Hilbert space methods in analysis

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I mostly referred in parts to the following texts:

- Jim Agler, John Edward McCarthy, Nicholas John Young, Operator Analysis: Hilbert Space Methods in Complex Analysis, Cambridge University Press, 2020, xvi+376 pp.
- Chavan, Sameer; Misra, Gadadhar, Notes on the Brown-Douglas-Fillmore theorem. Cambridge-IISc Series. Cambridge University Press, Cambridge, 2021. xi+246 pp
- Rudin, Walter Real and complex analysis. Third edition. McGraw-Hill Book Co., New York, 1987. xiv+416 pp

Power series representation and Parseval's formula

Theorem

If $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ *is convergent on* $\mathbb{D}_R(a)$, *then for any* $0 < r < R$,

$$a_n r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a + re^{i\theta}) e^{-in\theta} d\theta, \quad n \geq 0$$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 d\theta.$$

• The first part follows from the fact that the power series above converges uniformly on $\mathbb{D}_r(a)$, and hence one can interchange the infinite sum and the integral.

• For the second part, we need some bits of Hilbert space theory.

Corollary (Maximum modulus principle)

Let $f \in H(U)$ for some open connected set U. If $\overline{\mathbb{D}}_r(a) \subseteq U$, then

$$|f(a)| \leq \sup_{|z-a|=r} |f(z)|.$$

Moreover, equality occurs above if and only if f is constant.

Proof.

Suppose that $\sup_{|z-a|=r} |f(z)| \leq |f(a)|$. It follows from the last theorem that

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 d\theta \leq |f(a)|^2 = a_0^2.$$

This implies that $a_1 = 0, a_2 = 0, \ldots$. That is, f is constant on $\mathbb{D}_r(a)$. By the identity theorem, f is constant on U. $\square$

A subset $\{e_n\}_{n\geq 0}$ of H is said to be an *orthonormal basis* for a Hilbert space H if $\{e_n\}_{n\geq 0}$ is an orthonormal set and for every $h \in H$, there exist scalars α_n such that $\sum_{n=0}^{\infty} \alpha_n e_n$ converges to h.

Theorem (Parseval's Identity)

Let $\{e_n\}$ be an orthonormal basis of a Hilbert space (complete inner-product space) H. Then, for $h \in H$, $\|h\|^2 = \sum_{n=0}^{\infty} |\langle h, e_n\rangle|^2$.

Proof.

If $h = \sum_{n=0}^{\infty} \alpha_n e_n$ then for $m \geq 0$, by the orthogonality of $\{e_n\}_{n\geq 0}$,

$$\langle h, e_m\rangle = \sum_{n=0}^{\infty} \alpha_n \langle e_n, e_m\rangle = \alpha_m.$$

Note that for every $k \geq 0$, by the orthogonality of $\{e_n\}_{n\geq 0}$,

$$\|\sum_{n=0}^{k} \langle h, e_n\rangle e_n\|^2 = \sum_{n=0}^{k} |\langle h, e_n\rangle|^2$$

Let now $k \to \infty$ and use the continuity of the norm. $\square$

Let $\mathbb{T}$ denote the unit circle in the complex plane $\mathbb{C}$ and define $e_n(\theta) = e^{in\theta},\ \theta \in [-\pi, \pi],\ n \in \mathbb{Z}$.

Example

We claim that $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$.

- Note that $L^\infty(\mathbb{T})$ is contained in $L^2(\mathbb{T})$.
- For every $f \in L^2(\mathbb{T})$,

$$\|f\|_2^2 = \int_{[-\pi,\pi]} |f(t)|^2 dt \leqslant 2\pi \|f\|_\infty^2,$$

  where $\|\cdot\|_\infty$ denotes the essential sup norm of f.
- Since trigonometric polynomials are dense in $C(\mathbb{T})$ (Stone-Weierstrass theorem), the linear span of $\{e_n\}_{n=-\infty}^{\infty}$ is dense in the subspace of continuous functions in $L^2(\mathbb{T})$.
- However, continuous functions are dense in $L^2(\mathbb{T})$.

