FUBINI-TONELLI THEOREM: PROBLEMS

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1. Motivation

The following example shows that the business of computing iterated integrals could be quite tricky.

Problem 1.1 (Chapter 8, [3])**.** *For every positive integer n, let g_n denote a real-valued continuous function on $[0,1]$ with support contained in $\left(\frac{n-1}{n}, \frac{n}{n+1}\right)$ such that $\int g_n(t)dt = 1$. Define f by*

$$f(x,y) = \sum_{n=1}^{\infty} \big(g_n(x) - g_{n+1}(x)\big) g_n(y), \ \ (x,y) \in [0,1] \times [0,1].$$

Verify the following:

(1) *f maps $[0,1] \times [0,1]$ into $\mathbb{R}$.*
(2) *f is continuous except at $(1,1)$.*
(3) *$\int f(x,y)dydx = 1$, and $\int f(x,y)dxdy = 0$.*
(4) *$\int |f(x,y)|dydx = \infty$.*

Let us see how the issue of interchange of order of integration arises naturally in the following geometric problem. *Given a non-negative function $f : \mathbb{R} \to \mathbb{R}$, compute the area under f, that is, area of the set given by*

$$\mathcal{A} := \{(x,y) \in \mathbb{R} \times \mathbb{R} : 0 \leq y \leq f(x)\}.$$

Note that unless $\mathcal{A}$ is measurable, one can not talk about its area. In this context, it is reasonable to believe that the measurability of f would suffice to ensure measurability of $\mathcal{A}$. Keeping this problem aside for a moment, let us return to the problem of computing the area of $\mathcal{A}$. By experience, one would guess that its nothing but the area under the curve $y = f(x)$, that is, $\int f$. Here is one naive argument: If $\mathcal{A}_x$ denotes the slice $\{y \in \mathbb{R} : (x,y) \in \mathcal{A}\} = \{y \in \mathbb{R} : 0 \leq y \leq f(x)\}$ of $\mathcal{A}$ then

$$\begin{aligned} \text{Area}(\mathcal{A}) &= \int_{\mathbb{R}^2} \chi_{\mathcal{A}}(x,y)dxdy = \int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \chi_{\mathcal{A}}(x,y)dy \Big) dx \\ &= \int_{\mathbb{R}} \text{Length}(\mathcal{A}_x)dx = \int_{\mathbb{R}} f(x)dx, \end{aligned}$$

where we used measurability of slices of $\mathcal{A}$ and also the equality of the iterated integrals. Here we arrive naturally at somewhat related question of whether slices of a measurable set E are again measurable. It is then also natural to ask whether one can recover the measure of E from that of its slices. In other words, for any measurable set E, one may ask for the formula

$$m(E) = \int m(E^y)dy, \tag{1.1}$$

where m denotes the Lebesgue measure (same notation for all dimensional Lebesgue measures), and $E^y = \{x : (x, y) \in E\}$ denotes slice of E. We will see as a consequence of Fubini-Tonelli Theorem that both these questions have affirmative answers, where first one holds true for almost all slices in the usual measure-theoretic sense. However, the converse of the first assertion is far from being true:

Problem 1.2 (Sierpinski). *This example relies on the existence of an ordering $\prec$ of reals such that $\{x \in \mathbb{R} : x \prec y\}$ is countable for each $y \in \mathbb{R}$, which in turn relies on the continuum hypothesis (see,* [2, Chapter 2, Problem 5] *for details). Given this ordering, we let*

$$E := \{(x, y) \in [0, 1] \times [0, 1] : x \prec y\}.$$

Note that for each $y \in [0, 1]$, $E^y = \{x \in [0, 1] : x \prec y\}$ is countable, and hence it is measurable of measure 0. Also, E_x, being the complement of a countable subset of $[0, 1]$, is measurable with measure 1. However, by (1.1), $m(E) = 0$, which is not possible since by symmetry, we also have the formula $m(E) = \int m(E_x)dx$ (Why ?). This shows that E can not be measurable.

2. Fubini's Theorem in $\mathbb{R}^d$

Let us now state a special but significant case of Fubini's Theorem.

