a convergent sequence is bounded, and on the other hand, $|f(x_{n_k})| > n_k$. Not possible! $\square$

### Example

Consider the vector space $C[a, b]$ of real-valued continuous functions defined on $[a, b]$. Then $C[a, b]$ is a metric space with metric $d_\infty$ given by

$$d_\infty(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|. \quad f, g \in C[a, b].$$

# Metric topology

Let $X$ be a metric space with metric $d$ (denoted by $(X, d)$).

**Question** Given two metric spaces $(X, d_X)$ and $(Y, d_Y)$, when can we say that these are equivalent/isomorphic metric spaces ?

**Clue** We know that two sets $X$ and $Y$ are equivalent if there is a bijection between $X$ and $X'$. Similarly, two vector spaces $X$ and $Y$ are isomorphic if there is a bijective linear map between $X$ and $Y$.

**Guess** So, for two metric spaces to be equivalent, we need to impose some constraint on the given bijection that will take the metric structures into account (for instance, sequential continuity).

**Question** What do we mean by metric structures ?

**Clue** The metric $d$ is the distance function, which measures distance between two points. One may look for a metric analog of a disc/ball in the metric space.

# Metric balls

Let $(X, d)$ be a metric space. For $x_0 \in X$ and a real number $r > 0$, define the open ball and closed ball around $x_0$, respectively, by

$$B_r(x_0) = \{x \in X : d(x, x_0) < r\}, \quad \overline{B_r(x_0)} = \{x \in X : d(x, x_0) \leqslant r\}.$$

## Example (Discrete Geometry)

Let $X$ be a nonempty set. For every $x, y \in X$, define the <u>discrete metric</u> $d_0$ by $d_0(x, x) = 0$ and $d_0(x, y) = 1$ if $x \neq y$. Then

$$B_r(x_0) = \begin{cases} \{x_0\} & \text{if } r \leqslant 1, \\ X & \text{otherwise} \end{cases}, \quad \overline{B_r(x_0)} = \begin{cases} \{x_0\} & \text{if } r < 1, \\ X & \text{otherwise} \end{cases}.$$

The above calculations may be applied to $X = \mathbb{R}$.

## Example (One dimensional geometry)

Let $\mathbb{R}$ be the real line with $d_1(x, y) = |x - y|$, $x, y \in \mathbb{R}$. Then

$$B_r(x_0) = (x_0 - r, x_0 + r), \quad \overline{B_r(x_0)} = [x_0 - r, x_0 + r].$$

## Example (Plane geometry)

Let $\mathbb{R}^2$ be the real plane with $d_2(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2}$, $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. If $x_0 = (x_{01}, x_{02})$, then

$$B_r(x_0) = \{x \in \mathbb{R}^2 : |x_1 - x_{01}|^2 + |x_2 - x_{02}|^2 < r^2\} \text{ (circular region)},$$

$$\overline{B_r(x_0)} = \{x \in \mathbb{R}^2 : |x_1 - x_{01}|^2 + |x_2 - x_{02}|^2 \leqslant r^2\}.$$

What if $d_2$ is replaced by $d_\infty(x,y) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$ ?

$$B_r(x_0) = \{x \in \mathbb{R}^2 : |x_1 - x_{01}| < r, |x_2 - x_{02}| < r\} \text{ (square region)},$$

$$\overline{B_r(x_0)} = \{x \in \mathbb{R}^2 : |x_1 - x_{01}| \leqslant r, |x_2 - x_{02}| \leqslant r\}.$$

What if $d_2$ is replaced by $d_1(x,y) = |x_1 - y_1| + |x_2 - y_2|$ ? Why this metric is known as <u>taxi-cab metric</u> ?

**Notation** Although the notations $B_r(x_0)$, $\overline{B_r(x_0)}$ do not indicate dependence on the underlying metric, the dependence is evident.

## Problem

*Consider the metric space* $C[0,1]$ *with the sup metric* $d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|$, $f, g \in C[0,1]$. *Describe the open and closed balls* $B_r(f)$, $\overline{B_r(f)}$, *where* $f(x) = 0, x$ *and* $x^2$.

## Hint.

Note that

$$B_r(f) = \left\{ g \in C[0,1] : \sup_{x\in[0,1]} |g(x) - f(x)| < r \right\}$$

$$= \left\{ g \in C[0,1] : |g(x) - f(x)| < r \text{ for every } x \in [0,1] \right\}$$

$$= \{ g \in C[0,1] : g(x) \in (f(x) - r, f(x) + r) \text{ for every } x \in [0,1] \}.$$

Now draw the graphs of $f(x) - r$ and $f(x) + r$ in the above cases, and convince yourself that the "tube" enclosed between these two graphs is the open ball $B_r(f)$. Draw the diagram. $\square$

In the following problem, we need the fact that every continuous function on $[0, 1]$ is Riemann integrable.

### Problem

*Consider the metric space $C[0, 1]$ with the metric $d(f, g) = \int_{x \in [0,1]} |f(x) - g(x)| dx,\ f, g \in C[0, 1]$.*

1. *Describe the open ball $B_1(0)$.*
2. *Give an example of $f$ belonging to $B_1(0)$ and also an example belonging to $C[0, 1] \setminus B_1(0)$.*
3. *Verify that $B_r(f) = \{f + rg : g \in B_1(0)\}$.*
4. *Give an example of $g$ belonging to $B_r(f)$ and also an example belonging to $C[0, 1] \setminus B_r(f)$.*

### Hint.

Note that $f \in B_1(0)$ if and only if the area under the graph of $|f(x)|$ is less than 1. $\square$

## Problem

*Let $(X, d)$ be a metric space and let $A$ be a subset of $X$. Verify the following:*

1. *$A$ is a metric with metric $d_A$ given by $d_A(a, b) = d(a, b)$ for all $a, b \in A$.*
2. *Describe the open ball in the metric space $(A, d_A)$.*
3. *If $(X, d) = (\mathbb{R}, d_1)$ (resp. $(\mathbb{R}^2, d_2)$) and $A = \mathbb{Z}$ (resp. $\mathbb{Z} \times \mathbb{Z}$), then describe $d_A$ and the open ball in the metric space $(A, d_A)$.*

## Hint.

For the second part, verify that the open ball $B_r^A(a)$ centred at $a$ and of radius $r$ in the $d_A$ metric equals the intersection of $A$ with the open ball $B_r(a)$. For the last part, draw the lattice $\mathbb{Z} \times \mathbb{Z}$ and examine its intersection with open balls. □

# Hausdorff property

**Theorem**

*Let $(X, d)$ be a metric space and let $x, y$ be distinct points in $X$. Then there exist positive numbers $r, s$ such that $B_r(x) \cap B_s(y) = \emptyset$.*

**Proof.**

Since $x \neq y$, $d(x, y) > 0$. Let $r = d(x, y)/2 = s$. We must check that $B_r(x) \cap B_s(y) = \emptyset$, or equivalently,

$$B_r(x) \subseteq X \setminus B_s(y) = \{z \in X : d(z, y) \geqslant s\}.$$

If $z \in B_r(x)$, then by the symmetry and the triangle inequality,

$$d(z, y) \geqslant d(x, y) - d(x, z) > d(x, y) - r = s,$$

since $r + s = d(x, y)$. $\square$

There exist "topological spaces" without the Hausdorff property!

# Open and closed sets

### Definition

Let $(X, d)$ be a metric space and let $A$ be a subset of $X$. We say that $A$ is open in $X$ if either $A$ is an empty set or if for every $a \in A$, there exists $\epsilon > 0$ such that $B_\epsilon(a) \subseteq A$. We say that $A$ is closed if the complement of $A$ in $X$ is open.

- Both the empty set $\emptyset$ and $X$ are open, and hence closed.
- The open ball $B_r(x_0)$ is open. To see this, let $a \in B_r(x_0)$. Thus $d(a, x_0) = r - \delta$ for some $\delta > 0$. We must find an $\epsilon > 0$ such that $B_\epsilon(a) \subseteq A = B_r(x_0)$. For any $b \in B_\epsilon(a)$, by the triangle inequality,

$$d(b, x_0) \leqslant d(b, a) + d(a, x_0) \leqslant \epsilon + r - \delta,$$

which is less than $r$ provided $\epsilon < \delta$.

## Problem

*Show that the closed ball $\overline{B_r(x_0)}$ in a metric space $(X, d)$ is closed.*

## Hint.

Note that $X \setminus \overline{B_r(x_0)} = \{x \in X : d(x, x_0) > r\}$. Let $a \in X \setminus \overline{B_r(x_0)}$ and write $\delta = d(x, x_0) - r > 0$. Use the triangle inequality to find conditions on $\epsilon > 0$ (in terms of $\delta$), so that $B_\epsilon(a) \subseteq A = X \setminus \overline{B_r(x_0)}$. $\square$

The closed ball could be an open set in some metric spaces!

## Example (Discrete topology)

Consider $\mathbb{R}$ enodwed with the discrete metric $d_0$ given by $d_0(x, x) = 0$ and $d_0(x, y) = 1$ if $x \neq y \in \mathbb{R}$. Since single-tons are open balls ($\{x_0\} = B_1(x_0)$), any subset of $\mathbb{R}$ is open!

This can not happen in $\mathbb{R}^{\mathrm{d}}$ endowed with the metric $d_p$, $1 \leqslant p \leqslant \infty$ (to be seen later).

## Problem

*Answer the following:*

1. *Which of the sets are open/closed in* $\mathbb{R}$ *with metric* $d_1$ $x, y \in \mathbb{R}$ *?*
   1.1 $(0,1)$, $[0,1)$, $(0,1]$, $[0,1]$, $\mathbb{Z}$, $\mathbb{Q}$
2. *Which of the sets are open/closed in* $\mathbb{R}^2$ *with metric* $d_2$ *?*

   $(0,1)\times(0,1)$, $[0,1)\times[0,1)$, $(0,1]\times[0,1)$, $[0,1]\times[0,1]$, $\mathbb{Z}\times\mathbb{Z}$, $\mathbb{Q}\times\mathbb{Q}$, $\{(x,y)\in\mathbb{R}^2 : x^2+y^2=1\}$, $\{(x,y)\in\mathbb{R}^2 : xy\neq 0\}$

*Justify your answer.*

## Hint.

The answers in 1 are open, not open and not closed, not open and not closed, closed, closed, not open and not closed (in order).
The answers in 2 are open, not open and not closed, not open and not closed, closed, closed, not open and not closed, closed, open (in order). $\square$

Let $B[0,1]$ denote the metric space of bounded functions $f : [0,1] \to \mathbb{R}$ with the metric $d_\infty$ given by

$$d_\infty(f,g) = \sup_{x\in[0,1]} |f(x) - g(x)|, \quad f,g \in B[0,1].$$

## Problem

*Whether the subset $C[0,1]$ of $B[0,1]$ is open in $B[0,1]$ ? Justify your answer.*

## Hint.

No. To see this, given any $\epsilon > 0$, find $f \in B[0,1] \setminus C[0,1]$ such that $f \in B_\epsilon(0)$ ($\epsilon$-neighbourhood of the 0 function), that is, a bounded function $f$ discontinuous at least at a point in $[0,1]$ such that $\|f\|_\infty < \epsilon$. $\square$

## Problem

*Let $(X,d)$ be a metric space. For $a \in X$ and $r > 0$, show that $S_r(a) = \{x \in X : d(x,a) = r\}$ is closed in $X$.*

Let $(X, d)$ be a metric space and let $\mathcal{O}$ be the collection of all open subsets of $X$.

- $\mathcal{O}$ is nonempty: $\emptyset$ and $X$ belong to $\mathcal{O}$
- $\mathcal{O}$ is closed under arbitrary (finite or infinite) union: If $\{U_\alpha\} \subseteq \mathcal{O}$, then $\cup_\alpha U_\alpha$ is open (if $a \in \cup_\alpha U_\alpha$, then $a \in U_\alpha$ for some $\alpha$, and hence there is $r > 0$ such that $B_r(a) \subseteq U_\alpha \subseteq \cup_\alpha U_\alpha$)
- $\mathcal{O}$ need not be closed under countable intersection: Consider $X = \mathbb{R}$ with usual metric $d_1(x, y) = |x - y|$, $x, y \in \mathbb{R}$. Then for $U_n = (-1/n, 1/n) \in \mathcal{O}$, $\cap_{n \geqslant 1} U_n = \{0\} \notin \mathcal{O}$
- $\mathcal{O}$ is closed under finite intersection: If $U_1, \ldots, U_k \in \mathcal{O}$, then $\cap_{j=1}^k U_j \in \mathcal{O}$ (if $a \in \cap_{j=1}^k U_j$, then for some $r_j > 0$, $B_{r_j}(a) \subseteq U_j$, and hence $B_r(a) \subseteq \cap_{j=1}^k U_j$ with $r = \min\{r_1, \ldots, r_k\}$)

### Theorem

*For any metric space $(X, d)$, the collection $\mathcal{O}$ (known as topology of $X$) of open subsets of $X$ contains empty set and $X$. Further, $\mathcal{O}$ is closed under arbitrary union and finite intersection.*

# Open sets in $\mathbb{R}$

## Problem

*Let $U$ be an open set in $\mathbb{R}$ and let $x \in U$. Show that there exists largest open interval containing $x$ and contained in $U$.*

## Hint.

Let $a_x = -\infty$ if $\{a \in \mathbb{R} : a < x,\ (a, x) \subseteq U\}$ is not bounded below, and $b_x = \infty$ if $\{b \in \mathbb{R} : b > x,\ (x, b) \subseteq U\}$ is not bounded above. Otherwise, set

$$a_x = \operatorname{glb}\{a \in \mathbb{R} : a < x,\ (a, x) \subseteq U\},$$

$$b_x = \operatorname{lub}\{b \in \mathbb{R} : b > x,\ (x, b) \subseteq U\},$$

and consider the open interval $I_x = (a_x, b_x)$. Then $x \in I_x$ (otherwise either $x \leqslant a_x$ or $x \geqslant b_x$). We claim that the interval $I_x$ is contained in $U$. If $I_x \not\subseteq U$, then there is $y \in I_x \setminus U$, and hence either $(x, y)$ or $(y, x)$ intersects the complement of $U$, and hence $b_x \leqslant y$ or $a_x \geqslant y$. That's the contradiction since $y \in I_x$. $\square$

### Theorem

*Any open set $U$ in $\mathbb{R}$ can be expressed as the disjoint union of countably many open intervals.*

### Proof.

Let $I_x$ denote the largest open interval containing $x$ and contained in $U$. Then $U = \cup_{x \in U} I_x$. If $x, y \in U$, then either $I_x \cap I_y = \emptyset$ or $I_x = I_y$. Since rationals are countable and distinct intervals contains distinct rationals, the collection $\{I_x\}_{x \in U}$ contains at most countably many disjoint intervals whose union is $U$. □

The role of an open interval in $\mathbb{R}$ may be replaced by open rectangles $(a, b) \times (c, d)$ in $\mathbb{R}^2$.

**Question** Is it possible to express an open subset of $\mathbb{R}^2$ as disjoint union of countably many open rectangles ?

# Open sets in $\mathbb{R}^2$

## Example

Consider the open unit disc in $\mathbb{R}^2$:

$$\mathbb{D} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

We claim that $\mathbb{D}$ can not be written as disjoint union of countably many open rectangles. Indeed, if $\mathbb{D} = \cup_{n \geqslant 1} R_n$ is union of open rectangles $R_n = (a_n, b_n) \times (c_n, d_n) \subseteq \mathbb{D}$, then at least some point of the form $(a_n, y)$, $c_n < y < d_n$ will lie in $\mathbb{D}$, and hence $(a_n, y)$ must lie in some open rectangle $R_m$, $m \neq n$. In that case, $R_n \cap R_m \neq \emptyset$.

• Major difference between $\mathbb{R}$ and $\mathbb{R}^2$ is the "natural" order structure (and LUB property).

# Interior of a set

Let $(X, d)$ be a metric space.

• Given the closed ball $\overline{B_r(a)} = \{x \in X : d(x, a) \leqslant r\}$ in $X$, we can distinguish the points lying in $B_r(a) = \{x \in X : d(x, a) < r\}$ from the remaining points $\{x \in X : d(x, a) = r\}$.

**Question** Can we do the same for an arbitrary set ?

## Definition

Let $A$ be a subset of $X$ and let $a \in A$. We say that $a$ is an <u>interior point</u> of $A$ if there exists $r > 0$ such that $B_r(a) \subseteq A$.

Let $A^\circ = \{a \in A : a \text{ is an interior point of } A\}$.

• Note that $A$ is open in $X$ if and only if $A^\circ = A$.

Let $(X, d)$ be a metric space and let $a \in X$. We know that any point in $B_r(a)$ is an interior point of

$$\overline{B_r(a)} = \{x \in X : d(x, a) \leqslant r\}.$$

Example

In general, the interior of $\overline{B_r(a)}$ need not be $B_r(a)$. Indeed, if $x \in \overline{B_r(a)} \setminus B_r(a)$, then $x$ could be an interior point of $\overline{B_r(a)}$. To see this, let $X = \mathbb{R}$ with discrete metric $d_0$ and $a = 0$, $r = 1$. Then $B_1(0) = \{0\}$ and the interior of $\overline{B_1(0)} = \mathbb{R}$ is $\mathbb{R}$.

This does not happen in $X = \mathbb{R}^{\mathrm{d}}$ endowed with any metric $d_p$, $p \geqslant 1$, as the following example illustrates.

## Example

In $(\mathbb{R}^{\mathrm{d}}, d_p)$ the interior of $\overline{B_r(a)}$ is $B_r(a)$. We have seen that the $B_r(a)$ is contained in the interior of $\overline{B_r(a)}$. Let $b \in \overline{B_r(a)}$ be such that $d_p(b,a) = r$. We claim that for any $s > 0$, the ball $B_s(b)$ intersects $\mathbb{R}^{\mathrm{d}} \setminus \overline{B_r(a)}$.

The line joining $a$ and $b$ is given by $a + t(b-a)$, $t \geqslant 0$. Note that

$$d_p(a + t(b-a), b) = \Big( \sum_{j=1}^{d} |1-t|^p |a_j - b_j|^p \Big)^{1/p},$$

which can be made less than $s$ if $|1-t|$ is very small. Thus $a + t(b-a)$ belongs to $B_s(b)$ if $|1-t|$ is very small. Also, if $t > 1$,

$$d_p(a + t(b-a), a) = \Big( \sum_{j=1}^{d} t^p |a_j - b_j|^p \Big)^{p} > d_p(b,a) = r.$$

# A proper subset of $\mathbb{Q}$ which is open as well as closed

## Example

Consider the metric space $(\mathbb{Q}, d_1)$, where $d_1(x, y) = |x - y|$, $x, y \in \mathbb{Q}$. Let $\zeta \in \mathbb{R} \setminus \mathbb{Q}$ (e.g. $\zeta = \sqrt{2}$) and let $A = \{x \in \mathbb{Q} : x < \zeta\}$. We claim that $A$ is open and closed in $\mathbb{Q}$.

- $A$ is open since $A = (-\infty, \zeta) \cap \mathbb{Q}$ and $(-\infty, \zeta)$ is open in $\mathbb{R}$
- $A$ is closed since $\mathbb{Q} \setminus A = \{x \in \mathbb{Q} : x \geqslant \zeta\} = \{x \in \mathbb{Q} : x > \zeta\}$, which is again open since it is equal to $\mathbb{Q} \cap (\zeta, \infty)$

The phenomenon above can not occur in $\mathbb{R}$ ! This means that there are no proper open and closed subsets of $\mathbb{R}$ (to be seen later).

## Problem

*Find a proper subset of* $\mathbb{Q} \times \mathbb{Q}$ *with the metric* $d_2(x, y) = \|x - y\|_2$ $(x, y \in \mathbb{Q} \times \mathbb{Q})$, *which is open and closed.*

# Limit point of a set

Let $(X, d)$ be a metric space.

Definition

Let $A \subseteq X$ and let $a \in X$. We say that $a$ is a

- limit point[5] of $A$ if for every $r > 0$, $B_r(a) \cap A \neq \emptyset$.
- cluster point of $A$ if for every $r > 0$, $B_r(a) \cap (A \setminus \{a\}) \neq \emptyset$.

• A limit point need not belong to $A$ (0 is a limit point of $(0, 1)$)

• $a$ is not a limit point iff there exists some $r > 0$ such that $B_r(a) \cap A = \emptyset$ (2 is not a limit point of $(0, 1)$ for $B_1(2) \cap (0, 1) = \emptyset$)

• An interior point is a limit point (this follows from definition)

• A cluster point of $A$ is a limit point of $A$, but not conversely (e.g. $(\mathbb{R}, d_0)$, 0 is a limit point of $A = \{0\}$ but not a cluster point)

[5] Some authors define limit point as the point for which $B_r(a) \cap (A \setminus \{a\}) \neq \emptyset$

## Example

Let $X = \mathbb{R}$ and let $d_1(x, y) = |x - y|$, $x, y \in \mathbb{R}$. Let $\overline{A}$ denote the set of limit points and let $\mathrm{cl}(A)$ be the set of cluster points of $A$:

- If $A = [a, b)$, then $\overline{A} = [a, b] = \mathrm{cl}(A)$
- If $A = (a, b] \cup \{a + b\}$, then $\overline{A} = [a, b] \cup \{a + b\}$, $\mathrm{cl}(A) = [a, b]$
- If $A = \{1/n : n \in \mathbb{N}\}$, then $\overline{A} = A \cup \{0\}$, $\mathrm{cl}(A) = \{0\}$ (apply the Archimedean property)
- If $A = \mathbb{Z}$, then $\overline{A} = \mathbb{Z}$, $\mathrm{cl}(A) = \emptyset$

## Problem

*Let $(X, d) = (\mathbb{R}, d_1)$. Find all limit points and cluster points of $\mathbb{Q}$.*

## Hint.

Since rationals are dense in $\mathbb{R}$, any open interval contains infinitely many rationals. So every real number is a cluster point of $\mathbb{R}$. $\square$

# Closure of a set

**Theorem**

*Let $(X, d)$ be a metric space and let $A \subseteq X$. Then the set $\overline{A}$ of limit points of $A$ is closed in $X$.*

**Proof.**

We must check that $X \setminus \overline{A}$ is open. To see this, let $x \in X \setminus \overline{A}$. Thus, there exists $r > 0$ such that $B_r(x) \cap A = \emptyset$ or $B_r(x) \subseteq X \setminus A$.

We claim that $B_r(x) \subseteq X \setminus \overline{A}$. Otherwise, there exists $y \in B_r(x) \cap \overline{A}$, and hence $B_s(y) \cap A \neq \emptyset$ for every $s > 0$. However, for small $s > 0$, $B_s(y) \subseteq B_r(x)$ (since $B_r(x)$ is open), and hence $B_r(x)$ intersects with $A$, which is a contradiction.

Thus $B_r(x) \subseteq X \setminus \overline{A}$ and hence $X \setminus \overline{A}$ is open or $\overline{A}$ is closed. $\square$

- $\overline{A}$ is known as the <u>closure</u> of $A$.

## Problem

*Let $(X, d)$ be a metric space and let $A \subseteq X$. Then $\overline{A}$ is the smallest closed set containing $A$, that is, if $L \subseteq X$ is a closed set such that $A \subseteq L \subseteq \overline{A}$, then $L = \overline{A}$. Conclude that $A$ is closed iff $\overline{A} = A$.*

## Solution.

Let $x \in X \setminus L$. Since $L$ is closed, there exists $r > 0$ such that $B_r(x) \cap X \setminus L \subseteq X \setminus A$, or $B_r(x) \cap A = \emptyset$, so $x \in X \setminus \overline{A}$. Thus $X \setminus L \subseteq X \setminus \overline{A}$ or $\overline{A} \subseteq L$, and hence $L = \overline{A}$. □

## Problem

*Let $(X, d)$ be a metric space and let $A \subseteq X$. Then $A^\circ$ is the largest open set contained in $A$, that is, if $O \subseteq X$ is an open set such that $A^\circ \subseteq O \subseteq A$, then $O = A^\circ$.*

## Hint.

Show that $O$ is contained in $A^\circ$. □

- For any set $A$ of $X$, $A^\circ \subseteq A \subseteq \overline{A}$ (strict inclusion may hold).

# Convergence in metric spaces

By a sequence $\{x_n\}_{n\geqslant 1}$ in a metric space $X$, we understand a function from $\mathbb{N}$ into $X$, which maps $n$ to $x_n$.

Definition

Let $(X, d)$ be a metric space and let $\{x_n\}_{n\geqslant 1}$ be a sequence in $X$ We say that $\{x_n\}_{n\geqslant 1}$ is a <u>convergent sequence</u> in $X$ if there exists $x \in X$ (<u>limit</u> of $\{x_n\}_{n\geqslant 1}$) such that

$$d(x_n, x) \to 0 \text{ as } n \to \infty.$$

**Remark**

• The limit $x$ of a convergent sequence is unique: If $d(x_n, x) \to 0$ and $d(x_n, y) \to 0$, then $0 \leqslant d(x, y) \leqslant d(x, x_n) + d(x_n, y)$ $= d(x_n, x) + d(x_n, y) \to 0$ as $n \to \infty$.

### Theorem

*Let $(X, d)$ be a metric space, let $A$ be a subset of $X$ and let $x \in X$. Then $x$ is a limit point of $A$ if and only if there exists a sequence $\{a_n\}_{n\geqslant 1} \subseteq A$ such that $d(a_n, x) \to 0$ as $n \to \infty$.*

### Proof.

Let $x$ be a limit point of $A$.