Problem (Maximal Orthonormal Set)

Let $\{e_n\}$ be an orthonormal set in a Hilbert space H with the property: If $x \in H$ such that $\langle x, e_n \rangle = 0$ for all n then $x = 0$. Show that $\{e_n\}$ is an orthonormal basis.

Solution.

By the continuity of the inner-product,

$$\langle x - \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n, e_m \rangle = 0.$$

By hypothesis,

$$x - \sum_{n=0}^{\infty} \langle x, e_n \rangle e_n = 0,$$

and hence $\{e_n\}$ is an orthonormal basis. $\square$

Since the linear span of $\{e_n\}_{n=-\infty}^{\infty}$ is dense in $L^2(\mathbb{T})$, by the last problem, $\{e_n\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for $L^2(\mathbb{T})$.

Theorem

If $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ is convergent on $\mathbb{D}_R(a)$, then for any $0 < r < R$,

$$a_n r^n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(a+re^{i\theta})e^{-in\theta}d\theta, \quad n \geq 0$$

$$\sum_{n=0}^{\infty} |a_n|^2 r^{2n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(a+re^{i\theta})|^2 d\theta.$$

Proof.

Apply the Parseval's identity to

$$f(a+re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta},\ H = L^2(\mathbb{T}),\ e_n(\theta) = e^{in\theta},\ \theta \in [-\pi,\pi],$$

where we used the fact that $\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$. $\square$

In the entire analysis above, the most crucial fact we used is

Theorem

$\{e_n\}_{n=-\infty}^{\infty}$ is an orthonormal basis for the Hilbert space $L^2(\mathbb{T})$.

One can give a proof of this fact using the so-called spectral theorem, Although we do not provide all the details here, let us understand the "spectral theorem".

Let $p(z) = \sum_{k=0}^{n} a_k z^k$ be a complex polynomial of degree n in the variable z. If $T \in \mathcal{L}(\mathcal{H})$ then it is natural to define $p(T)$ by $p(T) = \sum_{k=0}^{n} a_k T^k$, which is obtained by replacing z by T in the expression for $p(z)$. Notice that $p(T)$ belongs to $\mathcal{L}(\mathcal{H})$, since $\mathcal{L}(\mathcal{H})$ is an algebra. The association $p \mapsto p(T)$ is an *algebra homomorphism*: For $\alpha \in \mathbb{C}$, and for complex polynomials p, q,

$$(p + \alpha q)(T) = p(T) + \alpha(q(T)) \text{ and } (pq)(T) = p(T)q(T).$$

We call this algebra homomorphism as the *polynomial functional calculus* for T.

Spectrum

The spectrum $\sigma(T)$ of $T \in \mathcal{L}(\mathcal{H})$ is defined as the subset

$$\{\lambda \in \mathbb{C} : T - \lambda I \text{ is not invertible in } \mathcal{L}(\mathcal{H})\}$$

of the complex plane $\mathbb{C}$. It turns out that $\sigma(T)$ is a bounded subset of $\mathbb{C}$. In fact,

$$\sigma(T) \subseteq \{z \in \mathbb{C} : |z| \leqslant \|T\|\},$$

where $\|T\| = \sup\{\|Tx\| : x \in \mathcal{H},\ \|x\| = 1\}$ is the operator norm of T. This inclusion may be derived from the following fact.

Problem

If $C \in \mathcal{L}(\mathcal{H})$ has norm less than 1, then show that $I - C$ is invertible and $\lim_{n\to\infty} \left\| \sum_{k=0}^{n} C^k - (I - C)^{-1} \right\| = 0$.

Solution.

For integers $m < n$, one has
$\left\| \sum_{k=m}^{n} C^k \right\| \leqslant \sum_{k=m}^{n} \|C^k\| \leqslant \sum_{k=m}^{n} \|C\|^k$, where we used the fact that $\|C^k\| \leqslant \|C\|^k$ for every non-negative integer k.