Theorem 2.1 (Theorem 3.1, Chapter 2, [2]). *Suppose $f(x, y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$. Then, for almost every $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$,*

1. *the slice $f^y(x) := f(x, y)$ is integrable on $\mathbb{R}^{d_1}$,*
2. *the function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)dx$ is integrable on $\mathbb{R}^{d_2}$,*
3. *the slice $f_x(y) := f(x, y)$ is integrable on $\mathbb{R}^{d_2}$,*
4. *the function defined by $\int_{\mathbb{R}^{d_2}} f_x(y)dy$ is integrable on $\mathbb{R}^{d_1}$.*

Moreover,

$$\int \left(\int f(x, y)dx \right) dy = \int f = \int \left(\int f(x, y)dy \right) dx. \tag{2.2}$$

The proof of Theorem 2.1 amounts basically to the understanding of the family $\mathcal{F}$ of functions integrable on $\mathbb{R}^d$ which satisfy (1)-(4), and (2.2). With this, the conclusion of Theorem 2.1 is equivalent to the assertion that $\mathcal{F} = L^1$. Note first that $\mathcal{F}$ is non-empty since $\chi_Q \in \mathcal{F}$ for a bounded open cube $Q := Q_1 \times Q_2 \subset \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. This can be seen as follows:

We already know that χ_Q is integrable. Note that $\chi_Q^y(x) = 0$ if $y \notin Q_2$, and $= \chi_{Q_1}(x)$ otherwise. Thus $\chi_Q^y(x) = \chi_{Q_1}(x)\chi_{Q_2}(y)$, which is integrable separately in x and y. This verifies (1) and (3). Next, note that

$$\int_{\mathbb{R}^{d_1}} \chi_Q^y(x)dx = \int_{\mathbb{R}^{d_1}} \chi_{Q_1}(x)\chi_{Q_2}(y)dx = m(Q_1)\chi_{Q_2}(y).$$

This yields (2). Along similar lines, one can see (4). Finally, integrating (2.3) with respect to y, we obtain the first equality in (2.2). We leave the remaining one to the reader.

Before we examine $\mathcal{F}$ closely, let us see some more members of the family $\mathcal{F}$.

Problem 2.2. *Show that for any measurable subset E of the boundary of a bounded, closed cube, χ_E belongs to $\mathcal{F}$.*

The family $\mathcal{F}$ has many interesting properties.

Proposition 2.3. *$\mathcal{F}$ is a subspace of L^1.*

Proof. Note that a slice of sum is sum of slices. Since finite union of null sets is a null set, the desired conclusion may be deduced from the definition of $\mathcal{F}$. $\square$

We immediately obtain the following.

Corollary 2.4. *$\mathcal{F}$ contains the subspace of all step functions.*

The following problem says that $\mathcal{F}$ is "complete" in the following sense:

Problem 2.5. *Suppose that $E \subseteq F$ are subsets of measure 0. If $\chi_F \in \mathcal{F}$ then $\chi_E \in \mathcal{F}$.*

Hint. Use (2.2) to show that almost all slices of F are of measure zero. Since $E_x \subset F_x$ and $E^y \subset F^y$, almost all slices of E are also of measure zero. It is now easy to see that χ_E satisfies (1)-(4) with all terms in (2.2) identically zero.

Here is another important property of $\mathcal{F}$, which is an application of Monotone Convergence Theorem.

Proposition 2.6. *Let $\{f_k\} \subseteq \mathcal{F}$ be a sequence of non-neagtive functions such that $f_k \uparrow f$ for an integrable function f on $\mathbb{R}^d$. Then $f \in \mathcal{F}$.*

Remark 2.7 : The conclusion of the last problem holds true even if the non-negativity of f_k is relaxed. Indeed, apply Proposition 2.6 to $\{f_k - f_1\}$ to conclude that $f - f_1 \in \mathcal{F}$. But then, by Proposition 2.3, $f \in \mathcal{F}$. Also, one can replace integrability of f by measurability (with (2.2) understood in the extended sense).

Corollary 2.8. *If E is an open set of finite measure then $\chi_E \in \mathcal{F}$.*

Proof. Recall that E can be written as almost disjoint union of countably many disjoint closed cubes $\{Q_n\}$. Define $P_n := \cup_{k=1}^n Q_n$ and apply Proposition 2.6 to $\{f_n\}$, where $f_n := \chi_{P_n}$. $\square$

Problem 2.9. *Let $\{f_k\} \subseteq \mathcal{F}$ be such that $f_k \downarrow f$ for an integrable function f on $\mathbb{R}^d$. Show that $f \in \mathcal{F}$.*

Problem 2.10. *If E is a G_δ set of finite measure then show that $\chi_E \in \mathcal{F}$.*

Hint. Use Corollary 2.8 and Exercise 2.9.