- If $x \in A$, then take the constant sequence $\{a_n = x\}_{n\geqslant 1}$.

- Suppose that $x \notin A$. Then $x$ is a cluster point. Thus for every $r > 0$, $B_r(x) \cap (A \backslash \{x\}) \neq \emptyset$. By induction, after letting $r = 1/n$, we obtain $a_n \in B_{1/n}(x) \cap (A \backslash \{x\})$, and since $d(a_n, x) < 1/n$, we have $d(a_n, x) \to 0$ as $n \to \infty$.

Conversely, assume that there exists a sequence $\{a_n\}_{n\geqslant 1} \subseteq A$ such that $d(a_n, x) \to 0$ as $n \to \infty$. Thus for every $r > 0$, there exists $N \geqslant 1$ such that $d(a_n, x) < r$ for every $n \geqslant N$. Thus $a_N \in B_r(x) \cap A$, and hence $x$ is a limit point of $A$. $\square$

# Cauchy sequences and complete metric spaces

### Definition

Let $(X, d)$ be a metric space and let $\{x_n\}_{n\geqslant 1}$ be a sequence in $X$. We say that $\{x_n\}_{n\geqslant 1}$ is a <u>Cauchy sequence</u> in $X$ if $d(x_m, x_n) \to 0$ as $m, n \to \infty$. A metric space is said to be <u>complete</u> if every Cauchy sequence is convergent.

**Remark**

- Every convergent sequence is Cauchy: $0 \leqslant d(x_m, x_n)$ $\leqslant d(x_m, x) + d(x, x_n) = d(x_m, x) + d(x_n, x) \to 0$ as $m, n \to \infty$

### Example

The metric space $(\mathbb{R}, d_1)$ is complete.

### Problem

*Which of the following are complete metric spaces:* $([a, b], d_1)$, $((a, b), d_1)$, $([a, b), d_1)$, *where* $d_1$ *is the absolute metric.*

### Theorem

*Let $(X, d)$ be a complete metric space. Then every closed subset $A$ of $X$ is complete when endowed with the (relative) metric*

$$d^A(x, y) = d(x, y), \quad x, y \in A.$$

### Proof.

Let $\{a_n\}_{n \geqslant 1}$ be a Cauchy sequence in $(A, d^A)$. Clearly, $\{a_n\}_{n \geqslant 1}$ is a Cauchy sequence in $(X, d)$. However, $X$ is complete, so that $\{a_n\}_{n \geqslant 1}$ converges to some $a \in X$. It follows that $a$ is a limit point of $A$. Since $A$ is closed, $a \in A$, and it follows that $\{a_n\}_{n \geqslant 1}$ is convergent in $A$. $\square$

### Corollary

*Any closed subset of $(\mathbb{R}, d_1)$ is complete*

## Example

Consider the metric space $(\mathbb{Q}, d_1)$, where $d_1(x, y) = |x - y|$, $x, y \in \mathbb{Q}$. Let $x_1 \in \mathbb{Q}$ such that $\sqrt{2} - 1 < x_1 < \sqrt{2}$ (by density of rationals). Next choose $x_2 \in \mathbb{Q}$ such that $\sqrt{2} - 1/2 < x_2 < \sqrt{2}$, and hence by induction, for every positive integer $n$, there exists $x_n \in \mathbb{Q}$ such that $\sqrt{2} - 1/n < x_n < \sqrt{2}$. Thus, as $m, n \to \infty$,

$$d_1(x_m, x_n) = |x_m - x_n| \leqslant |x_m - \sqrt{2}| + |\sqrt{2} - x_n| \leqslant 1/m + 1/n \to 0.$$

However, $\{x_n\}_{n \geqslant 1}$ is not convergent in $\mathbb{Q}$ (since the limit being unique is necessarily $\sqrt{2}$ and $\sqrt{2} \notin \mathbb{Q}$).

- $(\mathbb{Q}, d_1)$ is not complete.

## Problem

*Show that* $(\mathbb{R} \setminus \mathbb{Q}, d_1)$ *is not complete.*

### Problem

*What are the Cauchy sequences in the discrete metric space* $(X, d_0)$ *?*

### Solution.

Let $\{x_n\}_{n \geqslant 1}$ be a Cauchy sequence in $(X, d_0)$. Thus for $\epsilon = 1$, there exists $N \geqslant 1$ such that

$$d_0(x_n, x_m) < 1 \text{ for every } m, n \geqslant N.$$

However, for the metric $d_0$, the distance between two points is either 0 or 1. It follows that $x_n = x_m$ for every $m, n \geqslant N$. $\square$

### Problem

*Which of the following are complete metric spaces:* $([a, b], d_0)$, $((a, b), d_0)$, $([a, b), d_0)$, *where* $d_0$ *is the discrete metric.*

### Theorem

*Let $(X, d)$ be a metric space. Then a Cauchy sequence $\{x_n\}_{n\geqslant 1}$ is convergent if and only if it has a convegent subsequence.*

### Proof.

If $\{x_n\}_{n\geqslant 1}$ is convergent, then clearly any subsequence of $\{x_n\}_{n\geqslant 1}$ is convergent (follows from the definition).

Let $\{x_{n_k}\}_{k\geqslant 1}$ be a convergent subsequence of $\{x_n\}_{n\geqslant 1}$ and let $x \in X$ be its limit. Thus given $\epsilon > 0$, there exists $N \geqslant 1$ such that $d(x_{n_k}, x) < \epsilon/2$ for every $k \geqslant N$. Also, for some $N' \geqslant N$, $d(x_n, x_m) \leqslant \epsilon/2$ for all $m, n \geqslant N'$. Thus for any $n \geqslant N'$,

$$d(x_n, x) \leqslant d(x_n, x_{n_{N'}}) + d(x_{n_{N'}}, x) < \epsilon$$

(since $n_{N'} \geqslant N' \geqslant N$). □

# Bounded sets and diameter

Let $(X, d)$ be a metric space and let $\emptyset \neq A \subseteq X$. We say that $A$ <u>bounded</u> if there exists $r > 0$ and $x \in X$ such that $A \subseteq B_r(x)$.

- Every Cauchy sequence $\{x_n\}_{n \geqslant 1}$ is bounded:

  Indeed, for $\epsilon = 1$, there exists $N \geqslant 1$ such that $d(x_n, x_N) < 1$ for all $n \geqslant N$. Thus $\{x_n\}_{n \geqslant 1} \subseteq B_r(x)$, where $x = x_N$ and $r = \max\{1, d(x_1, x_N), \ldots, d(x_{N-1}, x_N)\} + 1$.

Consider $S = \{d(x, y) : x, y \in A\}$. The <u>diameter</u> of $A$ is defined as

$$\operatorname{diam}(A) = \begin{cases} \operatorname{lub} S & \text{if } S \text{ is bounded above,} \\ \infty & \text{otherwise.} \end{cases}$$

**Problem**

*Let $(X, d)$ be a metric space and let $\emptyset \neq A \subseteq X$. Show that $A$ is bounded if and only if $\operatorname{diam}(A) < \infty$.*

**Hint.**

$\operatorname{diam}(B_r(x)) = 2r$, and if $A \subseteq B$, then $\operatorname{diam}(A) \leqslant \operatorname{diam}(B)$. $\square$

# Normed linear spaces

### Definition

A normed linear space $X$ (denoted by $(X, \|\cdot\|)$) is a vector space over $\mathbb{R}$ with a function assigning $\|x\| \in \mathbb{R}$ to every $x \in X$ such that for every $x, y, z \in X$ and $\alpha \in \mathbb{R}$,

- N1 (Non-negativity) $\|x\| \geq 0$.
- N2 (Positive Definiteness) $\|x\| = 0$ if and only if $x = 0$.
- N3 (Dilation) $\|\alpha x\| = |\alpha|\|x\|$.
- N4 (Triangle Inequality) $\|x + y\| \leq \|x\| + \|y\|$.

**Remark** Every normed linear space is a metric space with metric $d(x, y) = \|x - y\|$, $x, y \in X$. This metric has the additional property that $d(\alpha x, 0) = |\alpha| d(x, 0)$ for every $\alpha \in \mathbb{R}$ and $x \in X$.

## Example

We contend that $\mathbb{R}^{\mathrm{d}}$ with the norm $\|\cdot\|_p$ is complete, where

$$\|x\|_p = \Big(\sum_{j=1}^{d} |x_j|^p\Big)^{1/p}, \quad x \in \mathbb{R}^{\mathrm{d}}.$$

To see this, let $\{x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)})\}_{n\geqslant 1}$ be a Cauchy sequence in $\mathbb{R}^{\mathrm{d}}$. Thus for $k = 1, \ldots, d$,

$$|x_k^{(m)} - x_k^{(n)}| \leqslant \Big(\sum_{j=1}^{d} |x_j^{(m)} - x_j^{(n)}|^p\Big)^{1/p} = \|x^{(m)} - x^{(n)}\|_p \to 0,$$

that is, $\{x_k^{(n)}\}_{n\geqslant 1}$ is a Cauchy sequence in $\mathbb{R}$. Since $(\mathbb{R}, d_1)$ is complete, there exists $x_k \in \mathbb{R}$ such that $|x_k^{(n)} - x_k| \to 0$ as $n \to \infty$. Clearly, $\|x^{(n)} - x\|_p \to 0$ as $n \to \infty$, where $x = (x_1, \ldots, x_d) \in \mathbb{R}^{\mathrm{d}}$.

## Problem

*Show that $\mathbb{R}^{\mathrm{d}}$ with the norm $\|\cdot\|_\infty$ is complete.*

## Problem

*Let* $\{x^{(n)} = \{(x_1^{(n)}, x_2^{(n)}, \cdots)\}_{n \geqslant 1}$ *be a Cauchy sequence in* $l^p$, $1 \leqslant p < \infty$. *For* $\epsilon > 0$, *verify the following:*

1. $\{x_i^{(n)}\}_{n \geqslant 1} \subseteq \mathbb{R}$ *converges to some* $x_i \in \mathbb{R}$ *for every* $i \geqslant 1$.
2. *For* $k \geq 1$, *there exists* $n_0 \geq 1$ (*independent of* $k$) *such that*
$$\sum_{i=1}^{k} |x_i^{(n)} - x_i|^p \leq \epsilon \text{ for all } n \geq n_0.$$
3. *For* $k \geq 1$, $\sum_{i=1}^{k} |x_i|^p \leq (\epsilon + \|x^{(n_0)}\|_p)^p$.
4. *The normed linear space* $l^p$ *is complete.*

## Hint.

For part 1, argue as in the last example. For part 2, use the definition of Cauchy sequence and part 1. For part 3, use the triangle inequality. By parts 2-3, $x \in l^p$ and $\|x^{(n)} - x\|_p \to 0$. $\square$

## Problem

*Let $p$ be such that $p \geqslant 1$. For $f \in C[0,1]$, define*

$$\|f\|_p = (\int_0^1 |f(x)|^p dx)^{1/p}.$$

*Show that $(C[0,1], \|\cdot\|_p)$ is an incomplete normed linear space.*

## Hint.

To see triangle inequality, apply Hölder's equality (see the argument on Page 35). To see that $C[0,1]$ is not complete, let

$$f(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2), \\ 0 & \text{if } x \in [1/2, 1], \end{cases}$$

$$f_n(x) = \begin{cases} 1 & \text{if } x \in [0, 1/2 - 1/n), \\ (-n/2)x + (n/2+1)/2 & \text{if } x \in [1/2 - 1/n, 1/2 + 1/n], \\ 0 & \text{if } x \in [1/2 + 1/n, 1]. \end{cases}$$

Verify that $\{f_n\}_{n \geq 3}$ converges to $f \notin C[0,1]$. □

# Dense sets and separable metric spaces

### Definition

Let $(X, d)$ be a metric space and let $A \subseteq X$. We say that $A$ is dense in $X$ if every open ball in $X$ intersects $A$, that is, for every $x \in X$ and every $r > 0$, $B_r(x) \cap A \neq \emptyset$.

**Remark** $A$ is dense in $X$ if and only if every $x \in X$ is a limit point of $A$, or equivalently, if and only if $\overline{A} = X$.

### Example

- $\mathbb{Q}$ and $\mathbb{R} \setminus \mathbb{Q}$ are dense in $(\mathbb{R}, d_1)$
- $\mathbb{Z}$ is not dense in $(\mathbb{Q}, d_1)$
- $\mathbb{Q}^{\mathrm{d}}$ is a countable dense in $(\mathbb{R}^{\mathrm{d}}, d_p)$

A metric space is <u>separable</u> if it has a countable dense subset.

## Theorem

*Let $1 \leq p < \infty$. Then $l^p$ is separable.*

## Proof.

For $k \geqslant 1$, let $A_k = \{(r_1, \ldots, r_k, 0, \ldots, ) : r_k \in \mathbb{Q}\}$.

- $A_k$ is countable: There is a bijection between $A_k$ and $\mathbb{Q}^k$
- $A = \cup_{k=1}^{\infty} A_k$ is countable: Countable union of countable sets is countable

We claim that $A$ is dense in $l^p$ : Let $r > 0$, $x = (x_1, x_2, \ldots, ) \in l^p$. Then $\sum_{k=1}^{\infty} |x_k|^p < \infty$, and hence for some integer $N \geqslant 2$, $(\sum_{k=N}^{\infty} |x_k|^p)^{1/p} < r/2$. For $j = 1, \ldots, N-1$, let $r_j \in \mathbb{Q}$ be such that $|x_j - r_j| < \frac{r}{2(N-1)^{1/p}}$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$). Then for $y = (r_1, \ldots, r_{N-1}, 0, \ldots, ) \in A$,

$$\|x - y\|_p \leqslant (\sum_{j=1}^{N-1} |x_j - r_j|^p)^{1/p} + (\sum_{k=N}^{\infty} |x_k|^p)^{1/p} < r/2 + r/2 = r.$$

Thus for every $x \in l^p$ and $r > 0$, $y \in B_r(x) \cap A$. $\square$

## Problem

*Show that the set $l^\infty$ of bounded sequences is complete normed linear space with norm $\|x\|_\infty = \sup_{j \geqslant 1} |x_j|$, $x \in l^\infty$.*

## Example

We claim that $l^\infty$ is not separable. Consider the uncountable set

$$\{0,1\}^{\mathbb{N}} = \{x \in l^\infty : x_n = 0 \text{ or } 1\}.$$

If $x, y \in \{0,1\}^{\mathbb{N}}$, then $\|x - y\|_\infty = 1$ if $x \neq y$. Thus

$$\bigcup_{x \in \{0,1\}^{\mathbb{N}}} B_{1/2}(x) \subseteq l^\infty \text{ (disjoint union).}$$

Now if $l^\infty$ has a countable dense set, then each $B_{1/2}(x)$ would contain at least one element from this set. However, there are uncountably disjoint balls of the form $B_{1/2}(x)$. and hence $l^\infty$ can not admit a countable dense subset.

**Question** Whether $(C[0,1], d_\infty)$ is separable ?

# Boundary of a set

### Definition

Let $(X, d)$ be a metric space and $A \subseteq X$. A point $x \in X$ is said to be a <u>boundary point</u> of A in X if for every $r > 0$,

$$B_r(x) \cap A \neq \emptyset \text{ and } B_r(x) \cap (X \setminus A) \neq \emptyset.$$

The <u>boundary</u> of $A$ in $X$ (denoted by $\partial A$) is the set of boundary points of $A$ in $X$.

**Remark** Every boundary point is a limit point, that is, $\partial A \subseteq \overline{A}$.

### Example

- If $(X, d) = (\mathbb{R}, d_1)$, then for $a, b \in \mathbb{R}$ such that $a < b$,

$$\partial[a, b] = \partial(a, b) = \partial[a, b) = \partial(a, b] = \{a, b\}.$$

- If $(X, d) = (\mathbb{R}^{\mathrm{d}}, d_p)$, then $\partial \overline{B_r(x)} = \{y \in \mathbb{R}^{\mathrm{d}} : d_p(x, y) = r\}$.

### Theorem

*Let $(X, d)$ be a metric space and $\emptyset \neq A \subseteq X$. Then $\partial A = \overline{A} \setminus A^\circ$.*

### Proof.

Clearly, $x \in \overline{A} \setminus A^\circ$ if and only if $x \in \overline{A}$ and $x \notin A^\circ$. Note that

- $x \in \overline{A}$ if and only if for every $r > 0$, $B_r(x) \cap A \neq \emptyset$.
- $x \notin A^\circ$ if and only if for every $r > 0$, $B_r(x) \cap (X \setminus A) \neq \emptyset$.

Thus $x \in \overline{A} \setminus A^\circ$ if and only if for every $r > 0$, $B_r(x) \cap A \neq \emptyset$ and $B_r(x) \cap (X \setminus A) \neq \emptyset$, that is, $x \in \partial A$. $\square$

### Problem

*Find the boundary of $A = \{(x, y) \in \mathbb{R}^2 : x > 0,\ y > 0\}$.*

### Hint.

Since $A$ is open, $A^\circ = A$. Also, $\overline{A} = \{(x, y) \in \mathbb{R}^2 : x \geqslant 0,\ y \geqslant 0\}$. It follows that $\partial A = \{(x, y) \in \overline{A} : xy = 0\}$. $\square$

- The boundary of $\mathbb{Q}$ is equal to $\mathbb{R}$.

### Definition

Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Let $f : X \to Y$ be a function and let $a \in X$. We say that $f$ is

C1 <u>continuous at $a$</u> if for every $\epsilon > 0$, there exists $\delta > 0$ such that $\left(x \in X,\ d(x, a) < \delta \Longrightarrow \rho(f(x), f(a)) < \epsilon\right)$.

C2 <u>s-continuous at $a$</u> (or <u>sequential continuous at $a$</u>) if $\left(d(x_n, a) \to 0 \text{ as } n \to \infty \Longrightarrow \rho(f(x_n), f(a)) \to 0 \text{ as } n \to \infty\right)$.

We say that $f$ is <u>continuous</u> (resp <u>s-continuous</u>) if it is continuous (resp $s$-continuous) at every $a \in X$.

**Remark**

- $f$ is continuous at $a$ if and only if for every $\epsilon > 0$, there exists $\delta > 0$ such that $f(B_\delta^d(a)) \subseteq B_\epsilon^\rho(f(a))$.
- $f$ is s-continuous at $a$ if and only if $f$ maps convergent sequences to convergent sequences.

## Problem

*Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Let $f : X \to Y$ be a function and let $a \in X$. Then $f$ is continuous at $a$ if and only if $f$ is s-continuous at $a$.*

## Hint.

$\Rightarrow$ follows from the definition: Given $\epsilon > 0$, there exists $\delta > 0$ such that C1 holds. Now find an integer $N \geqslant 1$ such that $d(x_n, a) < \delta$, so by continuity, $\rho(f(x_n), f(a)) < \epsilon$.

For $\Leftarrow$, argue by contradiction: Suppose that $C2$ holds but $C1$ fails to hold. So, for some $\epsilon > 0$, for $\delta = 1/n$, $n \geqslant 1$, find $x_n \in X$ such that $d(x_n, a) < 1/n$ but $\rho(f(x_n), f(a)) \geqslant \epsilon$. Then $d(x_n, a) \to 0$ but $\rho(f(x_n), f(a)) \not\to 0$, a contradiction. $\square$

## Problem

*Let $(X, d)$, $Y, \rho)$ and $(Z, \eta)$ be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then so is $g \circ f : X \to Z$.*

## Example

Let $p : \mathbb{R} \to \mathbb{R}$ be a polynomial, that is, for some $a_0, \ldots, a_k \in \mathbb{R}$,

$$p(x) = a_0 + a_1 x + \cdots + a_k x^k, \quad x \in \mathbb{R}.$$

Notice that its not easy to verify the continuity of $p$ right from its definition (given $\epsilon$, how to find $\delta$ ?). However, $s$-continuity of $p$ follows immediately from the facts that sum and product of convergent sequences is convergent:

$$x_n \to x, y_n \to y \Rightarrow x_n + y_n \to x + y,\ x_n^m \to x^m,\ m \geqslant 1.$$

Thus if $x_n \to x$, then $p(x_n) \to p(x)$ as $n \to \infty$. So $p$ is continuous.

## Theorem

*Let $(X, d)$ be a metric space. Then the set $C(X)$ of continuous functions $f : X \to \mathbb{R}$ forms an algebra, that is, $C(X)$ is a vector space over $\mathbb{R}$ equipped with the binary operation $(f, g) \to fg$ from $C(X) \times C(X)$ to $C(X)$, which is a binary form.*

## Proof.

The set of convergent sequences in $\mathbb{R}$ forms an algebra. □

Let $(X, d)$ be a metric space. Let $A$ be a non-empty subset of $X$, and for $x \in X$, let

$$d(x, A) = \operatorname{glb} S_x, \text{ where } S_x = \{d(x, a) : a \in A\}$$

(since $d(x, a) \geqslant 0$, $S_x$ is bounded from below. Also, since $A$ is nonempty, so is $S_x$. Hence glb of $S$ exists).

**Problem**

*Show that $f : X \to [0, \infty)$ given by $f(x) = d(x, A)$ is continuous.*

**Solution.**

To see this, for $x, y \in X$ and $a \in A$, note that

$$f(x) = \operatorname{glb} S_x \leqslant d(x, a) \leqslant d(x, y) + d(y, a)$$

$$f(x) - d(x, y) \leqslant d(y, a) \text{ for every } a \in A \Longrightarrow$$

$f(x) - d(x, y) \leqslant \operatorname{glb} S_y = f(y)$ or $f(x) - f(y) \leqslant d(x, y)$. Changing roles of $x$ and $y$, we obtain $|f(x) - f(y)| \leqslant d(x, y)$. Let $\delta = \epsilon$. $\square$

# Separation by a continuous function

### Theorem (Urysohn's Lemma)

*Let $(X, d)$ be a metric space. Given closed non-empty disjoint subsets $A$ and $B$ of $X$, there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.*

### Proof.

For $C \subseteq X$ closed and $x \in X$, let $d(x, C) = \inf\{d(x, a) : a \in C\}$.

- $d(x, C)$ is a continuous function of $x$.
- $d(x, C) = 0$ if and only if $x \in C$.
- $d(x, A) + d(x, B) > 0$ for every $x \in X$.

Define $f : X \to [0,1]$ by

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}, \quad x \in X.$$

Clearly, $f$ is a continuous function. Note that $f(a) = 0$ and $f(b) = 1$ for every $a \in A$ and $b \in B$. □

# A characterization

**Theorem**

*Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Let $f : X \to Y$ be a function. Then the following statements are equivalent:*

(a) *$f$ is continuous.*

(b) *For every open subset $O$ of $Y$, $f^{-1}(O) = \{a \in X : f(a) \in O\}$ is an open subset of $X$.*

**Proof.**

(a) $\Rightarrow$ (b). Let $O \subseteq Y$ be open and $a \in f^{-1}(O)$. Thus $f(a) \in O$. Since $O$ is open, there exists $\epsilon > 0$ such that $B^{\rho}_{\epsilon}(f(a)) \subseteq O$. By the continuity, there is $\delta > 0$ such that $f(B^{d}_{\delta}(a)) \subseteq B^{\rho}_{\epsilon}(f(a)) \subseteq O$. Thus $B^{d}_{\delta}(a) \subseteq f^{-1}(O)$, which shows that $f^{-1}(O)$ is open.

