NOTES AND PROBLEMS: INTEGRATION THEORY

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Abstract. These are the lecture notes prepared for participants of AFS-I to be conducted at Kumaun University, Almora from 1st to 27th December, 2014.

Contents

1. Space of Riemann Integrable Functions

In these notes, we will be concerned about the space $L^1(X)$ of integrable functions on X and two of its subspaces: the subspace $C_c(X)$ of continuous functions with compact support and the subspace $R[a,b]$ of Riemann integrable functions in case $X = [a,b]$. If not specified, then $X = \mathbb{R}^d$ or more generally a measurable subset of $\mathbb{R}^d$ of positive measure. It turns out that the subspace $C[a,b] := C_c[a,b]$ of $L^1[a,b]$ is properly contained in the space $R[a,b]$ of Riemann integrable functions on $[a,b]$. Indeed, by a result of Lebesgue, $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if f is continuous a.e. (for a proof, see [4, Problem 4, Chapter 1]). Moreover, $R[a,b]$ is a subspace of $L^1[a,b]$ as shown below.

Theorem 1.1. *Every Riemann integrable function on $[a,b]$ is Lebesgue integrable. Moreover, the Riemann integral of f is same as the Lebesgue integral of f.*

Proof. We give outline of the proof. Suppose $|f(x)| \leq M$ for all $x \in [a,b]$ and some $M \in \mathbb{R}$. Use the definition of Riemann integrability to find sequences $\{\phi_k\}$ and $\{\psi_k\}$ of step functions bounded by M such that $\phi_k \uparrow \phi$ and $\psi_k \downarrow \psi$ for some measurable functions ϕ and ψ such that $\phi \leq f \leq \psi$. Also, the limits of Riemann integrals of ϕ_k and ψ_k agree with that of f. Since Riemann integral and Lebesgue integral agree for step functions, by bounded convergence theorem, Lebesgue integrals of ϕ and ψ are same. Since $\phi \leq \psi$, we must have $\phi = f = \psi$ a.e. This shows that f is measurable. It is now easy to see that the Riemann integral of f is same as its Lebesgue integral. $\square$

Remark 1.2 : The set of Riemann integrable functions forms a subspace of $L^1[a,b]$.

In general, it is hard to compute Lebesgue integral right from the definition. The preceding result, in particular, shows that Lebesgue integral of continuous functions may be calculated using the methods from Riemann integration theory.

The pointwise limit of Riemann integrable functions need not be Riemann integrable as shown below.

Problem 1.3. *Consider the function $f_m(x) = \lim_{n\to\infty}(\cos(m!\pi x))^n$ for $x \in \mathbb{R}$. Find the set of discontinuities of f_m. Further, verify the following:*

1. *$\{f_m\}$ converges pointwise to f, where $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, and $f(x) = 1$ for $x \in \mathbb{Q}$.*
2. *f is discontinuous everywhere.*

Remark 1.4 : If $\{r_n\}_{n=1}^{\infty}$ is enumeration of rationals then the sequence $\{\chi_{\{r_1,\cdots,r_n\}}\}$ of Riemann integrable functions converges pointwise to the characteristic function of rationals, which is not Riemann integrable.

Since $R[a,b]$ is a subspace of $L^1[a,b]$, it is natural to ask whether it is closed in $L^1[a,b]$. The answer is No.

Problem 1.5. *Let $\hat{C}$ denote the Cantor-like set obtained by removing 2^{k-1} centrally situated open subintervals $I_{1k}, \cdots, I_{2^{k-1}k}$ of $I := [0,1]$ each of length $1/4^k$ at the kth stage, where $k = 1, 2, \cdots$. Let $F_k : I \to I$ be a continuous function such that $F_1 = 1$ on $I \setminus \cup_{i=1}^{2^{k-1}} I_{ik}$ and $F_1 = 0$ at the mid-points of $I_{1k}, \cdots, I_{2^{k-1}k}$. For $f_n := \prod_{i=1}^n F_i$ for $n \geq 1$, verify the following:*

1. *The sequence $\{f_n\}$ of continuous functions decreases to some $f : I \to I$ pointwise.*
2. *The limit function f is discontinuous at every point of $\hat{C}$.*

Conclude that the set $R[0,1]$ of Riemann integrable functions as a subspace of $L^1[0,1]$ is not closed.

Remark 1.6 : Consider the vector space $R[a,b]$ of Riemann integrable functions. Then $R[0,1]$ is a subspace of $L^1[a,b]$ with norm $\|f-g\|_1 := \int_a^b |f(t)-g(t)|dt$, which is not complete.