Corollary 2.11. *$\mathcal{F} = L^1$ if and only if $\mathcal{F}$ contains the subspace of simple functions in L^1.*

Proof. Since $f = f_+ - f_-$ for $f_+, f_- \geq 0$ a.e. WLOG, we assume that $f \geq 0$ a.e. By [3, Theorem 1.4], there exists a sequence of simple functions increasing to f pointwise. Now apply Proposition 2.6. $\square$

Now we complete proof of Fubini's Theorem.

Proof of Theorem 2.1. In view of the last corollary, we must check that $\mathcal{F}$ contains any non-negative simple measurable function. By an application of Proposition 2.3, it suffices to check that $\chi_E \in \mathcal{F}$ for a measurable subset of $\mathbb{R}^d$ of finite measure. This job can be further simplified if we recall basic fact from measure theory [2, Chapter 1, Corollary 3.5]: *Every measurable set E is contained in a G_δ set G such that $G \setminus E$ is a null set.*

Thus $\chi_E = \chi_G - \chi_F$, where G is a G_δ set and F is a null set. By Problem 2.10, $\chi_G \in \mathcal{F}$. So in view of Proposition 2.3, it suffices to check that $\chi_F \in \mathcal{F}$. By the aforementioned fact, F is contained in a G_δ set G', which is also a null set. By Problem 2.10, $\chi_{G'} \in \mathcal{F}$. Hence, by Problem 2.5, $\chi_F \in \mathcal{F}$. $\square$

Here is a variant of Fubini's Theorem, which is often handy in applications.

Theorem 2.12 (Theorem 3.2, [2])**.** *Suppose $f(x, y)$ is a non-negative measurable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$. Then, for almost every $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$,*

1. *the slice $f^y(x) := f(x, y)$ is measurable on $\mathbb{R}^{d_1}$,*
2. *the function defined by $\int_{\mathbb{R}^{d_1}} f^y(x)dx$ is measurable on $\mathbb{R}^{d_2}$,*
3. *the slice $f_x(y) := f(x, y)$ is measurable on $\mathbb{R}^{d_2}$,*
4. *the function defined by $\int_{\mathbb{R}^{d_2}} f_x(y)dy$ is measurable on $\mathbb{R}^{d_1}$.*

Moreover,

$$\int \left(\int f(x, y)dx \right) dy = \int f = \int \left(\int f(x, y)dy \right) dx, \tag{2.3}$$

where all equalities are understood in the extended sense.

Proof. Define measurable sets E_k by $\{(x, y) \in \mathbb{R}^d : \|(x, y)\|_2 < k, |f(x, y)| < k\}$, and let $f_k = f\chi_{E_k}$. Since E_k has finite measure, f_k is integrable. Hence, by Fubini's Theorem, $f_k \in \mathcal{F}$. Since $f_k \uparrow f$, the result follows from (the remark following) Proposition 2.6. $\square$

Problem 2.13. *Compute the improper integral* $\lim_{A\to\infty} \int_{[0,A]} \frac{\sin x}{x} dx$.

Hint. $1/x = \int_0^\infty e^{-xt}dt$ for every $x > 0$, and then use Fubini's Theorem and integration by parts.

3. Applications of Fubini's Theorem

Proposition 3.1. *Suppose E is a measurable subset of $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^d$. Then, for almost every $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$,*

1. *the slice $E^y := \{x \in \mathbb{R}^{d_1} : (x, y) \in E\}$ is measurable,*
2. *the function $m(E^y)$ is measurable on $\mathbb{R}^{d_2}$,*
3. *the slice $E_x := \{y \in \mathbb{R}^{d_2} : (x, y) \in E\}$ is measurable on $\mathbb{R}^{d_1}$,*
4. *the function E_x is measurable on $\mathbb{R}^{d_1}$.*

Moreover,

$$\int m(E^y)dy = m(E) = \int m(E_x)dx, \tag{3.4}$$

where all equalities are understood in the extended sense.