(b) $\Rightarrow$ (a). Let $a \in X$ and $\epsilon > 0$. Since $O = B^{\rho}_{\epsilon}(f(a))$ is an open subset of $Y$, $a \in f^{-1}(B^{\rho}_{\epsilon}(f(a)))$ is an open subset of $X$. Hence there exists $\delta > 0$ such that $B^{d}_{\delta}(a) \subseteq f^{-1}(B^{\rho}_{\epsilon}(f(a)))$, and hence $f$ is continuous. $\square$

# An application to matrices

**Problem**

*Consider the vector space $M_n(\mathbb{R})$ (over $\mathbb{R}$) of $n \times n$ matrices $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ with real entries. For $A \in M_n(\mathbb{R})$, set $\|A\| = \sum_{i,j=1}^{n} |a_{ij}|$. Verify the following:*

1. *$\| \cdot \|$ defines a norm on $M_n(\mathbb{R})$.*
2. *The determinant $\det : M_n(\mathbb{R}) \to \mathbb{R}$ is continuous.*
3. *The set $GL_n(\mathbb{R})$ of $n \times n$ invertible matrices is open in $M_n(\mathbb{R})$.*

**Hint.**

For 1, the triangle inequality follows from the triangle inequality of real numbers. For 2, note that determinant is a polynomial in $n^2$ variables, and hence det is s-continuous. For 3, note that $GL_n(\mathbb{R}) = \det^{-1}(\mathbb{R} \setminus \{0\})$ and $\mathbb{R} \setminus \{0\}$ is open. □

**Remark** Can you see that $GL_n(\mathbb{R})$ has a proper clopen subset ?

# Equivalent norms

We say that two norms $\|\cdot\|$ and $\|\cdot\|'$ on a normed linear space $X$ are *equivalent* if there exists $m, M > 0$ such that

$$m\|x\| \leqslant \|x\|' \leqslant M\|x\|, \quad x \in X.$$

## Problem

*Verify that the equivalence of norms is an equivalence relation. Conclude that the all norms $\|\cdot\|_p$, $p \geqslant 1$, on $\mathbb{R}^n$ are equivalent.*

## Hint.

The first part is a routine verification. To see the second part, by transitivity of norms, it suffices to check that $\|\cdot\|_p$ is equivalent to $\|\cdot\|_\infty$. This follows from $\|x\|_\infty \leq \|x\|_p \leq n^{1/p}\|x\|_\infty$. This in turn follows from $|x_j| \leqslant \sum_{j=1}^n |x_j|^p \leqslant n(max_{j=1}^n |x_j|)^p$, $j = 1, \ldots, d$. $\square$

## Problem

*Let $X$ be a normed linear space with two norms $\|\cdot\|$ and $\|\cdot\|'$. Then the norms $\|\cdot\|$ and $\|\cdot\|'$ are equivalent if and only if the identity map $I$ from $(X, \|\cdot\|)$ onto $(X, \|\cdot\|')$ is continuous at $0$ with inverse being continuous at $0$.*

## Hint.

For $\Rightarrow$, show that the identity map and its inverse are s-continuous. For $\Leftarrow$, note that the continuity of $I : (X, \|\cdot\|) \to (X, \|\cdot\|')$ at 0 implies that for $\epsilon = 1$, there exists $\delta > 0$ such that

$$\|x\| < \delta \Rightarrow \|x\|' < 1.$$

For $0 \neq x \in X$, let $y = \frac{\delta}{2}\frac{x}{\|x\|}$. Thus $\|y\| = \frac{\delta}{2}$, and hence $\|y\|' < 1$, that is, $\frac{\delta}{2}\frac{\|x\|'}{\|x\|} < 1$ or $\|x\|' \leq \frac{2}{\delta}\|x\|$ for every $x \in X$. Similarly, one may get the other inequality. $\square$

## Problem

*If two norms $\|\cdot\|$ and $\|\cdot\|'$ on $X$ are equivalent, show that $(X, \|\cdot\|)$ is complete if and only if $(X, \|\cdot\|')$ is complete.*

**Question** What are all norms on $\mathbb{R}^d$ (up to equivalence) ?

Let us analyze the case of $d = 1$:

### Problem

*Describe all norms on $\mathbb{R}$.*

### Solution.

Let $\|\cdot\|$ be any norm on $\mathbb{R}$. For any $x \in \mathbb{R}$, note that by the dilation property,

$$\|x\| = \|x \cdot 1\| = |x|\|1\|.$$

This means that any norm $\|\cdot\|$ is of the form $\alpha|\cdot|$, $\alpha > 0$. $\square$

• Note that all norms on $\mathbb{R}$ are equivalent. Indeed, if $\|\cdot\|$ and $\|\cdot\|'$ are two norms on $\mathbb{R}$, then for $m = \frac{\|1\|'}{\|1\|} = M$, we have

$$m\|x\| = \|x\|' = M\|x\|, \quad x \in \mathbb{R}.$$

What about the case of dimension $d > 1$ ?

## Theorem

*Consider* $(\mathbb{R}^{\mathrm{d}}, \|\cdot\|_\infty)$. *Let* $S = \{x \in \mathbb{R}^{\mathrm{d}} : \|x\|_\infty = 1\}$ *and let* $f : S \to \mathbb{R}$ *be a continuous function. Then there exists* $a \in S$ *such that* $f(x) \geqslant f(a)$ *for every* $x \in S$.

## Proof.

Let $A = \{f(x) : x \in S\}$ and let $\alpha = \inf A$. Thus there exists a sequence $\{x^{(n)}\}_{n \geqslant 1} \subseteq S$ such that $f(x^{(n)}) \to \alpha$ as $n \to \infty$. Now if $x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)})$, then $\{x_1^{(n)}\}_{n \geqslant 1}$, being a bounded sequence, has a convergent subsequence (by the Bolazano-Weierstrass Theorem), say, $\{x_1^{(n_j)}\}_{j \geqslant 1}$. Since $\{x_2^{(n_j)}\}_{j \geqslant 1}$ is bounded, it has a convergent subsequence, and continuing this, we obtain a convergent subsequence of $x^{(n)} = (x_1^{(n)}, \ldots, x_d^{(n)})$ converging to say $a$ in $\|\cdot\|_\infty$. It follows that $f(a) = \alpha$ and $f(x) \geqslant \inf A = f(a)$ for every $x \in S$. Since $S$ is closed, $a \in S$. □

# Norms on $\mathbb{R}^{\mathrm{d}}$

### Theorem

*All norms on $\mathbb{R}^{\mathrm{d}}$ are equivalent.*

### Proof.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^{\mathrm{d}}$.

(a) Let $e_1, \cdots, e_d$ denote the standard basis of $\mathbb{R}^{\mathrm{d}}$. By the triangle inequality, $\|x\| \leq \left(\sum_{i=1}^{\mathrm{d}} \|e_i\|\right) \|x\|_\infty$.

(b) By (a), the function $f : (\mathbb{R}^{\mathrm{d}}, \|\cdot\|_\infty) \to \mathbb{R}$ given by $f(x) = \|x\|$ is (sequential) continuous.

(c) $f$ attains its minimum on $S = \{x \in \mathbb{R}^{\mathrm{d}} : \|x\|_\infty = 1\}$. Thus there exists $a \in S$ such that $\|x\| \geq \|a\| > 0$ for every $x \in S$.

(d) By (a) and (c), the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.

The desired conclusion now follows from the transitivity of the equivalence of norms. $\square$

One can produce inequivalent norms on infinite-dimensional spaces.

### Example

Let $\mathbb{R}[x]$ denote the vector space over $\mathbb{R}$ of polynomials $p(x) = \sum_{n=0}^{k} a_n x^n$ in $x$. For $c := \{c_n\}_{n=0}^{\infty}$, define

$$\|p\|_c := \sum_{n=0}^{k} |c_n||a_n|.$$

- $\|\cdot\|_c$ defines a norm on $\mathbb{R}[x]$ if $c_n \neq 0$ for every $n \geqslant 0$.

This is a routine verification.

- $\|\cdot\|_c$ and $\|\cdot\|_d$ are not equivalent if $c_n = 1/(n+1)$, $d_n = n+1$.

This is far from being obvious.

## Example (Example continued ...)

If possible, then assume that there exists $m, M > 0$ such that

$$m\|p\|_c \leqslant \|p\|_d \leqslant M\|p\|_c, \quad p \in \mathbb{R}[x].$$

Thus, for any $p \in \mathbb{R}[x]$,

$$\|p\|_d = \sum_{n=0}^{k}(n+1)|a_n| < 1 \text{ whenever } \|p\|_c = \sum_{n=0}^{k} \frac{|a_n|}{n+1} < \frac{1}{M}.$$

Choose an integer $k$ large enough so that $\frac{k}{M} > 2$. Letting $a_0 = 0$ and $a_n = \frac{1}{2(n+1)M}$ $(n = 1, \ldots, k)$, we get

$$\sum_{n=0}^{k} \frac{|a_n|}{n+1} = \frac{1}{2M} \sum_{n=1}^{k} \frac{1}{(n+1)^2} < \frac{1}{M}.$$

However, $\sum_{n=0}^{k}(n+1)|a_n| = \frac{k}{2M} > 1$, which is a contradiction.

### Theorem

*Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Let $f : X \to Y$ be a function. Then the following statements are equivalent:*

(a) *$f$ is continuous.*

(b) *For every closed subset $U$ of $Y$, $f^{-1}(U) = \{a \in X : f(a) \in U\}$ is a closed subset of $X$.*

### Proof.

Recall that $\mathrm{f}^{-1}(\mathrm{Y} \setminus \mathrm{U}) = \mathrm{X} \setminus \mathrm{f}^{-1}(\mathrm{U})$. Now apply the fact that $f$ is continuous iff for every open subset $O$ of $Y$, $f^{-1}(O)$ is an open subset of $X$. □

### Example

Let $X$ be a metric space endowed with the discrete metric $d_0$. Then any $f : X \to Y$ is continuous. Indeed, since every subset of $X$ is closed, by the previous theorem, $f$ is continuous. In particular, if $\mathbb{Z}$ carries the (relative) metric induced from $(\mathbb{R}, d_1)$, then any function $f : \mathbb{Z} \to Y$ is continuous.

## Problem

*Consider the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices $A = (a_{i,j})_{1 \leqslant i,j \leqslant n}$ with real entries. Define $d : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ by $d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|$. Verify the following:*

1. *For every positive integer $k$, the map $p_k : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ given by $p_k(A) = A^k$ is continuous.*
2. *If $\mathcal{N}_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) : \text{there is } k \geqslant 1 \text{ such that } A^k = 0\}$, then $\mathcal{N}_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$.*

## Hint.

To see that $p_k$ is s-continuous, note that the entries of $A^k$ are polynomials in the entries of $A$ (please verify this for $k = 2$ and $n = 2$). To see the part 2, note that

$$\mathcal{N}_n(\mathbb{R}) = \cup_{k \geqslant 1} \ker p_k = \cup_{k=1}^{n} \ker p_k$$

(justify the second equality). Since kernel of a continuous map is closed, $\mathcal{N}_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$. $\square$

# Homeomorphisms

## Definition

Let $(X, d)$ and $(Y, \rho)$ be two metric spaces.

- A function $f : X \to Y$ is a homeomorphism if $f$ is continuous, one-one, and onto with a continuous inverse.
- We say that $X$ and $Y$ are homeomorphic if there exists a homeomorphism $f : X \to Y$. In this case, we say that $X \cong Y$.

**Remark** Let $f : X \to Y$ be a homeomorphism.

1. $f^{-1}$ is also a homeomorphism.
2. If $U$ is open in $X$ then $f(U)$ is open in $Y$.
3. If $A$ is a subset of $X$ then $A$ and $f(A)$ are homeomorphic.
4. Composition of homeomorphisms is again a homeomorphism. In particular, if $X$ is homeomorphic to $Y$, and $Y$ is homeomorphic to $Z$ then $X$ is homeomorphic to $Z$.

Note that $\cong$ is an equivalence relation.

## Example

Consider the subsets $A$ of the metric space $(\mathbb{R}, d_1)$ as the metric spaces with relative metric $d_1^A(x, y) = |x - y|$, $x, y \in A$.

- $(0, 1) \not\cong (0, 1]$ : If $f : (0, 1) \to (0, 1]$ is a homeomorphism, then $f(c) = 1$ for some $c \in (0, 1)$, and hence it follows that $f^{-1}$ maps $(0, 1)$ onto $(0, c) \cup (c, 1)$, which is not possible in view of the intermediate value property.

## Problem

*Show that the interval $(a, b) \subseteq \mathbb{R}$ is homeomorphic to any other interval $(c, d) \subseteq \mathbb{R}$.*

## Hint.

Try $\alpha(t - b) + \beta(t - a)$ for appropriate scalars $\alpha$ and $\beta$. $\square$

## Problem

*Show that $e^{-x}$ is a homeomorphism from $(0, \infty)$ onto $(0, 1)$.*

## Problem

*Which of spaces $X$ and $Y$ are homeomorphic:*

(1) $X = \mathbb{R}$ *and* $Y = [0, 1)$

(2) $X = \mathbb{R}$ *and* $Y = [0, 1]$

(3) $X = [1, \infty)$ *and* $Y = (0, 1]$

(4) $X = (-1, 0)$ *and* $Y = (-\infty, -1)$

(5) $X = \mathbb{R}$ *and* $Y = (0, 1)$

(6) $X = \mathbb{Q}$ *and* $Y = \mathbb{Z}$

**Hint.** For (1), (2), use intermediate value property (No). For (3), (4), check that $1/x$ is the desired homeomorphism (Yes). ~~For (5), write~~ $\cancel{X = (-\infty, 1) \cup [1, \infty)}$~~, and note that by (3),~~ $\cancel{[1, \infty) \cong (0, 1]}$~~. Also, by (4),~~ $\cancel{(-\infty, 1) \cong (-\infty, -1) \cong (-1, 0)}$ ~~(No).~~ [6] For (6), choose an open ball centred at an integer (which contains finitely many elements), and analyze its image in $\mathbb{Q}$ (No).

[6] The answer to (5) is **Yes**. Please try tan (thanks to Sudip and Satyam for catching a careless assertion!)

### Theorem

*$\mathbb{R}^n$ is homeomorphic to $\mathbb{R}$ iff $n = 1$.*

### Proof.

Suppose that for $n > 1$. Thus there is a continuous bijection $f : \mathbb{R}^n \to \mathbb{R}$, and hence $g = f|_X$ is a homeomorphism from $X := \mathbb{R}^n \setminus \{0\}$ onto $Y := \mathbb{R} \setminus \{y_0\}$ for some $y_0 \in \mathbb{R}$. Choose $y_1, y_2 \in Y$ such that $y_1 < y_0 < y_2$ and let $x_1, x_2 \in X$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Let $L$ denote the line segment connecting $x_1$ and $x_2$.

- If $L$ does not pass through 0 then let $\gamma(t) = (1 - t)x_1 + tx_2$.
- If $L$ passes through 0 then choose any point $x_3 \notin L$ and let

$$\gamma(t) = \begin{cases} (1 - 2t)x_1 + 2t\, x_3 \text{ if } 0 \leq t \leq 1/2, \\ 2(1 - t)x_3 + (2t - 1)x_2 \text{ if } 1/2 \leq t \leq 1. \end{cases}$$

Thus $\gamma : [0, 1] \to X$ is a continuous function (since $\gamma(1/2) = x_3$) such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$.

## Proof continued ...

We consider the continuous function $h : [0, 1] \to [y_1, y_2]$ by

$$h(t) = g(\gamma(t)), \quad t \in [0, 1].$$

By the intermediate value property, there exists $t_0 \in [0, 1]$ such that $h(t_0) = y_0$, that is, $g(\gamma(t_0)) = y_0$. This implies that $y_0$ belongs to the image of $g = f|_X$, which is a contradiction. $\square$

**Remark** It is highly non-trivial fact that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $m = n$ (beyond the scope of this course).

## Problem

*Is an open annulus* $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 2\}$ *homeomorphic to the unit circle* $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ *? Justify your answer.*

## Hint.

Argue as in the last theorem. $\square$

**Objective** To find when two metric spaces are homeomorphic, one needs to look for "invariants" which are preserved under homeomorphisms (e.g, existence of a proper clopen set).

## Problem

*Show that the function $g : (0,1) \to \mathbb{R}$ given below is continuous on irrationals and discontinuous on rationals:*

$$g(x) = \begin{cases} \frac{1}{q} & \textit{if } x \in \mathbb{Q} \cap (0,1) \textit{ and } x = \frac{p}{q} \textit{ in reduced form} \\ 0 & \textit{otherwise.} \end{cases}$$

**Question** Does there exist a function $g : (0,1) \to \mathbb{R}$ which is continuous on rationals and discontinuous on irrationals ?

Answer: No

A solution was first provided by **Vito Volterra**.

# $G$-delta sets

### Definition

A set is a $\underline{G_\delta \text{ set}}$ if it is countable intersection of open sets.

### Example

The irrationals $\mathbb{R} \setminus \mathbb{Q}$ form a $G_\delta$ set for $\mathbb{R} \setminus \mathbb{Q} = \bigcap_{r \in \mathbb{Q}} \mathbb{R} \setminus \{r\}$.

### Problem

*The rationals do not form a $G_\delta$ set.*

### Solution.

Suppose that $\mathbb{Q} = \cap_{n=1}^{\infty} U_n$ for open sets $U_n$. Since $\mathbb{Q} \subseteq U_n$, each $U_n$ is dense in $\mathbb{R}$. Now note that $\emptyset = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$ is countable intersection of open dense sets $U_n$, $n \geqslant 1$ and $\mathbb{R} \setminus \{r\}$, $r \in \mathbb{Q}$. This contradicts BCT (to be proven later). $\square$

### Theorem (Baire Category Theorem)

*A countable intersection of open dense sets in $\mathbb{R}$ is dense in $\mathbb{R}$.*

# A Theorem of Volterra Vito

**Theorem**

*Let $U$ be an open subset of $\mathbb{R}$ and let $f : U \to \mathbb{R}$ be a function defined on $U$. Then $A = \{a \in U : f \text{ is continuous at } a\}$ is a $G_\delta$ set.*

**Proof.**

For positive integer $n$, consider the set $A_n$ given by

$$\{x_0 \in U : \exists\, \delta > 0 \text{ such that } |f(x) - f(y)| < 1/n,\ x, y \in (x_0 - \delta, x_0 + \delta)\}.$$

Note that $A_n$ is open and $A = \cap_{n=1}^{\infty} A_n$. $\square$

Since rationals do not form a $G_\delta$ set, we obtain the following:

**Corollary (Volterra Vito)**

*There is no function $g : (0, 1) \to \mathbb{R}$ which is continuous on rationals and discontinuous on irrationals.*

# Baire category theorem

## Theorem (Baire Category Theorem)

*Let $(X, d)$ be a complete metric space and let $\{U_n\}_{n\geqslant 1}$ be a sequence of open dense subsets of $X$. Then $\cap_{n\geqslant 1} U_n$ is dense in $X$.*

## Proof.

Let $x \in X$ and $r > 0$. We claim that $B_r(x) \cap (\cap_{n\geqslant 1} U_n) \neq \emptyset$.

- Since $U_1$ is dense in $X$, there exists $x_1 \in B_r(x) \cap U_1$. Since $U_1$ is open, for some $0 < r_1 < 1$, $B_{r_1}(x_1) \subseteq B_r(x) \cap U_1$.
- Since $U_2$ is dense in $X$, there exists $x_2 \in B_{r_1}(x_1) \cap U_2$. Since $U_2$ is open, for some $0 < r_2 < 1/2$, $B_{r_2}(x_2) \subseteq B_{r_1}(x_1) \cap U_2$.

By induction on $n \geqslant 1$, there exist $\{x_n\}_{n\geqslant 1} \subseteq X$ and $0 < r_n < 1/n$ such that $B_{r_{n+1}}(x_{n+1}) \subseteq B_{r_n}(x_n) \cap U_{n+1}$ for every $n \geqslant 1$. Thus $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence (since $d(x_n, x_m) < 1/n$ for every $m \geqslant n$), and hence converges to some $x_0 \in X$ (since $X$ is complete). Note that $x_0 \in B_r(x) \cap (\cap_{n\geqslant 1} U_n)$. $\square$

### Problem

*Let $X$ be a complete metric space. If $X$ is a union of closed sets $A_1, A_2, \ldots,$, then show that there exists $N \geqslant 1$ such that $A_N$ has non-empty interior.*

### Hint.

Assume that $X = \cup_{n \geqslant 1} A_n$ and that $A_n^\circ = \emptyset$ for every integer $n \geqslant 1$. Then $\emptyset = \cap_{n \geqslant 1}(X \setminus A_n)$, not possible (since $\overline{X \setminus A_n} = X$). $\square$

**Remark** $\mathbb{R}^2$ can not be written as the countable union of lines. Also, $\mathbb{R}^3$ can not be written as the countable union of planes.

### Problem

*Let $X$ be a complete metric space. If $f : X \to X$ is surjective, then show that there exists an integer $n \geqslant 1$ and $x_0 \in X$ such that $\overline{f(B_n(x_0))}$ has nonempty interior.*

### Hint.

Apply the last problem to appropriate closed subsets $A_n$ of $X$. $\square$

# An application

**Theorem**

*Let $X$ be a complete normed linear space containing an infinite linearly independent sequence $\{x_n\}_{n\geqslant 1}$ in $X$. Then the linear span of $\{x_n\}_{n\geqslant 1}$ is a proper subspace of $X$.*

**Proof.**

For $m \geqslant 1$, let $Y_m := \text{linspan}\{x_1, \cdots, x_m\}$.

- $Y_m$ is a proper closed subspace of $X$, and hence $Y_m$ has empty interior (if $B_r(x) \subseteq Y_m$, then for any nonzero $y \in X$, $x + \frac{r}{2}\frac{y}{\|y\|} \in Y_m$ and hence $y \in Y_m$).
- The complement of $Y_m$ is open and dense. By the Baire Category Theorem, the intersection $\bigcap_{m\geqslant 1}(X \setminus Y_m)$ is dense in $X$.

If linspan $\{x_n\}_{n\geqslant 1} = X$, then any element in $\bigcap_{m\geqslant 1}(X \setminus Y_m)$ (which exists) belongs to $Y_N$ for large $N$, and hence it belongs to $Y_N$ and its complement $X \setminus Y_N$ simultaneously, a contradiction. $\square$

## Problem

*If $(X, d)$ is a complete metric space such that every $x \in X$ is a cluster point of $X \setminus \{x\}$, then show that $X$ is not countable.*

## Solution.

If possible, assume that $X$ is countable. So one can enumerate $X$ as a sequence $\{x_n\}_{n\geqslant 1}$. Note that for every integer $n \geqslant 1$, $X \setminus \{x_n\}$ is open and dense in $X$. By the Baire Category Theorem, $\cap_{n=1}^{\infty}(X \setminus \{x_n\})$ is dense in $X$. However, $\cap_{n=1}^{\infty}(X \setminus \{x_n\}) = \emptyset$, and hence $X$ must be uncoutable. $\square$

• $(\mathbb{Z}, d_0)$ is a complete metric space even if $\mathbb{Z}$ is countable ($\{n\}$ is open for every $n \in \mathbb{Z}$). Also, the above problem is applicable to $(\mathbb{R}, d_1)$ providing indirect verification of uncountability of $\mathbb{R}$.

## Theorem (Nested Set Theorem)

*Let $X$ be a complete metric space and let $\{J_n\}_{n\geqslant 1}$ be a decreasing sequence of nonempty closed sets in $X$ such that $diam(J_n) \to 0$ as $n \to \infty$. Then $\cap_{n\geqslant 1} J_n$ contains exactly one point.*

# Uniform continuity

### Definition

Let $(X, d)$ and $(Y, \rho)$ be two metric spaces. Let $f : X \to Y$ be a function. We say that $f$ is uniformly continuous if for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$\Big(x, y \in X,\ d(x, y) < \delta \Longrightarrow \rho(f(x), f(y)) < \epsilon\Big).$$

- Every uniformly continuous function is continuous at every point in $X$. The major difference is that unlike the definition of continuity at a point, single $\delta$ works for all points in $X$.
- $f(x) = 1/x$, $x \in (0, 1)$ is continuous but not uniformly continuous. Indeed, if $m, n \geqslant 1$ are integers such that $m > n > \delta/2$, then $|1/n - 1/m| < \delta$ & $|f(1/n) - f(1/m)| \geqslant 1$.
- For any subset $A$ of a normed linear space $X$, the function $f(x) = \text{dist}(x, A)$, $x \in X$ is uniformly continuous. Indeed, since $|f(x) - f(y)| \leqslant \|x - y\|$, $x, y \in X$, one may take $\delta = \epsilon$.

### Problem

*Let $(\mathbb{R}, d_1)$ and let $f : \mathbb{R} \to \mathbb{R}$ be uniformly continuous. Then*

1. *$f$ maps bounded sets to bounded sets.*
2. *$f$ maps Cauchy sequences to Cauchy sequences.*

### Hint.

To see 1, let $A$ be a bounded subset of $\mathbb{R}$. Thus $A \subseteq [a, b]$ for some $a, b \in \mathbb{R}$ such that $a < b$. We must check that $f([a, b])$ is bounded. Note that $[a, b]$ can be covered by finitely many open intervals of length less than $\delta$ (the one obtained from the definition of uniform continuity with $\epsilon = 1$). Part 2 follows from definition. □

### Problem

*Let $X, Y$ be normed linear spaces and let $T : X \to Y$ be such that is, $T(x + \alpha y) = T(x) + \alpha T(y)$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$. Show that $T$ is continuous at $0$ iff $T$ is uniformly continuous.*

### Hint.

For $0 \neq x \in X$, let $x' = \frac{\delta}{2} \frac{x}{\|x\|}$. Now use the continuity at 0. □

## Theorem

*Let $a, b \in \mathbb{R}$ be such that $a < b$. If $f : [a, b] \to \mathbb{R}$ is continuous, then $f$ is uniformly continuous.*

## Proof.

Let $\epsilon > 0$ and let $x \in [a, b]$. Then, by the continuity of $f$, there exists $\delta_x$ such that

$$\Big(y \in [a, b],\ |x - y| < \delta_x \implies |f(x) - f(y)| < \epsilon/2\Big).$$

However, we want single $\delta$, which will work for every $x \in [a, b]$. Note that $[a, b] \subseteq \cup_{x \in [a,b]}(x - \delta_x/2, x + \delta_x/2)$. It turns out that there are finitely many $x_1, \ldots, x_k \in [a, b]$ such that

$$[a, b] \subseteq \cup_{n=1}^{k}(x_n - \delta_{x_n}/2, x_n + \delta_{x_n}/2)$$

(**to be seen later**). Let $\delta = \min_{n=1}^{k} \delta_{x_n}/2$. If $|x - y| < \delta$, then $x \in (x_n - \delta_{x_n}/2, x_n + \delta_{x_n}/2)$ for some $n$, and hence $|y - x_n| < \delta_{x_n}$. It follows that $|f(x) - f(y)| < \epsilon$. $\square$

# Compactness

### Definition

Let $(X, d)$ be a metric space. We say that $X$ is compact if for any collection $\{U_\alpha : \alpha \in I\}$ of open subsets of $X$ such that $X = \cup_{\alpha \in I} U_\alpha$, there exists finitely many indices $\{\alpha_1, \ldots, \alpha_k\}$ in $I$ such that $X = \cup_{j=1}^{k} U_{\alpha_j}$.