Solution continued ....

Since $\|C\| < 1$, by the convergence of the geometric series for real numbers,

$$P(m,n) = \sum_{k=m}^{n} C^k \to 0 \text{ in the operator norm as } m,n \to \infty.$$

Thus $\{P(0,n)\}_{n \geqslant 0}$ is a Cauchy sequence in $\mathcal{L}(\mathcal{H})$. Since $\mathcal{L}(\mathcal{H})$ is complete in the operator norm, there exists $T \in \mathcal{L}(\mathcal{H})$ such that $\{P(0,n)\}_{n \geqslant 0}$ converges to T in the operator norm. Since $\|C\| < 1$, $\{C^k\}_{k \geqslant 0}$ converges to 0 in the operator norm. To conclude the proof, note that

$$(I - C)P(0,n) = I - C^{n+1} = P(0,n)(I - C)$$

converges to $(I - C)T,\ I,\ T(I - C)$ simultaneously in the operator norm. Since limit is unique, we must have $(I - C)T = I = T(I - C)$. $\square$

Problem

Show that $\sigma(T)$ is a compact subset of $\mathbb{C}$.[1]

Hint.

For $z_0 \in \mathbb{C} \backslash \sigma(T)$, let $r = \|(T - z_0)^{-1}\|^{-1}$ and let $z \in \mathbb{C}$ be such that $|z - z_0| < r$. Observe that

$$T - zI = (T - z_0 I)(I - (T - z_0)^{-1}(z - z_0)).$$

By last problem, $I - (T - z_0)^{-1}(z - z_0)$ and so is $T - zI$. □

Theorem

The spectral radius $r(T) = \sup\{|z| : z \in \sigma(T)\}$ of T is given by

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n}.$$

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[1] Moreover, $\sigma(T)$ is nonempty. This is an application of Liouville's theorem from complex analysis.

Theorem (Spectral mapping property for polynomials)

For any polynomial in the complex variable and for any A in $\mathcal{L}(\mathcal{H})$, we have $\sigma(p(A)) = p(\sigma(A))$.

Proof.

Replacing p by $p - \lambda$, it suffices to verify that $p(T)$ is invertible if and only if p is nowhere-vanishing on $\sigma(T)$. By Fundamental Theorem of Algebra,
$p(z) = \alpha_0(z - \alpha_1)(z - \alpha_2)\cdots(z - \alpha_n),\ \{\alpha_k\}_{k=0}^n \subseteq \mathbb{C}$. Suppose that p is nowhere-vanishing on $\sigma(T)$. It follows that $\alpha_k \notin \sigma(T)$ for $k = 1, \ldots, n$. Hence

$$p(T) = \alpha_0(T - \alpha_1 I)(T - \alpha_2 I)\cdots(T - \alpha_n I),\ \{\alpha_k\}_{k=1}^n \subseteq \mathbb{C}\backslash\sigma(T).$$

It is now clear that $p(T)$ is invertible.

Conversely, assume that $p(T)$ is invertible. Then there exists $S \in \mathcal{L}(\mathcal{H})$ such that $p(T)S = I = Sp(T)$. It follows that $\alpha_k \notin \sigma(T)$ for any $k = 1, \ldots, n$. Thus $p(z)$ is nonzero for any $z \in \sigma(T)$. □

Let us discuss the spectral theorem for self-adjoint operators on complex Hilbert spaces. Recall that a bounded linear operator A is self-adjoint if $A^* = A$ or equivalently, $\langle Ax, y\rangle = \langle x, Ay\rangle, \ x, y \in H$. Note that $\langle Ax, x\rangle$ is a real number for any $x \in H$.

Problem

Show that spectrum of a self-adjoint operator A is contained in $\mathbb{R}$.

Solution.