Proof. Apply the last theorem to χ_E. □

In the further applications, we need some facts pertaining to Borel measurable functions. Recall that f is *(Borel) measurable* if inverse image of any open set under f is (Borel) measurable. Clearly, a Borel measurable function is of course measurable. As expected, the converse is not true.

Problem 3.2. *Define $f : \mathbb{R} \to \mathbb{R}^2$ by $f(x) = (x, 0)$, where we consider $\mathbb{R}^n$ for $n = 1, 2$ as measurable space with Lebesgue measure. Let $\mathcal{N}$ be a non-measurable subset of $\mathbb{R}$. Verify:*

1. *$f(\mathcal{N})$ is measurable.*
2. *f is Borel measurable.*
3. *$f(\mathcal{N})$ is not Borel.*
4. *$\chi_{f(\mathcal{N})}$ is measurable, but not Borel measurable.*

Inspite of the preceding exercise, the situation is not at all bad from measure-theoretic view point as shown below.

Problem 3.3. *For any measurable function f, there exists a Borel measurable function F such that $f = F$ almost everywhere.*

Hint. A consequence of Lusin's Theorem says that any non-negative measurable function is a pointwise limit of Borel measurable functions.

Let $f, g \in L^1$. In view of last exercise, WLOG, we may assume that f, g are Borel measurable. Now it is easy to see that $F(x, y) := f(x - y)$ and $G(x, y) := g(y)$ are Borel measurable functions (Exercise), and so is their product $H(x, y) = F(x, y)G(x, y)$.

Remark 3.4 : It is clear from the discussion above that measurability of composition of measurable functions is rather subtle. These complications are perhaps due to the lack of symmetry in the standard definition of the measurable function. It is worth noting here that if one defines a measurable function as a function which sends inverse image of any measurable set to a measurable set, then this problem disappears. This "categorical" viewpoint has been taken up and discussed in [1, Section 1].

Corollary 3.5 (Corollary 3.8, [2])**.** *Let $f : \mathbb{R}^d \to \mathbb{R}$ be a non-negative function, and let $\mathcal{A} := \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : 0 \leq y \leq f(x)\}$. Then $\mathcal{A}$ is measurable in $\mathbb{R}^{d+1}$ if and only f is measurable. If this happens then*

$$\int f(x)dx = m(\mathcal{A}). \tag{3.5}$$

Proof. Note that

$$\mathcal{A} = \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : F(x, y) \geq 0\} \cap \{(x, y) \in \mathbb{R}^d \times \mathbb{R} : G(x, y) \leq 0\},$$

where $F(x, y) := y,\ G(x, y) := y - f(x)$. Since F and G are (Borel) measurable, $\mathcal{A}$ is measurable.

Conversely, suppose that $\mathcal{A}$ is measurable. Note that $\mathcal{A}_x = [0, f(x)]$. But then $f(x) = m(\mathcal{A}_x)$ is measurable by the preceding proposition. Consequently, by (3.4),

$$m(\mathcal{A}) = \int m(\mathcal{A}_x)dx = \int f(x)dx.$$

This completes the proof of the corollary. □

Proposition 3.6 (Theorem 8.14, [3])**.** *Let $f, g \in L^1(\mathbb{R})$. Then there exists a null set E such that $\int_{\mathbb{R}} |f(x-y)g(y)|dy < \infty$ for every $x \notin E$. For $x \in \mathbb{R} \setminus E$, define the convolution $f * g$ of f and g as*

$$f * g(x) := \int_{\mathbb{R}} f(x-y)g(y)dy.$$

Then $f * g \in L^1(\mathbb{R})$, and $\|f * g\|_1 \leq \|f\|_1\|g\|_1$.

