RIESZ-MARKOV REPRESENTATION THEOREM: PROBLEMS

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Abstract. These are the lecture notes prepared for AFS schools conducted at Kumaun University, Almora, December 2014 and Bhaskacharya Pratishthana, Pune, 2015.

1. Space of Riemann Integrable Functions

In these notes, we will be concerned about the space $L^1(X)$ of integrable functions on X and two of its subspaces: the subspace $C_c(X)$ of continuous functions with compact support and the subspace $R[a,b]$ of Riemann integrable functions in case $X = [a,b]$. If not specified, then $X = \mathbb{R}^d$ or more generally a measurable subset of $\mathbb{R}^d$ of positive measure. It turns out that the subspace $C[a,b] := C_c[a,b]$ of $L^1[a,b]$ is properly contained in the space $R[a,b]$ of Riemann integrable functions on $[a,b]$. Indeed, by a result of Lebesgue, $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if f is continuous a.e. (for a proof, see [3, Problem 4, Chapter 1]). Moreover, $R[a,b]$ is a subspace of $L^1[a,b]$ as shown below.

Theorem 1.1. *Every Riemann integrable function on $[a,b]$ is Lebesgue integrable. Moreover, the Riemann integral of f is same as the Lebesgue integral of f.*

Proof. We give outline of the proof. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$ and some $M \in \mathbb{R}$. Use the definition of Riemann integrability to find sequences $\{\phi_k\}$ and $\{\psi_k\}$ of step functions bounded by M such that $\phi_k \uparrow \phi$ and $\psi_k \downarrow \psi$ for some measurable functions ϕ and ψ such that $\phi \leq f \leq \psi$. Also, the limits of Riemann integrals of ϕ_k and ψ_k agree with the Riemann integral of f. Since Riemann integral and Lebesgue integral agree for step functions, by bounded convergence theorem, Lebesgue integrals of ϕ and ψ are same. Since $\phi \leq \psi$, we must have $\phi = f = \psi$ a.e. This shows that f is measurable. It is now easy to see that the Riemann integral of f is same as its Lebesgue integral. $\square$

Remark 1.2 : The set of Riemann integrable functions forms a subspace of $L^1[a,b]$.

In general, it is hard to compute Lebesgue integral right from the definition. The preceding result, in particular, shows that Lebesgue integral of continuous functions may be calculated using the methods from Riemann integration theory.

The pointwise limit of Riemann integrable functions need not be Riemann integrable as shown below.

Problem 1.3. *Consider the function $f_m(x) = \lim_{n\to\infty}(\cos(m!\pi x))^n$ for $x \in \mathbb{R}$. Find the set of discontinuities of f_m. Further, verify the following:*

(1) *$\{f_m\}$ converges pointwise to f, where $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, and $f(x) = 1$ for $x \in \mathbb{Q}$.*
(2) *f is discontinuous everywhere.*

Remark 1.4 : If $\{r_n\}_{n=1}^{\infty}$ is enumeration of rationals then the sequence $\{\chi_{\{r_1,\cdots,r_n\}}\}$ of Riemann integrable functions converges pointwise to the characteristic function of rationals, which is not Riemann integrable.

It is easy to produce an element in $L^1[0,1]$, which is not in $R[0,1]$. For example, take $1/\sqrt{x}$ on $(0,1]$ and 0 at 0. Note that f appearing in the last problem would not suffice since it belongs to the equivalence containing 0, so its the 0 element as a member in $L^1[0,1]$. Since $R[a,b]$ is a (proper) subspace of $L^1[a,b]$, it is natural to ask whether it is closed in $L^1[a,b]$. The answer is No.