Let $\lambda = \alpha + i\beta \in \mathbb{C}$ be such that $\beta \neq 0$. Then, for any $x \in H$,

$$\|(A - \lambda I)x\|^2 = \|Ax\|^2 - 2\mathrm{Re}\langle Ax, \lambda x\rangle + |\lambda|^2\|x\|^2.$$

However, $\langle Ax, \lambda x\rangle = \overline{\lambda}\langle Ax, x\rangle = (\alpha - i\beta)\langle Ax, x\rangle$ Thus

$$\|(A - \lambda I)x||^2 = \|Ax\|^2 - 2\langle Ax, x\rangle\alpha + (\alpha^2 + \beta^2)\|x\|^2 \geqslant \beta^2\|x\|$$

for every $x \in H$. Clearly, $A - \lambda$ is injective and the range is closed. Since $A^* = A$, $(A - \lambda)^*$ is also injective, so the range of $A - \lambda I$ is dense. Hence $A - \lambda I$ is invertible. $\square$

Theorem (Spectral theorem for self-adjoint operators-I)

*If A is a self-adjoint operator in $\mathcal{L}(\mathcal{H})$, then there is a unique positive, isometric, algebraic *-homomorphism map $\phi : C(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ such that $\phi(p) = p(A)$ for every polynomial $p \in C(\sigma(A))$.*

We present a proof based on the following:

Theorem (Stone-Weierstrass theorem)

Let $\mathscr{A}$ be an algebra of continuous functions $f : K \to \mathbb{R}$ with the following properties:

(1) *If $x \neq y \in K$, then there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$.*

(2) *For every $x \in K$, there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$.*

Then $\mathscr{A}$ is dense in the algebra $C_{\mathbb{R}}(K)$ of continuous real-valued functions on K endowed with the uniform norm.

We also need the following fact:

Lemma

For any operator $T \in \mathcal{L}(\mathcal{H})$ for which $T^*T \leqslant TT^*$, $\|T\|$ is equal to the spectral radius $r(T) = \max\{|z| : z \in \sigma(T)\}$ of T.

Proof.

Note that for any positive integer n and any $h \in \mathcal{H}$,

$$\|T^n h\|^2 = \langle T^*T^n h,\ T^{n-1}h\rangle \leq \|T^*T^n h\|\,\|T^{n-1}h\| \leqslant \|T^{n+1}h\|\,\|T^{n-1}h\|.$$

Thus we have $\|T^n\|^2 \leqslant \|T^{n+1}\|\,\|T^{n-1}\|$ for any positive integer n. We now check by induction on integers $n \geqslant 1$ that $\|T^n\| \geqslant \|T\|^n$. Assuming the inductive hypothesis for $k = 1, \ldots, n$, we obtain

$$\|T^{n+1}\|\,\|T\|^{n-1} = \|T^{n+1}\|\,\|T^{n-1}\| \geqslant \|T^n\|^2 = \|T\|^{2n},$$

which yields $\|T^{n+1}\| \geqslant \|T\|^{n+1}$ completing the proof of induction. By the spectral radius formula $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$, we get $r(T) \geqslant \|T\|$. Since $r(T) \leqslant \|T\|$ holds true for any operator T in $\mathcal{L}(\mathcal{H})$, we obtain the equality $r(T) = \|T\|$. $\square$

Recall that any bounded linear operator T satisfies the identity $\|T^*T\| = \|T\|^2$ (C^*-algebra identity).

Proof of Spectral theorem for self-adjoint operators-I.