- $\{U_\alpha : \alpha \in I\}$: open cover of $X$
- $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$: open subcover of $X$

**Remark** If $f : X \to Y$ is a continuous surjection and $X$ is compact, then $Y$ is also compact. Indeed, if $\{V_\alpha : \alpha \in I\}$ is an open cover of $Y$, then $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is an open cover of $X$, and since $X$ is compact, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ admits a finite subcover $\{f^{-1}(V_{\alpha_1}), \ldots, f^{-1}(V_{\alpha_k})\}$ of $X$. It follows that $Y = \cup_{j=1}^{k} V_{\alpha_j}$.

## Problem

*Show that none of the following metric spaces $(X, d)$ is compact.*

1. $X = (0, 1]$, $d =$ *relative metric induced by* $d_1$
2. $X = \mathbb{Z}$, $d = d_0$
3. $X = \mathbb{R}^{\mathrm{d}}$, $d = d_p$
4. $X =$ *normed linear space,* $d(x, y) = \|x - y\|$

## Solution.

For 1, consider the open cover $\{(1/n, 1] : n \geqslant 2\}$ of $(0, 1]$. For 2, consider the open cover $\{\{n\} : n \in \mathbb{Z}\}$. Part 3 follows from 4, and for 4, consider the open cover $\{B_n(0) : n \geqslant 1\}$. Verify that none of above open covers admits a finite subcover. $\square$

## Problem

*Let $X$ be an arbitrary set endowed with a discrete metric $d_0$. Then $X$ is compact if and only if $\cdots$ .*

## Example

We claim that the finite interval $[a, b]$ with relative metric $d_1$ is compact. If this is false, then there exists an open cover $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ of $[a, b]$, which has no finite subcover.

- Note that $\mathcal{U}$ is also an open cover for $[a, (a+b)/2]$ and $[(a+b)/2, b]$, and $\mathcal{U}$ does not admits finite subcover for at least one of subintervals, say, $[a_1, b_1]$. Clearly, $b_1 - a_1 = \frac{b-a}{2}$.
- Note that $\mathcal{U}$ is also an open cover for $[a_1, (a_1+b_1)/2]$ and $[(a_1+b_1)/2, b_1]$, and $\mathcal{U}$ does not admits finite subcover for at least one of subintervals, say, $[a_2, b_2]$. Clearly, $b_2 - a_2 = \frac{b-a}{4}$.
- By induction, for every integer $n \geqslant 1, \mathcal{U}$ does not admits finite subcover for $[a_n, b_n] \subseteq [a_{n-1}, b_{n-1}]$ and $b_n - a_n = \frac{b-a}{2^n}$.
- By the nested interval theorem, there exists $c \in \cap_{k=1}^{\infty}[a_k, b_k]$. Since $\mathcal{U}$ is an open cover for $[a_1, b_1]$, $c \in U_\alpha$ for some $\alpha \in I$.
- $c$ is an interior point of $U_\alpha$, $a_k \uparrow c$ and $b_k \downarrow c$, for large $k$, $[a_k, b_k] \subseteq U_\alpha$, that is, $[a_k, b_k]$ admits a finite subcover. $\Rightarrow\Leftarrow$

### Theorem (Heine-Borel Theorem)

*Any closed and bounded subset $A$ of $(\mathbb{R}, d_1)$ is compact.*

### Proof.

We claim that any closed subset $A$ of a compact metric space $(X, d)$ is compact. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of $A$.

- $\mathcal{U} \cup \{X \setminus A\}$ is an open cover of $X$.
- There exists a finite subcover $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\} \cup X \setminus A$ of $X$ (since $X$ is compact).
- $\{U_{\alpha_1}, \ldots, U_{\alpha_k}\}$ is an open subcover of $A$.

Since $A$ is bounded, $A \subseteq [a, b]$ (finite interval). Since $[a, b]$ is compact and $A$ is a closed subset of $[a, b]$, $A$ is also compact. $\square$

- Every closed subset of a compact metric space is compact.

**Question** Does there exist a closed and bounded set which is not compact ?

# Compactness in relative metric

**Question** Let $(X, d)$ be a metric space and $Y \subseteq X$ endowed with the relative metric $d^Y(x, y) = d(x, y)$, $x, y \in X$. If $K \subseteq Y$, then whether or not

$$(K, d) \text{ compact } \Leftrightarrow (K, d^Y) \text{ compact ?}$$

- If $\{U_\alpha : \alpha \in I\}$ is an open cover of $K$ in $X$ then $\{U_\alpha \cap Y : \alpha \in I\}$ is an open cover of $K$ in $Y$.
- If $\{V_\alpha : \alpha \in I\}$ is an open cover of $K$ in $Y$, then for every $\alpha \in I$, $V_\alpha = U_\alpha \cap K$ for some open set $U_\alpha$ in $X$. Note that $\{U_\alpha : \alpha \in I\}$ is an open cover of $K$ in $X$.

Conclude that $(K, d)$ is compact $\Leftrightarrow$ $(K, d^Y)$ is compact.

## Theorem

*A compact subset $K$ of a metric space $(X, d)$ is closed & bounded.*

## Proof.

To see that $K$ is closed, it suffices to check that $X \setminus K$ is open.

- Let $x \in X \setminus K$ and let $r_y = d(x, y)/2 > 0$ for every $y \in K$.
- $\{B_{r_y}(y) : y \in K\}$ is an open cover of $K$. So it admits a finite subcover $\{B_{r_{y_j}}(y_j) : j = 1, \ldots, N\}$ (since $K$ is compact).
- If $r = \min\{r_{y_1}, \ldots, r_{y_N}\}$, then $B_r(x) \subseteq X \setminus K$. Indeed, if $y \in K \cap B_r(x)$, then $y \in B_{r_{y_j}}(y_j) \cap B_r(x)$ for some $j$, and hence $d(y, y_j) < d(x, y_j)/2$, $d(x, y) < r \leqslant d(x, y_j)/2$, $\Rightarrow\Leftarrow$.

Thus $x$ is an interior point of $X \setminus K$, and hence $K$ is closed.

To see that $K$ is bounded, let $x \in X$. Consider the open cover $\{B_r(x) \cap K : r > 0\}$ of $K$. Since $K$ is compact, there exists $r > 0$ such that $K \subseteq B_r(x)$, and hence it is bounded. $\square$

## Example

Consider $X = \{r \in \mathbb{Q} : 1 < r^2 < 2\}$ with $d$ being the relative metric induced by $d_1$ is not compact. Then $X = \{r \in \mathbb{Q} : 1 < |r| < \sqrt{2}\} = \Big((1, \sqrt{2}) \cap \mathbb{Q}\Big) \cup \Big((-\sqrt{2}, -1) \cap \mathbb{Q}\Big)$, which is not closed in $(\mathbb{Q}, d_1)$.

## Problem

*Assume that* $n \geqslant 2$. *Show that* $M_n(\mathbb{R}) \setminus GL_n(\mathbb{R})$ *is not compact as a metric space endowed with the (relative) metric* $d : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ *by* $d(A, B) = \sum_{i,j=1}^{n} |a_{ij} - b_{ij}|$.

## Hint.

$M_n(\mathbb{R}) \setminus GL_n(\mathbb{R})$ is closed in $M_n(\mathbb{R})$ but not bounded. Indeed, there are diagonal matrices $B$ in $M_n(\mathbb{R}) \setminus GL_n(\mathbb{R})$ for which $d(0, B)$ is of arbitrarily large magnitude provided $n > 1$. $\square$

### Problem

*Consider $\mathbb{C}^2$ endowed with the metric*

$$d((z,w),(z',w')) = (|z-z'|^2+|w-w'|^2)^{1/2},\ (z,w),(z',w') \in \mathbb{C}^2.$$

*Show that $\{(z,w) \in \mathbb{C}^2 : z^2 + w^2 = 1\}$ is not compact.*

### Hint.

Find $w \in \mathbb{C}$ such that $w^2 = 1 + n^2$. Thus, for $\iota = \sqrt{-1}$, we have $(\iota n, \sqrt{1+n^2}) \in \{(z,w) \in \mathbb{C}^2 : z^2 + w^2 = 1\}$. $\square$

**Question** What if $z^2 + w^2 - 1$ is replaced by any nonconstant polynomial in $z, w$ ?

The answer is yes (this can be deduced from the Fundamental theorem of algebra).

## Theorem

*If $(X, d)$ is compact, then every sequence in $X$ has a convergent subsequence.*

## Proof.

We claim that an infinite subset $A$ of $X$ has a cluster point in $X$.

- Suppose $A$ has no cluster point in $X$, that is, for every $x \in X$, there exists $r_x > 0$ such that $B_{r_x}(x) \cap A \subseteq \{x\}$.
- $\{B_{r_x}(x) : x \in X\}$ is an open cover for $X$, and hence $X$ has finite subcover $\{B_{r_{x_j}}(x_j) : j = 1, \ldots, N\}$.
- $A = \cup_{j=1}^{N}(B_{r_{x_j}}(x_j) \cap A) \subseteq \{x_1, \ldots, x_N\}$ is a finite set.

Applying the above fact to $A = \{x_n\}_{n \geqslant 1}$, we conclude that $A$ has a cluster point in $X$. However, for every cluster point, there exists a sequence in $A$ converging to this cluster point (see slide 62). $\square$

Let $Y$ be a proper, closed subspace of a normed linear space $X$.

- Choose $x_1 \in X \setminus Y$, and note that $d(x_1, Y) > 0$.
- There exists $x_0 \in Y$ such that $\|x_1 - x_0\| < 2d(x_1, Y)$.
- Note that $\|x_1 - x_0\|y + x_0$ belongs to $Y$ for any $y \in Y$.
- If $x = \frac{x_1 - x_0}{\|x_1 - x_0\|}$ (unit vector), then for any $y \in Y$,

$$\|x - y\| = \frac{\|x_1 - x_0 + \|x_1 - x_0\|y\|}{\|x_1 - x_0\|} \geqslant \frac{1}{2}.$$

Thus there exists a unit vector $x \in X$ such that $d(x, Y) \geqslant 1/2$.

**Theorem**

*If $X$ is infinite dimensional, then unit sphere in $X$ is not compact.*

**Proof.**

- Let $x_1 \in X$ be a unit vector in $X$ and let $X_1 := \text{span}\{x_1\}$.
- There exists $x_2 \in X$ such that $\|x_2\| = 1$ and $d(x_2, X_1) \geq 1/2$.
- Note that $x_2 \notin X_1$. Let $X_2 := \text{span}\{x_1, x_2\}$.
- There exists $x_3 \in X_2$ such that $\|x_3\| = 1$ and $d(x_3, X_2) \geq 1/2$.

Continuing this, we get $\{x_n\}_{n \geqslant 1}$ with no cgt subsequence. $\square$

## Definition

A metric space $X$ is <u>sequentially compact</u> if every sequence in $X$ has a subsequence convergent in $X$.

- A compact metric space is sequentially compact (slide 117).
- If $X$ is sequentially compact and $f : X \to Y$ is a continuous surjection, then $Y$ is sequentially compact (if $\{y_n = f(x_n)\}_{n \geqslant 1}$ is given, then $\{x_n\}_{n \geqslant 1}$ has a convergent subsequence, and hence $\{f(x_n)\}_{n \geqslant 1}$ has a convergent subsequence).
- Every sequentially compact metric space is complete. Indeed, if $\{x_n\}_{n \geqslant 1}$ is a Cauchy sequence, then since $\{x_n\}_{n \geqslant 1}$ has a convergent subsequence, $\{x_n\}_{n \geqslant 1}$ is convergent (slide 67).

• A complete metric space is not necessarily sequentially compact. For example, $\{n\}_{n \in \mathbb{N}}$ in $(\mathbb{R}, d_1)$ has no convergent subsequence.

**Question** What are all complete metric spaces which are also sequentially compact ?

## Problem

*Consider the vector space $M_n(\mathbb{R})$ of $n \times n$ matrices $A = (a_{i,j})_{1\leqslant i,j\leqslant n}$ with real entries. For $X = (x_1, \ldots, x_n) \in \mathbb{R}^n$ (seen as a column vector), note that $AX \in \mathbb{R}^n$. Define*

$$\|A\| = \sup_{X\in\mathbb{R}^n, \|X\|_2=1} \|AX\|_2, \quad A \in M_n(\mathbb{R}).$$

*Verify the following:*

1. *$(M_n(\mathbb{R}), \|\cdot\|)$ is a complete normed linear space.*
2. *$ISO = \{A \in M_n(\mathbb{R}) : \|AX\|_2 = \|X\|_2$ for every $X \in \mathbb{R}^n\}$ is sequentially compact.*

## Hint.

Since $\|\cdot\|_2$ is a norm on $\mathbb{R}^n$, $\|\cdot\|$ defines a norm on $M_n(\mathbb{R}^n)$. To see that $M_n(\mathbb{R})$ is complete, note that $f : M_n(\mathbb{R}) \to (R^{n^2}, \|\cdot\|_2)$ given by $f(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn})$ is a **linear** homeomorphism (since $\|A_m - A\| \to 0$ if and only if $(i,j)$th entry of $A_m$ converges to $(i,j)$th entry of $A$ for every $1 \leqslant i,j \leqslant n$). To see 2, note that ISO is closed and bounded in $M_n(\mathbb{R})$. $\square$

## Theorem

*Let $(X, d)$ be a metric space and let $A \subseteq X$ be sequentially compact. Then, for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_N \in X$ such that $A \subseteq \cup_{j=1}^{N} B_\epsilon(x_j)$.*

## Proof.

Let $\epsilon > 0$ and let $x_1 \in A$.

- If $A \subseteq B_\epsilon(x_1)$, then we are done.
- Otherwise, there exists $x_2 \in A \setminus B_\epsilon(x_1)$. Thus $d(x_1, x_2) \geqslant \epsilon$.
- If $A \subseteq B_\epsilon(x_1) \cup B_\epsilon(x_2)$, then we are done.
- Otherwise, there exists $x_3 \in A \setminus (B_\epsilon(x_1) \cup B_\epsilon(x_2))$. Thus $d(x_1, x_3) \geqslant \epsilon$ and $d(x_2, x_3) \geqslant \epsilon$.

Continuing this, we either have $A \subseteq \cup_{j=1}^{N} B_\epsilon(x_j)$ for some integer $N \geqslant 1$ or there exists a sequence $\{x_n\}_{n \geqslant 1}$ such that $d(x_n, x_m) \geqslant \epsilon$ for all positive integers $m \neq n$. The latter case does not arise since $A$ is sequentially compact. $\square$

# Totally bounded spaces

## Definition

A metric space $X$ is said to be totally bounded if for every $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_N \in X$ such that

$$X \subseteq \cup_{j=1}^{N} B_\epsilon(x_j).$$

- Unlike compactness and sequential compactness, total boundedness is not preserved under homeomorphism. To see this, note that $(0, 1)$ is totally bounded. Indeed, if choose an integer $N \geqslant 1$ such that $N\epsilon > 1$, then

  $$(0,1) \subseteq (-\epsilon, \epsilon) \cup (\epsilon/2, 3\epsilon/2) \cup (\epsilon, 3\epsilon) \cup \cdots \cup (N\epsilon, (N+2)\epsilon).$$

  On the other hand, $\mathbb{R}$ is homeomorphic to $(0, 1)$, but it is not totally bounded.

$$\begin{tikzcd}
\text{Compact} \arrow[r] \arrow[dr] & \text{Sequentially compact} \arrow[d] \\
& \text{Complete and totally bounded}
\end{tikzcd}$$

(see slides 117, 119, 121)

**Question** Whether Sequentially Compact $\Rightarrow$ Compact ?

**Question** Whether Complete and Totally bounded $\Rightarrow$ Compact ?

- This will give complete the diagram!

## Problem

*Every totally bounded set is bounded.*

## Hint.

Note that union of finitely many open balls is contained in a single open ball of sufficiently large radius. $\square$

## Problem

*Show that a subset of a totally bounded metric space is again totally bounded.*

## Solution.

Suppose $X$ is totally bounded and $A \subseteq X$. Then, for every $\epsilon > 0$, there exist $x_1, \ldots, x_N \in X$ such that $A \subseteq \cup_{j=1}^{N} B_{\epsilon/2}(x_j)$. For each $j$, choose $a_j \in A \cap B_{\epsilon/2}(x_j)$, and note that by the triangle inequality, $A \subseteq \cup_{j=1}^{N} B_{\epsilon}(a_j)$. $\square$

## Problem

*Show that union of finitely many totally bounded sets is totally bounded.*

## Solution.

This follows from the definition. □

## Problem

*Show that every totally bounded metric space is separable.*

## Solution.

For every integer $k \geqslant 1$, there exist $x_1^{(k)}, \ldots, x_{N_k}^{(k)} \in X$ such that $X \subseteq \cup_{j=1}^{N_k} B_{1/k}(x_j^{(k)})$. Clearly, $Y = \cup_{k \geqslant 1}\{x_1^{(k)}, \ldots, x_{N_k}^{(k)}\}$ is countable. Verify that $Y$ is dense in $X$, that is, every ball of radius $\epsilon$ in $X$ intersects $Y$ non-trivially. □

## Theorem

*Let $(X, d)$ be a metric space. Let $\{x_n\}_{n\geqslant 1}$ be a sequence in $X$ and let $A = \{x \in X : x = x_n$ for some $n \geqslant 1\}$. Then*

(1) *If $\{x_n\}_{n\geqslant 1}$ is Cauchy, then $A$ is totally bounded.*

(2) *If $A$ is totally bounded, then $\{x_n\}_{n\geqslant 1}$ has a Cauchy subsequence.*

## Proof.

To see (1), let $\epsilon > 0$. Since $\{x_n\}_{n\geqslant 1}$ is Cauchy, there exists $N \geqslant 1$ such that $d(x_n, x_m) < \epsilon$ for all $m, n \geqslant N$. Thus

$$A = \{x_1\} \cup \cdots \cup \{x_{N-1}\} \cup \{x_n : n \geqslant N\} \subseteq B_\epsilon(x_1) \cup \cdots \cup B_\epsilon(x_{N-1}) \cup B_\epsilon(x_N).$$

To see (2), we may assume that $A$ is infinite (otherwise, $\{x_n\}_{n\geqslant 1}$ is eventually constant, and hence it has a convergent sequence).

- Cover $A$ by finitely many balls of radius 1, and at least one of these balls, say, $A_1$ contains infinitely many points in $A$.
- Cover $A_1$ by finitely many balls of radius $1/2$, and at least one of these balls, say, $A_2$ contains infinitely many points in $A$.

## Proof continued ...

Continuing this, we obtain a decreasing sequnece $\{A_n\}_{n\geqslant 1}$ of balls $A_n$ of radius $1/n$, where each $A_n$ contains infinitely many points of $A$. Choose now a sequence $\{x_{n_k}\}_{k\geqslant 1}$ such that $x_{n_k} \in A_k$ and check that $\{x_{n_k}\}_{k\geqslant 1}$ is a Cauchy sequence. □

## Corollary

*Let $(X, d)$ be a metric space. Then $X$ is totally bounded and complete if and only if it is sequentially compact.*

## Proof.

We have already seen $\Leftarrow$ (see slides 119 and 121). To see $\Rightarrow$, let $\{x_n\}_{n\geqslant 1}$ be given and and let

$$A = \{x \in X : x = x_n \text{ for some } n \geqslant 1\}.$$

Since $X$ is totally bounded, so is $A$ (see slide 124). By the last theorem, $\{x_n\}_{n\geqslant 1}$ has a Cauchy subsequence, say, $\{x_{n_k}\}_{k\geqslant 1}$. However, $X$ is complete, so that $\{x_{n_k}\}_{k\geqslant 1}$ is convergent in $X$, and hence $X$ is sequentially compact. □

### Example

Let $H$ denote the space of all sequences in $[0,1]$. Thus

$$H = \{x = \{x_n\}_{n \geqslant 1} : 0 \leqslant x_n \leqslant 1 \text{ for every } n \geqslant 1\}.$$

One may think of $H$ as the infinite product $\prod_{n=1}^{\infty}[0,1]$. Define $d : H \times H \to [0,\infty)$ by

$$d(x,y) = \sup_{n \geqslant 1} \left( \frac{|x_n - y_n|}{2^n} \right), \quad x, y \in H.$$

- $d$ is a metric. Indeed, $0 \leqslant d(x,y) \leqslant 1/2$, $d(x,y) = 0 \Leftrightarrow x = y$, and $d(x,z) \leqslant d(x,y) + d(y,z)$ for every $x, y, z \in H$.
- $d(x^{(n)}, x) \to 0$ as $n \to \infty \Leftrightarrow |x_k^{(n)} - x_k| \to 0$ as $n \to \infty$ for every $k \geqslant 1$. The part $\Rightarrow$ is clear. To see $\Leftarrow$, choose $N \geqslant 2$ such that $2^{-N} \leqslant \epsilon/4$, and choose $N_0 \geqslant 1$ such that $|x_k^{(n)} - x_k| < \epsilon/2$ for all $k = 1, \ldots, N-1$ and for all $n \geqslant N_0$.

Example (Continued ...)

- Let $r > 0$ and $x \in H$. Let $N \geqslant 2$ be such that $2^N r > 2$. Then

$$B_r(x) = \{y \in H : d(x,y) < r\} = \{y \in H : \sup_{n\geqslant 1}\left(\frac{|x_n - y_n|}{2^n}\right) < r\}$$

$$\subseteq \{y \in H : |x_n - y_n| < 2^n r \text{ for every } n \geqslant 1\}$$

$$= \{y \in H : |x_n - y_n| < 2^n r \text{ for } n = 1, \ldots, N-1\}.$$

$$\Longrightarrow B_r(x) = \{y \in H : |x_n - y_n| < 2^n r \text{ for } n = 1, \ldots, N-1\}.$$

Thus the open ball $B_r(x)$ in $H$ is of the form $B_{2r}(x_1) \times \cdots \times B_{2^{N-1}r}(x_{N-1}) \times [0,1] \times \cdots$ .

We claim that $H$ is sequentially compact. It suffices to check that $H$ is complete and totally bounded.

## Example (Continued ...)

- The Hilbert cube $H$ is totally bounded.

Let $\epsilon > 0$. Find an integer $N \geqslant 1$ such that $2^N \epsilon > 2$. Then, by the discussion on the previous slide, the ball $B_\epsilon(x)$ is of the form

$$B_{2\epsilon}(x_1) \times \cdots \times B_{2^{N-1}\epsilon}(x_{N-1}) \times [0,1] \times \cdots .$$

Since $[0,1]$ is totally bounded, for every $j = 1, \ldots, N-1$, it can be covered by finitely many balls (intervals) of the form $B_{2^j\epsilon}(x_{ij})$ (with finitely choices of $i = 1, \ldots, N_j$). If one takes the product

$$B_{2\epsilon}(x_{i_1 1}) \times \cdots \times B_{2^{N-1}\epsilon}(x_{i_{N-1} N-1}) \times [0,1] \times \cdots ,$$

then $H$ can be covered by finitely many balls of radius $\epsilon$.

## Example (Continued ...)

- The Hilbert cube $H$ is complete.

To see that $H$ is complete, let $\{x^{(n)}\}_{n\geqslant 1}$ be Cauchy in $H$.

- Note that $d(x^{(n)}, x^{(m)}) \to 0$ as $m, n \to \infty$ if and only if $|x_k^{(n)} - x_k^{(m)}| \to 0$ as $n \to \infty$ for every $k \geqslant 1$.
- Since $\mathbb{R}$ is complete, $\{x_k^{(n}\}_{n\geqslant 1}$ converges to some $x_k$ in $\mathbb{R}$ for every $k \geqslant 1$.
- Note that $x = \{x_k\}_{k\geqslant 1}$ belongs to $H$. It follows from the discussion on slide 128 that $d(x^{(n)}, x) \to 0$ as $n \to \infty$.

Since complete and totally bounded metric space is sequentially compact, the discussion above shows that the Hilbert cube $H$ is indeed sequentially compact.

### Problem

*Show that* $\{x = \{x_n\}_{n \geqslant 1} : -1 \leqslant x_n \leqslant 1$ *for every* $n \geqslant 1\}$ *as a subset of* $l^\infty$ *is not compact.*

### Problem

*Give an example of infinite sequentially compact subset of* $l^\infty$.

### Solution.

Let $K = \{x = \{x_n\}_{n \geqslant 1} : 0 \leqslant x_n \leqslant 1/2^n$ for every $n \geqslant 1\}$ and note that $f : H \to l^\infty$ given by

$$f(\{x_1, x_2, x_3, \ldots,\}) = \{x_1/2, x_2/4, x_3/8 \ldots,\}$$

satisfies $\|f(x) - f(y)\|_\infty = d(x, y)$. Thus $f$ is continuous, and since $H$ is sequentially compact, so is $f(H)$. However, $f(H) = K$ is a subset of $l^\infty$. $\square$

The above example provides an infinite compact subset of $l^\infty$.

### Theorem

*A sequentially compact metric space is compact.*

### Proof.

Let $\{U_\alpha : \alpha \in I\}$ be an open cover of $X$. For $x \in X$, let $r_x = \sup\{r \in \mathbb{R} : B_r(x) \subseteq U_\alpha \text{ for some } \alpha\}$ ($< \infty$ since $X$ is bounded).