Proof. We already recorded that $H(x, y) = f(x-y)g(y)$ is (Borel) measurable. By the translation-invariance of Lebesgue measure,

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |H(x, y)|dxdy = \int_{\mathbb{R}} |g(y)| \left(\int_{\mathbb{R}} |f(x-y)|dx\right) dy = \int_{\mathbb{R}} |g(y)|\|f\|_1 dy = \|f\|_1\|g\|_1.$$

Hence, by (2.3), $\int_{\mathbb{R}} |H(x, y)|dy < \infty$ for almost every x, and $G \in L^1(\mathbb{R}^2)$. But then by Theorem 2.1(1), $f * g \in L^1(\mathbb{R})$. Finally, one more application of Fubini's Theorem yields

$$\|f * g\|_1 \leq \int_{\mathbb{R}}\int_{\mathbb{R}} |H(x, y)|dydx = \int_{\mathbb{R}}\int_{\mathbb{R}} |H(x, y)|dxdy,$$

and hence $\|f * g\|_1 \leq \|f\|_1\|g\|_1$. □

Problem 3.7. *Show that for any $f, g, h \in L^1(\mathbb{R})$,*

(1) $f * g = g * f$.
(2) $f * (g * h) = (f * g) * h$.

Remark 3.8 : $L^1(\mathbb{R})$ is a Banach algebra with convolution as multiplication.

4. Fubini-Tonelli Theorem

To discuss further application, we need a general Fubini Theorem. Let us first describe product measure (without minor details).

Suppose $(X_i, \mathcal{M}_i, \mu_i)$ for $i = 1, 2$ be two σ-finite, complete measure spaces. Consider the collection $\mathcal{A}$ of finite unions of disjoint measurable rectangles $A \times B$, where $A \in \mathcal{M}_1$ and $B \in \mathcal{M}_2$. Check that $\mathcal{A}$ is an algebra. Recalling the definition of area of a rectangle, it is natural to define a candidate for product measure by $\mu_0(A \times B) := \mu_1(A)\mu_2(B)$.

Proposition 4.1 (μ_0 is a premeasure)**.** *If* $A \times B = \cup_{j=1}^{\infty} A_j \times B_j$ *(disjoint union) then* $\mu_0(A \times B) = \sum_{j=1}^{\infty} \mu_1(A_j)\mu_2(B_j)$*, where* $A, A_j \in \mathcal{M}_1$ *and* $B, B_j \in \mathcal{M}_2$ *for every integer* $j \geq 1$*.*

Proof. For $x_1 \in A$, note that $B = \cup_{\{j : x_1 \in A_j\}} B_j$ (disjoint union). Now use countable additivity of μ_2 to get

$$\chi_A(x_1)\mu_2(B) = \mu_2(B) = \sum_{\{j : x_1 \in A_j\}} \mu_2(B_j) = \sum_{j=1}^{\infty} \chi_{A_j}(x_1)\mu_2(B_j).$$

Now integrate with respect to $d\mu_1$, and apply monotone convergence theorem. □

By a general Caratheodary Extension Theorem [2, Theorem 1.5, Chapter 6], there exists a unique measure space $(X_1 \times X_2, \mathcal{M}, \mu)$ such that $\mathcal{A} \subseteq \mathcal{M}$ and $\mu|_{\mathcal{A}} = \mu_0$. We denote the measure μ by $\mu_1 \times \mu_2$ (to be referred to as *product measure*).

Remark 4.2 : Even μ_1, μ_2 are complete, $\mu_1 \times \mu_2$ need not be complete. For instance, let $\mu_1 = \mu_2$ be the Lebesgure measure on $\mathbb{R}$.

Now we are ready to state a general form of Fubini's Theorem.

Theorem 4.3 (Theorem 3.3, Chapter 6, [2])**.** *Suppose* $f(x_1, x_2)$ *is integrable on* $(X_1 \times X_2, \mu_1 \times \mu_2)$*. Then, for almost every* $x_1 \in \mathbb{X}_1$ *and* $x_2 \in X_2$*,*

1. *the slice* $f_2^x(x_1) := f(x_1, x_2)$ *is integrable on* (X_1, μ_1)*,*
2. *the function defined by* $\int_{X_1} f^{x_2}(x_1)d\mu_1$ *is integrable on* (X_2, μ_2)*,*
3. *the slice* $f_{x_1}(x_2) := f(x_1, x_2)$ *is integrable on* (X_2, μ_2)*,*
4. *the function defined by* $\int_{X_2} f_{x_1}(x_2)d\mu_2$ *is integrable on* (X_1, μ_1)*,*

Moreover,

$$\int \left(\int f(x_1, x_2)d\mu_1 \right) d\mu_2 = \int f d\mu_1 \times \mu_2 = \int \left(\int f(x_1, x_2)d\mu_2 \right) d\mu_1. \tag{4.6}$$

Problem 4.4. *Let* (X, Σ, μ) *be a* σ*-finite measure space and let* $f : X \to \mathbb{R}$ *be a non-negative measurable function. Show that*

$$\int_X f(x)d\mu(x) = \int_0^{\infty} \mu(\{x \in X : f(x) \geq t\})dt.$$

Hint. $f(x) = \int_0^{\infty} \chi_{[0, f(x)]}(t)dt$, and apply Fubini-Tonelli Theorem.