Since A is self-adjoint, for any complex polynomial p,

$$p(A)^* p(A) = |p|^2(A).$$

Hence, by the C^*-algebra identity and the last lemma,

$$\|p(A)\|^2 = \|p(A)^* p(A)\| = \||p|^2(A)\| = r(|p|^2(A)).$$

However, $\sigma(|p|^2(A)) = |p|^2(\sigma(A))$ (since $|p|^2$ is also a polynomial in the real variable), and hence

$$r(|p|^2(A)) = |p(z_0)|^2 \text{ for some } z_0 \in \sigma(A).$$

Thus $\|p(A)\| = \|p\|_{\infty,\sigma(A)}$ showing that $\phi(p) = p(A)$ defines a positive, isometric, algebraic *-homomorphism. Now apply Stone-Weierstrass Theorem to extend ϕ isometrically. $\square$

Square-root and polar decomposition

Let P be a positive operator, that is, P is self-adjoint and $\langle Px, x\rangle \geqslant 0$ for every $x \in H$.

- The spectrum of a positive operator is contained in $[0, \infty)$: Since $\sigma(P) \subseteq \mathbb{R}$, it suffices to check that $P - \lambda$ is invertible for every $\lambda \in (-\infty, 0)$. However, $\langle (P-\lambda)x, x\rangle \geqslant -\lambda \|x\|^2$ for every $x \in H$. Conclude that $P - \lambda$ is bijective.

Theorem

Every positive operator P has a positive square-root, that is, there exists a positive operator Q such that $P = Q^2$.

Proof.

By the spectral theorem, $\phi(p) = P$, where p is the polynomial $p(z) = z$. Note that *id* is defined on the spectrum of P. However, the spectrum of P is contained in $[0, \infty)$. So the function p has a positive continuous sqaure-root, that is, there exists positive element $q \in C(\sigma(P))$ such that $p = q^2$. Applying ϕ on both sides

Proof continued ....

gives $P = \phi(p) = \phi(q^2) = \phi(q)^2$. Note that $Q = \phi(q)$ is a positive operator satisfying $P = Q^2$. $\square$

Let T be a left-invertible bounded linear operator on H, that is, there exists a bounded linear operator L such that $LT = I$.

- Note that $\|x\| = \|LTx\| \leqslant \|L\|\|Tx\|$ and hence

$$\langle T^*Tx, x\rangle = \|Tx\|^2 \geqslant \|L\|^{-2}\|x\|^2, \quad x \in H.$$

- Thus $\sigma(T^*T) \subseteq [\frac{1}{\|L\|}, \infty)$ (Exercise).
- Define $U : H \to H$ by $U = T(T^*T)^{-1/2}$, where $(T^*T)^{-1/2}$ denote the positive square-root of $(T^*T)^{-1}$.
- Note that U is an isometry, that is, $U^*U = I$:

$$U^*U = (T(T^*T)^{-1/2})^*T(T^*T)^{-1/2}$$

$$= (T^*T)^{-1/2}T^*T(T^*T)^{-1/2} = I.$$

- $T = UP$, where U an isometry and $P = (T^*T)^{1/2}$ positive.

Theorem (Polar decomposition)

Every invertible $T \in \mathcal{L}(\mathcal{H})$ can be decomposed (uniquely) as $T = UP$ with positive invertible P and unitary U (that is, an invertible isometry).

Let $\mathcal{U}(\mathcal{H})$ denote the set of all unitary operators in $\mathcal{L}(\mathcal{H})$. Let $\mathcal{G}(\mathcal{H})$ denote the set of all invertible operators in $\mathcal{L}(\mathcal{H})$. Note that $\mathcal{G}(\mathcal{H})$ is a group under the operation of composition and that

$$\mathcal{U}(\mathcal{H}) \subsetneq \mathcal{G}(\mathcal{H}) \subsetneq \mathcal{L}(\mathcal{H}).$$

Let V be a normed linear space with the norm $\|\cdot\|$. A subset $\mathcal{O}$ of V is said to be *path-connected* if for any $v_1, v_2 \in \mathcal{O}$ there exists a map (to be referred to as *path*) $\gamma : [0, 1] \to \mathcal{O}$ such that

(1) $\gamma(0) = v_1$ and $\gamma(1) = v_2$, and

(2) $\lim_{n\to\infty} \|\gamma(t_n) - \gamma(t)\| = 0$ whenever $\lim_{n\to\infty} |t_n - t| = 0$ for any $t, t_n \in [0, 1]$.