The preceding remark raises the following question: What is the closure of $R[a,b]$ in $L^1[a,b]$?

2. Space of Lebesgue Integrable Functions

The measurable functions, in general, could be extended real-valued. The following simple observation shows that for members of L^1, WLOG, we may confine ourselves to real-valued functions.

Problem 2.1. *Let $f : \mathbb{R}^d \to \mathbb{R}\cup\{\infty\}$ be an extended real-valued function. If $f \in L^1$ then the set $\{x \in \mathbb{R}^d : |f(x)| = \infty\}$ is of measure zero.*

Problem 2.2. *Consider the intervals* $I_1 = [0,1], I_2 = [0,1/2], I_3 = [1/2,1], I_4 = [0,1/4], I_5 = [1/4,1/2], I_6 = [1/2,3/4], I_7 = [3/4,1]$ *and so on. For* $f_n = \chi_{I_n}$ *and* $f = 0$*, show that* $\|f_n - f\|_1 \to 0$*, but* $f_n(x) \nrightarrow f(x)$ *for any* $x \in [0,1]$*.*

Hint. Any $x \in [0,1]$ belongs to infinitely many I_n's and complements of infinitely many I_n's.

Although, the convergence in L^1 need not imply the pointwise convergence, the situation is not very bad.

Lemma 2.3. *If* $\{f_n\}$ *is a Cauchy sequence in* L^1*, then there exists a subsequence* $\{f_{n_k}\}$ *of* $\{f_n\}$ *such that* $\|f_{n_k+1} - f_{n_k}\| \leq 2^{-k}$ *for all non-negative integers* k *and* $f_{n_k}(x) \to f(x)$ *a.e. where*

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty}(f_{n_k+1}(x) - f_{n_k}(x)).$$

A normed linear space is *complete* if every Cauchy sequence is convergent.

Theorem 2.4. *The normed linear space* L^1 *is complete.*

Proof. Assume that $\{f_n\}$ is a Cauchy sequence in L^1. By the preceding lemma, there exists a subsequence $\{f_{n_k}\}$ satisfying the conditions given there. Check that $|f - f_{n_k}| \leq g$, where

$$g(x) = |f_{n_1}(x)| + \sum_{k=1}^{\infty}|f_{n_k+1}(x) - f_{n_k}(x)|.$$

Since $\|f_{n_k+1} - f_{n_k}\| \leq 2^{-k}$,

$$\sum_{k=1}^{\infty}\int |f_{n_k+1} - f_{n_k}| < 1.$$

By the monotone convergence theorem, $g \in L^1$. On the other hand, since $|f - f_{n_k}| \leq g$, by the dominated convergence theorem, $\|f_{n_k} - f\|_1 \to 0$ as $k \to \infty$. By general theory, a Cauchy sequence with a convergent subsequence is necessarily convergent. □

The Riesz-Fischer Theorem makes possible the application of abstract theory of Banach spaces to L^1. For instance, one may conclude that every absolutely convergent series in L^1 is convergent. An astute reader may notice, however, that this observation is implicitly there in the above proof of Riesz-Fischer theorem.

Problem 2.5. *Verify that a closed subspace of a complete normed linear space is complete. Conclude that* $C[a,b]$ *is not closed in* $L^1[a,b]$*.*

Hint. Suppose that $C[a,b]$ is closed in $L^1[a,b]$. By Riesz-Fischer theorem, $(C[a,b], \|\cdot\|)$ is a complete normed linear space. We apply the bounded inverse theorem to the mapping $f \to f$ from $(C[a,b], \|\cdot\|_\infty)$ onto $(C[a,b], \|\cdot\|_1)$ to get $C > 0$ such that $\|f\|_\infty \leq C\|f\|_1$ for every $f \in C[a,b]$.

The preceding exercise raises the question: what is the closure of $C[0,1]$ in $L^1[0,1]$? We will see in Section 3 that it is the whole of $L^1[0,1]$.

3. Some Dense subspaces of L^1

We say that a subset Y of X is *dense* in X if for every $f \in X$ and for every $\epsilon > 0$, there exists a g in Y such that $\|f - g\| < \epsilon$.

Many times it is practical to know "good" dense subsets of a given normed linear space. For instance, an inequality for all members of X could be deduced from the same inequality for members of Y (see Exercise 3.9 below for another application).

Problem 3.1. *Show that the set of simple functions is dense in L^1.*

Hint. Since $f = f_+ - f_-$, WLOG, we assume that $f \geq 0$ a.e. Then there exists a sequence of simple functions increasing to f pointwise. Now apply monotone convergence theorem.