- For $\epsilon = \inf\{r_x : x \in X\}$, there is $\{x_n\}_{n\geqslant 1}$ such that $r_{x_n} \to \epsilon$.
- By sequential compactness, $\{x_{n_k}\}_{k\geqslant 1}$ converges to $x \in X$.
- $x \in U_\alpha$ for some $\alpha$, and hence $B_r(x) \subseteq U_\alpha$ for some $r > 0$.
- If $d(x_{n_k}, x) < r/2$ for all large $k$, then $r_{x_k} > r/2$. Thus $\epsilon > 0$.
- Let $x_1 \in X$. Either $X = B_{\epsilon/2}(x_1) \subseteq U_\alpha$ for some $\alpha$ or there exists $x_2 \in X \setminus B_{\epsilon/2}(x_1)$.
- Either $X = B_{\epsilon/2}(x_1) \cup B_{\epsilon/2}(x_2) \subseteq U_\alpha \cup U_\beta$ for some $\alpha, \beta$ or there exists $x_3 \in X \setminus (B_{\epsilon/2}(x_1) \cup B_{\epsilon/2}(x_2))$.

One may continue this. However, this process ends in finitely many steps (otherwise we will get a sequence without a convergent subsequence contradicting sequential compactness). $\square$

## Problem

*Let $(X, d)$, $(Y, \rho)$ be two metric spaces. For $(x_j, y_j) \in X \times Y$ define $D((x_1, y_1), (x_2, y_2)) = \max\{d(x_1, x_2), \rho(y_1, y_2)\}$, Verify:*

1. *$D$ defines a metric on $X \times Y$.*
2. *If $X$ and $Y$ are compact, then so is $X \times Y$.*

## Hint.

For 2, it suffices to check that $X \times Y$ is sequentially compact.

- Let $\{(x_n, y_n)\}_{n \geqslant 1}$ be a sequence in $X \times Y$, Then $\{x_n\}_{n \geqslant 1} \subseteq X$ and $\{y_n\}_{n \geqslant 1} \subseteq Y$.
- Since $X$ is sequential compact, $\{x_{n_k}\}_{k \geqslant 1}$ is convergent.
- Now $\{y_{n_k}\}_{k \geqslant 1} \subseteq Y$, so by the sequential compactness of $Y$, $\{y_{n_{k_l}}\}_{l \geqslant 1}$ is convergent.

Thus $\{x_{n_{k_l}}\}_{l \geqslant 1}$ and $\{y_{n_{k_l}}\}_{l \geqslant 1}$ are convergent. Then $\{(x_{n_{k_l}}, y_{n_{k_l}})\}_{l \geqslant 1}$ is convergent in $X \times Y$. So $X \times Y$ is sequentially compact. $\square$

**Question** Can you say that $X$ and $Y$ are compact if so is $X \times Y$ ?

One may define sum of two subsets $A$ and $B$ of $\mathbb{R}^n$ as

$$A + B = \{a + b : a \in A, b \in B\}.$$

### Problem

*Let $A$ be closed, $B$ be compact in $\mathbb{R}^n$. Show that $A + B$ is closed.*

### Solution.

Let $\{x_n\}_{n \geqslant 1}$ be a sequence in $A + B$ such that $x_n \to x$ in $\mathbb{R}^n$.

- Thus $x_n = a_n + b_n$ for some $\{a_n\}_{n \geqslant 1} \subseteq A$ and $\{b_n\}_{n \geqslant 1} \subseteq B$.
- $\{b_{n_k}\}_{k \geqslant 1}$ converges to $b \in B$ ($B$ is sequentially compact).
- $\{x_{n_k}\}_{k \geqslant 1}$ converges to $x$, and $\{a_{n_k}\}_{k \geqslant 1}$ converges to $a \in A$ (difference of cgt sequences is cgt and $A$ is closed).

Thus $x = a + b$ for $a \in A$ and $b \in B$. Hence $A + B$ contains all its limit points showing that it is closed. $\square$

### Problem

*Let $A$ and $B$ be compact in $\mathbb{R}^n$. Show that $A + B$ is compact.*

### Problem

*Give an example of two closed sets whose sum is not closed.*

## Corollary (Lebesgue covering lemma)

*Let $(X, d)$ be a compact metric space. Let $\{U_\alpha : \alpha \in I\}$ be an open cover of $X$. Then there is a $\delta > 0$ such that if $A \subseteq X$ with diameter diam$(A) < \delta$, then there is $\alpha \in I$ such that $A \subseteq U_\alpha$.*

## Proof.

Suppose there is no $\delta > 0$ with the above property.

- For a positive integer $n$, letting $\delta = 1/n$, there exists $A_n \subseteq X$ with diameter $\mathrm{diam}(A_n) < 1/n$, then $A_n \not\subseteq U_\alpha$ for every $\alpha \in I$.
- Choose any $x_n \in A_n$ and note that $\{x_n\}_{n \geqslant 1}$ has a subsequence $\{x_{n_k}\}_{k \geqslant 1}$ converging to some $x \in X$ (by compactness of $X$).
- $x \in U_\alpha$ for some $\alpha \in I$ (since $\{U_\alpha : \alpha \in I\}$: open cover of $X$).
- Since $U_\alpha$ is open, $B_r(x) \subseteq U_\alpha$ for some $r > 0$.
- Find $k \geqslant 1$ such that $n_k r > 2$ and $x_{n_k} \in B_{r/2}(x)$.
- Now if $a \in A_{n_k}$, then $d(x, a) \leqslant d(x, x_{n_k}) + d(x_{n_k}, a) < r$.

This shows that $A_{n_k}$ is contained in $B_r(x) \subseteq U_\alpha$, $\Rightarrow\Leftarrow$ . $\square$

# Lebesgue number

## Definition

Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be an open cover of $(X, d)$. A real number $\delta > 0$ is said to be *Lebesgue number* for $\mathcal{U}$ if $A \subseteq X$ with diameter $\mathrm{diam}(A) < \delta$, then there is $\alpha \in I$ such that $A \subseteq U_\alpha$.

## Example

Consider the metric space $[0, 1]$ with relative metric induced by $d_1$. Let $\delta > 0$ be given. We claim that the Lebesgue number for the open covering $\mathcal{U} = \{[0, \delta)\} \cup \{(1/n, 1] : n \geqslant 1\}$ of $[0, 1]$ equals $\delta$. [7]

- If $N$ is a positive integer such that $1/N < \delta$ (which exists by AP), then $\{[0, \delta), (1/N, 1]\}$ is a finite open subcover of $[0, 1]$.
- Let $A$ be such that $\mathrm{diam}(A) < \delta$. Then either $A \subseteq [0, \delta)$ or $A \cap [\delta, 1] \neq \emptyset$. Let $a \in A$ be such that $a \geqslant \delta$.
- If $A \not\subseteq (1/n, 1]$ for any $n \geqslant 1$, (so $a > \delta$) then there is $a_M \in A$ such that $a_M \leqslant 1/M < a - \delta < a$. Thus $\mathrm{diam}(A) > \delta \Rightarrow\Leftarrow$

[7] I thought for a moment that there is a glitch but this calculation is fine.

# Continuous functions on compact metric spaces

### Theorem

*If $f$ is a continuous function from a compact metric space $(X, d)$ to any other metric space $(Y, \rho)$, then $f(X)$ is bounded.*

### Proof.

Fix $y_0 \in Y$ and write $Y = \cup_{n \geqslant 1} B_n^\rho(y_0)$.

- $f^{-1}(B_n^\rho(y_0))$ is open for every $n \geqslant 1$ (since $f$ is continuous).
- $X = \cup_{n \geqslant 1} f^{-1}(B_n^\rho(y_0))$.
- Since $X$ is compact, there exist positive integers $n_1, \ldots, n_k$ such that $X = \cup_{j=1}^k f^{-1}(B_{n_j}^\rho(y_0))$.

Let $N = \max\{n_1, \ldots, n_k\}$ and note that $X = f^{-1}(B_N^\rho(y_0))$. Thus $f(X) \subseteq B_N^\rho(y_0)$ is bounded. $\square$

• Let $f : X \to \mathbb{R}$ be a continuous function. Then $\sup_{x \in X} f(x)$ and $\inf_{x \in X} f(x)$ exist. Can we find $x_0$ and $x_1$ in $X$ such that $\sup_{x \in X} f(x) = f(x_0)$ and $\inf_{x \in X} f(x) = f(x_1)$ ?

Suppose that there is no $x_0 \in X$ such that $M = f(x_0)$, where $M = \sup_{x\in X} f(x)$. Thus $f(x') < \sup_{x\in X} f(x)$ for every $x' \in X$.

- $\{x' \in X : f(x') < M - 1/n\}$ is an open subset of $X$.
- $X = \cup_{n\geqslant 1}\{x' \in X : f(x') < M - 1/n\}$.
- There exist positive integers $n_1, \ldots, n_k$ such that

$$X = \cup_{j=1}^{k}\{x' \in X : f(x') < M - 1/n_j\}.$$

- $X = \{x' \in X : f(x') < M - 1/N\}$ with $N = \max\{n_1, \ldots, n_k\}$.

This would imply that $f(x) < M - 1/N$ for every $x \in X$, and hence $M < M - 1/N \Rightarrow\Leftarrow$

• Similarly, one can see that there exists $x_1 \in X$ such that $\inf_{x\in X} f(x) = f(x_1)$ (Exercise).

### Problem

*Let $X$ be a compact metric space. Then there exists no continuous map from $X$ onto $(0, 1)$.*

### Theorem (Criterion for homeomorphism)

*Let $X, Y$ be metric spaces and let $f : X \to Y$ be a bijective continuous map. If $X$ is compact, then $f$ is a homeomorphism.*

### Proof.

To check that $f^{-1}$ is continuous, let $A$ be a closed subset of $X$.

- $A$ is compact (being closed subset of a compact space $X$).
- $f(A)$ is compact (continuous image of a compact space).
- $f(A)$ is closed (compact subset of a metric space is closed).

Thus continuous image of a closed set is closed. Hence $f^{-1}$ is continuous. □

### Problem

*Let $X$ be a normed linear space with two norms $\|\cdot\|$ and $\|\cdot\|'$. Let $Y$ be a compact subset of $X$. If there exists $M > 0$ such that $\|x\| \leq M\|x\|'$ for every $x \in Y$, then show that $(Y, \|\cdot\|)$ and $(Y, \|\cdot\|')$ are homeomorphic.*

### Theorem

*Any continuous function $f$ from a compact metric space $(X, d)$ to any other metric space $(Y, \rho)$ is uniformly continuous.*

### Proof.

Since $f$ is continuous at $x$, given $\epsilon > 0$, there exists $\delta_x > 0$ such that $\rho(f(x), f(x')) < \epsilon/2$ whenever $x' \in X$ and $d(x, x') < \delta_x$.

- $\{B_{\delta_x}(x) : x \in X\}$ is an open cover of $X$.
- There is a $\delta > 0$ (Lebesgue number) such that if $A \subseteq X$ with diameter $\operatorname{diam}(A) < \delta$, then there is $x \in X$ such that $A \subseteq B_{\delta_x}(x)$ (by Lebesgue covering lemma).
- If $x_1, x_2 \in X$ are such that $d(x_1, x_2) < \delta$, then apply the above to $A = \{x_1, x_2\}$, we get $x \in X$ such that $\{x_1, x_2\} \subseteq B_{\delta_x}(x)$.

Thus $d(x_1, x) < \delta_x$ and $d(x_2, x) < \delta_x$. Hence, by continuity at $x$, $\rho(f(x_1), f(x_2)) \leqslant \rho(f(x_1), f(x)) + \rho(f(x_2), f(x)) < \epsilon$. $\square$

## Example

Let $f : (0,1) \to \mathbb{R}$ be a bounded, continuous function, which is monotonically increasing.

- $\inf_{x\in(0,1)} f(x)$ and $\sup_{x\in[0,1)} f(x)$ exist (since $f$ is bounded).
- Define $g : [0,1] \to \mathbb{R}$ by

$$g(x) = \begin{cases} f(x) & \text{if } x \in (0,1), \\ \inf_{x\in(0,1)} f(x) & \text{if } x = 0, \\ \sup_{x\in[0,1)} f(x) & \text{if } x = 1. \end{cases}$$

Note that $g : [0,1] \to \mathbb{R}$ is a continuous function (since $f$ is increasing), and hence it is uniformly continuous.

• One can not drop the assumption of boundedness in the above example (e.g. $f(x) = 1/x$, $x \in (0,1)$).

### Problem

*Given $(X, d)$, $(Y, \rho)$, such that $X$ is compact, verify:*

(1) $\sup_{x\in X} \rho(f(x), y) < \infty$ *for any* $y \in Y$.

(2) *Consider* $C(X, Y) = \{f : X \to Y : f \text{ is continuous}\}$. *Define*

$$D(f, g) = \sup_{x\in X} \rho(f(x), g(x)), \quad f, g \in C(X, Y).$$

*Then* $(C(X, Y), D)$ *is a metric space.*

### Hint.

For a positive integer $k$, define $U_k = \{x \in X : \rho(f(x), y) < k\}$. Verify that $\{U_k : k \geqslant 1\}$ is an open cover of $X$. Now (1) follows from the compactness of $X$.

By (1) and the triangle inequality, $D(f, g) < \infty$ for every $f, g \in C(X, Y)$. Since $\rho$ is a metric, so is $D$. $\square$

### Theorem

*Let $(X, d)$, $(Y, \rho)$ be metric spaces and $X$ be compact. If $Y$ is complete, then $C(X, Y)$ is complete.*

### Proof.

Let $\{f_n\}_{n \geqslant 1}$ be a Cauchy sequence in $C(X, Y)$, that is, given $\epsilon > 0$, there exists $N \geqslant 1$ such that $D(f_n, f_m) < \epsilon/3$ for every $m, n \geqslant N$.

- For every $x \in X$, $\{f_n(x)\}_{n \geqslant 1}$ is a Cauchy sequence in $Y$.
- For every $x \in X$, $\{f_n(x)\}_{n \geqslant 1}$ converges to some $f(x) \in Y$.
- $\rho(f_n(x), f_m(x)) < \frac{\epsilon}{3}$, $m, n \geqslant N \Rightarrow \rho(f(x), f_m(x)) \leqslant \frac{\epsilon}{3}$, $m \geqslant N$

To see that $f \in C(X, Y)$, note that for any $x, x' \in X$,

$$\rho(f(x), f(x')) \leqslant \rho(f(x), f_N(x)) + \rho(f_N(x), f_N(x')) + \rho(f_N(x'), f(x')),$$

which is $\leqslant 2\epsilon/3 + \rho(f_N(x), f_N(x'))$. However, $f_N$ is continuous and $X$ is compact, so that for some $\delta > 0$,

$$\rho(f_N(x), f_N(x')) < \epsilon/3 \text{ whenever } d(x, x') < \delta.$$

Thus $\rho(f(x), f(x')) < \epsilon$ whenever $d(x, x') < \delta$. $\square$

## Corollary

*Let $(X, d)$ be a compact metric space and consider*

$$C(X) = \{f : X \to \mathbb{R} : f \textit{ is continuous}\}.$$

*If* $\|f\|_\infty = \sup_{x \in X} |f(x)|$, $f \in C(X)$, *then* $C(X)$ *is a complete normed linear space with* $\| \cdot \|_\infty$.

The rest of the notes is devoted to the study of this metric space.

- What are compact subsets of $C(X)$ ? (Arzela-Ascoli Theorem)
- Whether $C(X)$ is separable ? (Weierstrass Theorem)
- Is there any proper clopen subset of $C(X)$ ?

Answer to the last question is No (Why ?)

# Totally bounded subsets of $C(X)$

**Theorem**

*Let $(X, d)$ be a compact metric space and let $\mathcal{F}$ be a totally bounded subset of $C(X)$. Then for each $\epsilon > 0$, there exists $\delta > 0$ such that $\sup_{f \in \mathcal{F}} |f(x) - f(y)| < \epsilon$ whenever $x, y \in X$ and $d(x, y) < \delta$.*

**Proof.**

Let $\epsilon > 0$. Then

- there exist $f_1, \dots, f_N \in \mathcal{F}$ such that $\mathcal{F} \subseteq \cup_{j=1}^{N} B_{\epsilon/3}(f_j)$.
- there exists $\delta_j > 0$ such that $|f_j(x) - f_j(y)| < \epsilon/3$ whenever $d(x, y) < \delta_j$ (since each $f_j$ is uniformly continuous; slide 141).

If $\delta = \min\{\delta_1, \dots, \delta_N\}$ and $d(x, y) < \delta$, then any $f \in \mathcal{F}$ lies in some ball $B_\epsilon(f_j)$, and hence
$|f(x) - f(y)| \leqslant |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < \epsilon$. $\square$

# Equicontinuity

### Definition

Let $(X, d)$ be a compact metric space and let $\mathcal{F}$ be a subset of $C(X)$. We say that $\mathcal{F}$ is <u>equicontinuous</u> if for each $\epsilon > 0$, there exists $\delta > 0$ such that $\sup_{f \in \mathcal{F}} |f(x) - f(y)| < \epsilon$ whenever $x, y \in X$ and $d(x, y) < \delta$.

• Every compact subset of $C(X)$ is equicontinuous (since compact metric space is totally bounded; see slides 121 and 133).

**Question** What are all compact subsets of $C(X)$ ?

• A compact subset of $C(X)$ is closed and bounded (see slide 114).

**Question** Are equicontinuous, closed and bounded subsets of $C(X)$ all compact subsets of $C(X)$ ?

## Theorem (Generalized Arzela-Ascoli Theorem)

*Suppose that $(X, d)$ is a compact metric space and let $\{f_n\}_{n\geqslant 1}$ be a sequence in $C(X)$ such that*

(A) $\sup_{n\geqslant 1} |f_n(x)| < \infty$ *for every* $x \in X$,

(B) *for each* $\epsilon > 0$ *and* $x \in X$, *there exists* $\delta_x > 0$ *such that* $\sup_{n\geqslant 1} |f_n(x) - f_n(y)| < \epsilon$ *whenever* $y \in X$ *and* $d(x, y) < \delta_x$.

*Then there exist a subsequence $\{f_{n_k}\}_{k\geqslant 1}$ of $\{f_n\}_{n\geqslant 1}$ and $f \in C(X)$ such that $\|f_{n_k} - f\|_\infty \to 0$ as $k \to \infty$.*

**Remark** Let $\mathcal{F}$ be a subset of $C(X)$.

- If $\mathcal{F}$ is bounded, then any sequence $\{f_n\}_{n\geqslant 1}$ in $\mathcal{F}$ satisfies (A).
- Condition (B) is the "equicontinuity" of $\{f_n\}_{n\geqslant 1}$ at $x$.
- If all sequences in $\mathcal{F}$ satisfy (A) and (B), then $\mathcal{F}$ is compact.

## Corollary (Arzela-Ascoli Theorem)

*A subset $\mathcal{F}$ of $C(X)$ is (sequentially) compact if and only if it is closed, bounded and equicontinuous.*

## Proof of Generalized Arzela-Ascoli Theorem.

The proof is divided into two steps:

**Step 1**

- Let $\{x_n\}_{n\geqslant 1}$ be a countable dense subset of $X$ (a compact metric space is separable; see slide 125).
- There exists a convergent subsequence $\{f_{n_k}(x_1)\}_{k\geqslant 1}$ of the bounded sequence $\{f_n(x_1)\}_{n\geqslant 1}$ (use (A) and H-B Theorem)
- There exists a convergent subsequence $\{f_{n_{k_l}}(x_2)\}_{l\geqslant 1}$ of the bounded sequence $\{f_{n_k}(x_2)\}_{k\geqslant 1}$ (use (A) and H-B Theorem)
- $\{f_{n_{k_l}}(x_1)\}_{l\geqslant 1}$ is convergent (subsequence of a convegent sequence is convergent)

Continuing this, we obtain a subsequence $\{f_{n_k}\}_{k\geqslant 1}$ such that $\{f_{n_k}(x_j)\}_{k\geqslant 1}$ is convergent for every $j \geqslant 1$ (form the set $\{n_k\}_{k\geqslant 1}$ by picking up first element in first subsequence, second element in second subsequence and so on). $\square$

## Proof of Generalized Arzela-Ascoli Theorem continued ...

**Step 2** Let $\delta_x$ be as given in (B).

- There exist $a_1, \ldots, a_m \in X$ such that $X = \cup_{k=1}^m B_{\delta_{a_k}}(a_k)$ (since $X = \cup_{a \in X} B_{\delta_a}(a)$ and $X$ is compact)
- Choose $y_k \in B_{\delta_{a_k}}(a_k) \cap \{x_n\}_{n \geqslant 1}$ and let $N \geqslant 1$ be such that $|f_{n_i}(y_k) - f_{n_j}(y_k)| < \epsilon$ for $i, j \geqslant N$ and $k = 1, \ldots, m$ (Step 1)
- If $x \in X$. then $x \in B_{\delta_{a_k}}(a_k)$ for some $k = 1, \ldots, m$.

Note that for $i, j \geqslant N$, (use (B) four times)

$$|f_{n_i}(x) - f_{n_j}(x)| \leqslant |f_{n_i}(x) - f_{n_i}(a_k)| + |f_{n_i}(a_k) - f_{n_i}(y_k)| + |f_{n_i}(y_k) - f_{n_j}(y_k)|$$

$$+ |f_{n_j}(y_k) - f_{n_j}(a_k)| + |f_{n_j}(a_k) - f_{n_j}(x)| \leqslant 5\epsilon.$$

Thus the sequence $\{f_{n_j}\}_{j \geqslant 1}$ is Cauchy in $C(X)$. Since $C(X)$ is complete, $\{f_{n_j}\}_{j \geqslant 1}$ is convergent. $\square$

## Problem

*Let $(X, d)$ be a compact metric space and let $F : X \times X \to \mathbb{R}$ be a continuous function. For $y \in X$, define $f_y(x) = F(x, y)$. Show that the family $\{f_y\}_{y \in X}$ is bounded and equicontinuous.*

## Solution.

Note that $\|f_y\|_\infty = \sup_{x \in X} |F(x, y)| \leqslant ||F||_\infty < \infty$ (since $F$ is continuous and $X \times X$ is compact; see slide 134). Thus $\sup_{y \in X} \|f_y\|_\infty < \infty$, and hence $\{f_y\}_{y \in X}$ is bounded.

Since $F$ is continuous on a compact metric space, $F$ is uniformly continuous. Thus, for given $\epsilon > 0$, there exists $\delta > 0$ such that $|f_y(x) - f_{y'}(x')| < \epsilon$ whenever $x, x', y, y' \in X$, $d(x, x') < \delta$ and $d(y, y') < \delta$. Let $y' = y$. $\square$

- By (Generalized) Arzela-Ascoli Theorem, any sequence in $\{f_y\}_{y \in X}$ has a subsequence convergent in $C(X)$.

# Connected metric spaces

### Definition

Let $(X, d)$ be a metric space. We say that $X$ is <u>connected</u> if the only subsets of $X$, which are both open and closed are $\emptyset$ and $X$.

### Example

We check that $\mathbb{R}$ is connected. Write $\mathbb{R} = A \sqcup B$ (disjoint union) for nonempty, open sets $A$ and $B$. Let $a \in A$, $b \in B$ with $a < b$.

- $[a, b] = A_0 \sqcup B_0$, where $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$.
- Let $c = \sup A_0 \in [a, b]$. Thus either $c \in A_0$ or $c \in B_0$.
- If $c \in A_0$ then either $c = a$ or $a < c < b$ (since $A$ is closed).
- Since $A$ is open, there is $r > 0$ such that $[c, c + r) \subseteq A$.

This contradicts that $c = \sup A_0$. Similarly, prove that $c \notin B_0$.

### Problem

*Show that for any $a \in \mathbb{R}$, $\mathbb{R} \setminus \{a\}$ is not connected.*

### Example

Any normed linear space $X$ over $\mathbb{R}$ is connected. To see this, assume that $X = A \sqcup B$ such that $A$ and $B$ are open subsets of $X$. We claim that one of the $A$ and $B$ is empty. If not, then for $a \in A$ and $b \in B$, consider $f : \mathbb{R} \to X$ given by

$$f(t) = (1-t)a + tb, \quad t \in \mathbb{R}.$$

Then $f(0) = a$ and $f(1) = b$. Since $f$ is continuous, $f^{-1}(A)$ and $f^{-1}(B)$ are open subsets of $\mathbb{R}$. Since these are proper and disjoint, we arrive at the conclusion that $\mathbb{R}$ is not connected $\Rightarrow\Leftarrow$

### Example

The space $\mathbb{Q}$ is not connected. In fact, for any irrational $x \in \mathbb{R}$, the set $[-x, x] \cap \mathbb{Q} = (-x, x) \cap \mathbb{Q}$ is both open and closed in $\mathbb{Q}$.

### Theorem

*Let $f : X \to Y$ be continuous. If $X$ is connected, then so is $f(X)$.*

### Proof.

Suppose $f(X)$ has a nonempty open and closed subset, say, $U$. Then $f^{-1}(U)$ is an open and closed subset of $X$. However, $X$ is connected, so $f^{-1}(U) = \emptyset$ or $f^{-1}(U) = X$. Since $f^{-1}(U) \neq \emptyset$, we conclude that $f^{-1}(U) = X$, and hence $U = f(X)$. □

• A path connecting points $x, y$ is a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$. Since $[a, b]$ is connected, so is $f([a, b])$.

### Problem

*Show that union of line segments $L_1, \ldots, L_m$ in a normed linear space is connected if $L_1 \cap L_2 \neq \emptyset$, $L_2 \cap L_3 \neq \emptyset$, $\ldots$, $L_m \cap L_1 \neq \emptyset$.*

## Example

Let us see that $\mathbb{R}^n \setminus \{0\}$ is connected. Suppose there exists a proper set $U$ which is both open and closed in $\mathbb{R}^n \setminus \{0\}$.