Theorem

$\mathcal{U}(\mathcal{H})$ is path-connected.

Corollary

$\mathcal{G}(\mathcal{H})$ is path-connected.

Proof.

It suffices to show that there exists a path joining any $T \in \mathcal{G}(\mathcal{H})$ and the identity operator I. Let $T \in \mathcal{G}(\mathcal{H})$ and consider its polar decomposition $T = UP$. Let $\gamma : [0, 1] \to \mathcal{U}(\mathcal{H})$ be a path joining U and I. Define $\delta : [0, 1] \to \mathcal{G}(\mathcal{H})$ by

$$\delta(t) = \gamma(t)((1 - t)P + tI), \quad t \in [0, 1].$$

Then δ is well-defined since $(1 - t)P + tI$ is invertible in view of

$$\sigma((1 - t)P_1 + tI) = \{(1 - t)\lambda + t : \lambda \in \sigma(P)\}$$

and $\sigma(P) \subset (0, \infty)$. Clearly, the continuity of δ follows from that of γ. Since $\delta(0) = T$ and $\delta(1) = I$, the proof is over. $\square$

For the proof of the connectedness of $\mathcal{U}(\mathcal{H})$ using spectral theorem, we need to go beyond continuous functional calculus.

Recall that a bounded linear operator T on a Hilbert space $\mathcal{H}$ is *cyclic* if there exists a vector $h \in \mathcal{H}$ (a *cyclic vector*) such that

$$\bigvee\{T^n h : n \text{ is a non-negative integer}\} = \mathcal{H}.$$

Lemma

Let A be a self-adjoint operator in $\mathcal{L}(\mathcal{H})$. If A is cyclic with cyclic vector $f \in \mathcal{H}$, then there exists a finite positive Borel measure μ_f and a unitary operator $U : \mathcal{H} \to L^2(\sigma(A), d\mu_f)$ such that

$$(UAU^{-1}g)(\lambda) = \lambda g(\lambda), \quad g \in L^2(\sigma(A), \mu_f).$$

Let ϕ denote the continuous functional calculus.

Proof.

Consider the bounded linear functional $\psi : C(\sigma(A)) \to \mathbb{C}$ by $\psi(g) = \langle \phi(g)f, f \rangle, \quad g \in C(\sigma(A))$. By Riesz Representation Theorem, there exists a finite positive Borel measure μ_f on $\sigma(A)$ such that

$$\psi(g) = \int_{\sigma(A)} g(t) d\mu_f(t), \quad g \in C(\sigma(A)).$$

Define U by $U\phi(g)f = g$ for $g \in C(\sigma(A))$, and note that

$$\|\phi(g)f\|^2 = \langle \phi(|g|^2)f, f \rangle = \int_{\sigma(A)} |g(t)|^2 d\mu_f(t) = \|g\|^2.$$

Since f is cyclic for A, U extends isometrically from $\mathcal{H}$ into $L^2(\sigma(A), \mu_f)$. Since the range of U contains continuous functions, U is surjective. Note that for any $g \in L^2(\sigma(A), \mu_f)$ and $\lambda \in \sigma(A)$,

$$(UAU^{-1}g)(\lambda) = (UA\phi(g)f)(\lambda) = (U\phi(zg)f)(\lambda) = \lambda g(\lambda).$$

This completes the proof. $\square$

Lemma

Let $\mathcal{H}$ be a separable Hilbert space. If $A \in \mathcal{L}(\mathcal{H})$, then there exists an orthonormal family $\{g_j\}_{j=1}^{N}$ with $N \in \mathbb{N}$ or $N = \infty$, such that $\mathcal{H} = \oplus_{j=1}^{N} \mathcal{H}_j$, where

$$\mathcal{H}_j = \bigvee \{A^{*k} A^l g_j : k, l \in \mathbb{N}\}, \quad j = 1, \ldots, N.$$

Proof.