Problem 3.2. *Show that for any step function $s : \mathbb{R} \to \mathbb{R}$ and $\epsilon > 0$, there exists a continuous function f such that $\|f - s\|_\infty < \epsilon$.*

Hint. First try a characteristic function of an interval.

Problem 3.3. *Show that for any step function $s : \mathbb{R}^d \to \mathbb{R}$ in L^1 and $\epsilon > 0$, there exists a continuous function f with compact support such that $\|f - s\|_\infty < \epsilon$.*

Hint. Characteristic function of a cube is a product of characteristic functions of intervals. Now apply the last exercise.

Problem 3.4. *For a measurable subset E of $\mathbb{R}^d$ of finite measure and $\epsilon > 0$, show that there exists an almost disjoint family of rectangles $\{R_1, \cdots, R_M\}$ with $m(E \triangle \cup_{j=1}^M R_j) \leq \epsilon$.*

Hint. Cover E by $\cup_{j=1}^\infty Q_j$ such that $\sum_{j=1}^\infty |Q_j| \leq m(E) + \epsilon/2$. There exists $N \geq 1$ such that $\sum_{j=N+1}^\infty |Q_j| \leq \epsilon/2$. Verify that $m(E \triangle \cup_{j=1}^N Q_j) \leq \epsilon$. Warning: $Q_1, \cdots, Q_N$ may not be almost disjoint.

Problem 3.5. *Let E be a measurable set of finite measure and let $\epsilon > 0$ be given. Show that there exists a step function g such that $\|\chi_E - g\|_1 < \epsilon$.*

Hint. By the last exercise, there exists an almost disjoint family of rectangles $\{R_1, \cdots, R_M\}$ with $m(E \triangle \cup_{j=1}^M R_j) \leq \epsilon/2$. Check that $\|\chi_E - \sum_{j=1}^M \chi_{R_j}\|_1 < \epsilon$.

Theorem 3.6. *The vector space of continuous functions with compact support is dense in L^1.*

Proof. Let $f \in L^1$ and $\epsilon > 0$ be given. By Problem 3.1, there exists a simple function s such that $\|f - s\| < \epsilon/3$. Since s is a finite linear combination of characteristic functions, by the preceding problem, there exists a step function g such that $\|s - g\|_1 < \epsilon/3$. Now by Problem 3.3, there exists a continuous function h with compact support such that $\|g - h\|_1 < \epsilon/3$. Finally, by triangle inequality, we obtain $\|f - h\|_1 < \epsilon$. $\square$

Remark 3.7 : $R[a, b]$ is dense in $L^1[a, b]$.

Corollary 3.8. *The space of polynomials on $[a, b]$ are dense in $L^1[a, b]$. In particular, $L^1[a, b]$ is separable.*

Let us discuss one application of the density theorem.

Problem 3.9. *For $f : \mathbb{R} \to \mathbb{R}$ and $y \in \mathbb{R}$, let $f_y(x) = f(x - y)$ denote the translate of f. If $f \in L^1$ then show that $\psi : \mathbb{R} \to L^1$ given by $\psi(y) = f_y$ is uniformly continuous.*

Hint. Let g be a continuous function in L^1 with support in $[-A, A]$ such that $\|f-g\|_1 < \epsilon$. Since g is uniformly continuous on $[-A, A]$, there exists $\delta < A$ such that $|g(s)-g(t)| < \epsilon/3A$ whenever $|s-t| < \delta$. Check that $\|g_s-g_t\|_1 < (2A+\delta)(\epsilon/3A) < \epsilon$, and hence $\|f_s - f_t\|_1 < 3\epsilon$ whenever $|s - t| < \delta$.

The following is known as Riemann-Lebesgue lemma:

Corollary 3.10. *If $f \in L^1$ then $\hat{f}(\zeta) := \int_{\mathbb{R}^d} f(x)e^{-2\pi i x\zeta}dx$ tends to 0 as $|\zeta| \to \infty$.*

Proof. By the preceding exercise, for $f \in L^1$, $\|f - f_h\| \to 0$ as $h \to \infty$. The desired conclusion follows from

$$\hat{f}(\zeta) = \frac{1}{2}\int_{\mathbb{R}^d}(f(x) - f_h(x))e^{-2\pi i x\zeta}dx,$$

where $h = \frac{1}{2}\frac{\zeta}{|\zeta|^2} \to 0$ as $|\zeta| \to \infty$. □

4. Space of Functions of Bounded Variation

For a function $F : [a, b] \to \mathbb{C}$ and a partition $P : \{t_0 < t_1 < \cdots < t_n\}$ of $[a, b]$, let $V(F, P) := \sum_{i=1}^n |F(t_i) - F(t_{i-1})|$.