- Let $x \in U$ and $y \in (\mathbb{R}^n \setminus \{0\}) \setminus U$.
- $x$ and $y$ can be connected by union $L$ of line segments.
- $L$ contains open and closed proper subset $L \cap U$.

However, $L$ being connected, $L \cap U = L$ or $L \cap U = \emptyset \Rightarrow\Leftarrow$

## Example

The unit sphere $\mathbb{S}^n = \{x \in \mathbb{R}^n : \|x\|_2 = 1\}$ in $\mathbb{R}^n$ is connected. Indeed, $\mathbb{R}^n \setminus \{0\}$ is connected and $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^n$ given by

$$g(x) = x/\|x\|_2, \quad x \in \mathbb{R}^n \setminus \{0\}$$

is a continuous surjection.

### Theorem

*Let $(X, d)$ be a metric space and let $A \subseteq X$. If $A$ is connected and $B$ is a set such that $A \subseteq B \subseteq \overline{A}$, then $B$ is also connected.*

### Proof.

Let $B = C \sqcup D$ for open sets $C$ and $D$.

- $A = (A \cap C) \sqcup (A \cap D)$ for open sets $A \cap C$ and $A \cap D$.
- Since $A$ is connected, either $A \cap C = \emptyset$ or $A \subseteq C$.
- Thus either $A \subseteq C$ or $A \subseteq D$.
- Since $C$ and $D$ are closed in B, we obtain $\overline{A} \cap B \subseteq C$ or $\overline{A} \cap B \subseteq D$

It follows that either $C = B$ or $D = B$. However, in this case, either $C = \emptyset$ or $D = \emptyset$, which shows that $B$ is connected. $\square$

### Corollary

*The closure of a connected set is connected.*

### Example (Topologist's sine curve)

Consider the continuous function $s : (0,1] \to [0,1]$ given by

$$s(x) = \sin(1/x), \quad x \in (0,1].$$

Let $S$ denote the graph of $s$ given by

$$S = \{(x, s(x)) \in \mathbb{R}^2 : x \in (0,1]\}.$$

Note that $S = f((0,1])$, where $f(x) = (x, s(x))$ is a continuous function. Thus $S$ is connected, and hence $\overline{S}$ is also connected.

The metric space $\overline{S}$ (with relative metric induced by $d_2$) is commonly known as the <u>topologist's sine curve</u>.

# Path-connected metric spaces

### Definition

Let $(X, d)$ be a metric space. We say that $X$ is path-connected if for any two points $x, y \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

**Remark** If $U$ is path-connected then for open sets $V, W$ of $U$ such that $U = V \sqcup W$ and for any path $f$ in $U$, the range of $f$ being the continuous image of the connected set $[0, 1]$ is connected, and hence lies entirely either in $V$ or $W$. This shows that one of $V, W$ must be empty, that is, $U$ is connected.

### Problem

*Show that the complement of any countable subset $\mathcal{C}$ in $\mathbb{R}^2$ is path-connected.*

### Hint.

Given $p, q \in \mathbb{R}^2 \setminus \mathcal{C}$, consider the uncountable set $\mathcal{F}$ given by $\{f : f \text{ is a path connecting } p \text{ and } q\}$. $\square$

## Example (Comb space)

Let $K$ denote the set $\{1/n : n \geq 1\}$ and consider

$$E = ([0,1] \times \{0\}) \ \cup \ (K \times [0,1]) \subseteq \mathbb{R}^2.$$

- $E$ is path-connected (see diagram).

The comb space $\mathcal{C}$ is defined to be the space $E \cup (\{0\} \times [0,1])$.

- $\mathcal{C}$ is also connected (since $E$ is connected and $\overline{E} = \mathcal{C}$)

The deleted comb space $\mathcal{C}_0$ is defined as $E \cup \{(0,1)\}$.

- $\mathcal{C}_0$ is connected (since $E \subseteq \mathcal{C}_0 \subseteq \overline{E}$)
- $\mathcal{C}_0$ is not path-connected (since there is no path which connects the points $p = (0,1)$ and $q = (1,0)$)

### Theorem

*The deleted comb space is not path-connected.*

### Proof.

Suppose, contrary to this, that there is a path $\gamma : [0,1] \to \mathcal{C}_0$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

- $\gamma^{-1}(\{p\})$ is a closed subset of $[0,1]$. and hence it is compact. Let $t_0 \in [0,1]$ be its maximum.
- Consider the projection $P_1(x,y) = x$ of $\mathbb{R}^2$ onto the $X$-axis.
- Let $\{t_n\}_{n \geqslant 1} \subseteq (t_0, 1]$ be a sequence converging to $t_0$.

If, for every $n \geqslant 1$, there exists $t_0 < s_n < t_n$ such that $\gamma(s_n) = (x_n, 0)$ for some $x_n \in [0,1] \setminus K$, then $\{s_n\}_{n \geqslant 1}$ converges to $t_0$, by the continuity, $(x_n, 0) = \gamma(s_n) \to \gamma(t_0) = p = (0,1) \Rightarrow\Leftarrow$

- There exists $t_1 \in (t_0, 1]$ such that $(P_1 \circ \gamma)(t_0, t_1) \subseteq K$.
- $(P_1 \circ \gamma)(t_0, t_1)$ is a connected subset of $K$ containing 1 (wlog)

Thus $(P_1 \circ \gamma)(t_0, t_1) = \{1\}$. By continuity, $(P_1 \circ \gamma)[t_0, t_1) = \{1\}$, which is impossible since $(P_1 \circ \gamma)(t_0) = 0$. $\square$

### Theorem

*Let $f : X \to Y$ be a continuous surjection. If $X$ is path-connected, then so is $Y$.*

### Proof.

Let $y_0, y_1 \in Y$. Let $x_0, x_1 \in X$ be such that $f(x_0) = y_0$ and $f(x_1) = y_1$. If $\gamma$ is a path connecting $x_0$ and $x_1$, then $f \circ \gamma$ is a path connecting $y_0$ and $y_1$. $\square$

If $p$ is a polynomial in the real variables $x_1, \cdots, x_n$ and $Z(p) = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^d : p(x) = 0\}$, then $\mathbb{R}^n \setminus Z(p)$ is not necessarily path-connected.

- If $p(x, y) = x^2 + y^2 - 1$, then $Z(p)$ is equal to

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}.$$

Clearly, $\mathbb{R}^2 \setminus Z(p)$ is not connected.

## Corollary

*If $p$ is a polynomial in the complex variables $z_1, \cdots, z_n$ and $Z(p) = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : p(z) = 0\}$, then $\mathbb{C}^n \setminus Z(p)$ is path-connected.*

## Proof.

Let $z, w \in \mathbb{C}^n \setminus Z(p)$. Consider the straight-line path

$$\gamma(t) = (1-t)z + tw, \quad t \in \mathbb{C}.$$

- $Z = \{t \in \mathbb{C} : \gamma(t) \in Z(p)\}$ is the set of zeros of $p \circ \gamma$
- $Z$ is a finite subset of $\mathbb{C}$ ($p \circ \gamma$ is a polynomial in one variable)
- $\mathbb{C} \setminus Z$ is path-connected (see slide 158)
- $\gamma$ maps $\mathbb{C} \setminus Z$ continuously into $\mathbb{C}^n \setminus Z(p)$

In particular, $z$ and $w$ belong to the path-connected subset $\gamma(\mathbb{C} \setminus Z)$ of $\mathbb{C}^n \setminus Z(p)$. □

### Example

The general linear group $GL_n(\mathbb{C})$ of all invertible $n \times n$ matrices with complex entries is path-connected.

- Define $f : GL_n(\mathbb{C}) \to \mathbb{C}^{n^2}$ by

$$f(A) = (a_{11}, \ldots, a_{1n}, \ldots, a_{n1}, \ldots, a_{nn}), \quad A \in GL_n(\mathbb{C}).$$

- $f$ is a (linear) homeomorphism
- $f$ maps $GL_n(\mathbb{C})$ onto $\mathbb{C}^{n^2} \setminus Z(\det)$, where det is the complex polynomial in the variables $a_{i,j}$, $1 \leqslant i, j \leqslant n$, which sends $f(A)$ to the determinant of $A$

Thus $GL_n(\mathbb{C})$ is path-connected (since so is $\mathbb{C}^{n^2} \setminus Z(\det)$).

### Problem

*Show that $GL_n(\mathbb{R})$ is not connected.*

## Problem

*For an open subset $U$ of $\mathbb{R}^n$, show that $U$ is connected if and only if $U$ is path-connected.*

## Solution.

We have already seen that path-connected space is connected. To see the converse, consider for any $p \in U$, the set

$$S = \{x \in U : \text{there is a path connecting } p \text{ and } x\}.$$

We claim that $S$ is the whole of $U$.

- $S$ is nonempty (since $p \in U$)
- $S$ is open (if $x \in S$ and $B_r(x) \subseteq U$ for some $r > 0$, then any $y \in B_r(x)$ can be connected to $x$ and $x$ can be connected to $p$, so $y$ can be connected to $p \Rightarrow B_r(x) \subseteq S$)
- $S$ is closed (let $\{x_n\}_{n \geqslant 1}$ be a sequence in $S$ converging to $x$, connect $x$ and $x_N$ (for large $N$) and $x_N$ to $p$)

Since $U$ is connected, $S = U$. $\square$

# Pointwise convergence

## Example

For $n \geqslant 1$, consider the function $f_n(x) = x^n$ for $x \in [0,1]$. Note that $\{f_n\}_{n \geqslant 1}$ converges pointwise to $f$, where $f(x) = 0$ for $x \in [0,1)$ and $f(1) = 1$. Thus the pointwise limit of a sequence of continuous functions is not necessarily continuous.

## Problem

*For $m \geqslant 1$, consider the function $f_m(x) = \lim_{n\to\infty}(\cos(m!\pi x))^n$ for $x \in \mathbb{R}$. Verify the following:*

(1) *$\{f_m\}_{m \geqslant 1}$ converges pointwise to $f$, where $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, and $f(x) = 1$ for $x \in \mathbb{Q}$.*

(2) *$f$ is discontinuous everywhere.*

## Hint.

If $x = p/q \in (0,1)$, $q \neq 0$ then $\lim_{n\to\infty}(\cos(m!\pi x))^n = 1$ for every $m \geqslant q$, and hence $f(p/q) = 1$. If $x \notin \mathbb{Q}$, then $\cos(m!\pi x) < 1$, and hence $\lim_{n\to\infty}(\cos(m!\pi x))^n = 0$ and $f(x) = 0$. $\square$

For a metric space $(X, d)$, consider the vector space *problem* of all bounded functions $f : X \to \mathbb{R}$.

- $B(X)$ is a normed linear space with norm $\|f\|_\infty = \sup_{x\in X} |f(x)|$

## Theorem

*$B(X)$ is a complete normed linear space.*

## Proof.

Let $\{f_n\}_{n\geqslant 1}$ be a Cauchy sequence in $B(X)$.

- $\sup_{n\geqslant 1} \|f_n\|_\infty < \infty$ (every Cauchy sequence is bounded)
- For any $x \in X$, $\{f_n(x)\}_{n\geqslant 1}$ is Cauchy. Indeed,

$$|f_m(x) - f_n(x)| \leqslant \|f_m - f_n\|_\infty, \quad m, n \geqslant 1$$

- Define $f(x) = \lim_{n\to\infty} f_n(x)$, $x \in X$
- $f \in B(X)$ (since $\|f\|_\infty \leqslant \|f_m - f_n\|_\infty + \|f_n\|_\infty$)

Given $\epsilon > 0$, there exists $N \geqslant 1$ such that $\|f_m - f_n\| < \epsilon$ for all $m, n \geqslant N$. Thus for every $x \in X$, $|f_m(x) - f_n(x)| \leqslant \epsilon$ for every $m, n \geqslant N$. Now let $n \to \infty$ and take supremum over $X$. $\square$

## Problem

*Let $(X, d)$ be a metric space and fix $z \in X$. For every $x \in X$, define $f_x(y) = d(x, y) - d(y, z)$. Verify the following:*

(1) *For each $x \in X$, $f_x \in B(X)$.*

(2) *$F : X \to B(X)$ by $F(x) = f_x$ satisfies $\|F(x)\|_\infty \leqslant d(x, z)$.*

(3) *For every $x, y \in X$, $\|F(x) - F(y)\|_\infty = d(x, y)$.*

*Conclude that $X$ is homeomorphic to $(F(X), \|\cdot\|_\infty)$.*

## Solution.

(1) follows from $|f_x(y)| \leqslant d(x, z)$, while (2) follows from (1). To see (3), note that for any $x, y \in X$, by the triangle inequality,

$$\|F(x) - F(y)\|_\infty = \sup_{w \in X} |d(x, w) - d(y, w)| \leqslant d(x, y).$$

Since $|f_x(x) - f_y(x)| = |-d(x, z) - d(y, x) + d(x, z)| = d(x, y)$, (3) follows. $\square$

- The subspace $\overline{F(X)}$ of $B(X)$ is said to be the <u>completion</u> of $X$.

# Uniform convergence

Let $(X, d)$ be a metric space. For $n \geqslant 1$, let $f_n, f : X \to \mathbb{R}$ be such that $f_n - f \in B(X)$. The sequence $\{f_n\}_{n \geqslant 1}$ converges <u>uniformly</u> to $f$ if

$$\|f_n - f\|_\infty = \sup_{x \in X} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

**Theorem**

*Let $(X, d)$ be a compact metric space. Let $\{f_n\}_{n \geqslant 1}$ be a sequence of continuous functions on $X$. If $\{f_n\}_{n \geqslant 1}$ converges uniformly to $f$ on $X$, then $f$ is continuous.*

**Proof.**

This follows from the fact that $C(X)$ is complete metric space. $\square$

- In general, pointwise convergence $\not\Rightarrow$ uniform convergence

## Theorem

*Let $\{f_n\}_{n\geqslant 1}$ be a sequence of continuous functions. If $\{f_n\}_{n\geqslant 1}$ converges uniformly to $f$ on $[a, b]$ then*

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

## Proof.

Note that $|\int_a^b f_n(x)dx - \int_a^b f(x)dx| = |\int_a^b (f_n(x) - f(x))dx| \leqslant \int_a^b |f_n(x) - f(x)|dx \leqslant \|f_n - f\|_\infty (b - a)$. □

## Problem

*For $n \geqslant 1$, consider the function $f_n(x) = nx(1 - x^2)^n$ for $x \in [0, 1]$. Verify that $\{f_n\}_{n\geqslant 1}$ converges pointwise to $f$, where $f(x) = 0$ for all $x \in [0, 1]$, but $\{f_n\}_{n\geqslant 1}$ does not converge uniformly to $f$.*

## Hint.

For second part, use $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$. □

# Dini's Theorem

Sometimes pointwise convergence $\Rightarrow$ uniform convergence.

**Theorem**

*Let $(X, d)$ be a compact metric space. Let $\{f_n\}_{n \geqslant 1}$ be a sequence in $C(X)$ converging pointwise to a continuous function $f$. If $\{f_n(x)\}_{n \geqslant 1}$ is decreasing for all $x \in X$, then $\{f_n\}_{n \geqslant 1}$ converges uniformly to $f$.*

**Proof.**

Let $g_n = f_n - f \geqslant 0$. For $\epsilon > 0$, let $K_n = \{x \in X : g_n(x) \geqslant \epsilon\}$.

- $K_n$ is compact and $K_{n+1} \subseteq K_n$ (since $K_n$ is closed, $g_n \geqslant g_{n+1}$)
- If each $K_n \neq \emptyset$, then so is finitely many sets from $\{K_n\}_{n \geqslant 1}$
- If each $K_n \neq \emptyset$, then by the finite intersection property (see [Assignment 7, Exercise 2]), $\cap_{n=1}^{\infty} K_n \neq \emptyset$

However, if $x \in X$, then since $g_n(x) \to 0$, $x \notin K_n$ for sufficiently large $n$. Hence $K_N$ is empty for some $N$, that is, $0 \leqslant g_n(x) < \epsilon$ for every $x \in X$ and for every $n \geqslant N$. $\square$

### Example

Define a sequence $\{p_n\}_{n \geqslant 0}$ of polynomials by $p_0(x) = 0$, and

$$p_{n+1}(x) = p_n(x) + (x^2 - p_n(x)^2)/2, \quad n \geqslant 0.$$

A routine calculation shows that

$$|x| - P_{n+1}(x) = (|x| - P_n(x))(1 - (|x| + P_n(x))/2), \quad n \geqslant 0.$$

One may now verify inductively that

$$0 \leqslant p_n(x) \leqslant p_{n+1}(x) \leqslant |x|, \quad x \in [-1, 1],\ n \geqslant 0.$$

In particular, $\{p_n(x)\}_{n \geqslant 0}$ converges pointwise to $|x|$. Now apply Dini's Theorem to $f_n = -p_n$, $n \geqslant 0$ to conclude that $\{p_n\}_{n \geqslant 0}$ converges uniformly to $f(x) = |x|$ on $[-1, 1]$.

## Problem

*Show that the function* $g : \mathbb{R} \to \mathbb{R}$ *given by*

$$g(x) = \begin{cases} 0 & \text{if } x \leqslant 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

*Show that for any* $\alpha > 0$, $g$ *can be uniformly approximated by polynomials on* $[-\alpha, \alpha]$.

## Hint.

Note that $g(x) = \frac{1}{2}(x + |x|)$. $\square$

## Problem

*Let* $\{p_n\}_{n \geqslant 1}$ *be a sequence of polynomials of degree* $d_n$. *Suppose that* $\|p_n - f\|_\infty \to 0$ *as* $n \to \infty$ *for some* $f \in C([a, b])$. *If* $f$ *is not a polynomial, then show that* $d_n \to \infty$ *as* $n \to \infty$.

## Hint.

The subspace of polynomials of degree less than or equal to $d$ is finite-dimensional, and hence it is closed in $C([a, b])$. $\square$

### Theorem

- $\overline{\mathcal{P}} = \{f \in B(X) : \exists\ \{p_n\}_{n\geqslant 1}$ *such that* $\|p_n - f\|_\infty \to 0\} \subseteq C(X)$.
- *There are compact metric spaces* $X$ *with* $\overline{\mathcal{P}} \subsetneq C(X)$.

The first part follows from the completeness of $C(X)$.

### Example (Failure of polynomial approximation)

Let $X = \{z \in \mathbb{C} : |z| = 1\}$ with $d(z, w) = |z - w|$, $z, w \in \mathbb{C}$.

- $X$ is a compact metric space ($X \subseteq \mathbb{C}$ is closed and bounded)
- Consider the continuous function $f(z) = \overline{z}$ ($\mathbb{C}$-conjugate)
- If $\gamma(t) = e^{it}$, $0 \leqslant t \leqslant 2\pi$, then $\int_0^{2\pi} p(\gamma(t))\gamma'(t)dt = 0$ for any polynomial $p$ and $\int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = 1$
- If $\{p_n\}_{n\geqslant 1}$ is a sequence of complex polynomials $p_n$ in $z$ such that $\|p_n - f\|_\infty \to 0$, then
$$\int_0^{2\pi} p_n(\gamma(t))\gamma'(t)dt \to \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt \text{ as } n \to \infty \quad \Rightarrow\Leftarrow$$

Thus $f \in C(X) \setminus \overline{\mathcal{P}}$.

### Lemma

*Let $(X, d)$ be a compact metric space. The following are true:*

(1) *$\overline{\mathcal{P}}$ is a complete normed linear space.*

(2) *Given $f \in C(X)$, if, for every $\epsilon > 0$, there exists $g_\epsilon \in \overline{\mathcal{P}}$ such that $\|f - g_\epsilon\|_\infty < \epsilon$, then $f \in \overline{\mathcal{P}}$.*

### Proof.

(1): Clearly, linear combination of functions approximated uniformly by polynomials are again in $\overline{\mathcal{P}}$. Since $\overline{\mathcal{P}}$ is a closed subset of $C(X)$ and $C(X)$ is complete, $\overline{\mathcal{P}}$ is complete.

(2): Every ball $B_\epsilon(f)$ intersects $\overline{\mathcal{P}}$, and hence $f$ is a limit point of $\overline{\mathcal{P}}$. Thus $f \in \overline{\mathcal{P}}$. $\square$

### Theorem (Weierstrass' approximation theorem)

*Any $f \in C([a, b])$ can be approximated uniformly by polynomials.*

## Lebesgue's Proof of Weierstrass' Theorem*[8].

Since $f$ is uniformly continuous on $[a, b]$, there exists an integer $N \geqslant 1$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < 1/N$. For $x_i := a + (b-a)(i/N)$ $(i = 0, \cdots, N)$, consider the function $h(x)$ with graph equal to a polygon (see diagram) of vertices at

$$(a, f(a)), (x_1, f(x_1)), \cdots, (x_{n-1}, f(x_{n-1})), (b, f(b)).$$

- $\|f - h\|_\infty < \epsilon$. Indeed, if $x \in (x_i, x_{i+1})$ then

$$f(x) - h(x) = f(x) - ((1-t)f(x_i) + tf(x_{i+1}))$$

$$= (1-t)(f(x) - f(x_i)) + t(f(x) - f(x_{i+1})) \Rightarrow |f(x) - h(x)| < \epsilon$$

- If $h(x) = f(a) + \sum_{i=0}^{N-1} c_i g(x - x_i)$ $(x \in [a, b])$ for some scalars $c_0, \cdots, c_{N-1}$ and $g(x) = \frac{1}{2}(x + |x|)$

Since $g$ can be approximated uniformly by polynomials (see slide 172), $h \in \overline{\mathcal{P}}$, and hence $f \in \overline{\mathcal{P}}$ (see the lemma on last slide). $\square$

[8] J. Burkill, Lectures on Approximation by Polynomials, Lecture Notes, TIFR, Bombay, 1959

### Problem

*Use Weierstrass' Theorem to show that* $C([a,b])$ *is separable.*

### Hint.

Polynomials with rational coefficients are countable and dense. $\square$

### Problem

*Let* $f \in C[a,b]$ *be such that* $\int_a^b t^n f(t)dt = 0$ *for all non-negative integers* $n$*. Show that* $f(t) = 0$ *for every* $t \in [a,b]$*.*

### Solution.

Note that $\int_a^b p(t)f(t)dt = 0$ for any polynomial $p$. If $\{p_n\}$ is a sequence converging uniformly to $f$, then $\int_a^b f(t)^2 dt = 0$, and hence $f = 0$. $\square$

### Problem

*Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Show that there exists a sequence $\{p_n\}_{n \geqslant 1}$ of polynomials such that*

$$\int_a^b |p_n(x) - f(x)|^p dx \to 0 \text{ as } n \to \infty.$$

Recall that $\chi_A$ is 1 on $A$ and 0 outside $A$.

### Problem

*For $0 \leq x < y \leq 1$, consider $\chi_{[x,y]} : [0, 1] \to \mathbb{R}$. Show that there exists a sequence $\{p_n\}$ of polynomials such that*

$$\int_0^1 |p_n(x) - \chi_{[x,y]}(x)| dx \to 0 \text{ as } n \to \infty.$$

*Conclude that finite linear combination of indicator functions of subintervals of $[0, 1]$ can be approximated by polynomials.*

### Hint.

Approximate $\chi_{[x,y]}$ by continuous functions. $\square$

## Problem

*Let $f : [a, b] \to \mathbb{R}$ be a continuouly differentiable function. Show that there exists a sequence $\{r_n\}_{n \geqslant 1}$ of polynomials such that*

$$\|r_n - f\|_\infty \to 0 \text{ and } \|r_n' - f'\|_\infty \to 0 \text{ as } n \to \infty.$$

*Conclude that $C^1[a, b]$ (the space of continuously differentiable functions on $[a, b]$) is a separable normed linear space with norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$.*

## Solution.

Let $g(x) = f(x) - f(a)$ and note that $g' = f'$. Find a sequence $\{q_n\}_{n \geqslant 1}$ of polynomials such that $\|q_n - g'\|_\infty \to 0$. Set $p_n(x) := \int_a^x q_n(t)dt$. Note that $p_n' = q_n$, and hence $\|p_n' - g'\|_\infty \to 0$. Also,

$$|p_n(x) - g(x)| = \left| \int_a^x q_n(t)dt - \int_a^x g'(t)dt \right| \leq (b - a)\|q_n - g'\|_\infty.$$

Let $r_n(x) := p_n(x) + f(a)$. $\square$

# Pointwise limit of a sequence of continuous functions

• We have seen that the pointwise limit of sequence of functions can be discontinuous at every point (see slide 165)

**Question** Can the pointwise limit of sequence of continuous functions be discontinuous at every point ?

**Answer** No

A subset $A$ of a metric space $(X, d)$ is said to be nowhere dense if the interior of the closure of $A$ is empty, that is, $(\overline{A})^\circ = \emptyset$.

### Theorem (Baire-Osgood Theorem)

*If $f : [a, b] \to \mathbb{R}$ is a pointwise limit of a sequence of continuous functions on $[a, b]$, then the set $D(f)$ of discontinuities of $f$ is a countable union of closed nowhere dense sets.*

For a proof of this theorem, we need some preliminaries.