Let $\{e_j\}_{j \in \mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Let $g_1 = e_1$. If $\mathcal{H}_1 = \mathcal{H}$, then let $N = 1$. Otherwise, let k_1 be the smallest positive integer such that $P_{\mathcal{H}_1^\perp} e_{k_1} \neq 0$ and let $g_2 = \frac{P_{\mathcal{H}_1^\perp} e_{k_1}}{\|P_{\mathcal{H}_1^\perp} e_{k_1}\|}$. Since $\langle g, g_2 \rangle = 0$ for every $g \in \mathcal{H}_1$, the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal. Further, $\{e_j\}_{j=1}^{k_1} \subseteq \mathcal{H}_1 \oplus \mathcal{H}_2$. Now proceed by induction. $\square$

Theorem (Spectral Theorem for Self-adjoint Operators)

Let $\mathcal{H}$ be a separable Hilbert space and let $A \in \mathcal{L}(\mathcal{H})$. If A is a self-adjoint operator, then there exist finite positive Borel measures $\mu_1, \ldots, \mu_N$ ($N \in \mathbb{N}$ or $N = \infty$) and a unitary operator $U : \mathcal{H} \to \oplus_{n=1}^{N} L^2(\sigma(A), \mu_n)$ such that

$$(UAU^{-1}g)_n(\lambda) = \lambda g_n(\lambda), \quad g = (g_n)_{n=1}^{N} \in \oplus_{n=1}^{N} L^2(\sigma(A), \mu_n).$$

*In particular, there exists a unique positive, contractive, algebraic *-homo-morphism $\phi : B_\infty(\sigma(A)) \to \mathcal{L}(\mathcal{H})$ such that*

$$\phi(f) = f(A), \quad f \in C(\sigma(A)) \text{ is a polynomial.}$$

Moreover, ϕ is isometric on $C(\sigma(A))$.

Proof.

There exist invariant subspaces $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_N$ of A such that

$$\mathcal{H} = \oplus_{n=1}^{N} \mathcal{H}_n, \text{ and } A|_{\mathcal{H}_n} \text{ is cyclic.}$$

The first part now follows from Lemma 20. To see the second part, let $f_n \in L^2(\sigma(A), \mu_n)$ such that $\|f_n\| = 2^{-n}$, and let M denote the disjoint union of N copies of $\sigma(A)$. If μ be the restriction of μ_n to nth copy of $\sigma(A)$, then $\langle M,\, \mu\rangle$ is the desired finite measure space. Moreover, A is unitarily equivalent to the operator M_λ of multiplication by λ on $L^2(M, \mu)$. The Borel functional calculus of A now follows from that of M_λ. This completes the proof. $\square$

Theorem (L^2-version of the Müntz-Szász Theorem)

If $\{n_k\}_{k\geqslant 1}$ is a strictly increasing sequence of nonnegative integers, then closed linear span$\{t^{n_k} : k \geqslant 1\} = L^2[0,1] \Leftrightarrow \sum_{k=1}^{\infty} \frac{1}{n_k+1} = \infty$.

Corollary

The closed linear span$\{t^k : k \geqslant 1\} = L^2[0,1]$.

For a complex number $z = x + iy$, note that if $2x+1>0$, then

$$\int_0^1 |t^z|^2 dt = \int_0^1 t^{2x} dt = \frac{t^{2x+1}}{2x+1}\Big|_0^1.$$

Also, if $2x+1 \leqslant 0$, then $t^z \notin L^2[0,1]$. Thus if $\Omega = \{z \in \mathbb{C} : \mathrm{Re}(z) > -\frac{1}{2}\}$. then

$$t^z \in L^2[0,1] \ \Leftrightarrow z \in \Omega.$$

- $z = x + iy \in \Omega$ iff $x > \frac{1}{2}$ iff $\frac{x^2+y^2}{(x+1)^2+y^2} < 1$ iff $\left|\frac{z}{z+1}\right| < 1$.
- The map $\phi : \Omega \to \mathbb{D}$ given by $\phi(z) = \frac{z}{z+1}$ is bijective.