Problem 1.5. *Let $\hat{C}$ denote the Cantor-like set obtained by removing 2^{k-1} centrally situated open subintervals $I_{1k}, \cdots, I_{2^{k-1}k}$ of $I := [0,1]$ each of length $1/4^k$ at the kth stage, where $k = 1, 2, \cdots$. Let $F_k : I \to I$ be a continuous function such that $F_k = 1$ on $I \setminus \cup_{i=1}^{2^{k-1}} I_{ik}$ and $F_k = 0$ at the mid-points of $I_{1k}, \cdots, I_{2^{k-1}k}$. For $f_n := \prod_{i=1}^{n} F_i$ for $n \geq 1$, verify the following:*
(1) *The sequence $\{f_n\}$ of continuous functions decreases to some $f : I \to I$ pointwise.*
(2) *$\|f_n - f\|_1 \to 0$ as $n \to \infty$.*
(3) *The limit function f is discontinuous at every point of $\hat{C}$.*
(4) *The set $R[0,1]$ of Riemann integrable functions as a subspace of $L^1[0,1]$ is not closed (**Hint.** The measure of $\hat{C}$ is positive).*

Remark 1.6 : Consider the vector space $R[a,b]$ of Riemann integrable functions. Then $R[0,1]$ is a subspace of $L^1[a,b]$ with norm $\|f-g\|_1 := \int_a^b |f(t)-g(t)|dt$, which is not complete.

The preceding remark raises the following question: What is the closure of $R[a,b]$ in $L^1[a,b]$? We answer this in the next section.

2. Space of Lebesgue Integrable Functions

The measurable functions, in general, could be extended real-valued. The following simple observation shows that for members of L^1, WLOG, we may confine ourselves to real-valued functions.

Problem 2.1. *Let $f : \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be an extended real-valued function. If $f \in L^1$ then the set $\{x \in \mathbb{R}^d : |f(x)| = \infty\}$ is of measure zero.*

Problem 2.2. *Consider the intervals $I_1 = [0,1], I_2 = [0,1/2], I_3 = [1/2,1], I_4 = [0,1/4], I_5 = [1/4,1/2], I_6 = [1/2,3/4], I_7 = [3/4,1]$ and so on. For $f_n = \chi_{I_n}$ and $f = 0$, show that $\|f_n - f\|_1 \to 0$, but $f_n(x) \nrightarrow f(x)$ for any $x \in [0,1]$.*

Hint. Any $x \in [0,1]$ belongs to infinitely many I_n's and complements of infinitely many I_n's.

Although, the convergence in L^1 need not imply the pointwise convergence, the situation is not very bad.

Lemma 2.3. *If $\{f_n\}$ is a Cauchy sequence in L^1, then there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\|f_{n_k+1} - f_{n_k}\| \leq 2^{-k}$ for all non-negative integers k.*

A normed linear space is *complete* if every Cauchy sequence is convergent.

Theorem 2.4. *The normed linear space L^1 is complete.*

Proof. Assume that $\{f_n\}$ is a Cauchy sequence in L^1. By the preceding lemma, there exists a subsequence $\{f_{n_k}\}$ satisfying the condition given there. We claim that $f_{n_k}(x) \to f(x)$ a.e. where

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x)).$$

Note that $|f| \leq g$, where

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_k+1}(x) - f_{n_k}(x)|.$$

By the monotone convergence theorem,

$$\|g\|_1 \leq \|f_{n_1}\|_1 + \sum_{k=1}^{\infty}\|f_{n_k+1} - f_{n_k}\|_1 < \infty,$$

and hence $f \in L^1$. By Problem 2.1, the series $\sum_{k=1}^{\infty}(f_{n_{k+1}}(x) - f_{n_k}(x))$ converges (absolutely) for a.e. x. But then

$$f(x) = f_{n_1}(x) + \lim_{l\to\infty}\sum_{k=1}^{l-1}(f_{n_{k+1}}(x) - f_{n_k}(x)) = \lim_{l\to\infty} f_{n_l}(x)$$

for a.e. x, and the claim stands verified.

Finally, $\|f_{n_l} - f\|_1 \leq \sum_{k=l}^{\infty}\|f_{n_{k+1}} - f_{n_k}\|_1 \to 0$. By general theory, a Cauchy sequence with a convergent subsequence is necessarily convergent. □

The Riesz-Fischer Theorem makes possible the application of abstract theory of Banach spaces to L^1. For instance, one may conclude that every absolutely convergent series in L^1 is convergent. An astute reader may notice, however, that this observation is implicitly there in the above proof of Riesz-Fischer theorem.