We will not discuss proof of Fubini-Tonelli Theorem as it is along the lines of that of Theorem 2.1. In the remaining part of these notes, we will discuss its applications to integration theory.

Any non-zero vector x in $\mathbb{R}^d$ can be rewritten as $x = r\gamma$, where $r = \|x\|_2 \in (0, \infty)$ and $\gamma = x/\|x\|_2 \in S^{d-1}$. Here S^{d-1} denotes the unit sphere $\{x \in \mathbb{R}^d : \|x\|_2 = 1\}$ in $\mathbb{R}^d$. The polar coordinates of x are thus given by the pair (r, γ).

The Integration Formula for Polar Co-ordinates provides a handy formula to compute integral over $\mathbb{R}^d$ as iterated integrals one with respect to a certain weighted Lebesgue measure over $(0, \infty)$ and second with respect to the surface area measure over unit sphere. Before we present the precise formula, let us define the so-called surface area measure on the unit sphere. For a subset E of S^{d-1}, let

$$\tilde{E} := \{x \in \mathbb{R}^d : 0 < \|x\|_2 < 1,\ x/\|x\|_2 \in E\}.$$

Consider $\mathcal{S} := \{E : \tilde{E} \text{ is Lebesgue measurable}\}$, and note that $\mathcal{S}$ is a σ-algebra. Define the *surface area measure* on S^{d-1} by setting $\sigma(E) = d\ m(\tilde{E})$ for $E \in \mathcal{S}$, where m is the Lebesgue measure on $\mathbb{R}^d$. Since m is σ-finite and complete, so is σ.

Theorem 4.5 (Integration Formula for Polar Coordinates). *Suppose f is an integrable function on $\mathbb{R}^d$. Then for almost every $\gamma \in S^{d-1}$, the slice $f^\gamma(r) = f(r\gamma)$ is an integrable function with respect to the measure space $((0,\infty), \mathcal{N}, r^{d-1}dr)$, where $\mathcal{N}$ denotes the σ-algebra of Lebesgue measurable subsets of $(0,\infty)$. Moreover, $\int_0^\infty f^\gamma(r) r^{d-1} dr$ is integrable with respect to $(S^{d-1}, \mathcal{S}, \sigma)$, and the following identity holds:*

$$\int_{\mathbb{R}^d} f(x)dx = \int_{S^{d-1}} \left(\int_0^\infty f(r\gamma) r^{d-1} dr\right) d\sigma(\gamma). \tag{4.7}$$

Remark 4.6 : In case $d = 1$, the formula (4.7) holds trivially.

Corollary 4.7. *Suppose f is an integrable function on $\mathbb{R}^2$. Then*

$$\int_0^{2\pi} \left(\int_0^\infty f(re^{i\theta}) r dr\right) d\theta = \int_{\mathbb{R}^2} f(x,y)dxdy = \int_0^\infty \left(\int_0^{2\pi} f(re^{i\theta}) d\theta\right) r dr.$$

Problem 4.8. *Write down explicitly the formula* (4.7) *in case $d = 3$.*

To prove the formula (4.7), we must understand the product space $((0,\infty) \times S^{d-1}, \mathcal{M}, d\mu := r^{d-1}dr \times d\sigma)$. Although this does not agree with the measure space $(\mathbb{R}^d \setminus \{0\}, \mathcal{L}, dm)$ (even after taking into consideration identification $(0,\infty) \times S^{d-1}$ with $\mathbb{R}^d \setminus \{0\}$) both these σ-algebras $\mathcal{M}$ and $\mathcal{L}$ contain the σ-algebra of Borel measurable sets, where both μ and m agree. We begin with some simple observations.