For $z \in \Omega$, define $v_z : \mathbb{D} \to \mathbb{C}$ by

$$v_z(\eta) = \frac{1}{(1+z)(1-\phi(z)\eta)}, \quad \eta \in \mathbb{D}.$$

Consider the Hilbert space $H^2(\mathbb{D})$ of all holomorphic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on $\mathbb{D}$ endowed with the inner-product

$$\langle f, g \rangle_{H^2(\mathbb{D})} = \sum_{n=0}^{\infty} a_n \overline{b}_n.$$

Note that $\langle f, v_z \rangle_{H^2(\mathbb{D})} = \frac{1}{1+\overline{z}} \langle f, \sum_{n=0}^{\infty} \phi(z)^n \eta^n \rangle = \frac{1}{1+\overline{z}} \sum_{n=0}^{\infty} a_n \overline{\phi(z)}^n = \frac{1}{1+\overline{z}} f(\overline{\phi(z)})$. Since ϕ is surjective, we obtain

Lemma

The closed linear span of $\{v_z : z \in \Omega\}$ is equal to $H^2(\mathbb{D})$.

Lemma

For every $z, w \in \Omega$,

$$\langle v_z,\ v_w \rangle_{H^2(\mathbb{D})} = \langle t^z,\ t^w \rangle_{L^2[0,1]}.$$

Proof.

A calculation using the geometric series shows that

$$\langle v_z,\ v_w \rangle_{H^2(\mathbb{D})} = \frac{1}{(1+z)(1+\overline{w})} \langle \frac{1}{1-\phi(z)\eta},\ \frac{1}{1-\phi(w)\eta} \rangle_{H^2(\mathbb{D})}$$

$$= \frac{1}{(1+z)(1+\overline{w})} \frac{1}{1-\phi(z)\overline{\phi(w)}} \cdot \frac{1}{(1+z)(1+\overline{w})} \frac{1}{1-\frac{z}{z+1}\frac{\overline{w}}{\overline{w}+1}}$$

$$= \frac{1}{(1+z)(1+\overline{w}) - z\overline{w}} = \frac{1}{1+z+\overline{w}} = \langle t^z,\ t^w \rangle.$$

This completes the proof. □

Define $V : H^2(\mathbb{D}) \to L^2[0,1]$ by setting

$$V(v_z) = t^z, \quad z \in \Omega,$$

and extending linearly and continuously to $H^2(\mathbb{D})$. Note that V is an isometry. Hence it has closed range. However, the range is dense (since polynomials are dense in $L^2[0,1]$), so V is unitary.

The above argument shows the following:

Lemma

For any strictly increasing sequence $\{n_k\}_{k\geqslant 1}$, the following are equivalent:

(i) *the closed linear span of $\{v_{n_k}\}_{k\geqslant 1}$ is $H^2(\mathbb{D})$,*

(ii) *the closed linear span of $\{t^{n_k}\}_{k\geqslant 1}$ is $L^2[0,1]$,*

(iii) *the closed linear span of $\{\frac{1}{1-\phi(n_k)\eta}\}_{k\geqslant 1}$ is $H^2(\mathbb{D})$.*

We also need a fact from complex analysis.

Lemma

Let $\{a_n\}_{n\geqslant 1}$ be a sequence of points in $\mathbb{D}$. There exists a non-zero function $f \in H^2(\mathbb{D})$ that vanishes at each point a_n if and only if $\sum_{n=1}^{\infty}(1-|a_n|) < \infty$.

Proof of the L^2-version of the Müntz-Szász Theorem.

Note that

$$\sum_{n=1}^{\infty}(1-|\phi(n_k)|) = \sum_{k=1}^{\infty} \frac{1}{n_k+1},$$

and apply all the lemmas. $\square$