Problem 2.5. *Verify that a closed subspace of a complete normed linear space is complete. Conclude that $C[a,b]$ is not closed in $L^1[a,b]$.*

Hint. Suppose that $C[a,b]$ is closed in $L^1[a,b]$. By Riesz-Fischer theorem, $(C[a,b], \|\cdot\|)$ is a complete normed linear space. We apply the bounded inverse theorem to the mapping $f \to f$ from $(C[a,b], \|\cdot\|_\infty)$ onto $(C[a,b], \|\cdot\|_1)$ to get $C > 0$ such that $\|f\|_\infty \leq C\|f\|_1$ for every $f \in C[a,b]$. Can you give a less elementary verification of this fact ?

Problem 2.6. *Show that the convergence in L^1 need not imply that in L^∞.*

The preceding exercise raises the question: what is the closure of $C[0,1]$ in $L^1[0,1]$? We will see in Section 3 that it is the whole of $L^1[0,1]$.

3. Some Dense subspaces of L^1

We say that a subset Y of X is *dense* in X if for every $f \in X$ and for every $\epsilon > 0$, there exists a g in Y such that $\|f - g\| < \epsilon$.

Many times it is practical to know "good" dense subsets of a given normed linear space. For instance, an inequality for all members of X could be deduced from the same inequality for members of Y (see Exercise 3.9 below for another application).

Problem 3.1. *Show that the set of simple functions is dense in L^1.*

Hint. Since $f = f_+ - f_-$ for $f_+, f_- \geq 0$ a.e. WLOG, we assume that $f \geq 0$ a.e. Then there exists a sequence of simple functions increasing to f pointwise. Now apply monotone convergence theorem.

Problem 3.2. *Show that for any step function $s : \mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$, there exists a continuous function f such that $\|f - s\|_\infty < \epsilon$.*

Hint. First try a characteristic function of an interval.

Problem 3.3. *Show that for any step function $s : \mathbb{R}^d \to \mathbb{R}$ in L^1 and $\epsilon > 0$, there exists a continuous function f with compact support such that $\|f - s\|_1 < \epsilon$.*

Hint. Characteristic function of a cube is a product of characteristic functions of intervals. Now apply the last exercise.

Problem 3.4. *For a measurable subset E of $\mathbb{R}^d$ of finite measure and $\epsilon > 0$, show that there exists an almost disjoint family of rectangles $\{R_1, \cdots, R_M\}$ with $m(E \triangle \cup_{j=1}^M R_j) \leq \epsilon$.*

Hint. Cover E by $\cup_{j=1}^\infty Q_j$ such that $\sum_{j=1}^\infty |Q_j| \leq m(E) + \epsilon/2$. There exists $N \geq 1$ such that $\sum_{j=N+1}^\infty |Q_j| \leq \epsilon/2$. Verify that $m(E \triangle \cup_{j=1}^N Q_j) \leq \epsilon$. Warning: $Q_1, \cdots, Q_N$ may not be almost disjoint.

Problem 3.5. *Let E be a measurable set of finite measure and let $\epsilon > 0$ be given. Show that there exists a step function g such that $\|\chi_E - g\|_1 < \epsilon$.*

Hint. By the last exercise, there exists an almost disjoint family of rectangles $\{R_1, \cdots, R_M\}$ with $m(E \triangle \cup_{j=1}^M R_j) \leq \epsilon/2$. Check that $\|\chi_E - \sum_{j=1}^M \chi_{R_j}\|_1 < \epsilon$.

Recall that the *support* of $f : X \to \mathbb{R}$ is the closure of $\{x \in X : f(x) \neq 0\}$.

Theorem 3.6. *The vector space of continuous functions with compact support is dense in L^1.*

Proof. Let $f \in L^1$ and $\epsilon > 0$ be given. By Problem 3.1, there exists a simple function s such that $\|f - s\| < \epsilon/3$. Since s is a finite linear combination of characteristic functions, by the preceding problem, there exists a step function g such that $\|s - g\|_1 < \epsilon/3$. Now by Problem 3.3, there exists a continuous function h with compact support such that $\|g - h\|_1 < \epsilon/3$. Finally, by triangle inequality, we obtain $\|f - h\|_1 < \epsilon$. □

Remark 3.7 : $R[a,b]$ is dense in $L^1[a,b]$.