Lemma 4.9. $m((0,1) \times B) = \mu((0,1) \times B)$ *for any measurable subset B of S^{d-1}.*

Proof. Note that $m((0,1) \times B) = m(\tilde{B})$. Note also that

$$\mu((0,1) \times B) = \left(\int_0^1 r^{d-1} dr\right) \sigma(B),$$

which is nothing but $m(\tilde{B})$. ☐

Remark 4.10 : Since m is dilation-invariant, $m((0,b) \times B) = b^d\ m((0,1) \times B) = b^d\ \mu((0,1) \times B) = \mu((0,b) \times B)$.

Lemma 4.11. $m((a,b) \times B) = \mu((a,b) \times B)$ *for any measurable subset B of S^{d-1}.*

Proof. Note that $m((0,a] \times B) = m((0,a) \times B)$ and $\mu((0,a] \times B) = \mu((0,a) \times B)$. Since $(0,b) = (0,a] \cup (a,b)$ (disjoint union), by additivity of measures and the preceding lemma,

$$\begin{aligned} m((0,a] \times B) + m((a,b) \times B) &= m((0,b) \times B) = \mu((a,b) \times B) \\ &= \mu((0,a] \times B) + \mu((a,b) \times B), \end{aligned}$$

which, after cancellation, gives the desired equality. □

Remark 4.12 : $m(A \times B) = \mu(A \times B)$ for any open subset A of $(0, \infty)$ and any measurable subset B of S^{d-1}. This follows from countable additivity of m and μ. The same equality holds for all closed subsets A of $(0, \infty)$.

Outline of proof of Theorem 4.5. We only check that the formula (4.7). Again since any integrable function is a monotone pointwise limit of a sequence of simple measurable functions, by Monotone Convergence Theorem, it suffices to check (4.7) for $f = \chi_E$, where E is a Lebesgue measurable subset of $\mathbb{R}^d$. Here, there are two parts in the proof. The first is the verification that $\mathcal{M} = \mathcal{L}$ and the second is the equality of μ and m.

We claim that $m = \mu$ agrees on the algebra $\mathcal{A}$ of finite unions of disjoint measurable rectangles in $\mathcal{M}$. Let $E_1 \times E_2$ be a measurable rectangle in $\mathcal{A}$, where $0 < \sigma(E_2) < \infty$. Fixed $\epsilon > 0$. By the definition of measurable set, we can find sets $F_1 \subset E_1 \subset O_1$ with F_1 closed and O_1 open, such that

$$m(O_1) - \epsilon \leq m(E_1) \leq m(F_1) + \epsilon.$$

By the remark prior to the proof,

$$\begin{aligned} m(O_1 \times E_2) - \epsilon' &= \mu(O_1 \times E_2) - \epsilon' \leq \mu(E_1 \times E_2) \\ &\leq \mu(F_1 \times E_2) + \epsilon' = m(F_1 \times E_2) + \epsilon', \end{aligned}$$

where $\epsilon' := \epsilon\ \sigma(E_2)$. Also, $m(F_1 \times E_2) \leq m(E_1 \times E_2) \leq m(O_1 \times E_2)$, and hence $|m(E_1 \times E_2) - \mu(E_1 \times E_2)|$ is at most $2\epsilon'$. Since ϵ is arbitrary, we obtained the equality of μ and m for a measurable rectangle, and hence by additivity the same for members of $\mathcal{A}$. Thus the claim stands verified. By the uniqueness in the general Carathéodory Extension, $m = \mu$ on $\mathcal{M}$. Thus for any $E \in \mathcal{M}$, $m(E) = \mu(E)$, that is, the formula (4.7) holds for $f = \chi_E$. It now suffices to check the formula (4.7) for $f = \chi_E$, where E is Lebesgue measurable. Since any open set in $\mathbb{R}^d \setminus \{0\}$ can be rewritten as a countable union of measurable rectangles in $\mathcal{M}$, the formula holds for open E, and hence for any Borel set E. Since any measurable set is union of a Borel set and a null set contained in a Borel null set, the proof is over. □

Problem 4.13 (Exercise 5, Page 313, [2])**.** *Use polar coordinates to show that the volume v_d of the unit ball B in $\mathbb{R}^d$ is given by $v_d(B) = \pi^{d/2}/\Gamma(d/2 + 1)$, where Γ denotes the Gamma function.*

Remark 4.14 : Since factorial growth is faster than the exponential growth,

$$\lim_{d \to \infty} v_d(B) = 0.$$

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