# Oscillation

Let $f : [a, b] \to \mathbb{R}$ and let $I(c, r) = (c - r, c + r)$ for $c \in [a, b]$ and $r > 0$. Define the oscillation of $f$ on $I(c, r)$ by

$$\operatorname{osc}(f, c, r) = \sup_{x,y \in [a,b] \cap I(c,r)} |f(x) - f(y)|.$$

- $\operatorname{osc}(f, c, r)$ exists if $f$ is bounded
- $\operatorname{osc}(f, c, r) \geqslant 0$ and $\operatorname{osc}(f, c, r)$ is decreasing in $r$

Define the oscillation of $f$ at $c$ by

$$\operatorname{osc}(f, c) = \begin{cases} \lim_{r \to 0} \operatorname{osc}(f, c, r) & \text{if f is bounded near } c \\ \infty & \text{otherwise.} \end{cases}$$

**Problem**

*Show that $f$ is continuous at $c$ if and only if $\operatorname{osc}(f, c) = 0$.*

**Hint.**

The delta in the definition of continuity plays here the role of $r$. $\square$

For $\epsilon > 0$, consider the set

$$A_\epsilon = \{c \in [a, b] : \text{osc}(f, c) \geqslant \epsilon\}.$$

- The set $D(f)$ of discontinuities of $f$ is equal to $\cup_{\epsilon>0} A_\epsilon$.

## Lemma

*For every $\epsilon > 0$, $A_\epsilon$ is a compact subset of $[a, b]$.*

## Proof.

Let $\{c_n\}_{n\geqslant 1}$ be a sequence in $A_\epsilon$ converging to $c \in [a, b]$.

- If $c \notin A_\epsilon$, then $\delta = \epsilon - \text{osc}(f, c) > 0$ and hence $\text{osc}(f, c, r) < \epsilon - \delta/2$ for some $r$
- If $|c_n - c| < r/2$, then $I(c_n, r/2) \subseteq I(c, r)$, and hence $\text{osc}(f, c_n, r/2) = \sup\limits_{x,y\in[a,b]\cap I(c_n,r/2)} |f(x) - f(y)| < \epsilon$ (for $c \notin A_\epsilon$)

Since $\text{osc}(f, c_n) \leqslant \text{osc}(f, c_n, r/2)$, $\text{osc}(f, c_n) < \epsilon \Rightarrow\Leftarrow$ (for $c_n \in A_\epsilon$)

Thus $A_\epsilon$ is closed and hence it is compact. $\square$

## Proof of Baire-Osgood Theorem.

Note that $D(f) = \cup_{n \geqslant 1} A_{1/n}$. Since $A_\epsilon$ is closed for every $\epsilon > 0$, it suffices to check that $A_\epsilon^\circ = \emptyset$.

**Claim** For any closed interval $J \subseteq [a, b]$, $J \nsubseteq A_\epsilon$

- $E_n = \cap_{i,j \geqslant n} \{x \in [a, b] : |f_i(x) - f_j(x)| \leqslant \epsilon/5\}$ is closed (since $f_i - f_j$ is continuous for every $i, j \geqslant n$)
- $\cup_{n \geqslant 1} E_n = [a, b]$ (since $\lim_{n \to \infty} f_n(x) = f(x)$, $x \in [a, b]$)
- $J = \cup_{n=1}^{\infty} (E_n \cap J)$
- There exists $N \geqslant 1$ such that $E_N \cap J$ has nonempty interior (since $J$ is complete, BCT is applicable, see slide 103)

Thus there exists an open interval $K$ contained in $E_N \cap J$.

**SubClaim**: $K \subseteq [a, b] \setminus A_\epsilon$ ($\Rightarrow J \nsubseteq A_\epsilon$, and **Claim** follows)

- $|f_i(x) - f_j(x)| \leqslant \epsilon/5$ for all $x \in K$ and $i, j \geqslant N$ (since $K \subseteq E_N$)
- $|f_N(x) - f(x)| \leqslant \epsilon/5$ for all $x \in K$ (let $i = N$, $f_j(x) \to f(x)$)

## Proof of Baire-Osgood Theorem continued ...

- For $x_0 \in K$, there exists $\delta > 0$ such that $|f_N(x) - f_N(x_0)| < \epsilon/5$ whenever $x \in K$ and $|x - x_0| < \delta$ (by the continuity of $f_N$ at $x_0$)
- $|f(x) - f_N(x_0)| \leqslant 2\epsilon/5$ for every $x \in K$ such that $|x - x_0| < \delta$
- $|f(x) - f(y)| \leqslant 4\epsilon/5$ for every $x, y \in K$ such that $|x - y| < \delta$

This shows that $\mathrm{osc}(f, x_0, \delta) \leqslant 4\epsilon/5$, and hence $\mathrm{osc}(f, x_0) < \epsilon$. Thus the claim $K \subseteq [a, b] \setminus A_\epsilon$ stands verified. $\square$

## Corollary

*If $f : [a, b] \to \mathbb{R}$ is a pointwise limit of a sequence of continuous functions on $[a, b]$, then the set of continuities of $f$ (that is, $[a, b] \setminus D(f)$) is a dense subset of $[a, b]$.*

## Proof.

Since $[a, b] \setminus D(f) = \cap_{n \geqslant 1}([a, b] \setminus A_{1/n})$ and each $[a, b] \setminus A_{1/n}$ is dense in $[a, b]$, BCT yields the desired conclusion. $\square$

### Example

Consider the function $f : [a, b] \to \mathbb{R}$ such that $f = 1$ on rationals and $f = 0$ on irrationals. Note that $D(f) = [a, b]$. It follows from the Baire-Osgood Theorem that $f$ can not be a pointwise limit of a sequence of continuous functions.

### Problem

*If $f : [a, b] \to \mathbb{R}$ is continuous and $f : (a, b) \to \mathbb{R}$ is differentiable, then show that the set of continuities of $f'$ is dense in $(a, b)$.*

### Solution.

For any $x \in (a, b)$, note that $f'(x) = \lim_{n\to\infty} \frac{f(x+1/n)-f(x)}{1/n}$. Thus $f'$ is a pointwise limit of continuous functions. By Baire-Osgood Theorem, the set of continuities of $f'$ is dense in any closed interval $[c, d]$ contained in $(a, b)$. Hence it is dense in $(a, b)$. $\square$

# Convergence of series of functions

Let $(X, d)$ be a metric space. Let $\{f_n\}_{n\geqslant 1}$ be a sequence of bounded functions and let $f : X \to \mathbb{R}$ be a bounded function. We say that the series $\sum_{n=1}^{\infty} f_n$ <u>converges uniformly</u> to $f$ if the sequence $\{\sum_{n=1}^{k} f_n\}_{k\geqslant 1}$ converges uniformly to $f$, that is,

$$\|\sum_{n=1}^{k} f_n - f\|_\infty = \sup_{x\in X} |\sum_{n=1}^{k} f_n(x) - f(x)| \to 0 \text{ as } k \to \infty.$$

In this case, we write $f = \sum_{n=1}^{\infty} f_n$.

**Theorem**

*Let $(X, d)$ be compact. Let $\{f_n\}_{n\geqslant 1}$ be a sequence of functions in $C(X)$. If $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$, then $f \in C(X)$.*

**Proof.**

If each $f_n$ is continuous, then so is $\sum_{n=1}^{k} f_n$. Now use the fact that uniform limit of continuous functions is continuous. $\square$

### Problem

*Let $(X, d)$ be a compact metric space and let $\{f_n\}_{n\geqslant 1}$ be a sequence in $C(X)$. Assume that $\sum_{n\geqslant 1} f_n(x)$ converges pointwise to some $f \in C(X)$ for every $x \in X$. If $f_n(x) \geqslant 0$ for every $x \in X$ and every $n \geqslant 1$, then $\sum_{n\geqslant 1} f_n$ is uniformly convergent.*

### Hint.

Use Dini's Theorem. $\square$

### Problem

*Let $(X, d)$ be a compact metric space and let $A$ be a closed subset of $X$. Assume that $X \setminus A = \sqcup_{n=1}^{\infty} X_n$ (disjoint union), where each $X_n$ is a clopen set in $X$. Let $f \in C(X)$ be such that $f(a) = 0$ for every $a \in A$. Show that $\sum_{n=1}^{\infty} f\chi_{X_n}$ converges uniformly to $f$.*

### Hint.

Note that each $\chi_{X_n}$ is continuous. Now use the last problem. $\square$

# Weierstrass M-test

## Theorem

*Let $(X, d)$ be a metric space and let $\{f_n\}_{n\geqslant 1}$ be a sequence of bounded functions $f_n : X \to \mathbb{R}$. If $\sum_{n\geqslant 1} \|f_n\|_\infty < \infty$ (convergence in $\mathbb{R}$), then $\sum_{n\geqslant 1} f_n$ is uniformly convergent. Moreover,*

$$\|\sum_{n\geqslant 1} f_n\|_\infty \leqslant \sum_{n\geqslant 1} \|f_n\|_\infty.$$

## Proof.

For $n \geqslant 1$, let $g_n = \sum_{k=1}^n f_n$ and $g_0 = 0$. For a positive integers $m < n$, note that by triangle inequality,

$$\|g_n - g_m\|_\infty = \|\sum_{k=m+1}^{n} f_k\|_\infty \leqslant \sum_{k=m+1}^{n} \|f_k\|_\infty, \tag{1}$$

and hence $\{g_n\}_{n\geqslant 1}$ is Cauchy in $B(X)$. Since $B(X)$ is complete (see slide 166), $\{g_n\}_{n\geqslant 1}$ is uniformly convergent. Let $m = 0$, $n \to \infty$ in (1) to get the remaining part. $\square$

# Power Series

## Definition

A <u>power series</u> is an expansion of the form

$$\sum_{n=0}^{\infty} a_n x^n, \text{ where } a_n \in \mathbb{R}.$$

$\sum_{n=0}^{\infty} a_n x^n$ <u>converges absolutely</u> if $\sum_{n=0}^{\infty} |a_n||x|^n < \infty$.

## Definition (Domain of Convergence)

$D = \{w \in \mathbb{R} : \sum_{n=0}^{\infty} |a_n||w|^n < \infty\}$

Note that

- $w_0 \in D \implies \pm w_0 \in D$ for any $\theta \in \mathbb{R}$
- $w_0 \in D \implies w \in D$ for any $w \in \mathbb{R}$ with $|w| \leqslant |w_0|$

Conclude that $D$ is either $\mathbb{R}$, $(-R, R)$ or $[-R, R]$ for some $R \geqslant 0$.

## Definition

The radius of convergence (for short, RoC) of $\sum_{n=0}^{\infty} a_n x^n$ is defined as

$$R = \sup\{|x| : \sum_{n=0}^{\infty} |a_n||x|^n < \infty\}.$$

**Remark** The series $\sum_{n=0}^{\infty} a_n x^n$ converges uniformly on $[-r, r]$ for any $0 < r < R$.

## Example (Geometric series)

Consider the series $\sum_{n=0}^{\infty} x^n$, $x \in \mathbb{R}$.

- This series converges absolutely to $\frac{1}{1-x}$ for any $|x| < 1$. Indeed,

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}, \quad x^{n+1} \to 0 \text{ as } n \to \infty$$

- Thus $R \geqslant |x|$ for every $|x| < 1$, and hence $R \geqslant 1$
- $R = 1$ (since $\sum_{n=0}^{\infty} x^n$ diverges at $x = 1$)

## Theorem (Cauchy-Hadamard Formula)

*The RoC of $\sum_{n=0}^{\infty} a_n x^n$ is given by*

$$R = \frac{1}{\limsup |a_n|^{1/n}},$$

*where we use the convention that $1/0 = \infty$ and $1/\infty = 0$.*

## Proof.

Assume $R < \infty$. If $r > R$, then $\limsup |a_n|^{1/n} > 1/r$, and hence $\lim_{k\to\infty} \sup_{n\geqslant k} |a_n|^{1/n} > 1/r$. Thus there exists a subsequence $\{n_k\}_{k\geqslant 1}$ such that $|a_{n_k}|^{1/n_k} > 1/r$, that is, $r^{n_k}|a_{n_k}| \nrightarrow 0$, and hence $\sum_{n=0}^{\infty} a_n r^n$ is divergent. Thus RoC $\ngtr R = \limsup |a_n|^{1/n}$.

If $r < R$, then $|a_n| r^n < 1$ for all integers $n \geqslant N$. Thus, for $|x| < r$,

$$\sum_{n=N}^{\infty} |a_n||x|^n \leqslant \sum_{n=N}^{\infty} \left(\frac{|x|}{r}\right)^n \leqslant \frac{1}{1-|x|/r} < \infty.$$

Thus RoC $\geqslant r$ for any $r < R$ or RoC $\geqslant R = \limsup |a_n|^{1/n}$. $\square$

# Examples

- $\sum_{n=0}^{k} a_n x^n$, $a_n = 0$ for $n > k$, $R = \infty$
- $\sum_{n=0}^{\infty} \frac{x^n}{n!}$, $a_n = \frac{1}{n!}$, $R = \infty$
- $\sum_{n=0}^{\infty} x^n$, $a_n = 1$, $R = 1$
- $\sum_{n=0}^{\infty} n! x^n$, $a_n = n!$, $R = 0$

The coefficients of a power series may not be given by a single formula.

## Example

Consider the power series $\sum_{n=0}^{\infty} x^{n^2}$ . Then

$$a_k = 1 \text{ if } k = n^2, \text{ and } 0 \text{ otherwise.}$$

Clearly, $\limsup |a_n|^{1/n} = 1$, and hence $R = 1$.

Sometimes RoC can be computed without knowing the coefficients explicitly.

### Example

Consider the power series $\sum_{n=0}^{\infty} a_n x^n$, where $a_n$ is number of divisors of $n^{1111}$. Note that

$$1 \leq a_n \leq n^{1111}.$$

Note that $1 \leq \limsup |a_n|^{1/n} \leq \limsup (n^{1111})^{1/n} = 1$, and hence the RoC of $\sum_{n=0}^{\infty} a_n x^n$ equals 1.

### Theorem

*If the RoC of $\sum_{n=0}^{\infty} a_n x^n$ is $R$ then the RoC of the power series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ is also $R$.*

### Proof.

Since $\lim_{n\to\infty} n^{1/n} = 1$, $R = \frac{1}{\limsup |n a_n|^{1/n}} = \frac{1}{\limsup |a_n|^{1/n}}$. $\square$

# Power series is infinitely differentiable

**Theorem**

*If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a power series of radius $R > 0$, then $f$ is infinitely differentiable with $f'(x) = g(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$.*

**Proof.**

Let $x_0 \in (-R, R)$, $h \in \mathbb{R}$, $r > 0$ with $\max\{|x_0|, |x_0 + h|\} < r < R$.

- $S_k(x) = \sum_{n=0}^{k} a_n x^n$, $E_k(x) = \sum_{n=k+1}^{\infty} a_n x^n$
- $\frac{f(x_0+h)-f(x_0)}{h} - g(x_0) = A + (S_k'(x_0) - g(x_0)) + B$, where

$$A = \left(\frac{S_k(x_0 + h) - S_k(x_0)}{h} - S_k'(x_0)\right), B = \left(\frac{E_k(x_0 + h) - E_k(x_0)}{h}\right)$$

- $$|B| \leqslant \sum_{n=k+1}^{\infty} |a_n| \left|\frac{(x_0 + h)^n - x_0^n}{h}\right| \leqslant \sum_{n=k+1}^{\infty} |a_n| n r^{n-1}$$

Since $f'$ is a power series, $f$ is infinitely differentiable on $(-R, R)$.

□

# A nowhere differentiable continuous function

• Any continuous function on $[0,1]$ is a uniform limit of infinitely differentiable functions (Weierstrass' Theorem).

It is quite striking that uniform limit of infinitely real differentiable functions could be nowhere differentiable.

- Let $\phi(x) = |x|$ for $x \in [-1,1]$, which is extended periodically (with period 2) to $\mathbb{R}$ by setting $\phi(x+2) = \phi(x)$, $x \in \mathbb{R}$
- $\phi$ is a continuous function such that $0 \le \phi(x) \le 1$, $x \in \mathbb{R}$
- Plot graphs of $\phi(x)$, $\phi(4x)$ (period $1/2$), $\phi(16x)$ (period $1/8$)
- Define the function $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$

Note that $f$ is continuous. Indeed,

$$\left| f(x) - \sum_{n=0}^{k} \left(\frac{3}{4}\right)^n \phi(4^n x) \right| \le \sum_{n=k+1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x) \le \sum_{n=k+1}^{\infty} \left(\frac{3}{4}\right)^n \to 0.$$

Since $\phi$ is continuous, so is $f$.

## Theorem

*For any $x \in \mathbb{R}$, there exists $\{\delta_m\}_{m \geqslant 1}$ converging to 0 such that*

$$|(f(x+\delta_m) - f(x))/\delta_m| \to \infty \text{ as } m \to \infty.$$

*In particular, $f$ is not differentiable at any point in $\mathbb{R}$.*

## Proof.

For an integer $m \geqslant 1$, set $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m(x+\delta_m)$. Define $\gamma_n = (\phi(4^n(x+\delta_m)) - \phi(4^n x))/\delta_m$.

- If $n > m$, then $\phi(4^n(x+\delta_m)) = \phi(4^n x \pm 4^{n-m}/2) = \phi(4^n x)$, and hence $\gamma_n = 0$.
- $|\gamma_m| = 4^m$ (since $4^n(x+\delta_m)$ and $4^n x$ are both $\geqslant 0$ or $< 0$)
- When $0 \leq n < m$,

$$|\gamma_n| = \frac{||4^n(x+\delta_m)| - |4^n x||}{|\delta_m|} \leq \frac{|4^n\delta_m|}{|\delta_m|} = 4^n$$

Now we complete the argument.

## Proof continued ...

Since $\gamma_n = 0$ for $n > m$, $|\gamma_m| = 4^m$ and $|\gamma_n| \leqslant 4^m$, $0 \leq n < m$,

$$|(f(x+\delta_m) - f(x))/\delta_m| = \left|\sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \gamma_n\right| = \left|\sum_{n=0}^{m} \left(\frac{3}{4}\right)^n \gamma_n\right|$$

$$= \left|\sum_{n=0}^{m-1} \left(\frac{3}{4}\right)^n \gamma_n \pm 3^m\right| \geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m + 1),$$

where we used $\sum_{n=0}^{m-1} 3^n = (3^m - 1)/2$.

Thus $(f(x+\delta_m) - f(x))/\delta_m$ blows up to $\infty$ as $m \to \infty$. $\square$

Geometrically, a continuous nowhere differentiable function has a continous graph with "corner" at every point!

• The set of nowhere differentiable continuous functions turns out to be dense in $C[0,1]$ (this may be deduced from BCT).

# Riemann integral

Let $I = [a, b]$ and $f : I \to \mathbb{R}$ be a bounded function.

- $P = \{x_0 = a, \dots, x_n = b\}$ (partition)
- $I_i = [x_i, x_{i+1}]$, $[a, b] = \cup_{i=0}^{n-1} I_i$, $\ell(I_i)$ (length of $I_i$)
- $\|P\| = \max\limits_{0\leqslant i\leqslant n-1} \ell(I_i)$ (width of $P$)
- $m_i = \inf\limits_{x\in I_i} f(x)$, $M_i = \sup\limits_{x\in I_i} f(x)$
- $L(P, f) = \sum\limits_{i=0}^{n-1} m_i\,\ell(I_i)$ (lower Riemann sum)
- $U(P, f) = \sum\limits_{i=0}^{n-1} M_i\,\ell(I_i)$ (upper Riemann sum)
- $\underline{\int} f(x)dx = \sup\limits_{P} L(P, f)$ (lower Riemann integral)
- $\overline{\int} f(x)dx = \inf\limits_{P} U(P, f)$ (upper Riemann integral)

## Definition

A bounded function $f : I \to \mathbb{R}$ is <u>Riemann integrable</u> (or <u>Darboux integrable</u>) if $\int_{-} f(x)dx = \int^{-} f(x)dx$, say $\int_I f(x)dx$.

**Interpretation** Area of the region $R$ enclosed by the lines $x = a$, $x = b$, and the curve $y = f(x) \geqslant 0$.

## Example

For $x_0 \in I$, consider the function $f : I \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & \text{if } x \neq x_0, \\ 0 & \text{if } x = x_0. \end{cases}$$

For $\epsilon > 0$, let $P$ be a partition with $\|P\| = \max_{0 \leqslant i \leqslant n-1} \ell(I_i) < \epsilon$.

- If $x_0 \in I_{i_0}$ is an interior point, then $U(P,f) - L(P,f) = \ell(I_{i_0}) < \epsilon$
- If $x_0 \in I_{i_0}$ is an end point, then $U(P,f) - L(P,f) < 2\epsilon$
- $L(P,f) \leqslant \int_{-} f(x)dx \leqslant \int^{-} f(x)dx \leqslant U(P,f)$ (Exercise)

Conclude that $f$ is Riemann integrable.

### Theorem

*If $f : [a, b] \to \mathbb{R}$ is a bounded function, then $f$ is Riemann integrable if and only if for every $\epsilon > 0$, there exists a partition $P$ of $[a, b]$ such that $U(P, f) - L(P, f) < \epsilon$.*

### Proof.

The sufficiency part $\Leftarrow$ follows from

$$L(P, f) \leqslant \underline{\int} f(x)dx \leqslant \overline{\int} f(x)dx \leqslant U(P, f).$$

To see the necessity part $\Rightarrow$, find partitions $P$ and $Q$ of $[a, b]$ such that $U(P, f) - \overline{\int} f(x)dx < \epsilon/2$ and $\underline{\int} f(x)dx - L(Q, f) < \epsilon/2$. Let $R$ be a partition obtained from $P \cup Q$, and note that

$$L(Q, f) \leqslant L(R, f) \leqslant U(R, f) \leqslant U(P, f).$$

Then $U(R, f) - L(R, f) \leqslant U(P, f) - L(Q, f) =$
$U(P, f) - \overline{\int} f(x)dx + \underline{\int} f(x)dx - L(Q, f) < \epsilon.$ □

## Corollary

*Every continuous function $f : I \to \mathbb{R}$ is Riemann integrable.*

## Proof.

Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x')| < \epsilon$ whenever $x \in I$, $|x - x'| < \delta$. Thus for partition $P$ with $\|P\| < \delta$,

$$U(P, f) - L(P, f) \leqslant \sum_{i=0}^{m-1} |M_i - m_i| \ell(I_i).$$

However, $M_i = f(a_i)$ and $m_i = f(b_i)$ for some $a_i, b_i \in I_i$. Thus $U(P, f) - L(P, f) = \sum_{i=0}^{m-1} |f(a_i) - f(b_i)| \ell(I_i) \leqslant \epsilon \ell(I)$. Hence $f$ is Riemann integrable. $\square$

## Problem

*Show that the indicator function $\chi_{[0,1/2]} : [0, 1] \to \mathbb{R}$ (which is 1 on $[0, 1/2]$ and 0 outside) is Riemann integrable.*

## Problem

*Show that a bounded function, which is continuous except at a finite subset $F$ of $[a, b]$, is Riemann integrable.*

## Hint.

Consider first the case in which $F = \{x_0\}$. Assuming $f \neq 0$, let $I_\epsilon = (x_0 - \frac{\epsilon}{4\|f\|_\infty}, x_0 + \frac{\epsilon}{4\|f\|_\infty})$. Apply the last corollary to $I = [a, b] \setminus I_\epsilon$ to find a partition $P$ of $I$ such that $U(P, f) - L(P, f) < \epsilon/2$. Let $\tilde{P}$ be the partition formed by taking union of $P$ and the interval $\bar{I}_\epsilon$. Check that $U(P, f) - L(P, f) < \epsilon$. Extend this argument to any finite set $F$. $\square$

**Question** Does there exist a Riemann integrable function, which is discontinuous at countably infinite points ?

**Answer** There are plenty of such functions!

## Theorem

*Every bounded monotone function $f : [a, b] \to \mathbb{R}$ is integrable.*

## Proof.

Let $f$ be increasing, $P$ be a partition with $\ell(I_i) = \frac{b-a}{m}$ for all $i$.

- $U(P, f) = \sum_{i=0}^{m-1} M_i\, \ell(I_i) = \sum_{i=0}^{m-1} f(x_{i+1})\, \ell(I_i)$ and $L(P, f) = \sum_{i=0}^{m-1} m_i\, \ell(I_i) = \sum_{i=0}^{m-1} f(x_i)\, \ell(I_i)$

$U(P, f) - L(P, f) = (f(x_m) - f(x_0))\frac{b-a}{m} = \frac{f(b)-f(a)}{b-a}\frac{1}{m} \to 0.$ $\square$

## Problem

*Consider the Zeno's staircase function $Z : [0, 1] \to \mathbb{R}$ given by*

$$Z(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \le x < \frac{1}{2}, \\ \frac{3}{4} & \text{if } \frac{1}{2} \le x < \frac{3}{4}, \\ \frac{7}{8} & \text{if } \frac{3}{4} \le x < \frac{7}{8}, \\ \vdots & \\ \frac{2^k-1}{2^k} & \text{if } \frac{2^{k-1}-1}{2^{k-1}} \le x < \frac{2^k-1}{2^k}, \quad k \geqslant 1. \end{cases}$$

*Then $Z$ is integrable with countably infinite discontinuites.*

# Sets of measure 0

**Question** Does there exist a Riemann integrable function with uncountably many discontinuities ?

**Question** Does there exist a Riemann integrable function with discontinuities containing an open interval ?

### Definition

Let $E$ be a subset of $\mathbb{R}$. We say that $E$ is a set of measure 0 if for every $\epsilon > 0$, there exists a countable family of open intervals $\{I_k\}_{k\geqslant 1}$ such that $E \subseteq \cup_{k=1}^{\infty} I_k$ and $\sum_{k=1}^{\infty} \ell(I_k) < \epsilon$.