Corollary 3.8. *The space of polynomials on $[a,b]$ is dense in $L^1[a,b]$. In particular, $L^1[a,b]$ is separable.*

Let us discuss one application of Theorem 3.6.

Problem 3.9. *For $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, let $f_y(x) = f(x-y)$ denote the translate of f. If $f \in L^1$ then show that $\psi : \mathbb{R} \to L^1$ given by $\psi(y) = f_y$ is uniformly continuous.*

Hint. Let g be a continuous function in L^1 with support in $[-A, A]$ such that $\|f-g\|_1 < \epsilon$. Since g is uniformly continuous on $[-A, A]$, there exists $\delta < A$ such that $|g(s)-g(t)| < \epsilon/3A$ whenever $|s-t| < \delta$. Check that $\|g_s-g_t\|_1 < (2A+\delta)(\epsilon/3A) < \epsilon$ (since $\min\{-s-A, -t-A\} \leq x \leq \max\{A-s, A-t\}$), and hence $\|f_s - f_t\|_1 < 3\epsilon$ whenever $|s-t| < \delta$.

Corollary 3.10 (Riemann-Lebesgue Lemma)**.** *If $f \in L^1$ then*

$$\hat{f}(\zeta) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\zeta}dx \to 0 \ \text{as } |\zeta| \to \infty.$$

Proof. By the preceding exercise, for $f \in L^1$, $\|f - f_h\|_1 \to 0$ as $h \to 0$. The desired conclusion follows from

$$\hat{f}(\zeta) = \frac{1}{2}\int_{\mathbb{R}^d} (f(x) - f_h(x))e^{-2\pi i x\zeta}dx,$$

where $h = \frac{1}{2}\frac{\bar{\zeta}}{|\zeta|^2} \to 0$ as $|\zeta| \to \infty$. □

The density of the space $C_c(X)$ of continuous functions with compact support holds for $L^1(X,\mu)$, where X is any locally compact Hausdorff space and μ is any positive measure on X [4, 3.14 Theorem]. One can use this to obtain the following.

Problem 3.11. *Let μ be a positive measure on a locally compact subset X of $\mathbb{R}^d$ such that $\int f d\mu = 0$ for all $f \in C_c(X)$ then show that $\mu = 0$.*

4. Riesz-Markov Representation Theorem

This section is motivated by the following question:

Question. What are (all) examples of linear, positive functionals on $C_c(X)$?

Before we answer this question, let us see a class of examples of linear, positive functionals on $C_c(X)$.

Problem 4.1. *Let X be a locally compact space and let μ be a positive measure on X. Verify the following:*

1. *$\psi(f) = \int f d\mu$ is a linear functional on $C_c(X)$, that is, $\psi(f+\alpha g) = \psi(f) + \alpha\psi(g)$ for all $f, g \in C(X)$ and $\alpha \in \mathbb{C}$.*
2. *ψ is positive, that is, $\psi(f) \geq 0$ whenever $f \geq 0$ a.e.*

Problem 4.2. *Let μ be a positive measure on a compact metric space X and let ψ be a bounded linear functional on $C(X)$. For a positive measure μ, verify that the following are equivalent:*

(1) *$\psi(f) = \int f d\mu$ for all $f \in C(X)$.*
(2) *$\psi(f) = \int f d\mu$ for all real-valued $f \in C(X)$.*
(3) *$\psi(f) \leq \int f d\mu$ for all real-valued $f \in C(X)$.*
(4) *$\psi(f) \leq \int f d\mu$ for all non-negative real-valued $f \in C(X)$.*

Hint. For (3) implies (2), replace f by $-f$ in (3). For (4) implies (3), note that for any real-valued $f \in C(X)$, $f + \|f\|_\infty \geq 0$.

In case $X = [0, 1]$, the length (or measure) of (a, b) is intimately related to the values of the functional ψ as observed below.