• In case $E$ is compact, there exists a finite family of open intervals $\{I_k\}_{k=1}^{N}$ satisfying the conditions above.

• If $A$ is of measure 0 and $B \subseteq A$, then $B$ is of measure 0.

### Problem

*Show that the Cantor set is of measure* 0.

### Hint.

$C = \cap_{n\geqslant 1} C_n$ (see slide 28) and $\ell(C_n) \to 0$ as $n \to \infty$. □

## Problem

*Show that a countable union of sets of measure* 0 *is of measure* 0.

## Solution.

Let $E_1, E_2, \ldots,$ be countably many sets of measure 0. Let $\epsilon > 0$.

- Let $\{I_k^{(1)}\}_{k\geqslant 1}$ be a sequence of open intervals such that $E_1 \subseteq \cup_{k=1}^{\infty} I_k^{(1)}$ and $\sum_{k=1}^{\infty} \ell(I_k^{(1)}) < \epsilon/2$.
- Let $\{I_k^{(2)}\}_{k\geqslant 1}$ be a sequence of open intervals such that $E_2 \subseteq \cup_{k=1}^{\infty} I_k^{(2)}$ and $\sum_{k=1}^{\infty} \ell(I_k^{(2)}) < \epsilon/4$.

Continue like this to get for every $j \geqslant 1$, $\{I_k^{(j)}\}_{k\geqslant 1}$ such that $E_j \subseteq \cup_{k=1}^{\infty} I_k^{(j)}$ and $\sum_{k=1}^{\infty} \ell(I_k^{(j)}) < \epsilon/2^j$. Thus $\cup_{j=1}^{\infty} E_j \subseteq \cup_{j,k=1}^{\infty} I_k^{(j)}$ (countable union) and $\sum_{j,k=1}^{\infty} \ell(I_k^{(j)}) < \sum_{j=1}^{\infty} \epsilon/2^j = \epsilon$. □

## Problem

*Show that any countable subset of* $\mathbb{R}$ *is of measure* 0.

## Proof.

Any finite set (and hence single-ton) is of measure 0. □

Let $f : [a,b] \to \mathbb{R}$ be bounded with $M = \sup_{x\in[a,b]} |f(x)| < \infty$.

- $A_\epsilon = \{c \in [a,b] : \operatorname{osc}(f,c) \geqslant \epsilon\}$ is compact (see slide 181)

## Lemma

*If $A_\epsilon$ is a set of measure* 0*, then there exists a partition $P$ of $[a,b]$ such that $U(P,f) - L(P,f) < (2M + b - a)\epsilon$.*

## Proof.

Since $A_\epsilon$ is a compact set of measure 0, there exists open intervals $\{I_k\}_{k=1}^N$ such that $A_\epsilon \subseteq \cup_{k=1}^N I_k$ and $\sum_{k=1}^N \ell(I_k) < \epsilon$.

- $K = [a,b] \setminus (\cup_{k=1}^N I_k)$ is a compact subset of $[a,b]$
- For any $c \in K$, there exists an interval, open in $J_c \subseteq [a,b] \setminus A_\epsilon$ such that $c \in J_c$ ($\Rightarrow \sup_{x,y\in J_c} |f(x) - f(y)| \leqslant \epsilon$)
- There exists $J_1, \ldots, J_{N'}$ such that $K \subseteq \cup_{j=1}^{N'} J_j$ ($K$ is compact)

If $P$ : partition formed by end-points of $I_1, \ldots, I_N, J_1, \ldots, J_{N'}$, then

$$U(P,f) - L(P,f) \leqslant 2M \sum_{j=1}^{N} \ell(I_j) + \epsilon(b-a) < (2M + b - a)\epsilon. \qquad \square$$

- $A_\epsilon$ is a subset of $D(f)$ (the set of discontinuities of $f$)
- If $D(f)$ is of measure 0, then so is $A_\epsilon$ for every $\epsilon > 0$
- If $D(f)$ is of measure 0, then $f \in R[a,b]$ (apply Lemma)

## Theorem (Lebesgue's criterion for Riemann integrability)

*If $f : [a,b] \to \mathbb{R}$ is bounded, then $f \in R[a,b]$ if and only if $D(f)$ is of measure* 0.

## Proof.

To see ($D(f)$ is of measure $0 \Leftarrow f \in R[a,b]$), let $f \in R[a,b]$.

• Countable union of sets of measure zero is of measure 0 (see slide 204) and $D(f) = \cup_{n \geqslant 1} A_{1/n}$ **Claim** Each $A_{1/n}$ is of measure 0

- Let $\epsilon > 0$ and choose a partition $P = \{x_0, \ldots, x_N\}$ such that $U(P,f) - L(P,f) < \epsilon/n$ (since $f$ is Riemann integrable)
- $I_j = (x_{j-1}, x_j)$ and $I_{j_1}, \ldots, I_{j_{N'}}$ be sets intersecting with $A_{1/n}$
- $\sup\limits_{x \in I_{j_k}} f(x) - \inf\limits_{x \in I_{j_k}} f(x) \geqslant \sup\limits_{x \in I_{j_k} \cap A_{1/n}} f(x) - \inf\limits_{x \in I_{j_k} \cap A_{1/n}} f(x) \geqslant 1/n$

Finally, $\frac{1}{n} \sum_{k=1}^{N'} \ell(I_{j_k}) \leqslant U(P,f) - L(P,f) < \frac{\epsilon}{n}$. Choose $I'_{j_k} \supseteq I_{j_k}$ so that $\sum_{k=1}^{N'} \ell(I_{j_k}) < 2\epsilon$ and $A_{1/n} \subseteq \cup_{k=1}^{N'} I'_{j_k}$. $\square$

### Example

Consider the function $g : [0,1] \to \mathbb{R}$ given by

$$g(x) = \begin{cases} \frac{1}{q} & \text{if } x \in \mathbb{Q} \cap (0,1) \text{ and } x = \frac{p}{q} \text{ in reduced form} \\ 0 & \text{otherwise.} \end{cases}$$

Then $g \in R[0,1]$ since $D(g) = \mathbb{Q} \cap (0,1]$ is of measure 0.

### Problem

*Show by an example that composition of Riemann integrable functions need not be Riemann integrable.*

### Hint.

Take $f = \chi_{(0,1]}$ and $g$ as above. $\square$

### Problem

*Let $f : [a,b] \to [c,d]$ be Riemann integrable. If $\phi : [c,d] \to \mathbb{R}$ is continuous, then $\phi \circ f$ is Riemann integrable.*

### Hint.

$D(\phi \circ f) \subseteq D(f)$ is of measure 0. $\square$

## Problem

*Show that sum and product of Riemann integrable functions is Riemann integrable.*

## Hint.

$D(f+g) \subseteq D(f) \cup D(g)$ and $D(fg) \subseteq D(f) \cup D(g)$. $\square$

## Theorem

*The set $R[a,b]$ of Riemann integrable functions $f : [a,b] \to \mathbb{R}$ is a vector space over $\mathbb{R}$. Morever, for every $f, g \in R[a,b]$ and $\alpha \in \mathbb{R}$, $\int_a^b (f(x) + \alpha g(x)) dx = \int_a^b f(x) dx + \alpha \int_a^b g(x) dx$.*

For $f \in R[a,b]$, define $\|f\|_1 = \int_a^b |f(x)| dx$.

**Question** Is $\|\cdot\|_1$ a norm on $R[a,b]$ ?

- Clearly, $\|f\|_1 \geqslant 0$, $\|\alpha f\|_1 = |\alpha| \|f\|_1$ and $\|f+g\|_1 \leqslant \|f\|_1 + \|g\|_1$ for every $f, g \in R[a,b]$ and $\alpha \in \mathbb{R}$

- If $\|f\|_1 = 0$ for $f \in R[a,b]$, then $f$ need not be 0

- $\chi_{[0,1/2]} - \chi_{[0,1/2)}$ is Riemann integrable, $\|\chi_{[0,1/2]} - \chi_{[0,1/2)}\|_1 = 0$ but $\chi_{[0,1/2]} \neq \chi_{[0,1/2)}$

# Discontinuities of positive measure

### Example

Let $\hat{C}$ denote the Cantor-like set obtained by removing $2^{k-1}$ centrally situated open subintervals $I_{1k}, \cdots, I_{2^{k-1}k}$ of $I = [0, 1]$ each of length $1/4^k$ at the $k$th stage, where $k = 1, 2, \cdots$.

- $\ell(\hat{C}) = 1 - \sum_{k=1}^{\infty} \frac{2^{k-1}}{4^k} = 1 - \sum_{k=1}^{\infty} \frac{1}{2^{k+1}} = 1/2$
- Let $F_k : I \to I$ be a continuous function such that $F_k = 1$ on $I \setminus \cup_{i=1}^{2^{k-1}} I_{ik}$ and $F_k = 0$ at the mid-points of $I_{1k}, \cdots, I_{2^{k-1}k}$
- If $f_n = \prod_{i=1}^{n} F_i$, then $f_{n+1}(x) \leqslant f_n(x)$ for every $x \in I$, $n \geqslant 1$
- Let $f : I \to I$ be the pointwise limit of $\{f_n\}_{n \geqslant 1}$
- For $x \in \hat{C}$, there exists a sequence $\{x_n\}_{n \geqslant 1}$ converging to $x$ such that $f(x_n) = 0$
- $f$ is discontinuous on $\hat{C}$ (since $f(x) = 1$ for every $x \in \hat{C}$)

Conclude that $f$ is not Riemann integrable.

- It turns out that $\|f_n - f\|_1 \to 0$ as $n \to \infty$!

## Problem

*For $f, g \in R[a,b]$, define $f \sim g$ if $f = g$ outside a set of measure* 0*. Verify the following:*

(1) $\sim$ *defines an equivalence relation.*

(2) *If* $[f]$ *denotes the equivalence relation containing* $f$ *and* $\|[f]\|_1 = \int_a^b |f(x)|dx$*, then* $\|[f]\|_1 = 0$ *if and only if* $f \sim 0$.

(3) $\mathcal{R} = \{[f] : f \in R[a,b]\}$ *is a normed linear space endowed with the norm* $\|\cdot\|_1$.

(4) $(\mathcal{R}, \|\cdot\|_1)$ *is incomplete.*

## Problem

*Consider the intervals* $I_1 = [0,1], I_2 = [0,1/2], I_3 = [1/2,1], I_4 = [0,1/4], I_5 = [1/4,1/2], I_6 = [1/2,3/4], I_7 = [3/4,1]$ *and so on. For* $f_n = \chi_{I_n}$ *and* $f = 0$*, show that* $f_n(x) \nrightarrow f(x)$ *for any* $x \in [0,1]$.

## Hint.

Any $x \in I_1$ lies in infinitely many $I_n$ & infinitely many $[0,1] \setminus I_n$. □

- $\int_{[0,1]} |f_n(x) - f(x)|dx = \ell(I_n) \to 0$ as $n \to \infty$.

## Problem (Integrable function with uncountable discontinuities)

*Consider the indicator function $\chi_C$ of the Cantor set, that is,*

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C, \\ 0 & \text{if } x \in [0,1] \setminus C. \end{cases}$$

*Show that $\chi_C$ is discontinuous precisely at every $x \in C$. Conclude that $\chi_C \in R[0,1]$.*

### Hint.

Note that $C$ is closed and nowhere dense, and hence $[0,1] \setminus C$ is dense. Thus for every $x \in C$, there exists a sequence of points $x_n \in [0,1] \setminus C$ such that $x_n \to x$. Clearly, $\chi_C(x_n) = 0$ does not converge to $\chi_C(x) = 1$, and hence $C \subseteq D(\chi_C)$.

Let $x \in [0,1] \setminus C$. Then, some neighborhood of $x$ does not intersect $C$ (otherwise, $x$ is a limit point of $C$), and hence $\chi_C$ is sequentially continuous at $x$. Thus $D(\chi_C) = C$. Finally, since $C$ is of measure 0, by Lebesgue's criterion, $\chi_C \in R[0,1]$. $\square$

## Theorem

*The uniform limit $f$ of a sequence of Riemann integrable functions $f_n$ is Riemann integrable and* $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

## Proof.

Let $\{f_n\}_{n\geqslant 1} \subseteq R[a,b]$, $f : [a,b] \to \mathbb{R}$ be such that $\|f_n - f\|_\infty \to 0$.

- $f$ is bounded (since $\|f\|_\infty \leqslant \|f_n - f\|_\infty + \sup_{n\geqslant 1} \|f_n\|_\infty$)
- Given $\epsilon > 0$, there exists $N \geqslant 1$ such that for every $n \geqslant N$ and for every $x \in [a,b]$, $f_n(x) - \epsilon < f(x) < f_n(x) + \epsilon$
- For partitions $P, Q$ of $[a,b]$, $L(f_n - \epsilon, P) \leqslant L(f, P) \leqslant \underline{\int} f(x)dx \leqslant \overline{\int} f(x)dx \leqslant U(Q, f) \leqslant U(f_n + \epsilon, Q)$

Now take supremum over LHS and infinimum over RHS to get

$$\int_a^b (f_n(x) - \epsilon)dx \leqslant \underline{\int} f(x)dx \leqslant \overline{\int} f(x)dx \leqslant \int_a^b (f_n(x) + \epsilon)dx.$$

Thus $0 \leqslant \overline{\int} f(x)dx - \underline{\int} f(x)dx \leqslant 2\epsilon$, or $f \in R[a,b]$ and $\left| \int_a^b f_n(x)dx - \int_a^b f(x)dx \right| \leqslant (b-a)\epsilon$ for every $n \geqslant N$. $\square$

- A power series can be integrated termwise

## Problem

*Let* $f(x) = \sum_{n=0}^{\infty} a_n x^n$ *be a power series of radius of convergence* $R > 0$. *Show that for any* $-R < a < b < R$,

$$\int_a^b f(x)dx = \sum_{n=0}^{\infty} a_n \Big( \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1} \Big).$$

## Hint.

The partial sum of power series is continuous and it converges to $f(x)$ uniformly on $[a, b]$. Now apply last problem. $\square$

- One may use the last problem to compute $\int_a^b \frac{1}{x} dx$ for every $1 < a < b < 2$. Indeed, $\frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n$ converges uniformly on $|1-x| \leqslant 1-\epsilon$ for every $\epsilon > 0$, and hence on $[a, b]$. Thus

$$\int_a^b \frac{1}{x} dx = \sum_{n=0}^{\infty} \Big( \frac{(1-b)^{n+1}}{n+1} - \frac{(1-a)^{n+1}}{n+1} \Big) = \log b - \log a.$$

Let $f : [a, b] \to \mathbb{R}$ be Riemann integrable and let $x_0 \in [a, b] \setminus D(f)$.

- Given $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$
- Let $J$ be an interval of length $\ell(J) < \delta/2$ that contains $x_0$
- $f(x_0) - \frac{1}{\ell(J)} \int_J f(x)dx = \frac{1}{\ell(J)} \int_J (f(x_0) - f(x))dx$
- $\left| f(x_0) - \frac{1}{\ell(J)} \int_J f(x)dx \right| \leqslant \frac{1}{\ell(J)} \int_J |f(x_0) - f(x)|dx < \epsilon$ (see [9])

This proves the following:

### Theorem (Fundamental Theorem of Calculus-I)

*For every $f \in R[a, b]$ and every $x_0 \in [a, b] \setminus D(f)$,*

$$\lim_{\substack{\ell(J) \to 0 \\ x_0 \in J}} \frac{1}{\ell(J)} \int_J f(x)dx = f(x_0).$$

**Remark** The quantity $\frac{1}{\ell(J)} \int_J f(x)dx$ is the "average of $f$ over $J$".

[9] We used $|\int_a^b g(x)dx| \leqslant \int_a^b |g(x)|dx$, $g \in R[a, b]$ (Exercise)

## Corollary

*Let* $f \in R[a,b]$*. Define* $F:[a,b] \to \mathbb{R}$ *by* $F(x_0) = \int_a^{x_0} f(x)dx$, $x_0 \in [a,b]$*. Then*

(1) $F$ *is continuous at every point in* $[a,b]$,

(2) $F$ *is differentiable at every* $x_0 \in [a,b] \setminus D(f)$ *&* $F'(x_0) = f(x_0)$.

## Proof.

The first part follows from the estimate

$$|F(x) - F(x_0)| = \left| \int_x^{x_0} f(t)dt \right| \leqslant \Big( \sup_{t \in [a,b]} |f(t)| \Big) |x - x_0|, \quad x \in [a,b].$$

To see the second part, let $J = [x_0, x_0 + h]$ in FTC-I to obtain

$$\frac{F(x_0 + h) - F(x_0)}{h} = \frac{1}{h} \int_{x_0}^{x_0+h} f(x)dx \to f(x_0) \text{ as } h \to 0.$$

This also shows that $F'(x_0) = f(x_0)$. $\square$

### Corollary

*Let $f \in R[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by $F(x_0) = \int_a^{x_0} f(x)dx$, $x_0 \in [a, b]$. If $D(f)$ denotes the set of points of discontinuities of $f$ and $\mathit{Diff}(F)$ denotes the set of points at which $F$ is differentiable, then $[a, b] \setminus D(f) \subseteq \mathit{Diff}(F)$.*

### Problem

*Give an example of a continuous function on $[0, 1]$, which is differentiable at irrationals in $[0, 1]$ (we are not asking for a function differentiable precisely at irrationals in $[0, 1]$).*

### Hint.

Use the last corollary. $\square$

- There exist functions differentiable precisely at rationals [10]

[10] A Continuous Function That Is Differentiable Only at the Rationals, Mark Lynch, Mathematics Magazine , Vol. 86, No. 2 (April 2013), pp. 132-135

### Definition

Given a partition $P = \{a = x_0 < x_1 \cdots < x_n = b\}$ of $[a, b]$ and $g : [a, b] \to \mathbb{R}$, the variation of $g$ over $P$ is given by

$$V(g, P) = \sum_{j=1}^{n} |g(x_j) - g(x_{j-1})|.$$

• $g$ is of bounded variation on $[a, b]$ if the total variation $V_a^b g = \sup_P V(g, P)$ of $g$ over $[a, b]$ is finite.

### Theorem

*Let $f \in R[a, b]$. Define $F : [a, b] \to \mathbb{R}$ by $F(x_0) = \int_a^{x_0} f(x)dx$, $x_0 \in [a, b]$. Then $F$ is of bounded variation.*

### Proof.

Since $V(F, P) = \sum_{j=1}^{n} |F(x_j) - F(x_{j-1})| \leqslant \sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} |f(x)|dx$, $V_a^b F$ is at most $(b - a) \sup_{x \in [a,b]} |f(x)|$. $\square$

## Theorem (Fundamental Theorem of Calculus-II)

*Let $f \in R[a,b]$. If there is a continuous function $F : [a,b] \to \mathbb{R}$ differentiable on $(a,b)$ such that $F'(x) = f(x)$, $x \in (a,b)$, then $\int_a^b f(x)dx = F(b) - F(a)$.*

## Proof.

Let $\epsilon > 0$ and $P = \{a = x_0 < x_1 \cdots < x_n = b\}$ be a partition of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$.

- For $i = 1, \ldots, n$, there exists $c_i \in [x_i, x_{i-1}]$ such that $F(x_i) - F(x_{i-1}) = f(c_i)(x_i - x_{i-1})$ (Mean Value Theorem)
- $\sum_{i=1}^n f(c_i)(x_i - x_{i-1}) = \sum_{i=1}^n (F(x_i) - F(x_{i-1})) = F(b) - F(a)$
- $L(P,f) \leqslant \sum_{i=1}^n f(c_i)(x_i - x_{i-1}) \leqslant U(P,f)$ $(m_i \leqslant f(c_i) \leqslant M_i)$
- $L(P,f) \leqslant \int_a^b f(x)dx \leqslant U(P,f)$

Thus $\int_a^b f(x)dx$ and $F(b) - F(a)$ lie in interval $[L(U,f), U(P,f)]$ of length less than $\epsilon$. Hence $|F(b) - F(a) - \int_a^b f(x)dx| < \epsilon$. $\square$

The Fundamental Theorem of Calculus-II can be used to compute integrals provided derivatives are known.

- $\int_0^x t^n dt = \frac{x^{n+1}}{n+1}$ (since the derivative of $F(x) = \frac{x^{n+1}}{n+1}$ equals $x^n$)
- $\int_1^x \frac{1}{t} dt = \log x$ (since $(\log x)' = 1/x$)

## Definition

We say that a function $G : [a, b] \to \mathbb{R}$ is an anti-derivative of $g \in R[a, b]$ if $G$ is differentiable on $[a, b]$ and $G'(x) = g(x)$ at every $x \in [a, b]$.

## Example

The jump function $g = \chi_{(0,1]}$ does not have an anti-derivative $G$. Indeed, if there exists a differentiable function $G$ such that $G' = g$, then by the Fundamental Theorem of Calculus-II,

$$1 - x = \int_x^1 g(x)dx = G(1) - G(x), \quad 0 < x < 1.$$

This shows that $G(x) = G(1) - 1 + x$ for every $x \in (0, 1]$, and hence by continuity, $G(x) = G(1) - 1 + x$ for every $x \in [0, 1]$. However, $G'(0) = 1 \neq g(0) \Rightarrow\Leftarrow$

## Corollary (Integration by parts)

*Suppose that $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable in $[a, b]$, and $f', g'$ are integrable on $[a, b]$. Then*

$$\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx.$$

## Proof.

Define $F(x) = f(x)g(x)$, $x \in [0, 1]$. Then $F$ is continuous on $[a, b]$, $F$ is differentiable and

$$F'(x) = f'(x)g(x) + f(x)g'(x), \quad x \in (a, b).$$

By the Fundamental Theorem of Calculus-II,

$$\int_a^b \Big(f'(x)g(x) + f(x)g'(x)\Big)dx = F(b) - F(a) = f(b)g(b) - f(a)g(a).$$

This completes the proof. $\square$

• There exists $f \in R[a,b]$ such that $F' = f$ outside a set of measure 0, yet, $\int_a^b f(x)dx \neq F(b) - F(a)$.

## Example (Cantor Function)

Let $C$ denote the Cantor set obtained by removing $2^{n-1}$ centrally situated disjoint open subintervals $U_{1,n}, \cdots, U_{2^{n-1},n}$ of $[0,1]$ each of length $1/3^n$ at the $n$th stage, where $n = 1, 2, \cdots$ . Thus

$$C = \cap_{n\geqslant 1} C_n, \quad \text{where } C_n = [0,1] \setminus \big( \cup_{k=1}^{n} \cup_{j=1}^{2^{n-1}} U_{k,j} \big).$$

- $F_1 : [0,1] \to \mathbb{R}$ a continuous increasing function so that $F_1(0) = 0$, $F_1 = 1/2$ on $[1/3, 2/3]$, $F_1(1) = 1$, $F_1$ linear on $C_1$
- $F_2 : [0,1] \to \mathbb{R}$ a continuous increasing function so that $F_2(0) = 0$, $F_2 = 1/4$ on $[1/9, 2/9]$, $F_2 = 1/2$ on $[1/3, 2/3]$, $F_2 = 3/4$ on $[7/9, 8/9]$, $F_1(1) = 1$, $F_2$ linear on $C_2$

Continuing this, we obtain a sequence of continuous increasing functions $\{F_n\}_{n\geqslant 1}$ such that $|F_{n+1}(x) - F_n(x)| \leqslant 2^{-n-1}$.

## Example (Example continued ...)

Check that $\{F_n\}_{n \geqslant 1}$ is Cauchy in $C[0,1]$. Indeed, for $m > n$,

$$|F_m(x) - F_n(x)| \leqslant \sum_{j=n+1}^{m} |F_j(x) - F_{j-1}(x)| \leqslant \sum_{j=n+1}^{m} 2^{-j},$$

and hence $\{F_n\}_{n \geqslant 1}$ converges uniformly to $F \in C[0,1]$.

- We refer to $F$ as <u>Cantor function</u> or <u>devil's staircase function</u>.
  - $F$ is increasing (since so is $F_n$ for every $n \geqslant 1$)
  - $F' = 0$ on $[0,1] \setminus C$ (since $F$ is constant on each interval of the complement of $C$)
  - $F' = 0$ outside a set of measure 0 (since $C$ is of measure 0)

This shows that $\int_0^1 F'(x)dx = 0$ (why?) and $F(1) - F(0) = 1$, so

$$\int_0^1 F'(x)dx \neq F(1) - F(0).$$

- $F$ is non-constant, yet, $F' = 0$ outside a set of measure 0!

Let us summarize the discussion above:

- If $f \in R[a,b]$, then $F(x) = \int_a^x f(x)dx$ is differentiable outside the set $D(f)$ of measure 0.
- In general, $\int_a^b F'(x)dx \neq F(b) - F(a)$.

This raises the following question:

**Question** What conditions on a function F on $[a,b]$ guarantee that $F'(x)$ exists (outside a set of measure 0), that this function is integrable, and that moreover $\int_a^b F'(x)dx = F(b) - F(a)$ ?

This problem is closely related to the "averaging problem":

**Question** What are conditions on a function $f$ on $[a,b]$ for which

$$\lim_{\substack{\ell(J)\to 0 \\ x_0 \in J}} \frac{1}{\ell(J)} \int_J f(x)dx = f(x_0)$$

holds for $x_0 \in [a,b]$ outside a set of measure 0 ?

To answer these questions, one needs to venture into the theory of Lebesgue integration!