Problem 4.3. *Define $\psi(f) := \int_{[0,1]} f(x)dx$ for $f \in C[0, 1]$. Consider a segment $(a, b) \subseteq [0, 1]$ and consider the class of all $f \in C[0, 1]$ such that $0 \leq f(x) \leq 1$ for $x \in [0, 1]$ and $f(x) = 0$ for all x not in (a, b). Show that $\psi(f) < b - a$ for all such f, but we can choose a sequence $\{f_n\}$ so that $\psi(f_n) \to b - a$.*

It turns out that examples given in Problem 4.1 are the only linear, positive functionals on $C_c(X)$.

Theorem 4.4 (Riesz-Markov Representation Theorem)**.** *Let X be a locally compact Hausdorff space, and let ψ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathcal{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathcal{M}$ (to be referred as the representing measure of ψ) such that*

$$\psi(f) = \int_X f \ d\mu \text{ for every } f \in C_c(X).$$

Moreover, we have the following:

(1) *μ is outer regular, that is, for every $E \in \mathcal{M}$, we have*

$$\mu(E) = \inf\{\mu(V) : E \subseteq V,\ V \text{ open}\}.$$

(2) *μ is inner regular, that is, for every open E, and for every $E \in \mathcal{M}$ with $\mu(E) < \infty$, we have*

$$\mu(E) = \sup\{\mu(K) : K \subseteq E,\ K \text{ compact}\}.$$

(3) *μ is complete, that is, if $E \in \mathcal{M}$ and $A \subseteq E$ such that $\mu(E) = 0$, then $A \in \mathcal{M}$.*

Corollary 4.5. *Let μ be the representing measure of ψ as given in the Riesz-Markov Representation Theorem (for short, RRT). Then $\mu(K) < \infty$ for every compact subset K of X.*

Proof. By Urysohn's Lemma, there exists $f \in C_c(X)$ such that $0 \leq f(x) \leq 1$ for all $x \in X$, $f(x) = 1$ on K, and the support of f lies in an open set V containing K. Since $\chi_K \leq f$, we have $\mu(K) \leq \psi(f) < \infty$. $\square$

Let us understand RRT in finite dimensional case.

Problem 4.6. *What does the RRT say in case $X = \{1, \cdots, n\}$ for either a positive integer n or $n = \infty$? How does the representing measure μ look like in these cases ? Can you prove both these statements without an appeal to the RRT ? Isn't an exercise from Linear Algebra/Analysis ?*

Problem 4.7. *For a non-negative integer n, find the representating measure of the linear functional $\psi_n(f) = f(0) + f(1) + \cdots + f(n)$ for $f \in C_c(\mathbb{R})$.*

Here is a way to recover the representing measure from the linear functional.

Problem 4.8 (J. Feldman)**.** *Let V be an open subset of a compact Hausdorff metric space and let χ_V be the indicator function of V. Consider the sequence $\{f_n\}$ of functions from X given by*

$$f_n(x) := \begin{cases} 0 & \textit{if } x \in X \setminus V, \\ nd(x, X \setminus V) & \textit{if } x \in V \textit{ and } d(x, X \setminus V) \leq 1/n \\ 1 & if x \in V \textit{ and } d(x, X \setminus V) \geq 1/n. \end{cases}$$

For a linear functional ψ on $C(X)$ with representing measure μ, verify:

(1) *f_n is a continuous function on X such that $0 \leq f_n(x) \leq 1$ for every $x \in X$.*
(2) $\lim_{n\to\infty} f_n(x) = \chi_V(x)$ *for every $x \in X$.*
(3) $\lim_{n\to\infty} \psi(f_n) = \mu(V)$.

Conclude that μ is completely determined by the action of ψ.

Remark 4.9 : This establishes the uniqueness part of RRT (modulo (1)).

We refer the reader to [1] for a short proof of RRT in case X is a compact metric space. Finally, we mention that there are numerous applications of RRT, which include in particular the construction of Lebesgue measure [4, Section 2.19] and spectral theorem for self-adjoint operators [2].

References

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