Remark 4.1 : If $c \in (a, b)$ and P_c denote the refinement of P obtained by adjoining c to P, then $V(F, P) \leq V(F, P_c)$.

The *total variation* T_F of a function $F : [0, 1] \to \mathbb{C}$ is defined as

$$T_F := \sup_P V(F, P),$$

where sup is taken over all partitions $P : \{t_0 < t_1 < \cdots < t_n\}$ of $[0, 1]$. We say that F *is of bounded variation* if its total variation T_F is finite.

Remark 4.2 : A function of bounded variation is necessarily bounded. Just consider the partition $P : \{a, x, b\}$, and note that $V(P, F)$ is finite.

Let us see some examples of functions of bounded variation.

Problem 4.3. *Show that every bounded monotonic real-valued function is of bounded variation.*

Hint. $V(F, P) = |F(a) - F(b)|$ for any partition P.

Problem 4.4. *Show that every differentiable function with bounded derivative is of bounded variation.*

Hint. Apply mean value theorem to real and imaginary parts of F.

A differentiable function of bounded variation need not have bounded derivative.

Problem 4.5. *Show that the function $F : [0,1] \to \mathbb{R}$ given by*

$$\begin{aligned} F(x) &= x^{3/2}\sin(1/x), \ \textit{for } 0 < x \leq 1, \\ &= 0, \ \textit{if } x = 0 \end{aligned}$$

is of bounded variation.

Here is an example of continuous monotone function of bounded variation whose derivative vanishes almost everywhere.

Example 4.6 : Let C be the Cantor set obtained by removing 2^{k-1} centrally situated open subintervals $I_{1k}, \cdots, I_{2^{k-1}k}$ of $I := [0,1]$ each of length $1/3^k$ at the kth stage, where $k = 1, 2, \cdots$. Let $F_1 : I \to I$ be a continuous increasing function such that $F_1(0) = 0$, $F_1 = 1/2$ on I_{11}, $F_1(1) = 1$, and linear on remaining part. Similarly, let $F_2 : I \to I$ be a continuous increasing function such that $F_2(0) = 0$, $F_2 = 1/4$ on I_{12}, $F_2 = 1/2$ on I_{11}, $F_2 = 3/4$ on I_{22}, $F_1(1) = 1$, and linear on remaining part. One can inductively define the sequence $\{F_k\}$ of continuous increasing functions so that $F_{k+1}(0) = 0$, $F_{k+1} = F_j$ on $I_{1j}, \cdots, I_{2^{j-1}j}$ for $1 \leq j \leq k$, the values of F_{k+1} on consecutive I_{ij} differ by $1/2^{k+1}$, and $F_{k+1}(1) = 1$. Note that

$$|F_{k+1}(x) - F_k(x)| \leq \frac{1}{2^{k+1}} \text{ for every } x \in [0,1].$$

It is easy to see that $\|F_m - F_n\|_\infty \to 0$, and hence $\{F_k\}$ converges uniformly to a continuous function, say, F. The continuous increasing function F is known as *Cantor-Lebesgue function.* By Problem 4.3, F is of bounded variation. Note further that F is constant on each interval of the complement of C. Since C has measure 0, $F'(x) = 0$ almost everywhere.

Remark 4.7 : Note that $\int_{[0,1]} F'(x)dx \neq F(1) - F(0)$.

Let $F : [a,b] \to \mathbb{C}$ and a partition $P : \{t_0 < t_1 < \cdots < t_n\}$ of $[a,b]$ be given. If $c \in (a,b)$, then we have partitions P_1 and P_2 of $[a,c]$ and $[c,b]$ respectively such that $P_c = P_1 \cup P_2$. Moreover,

$$V(F, P_c) = V(F|_{[a,c]}, P_1) + V(F|_{[c,b]}, P_2).$$

By Remark 4.1, we have $T_{F|_{[a,b]}} = T_{F|_{[a,c]}} + T_{F|_{[c,b]}}$. Replacing b by v and c by u, for $a \leq u < v \leq b$, we obtain

$$T_{F|_{[a,v]}} - T_{F|_{[a,u]}} = T_{F|_{[u,v]}}. \tag{4.1}$$

Remark 4.8 : If F is of bounded variation then the mapping $x \to T_{F_{[a,x]}}$ increasing.

The following decomposition of functions of bounded variation is due to Jordan.

Theorem 4.9. *Any real-valued function of bounded variation on $[a,b]$ is a difference of two bounded, increasing functions.*

Proof. We decompose $F(x)$ as $[F(x) + T_{F_{[a,x]}}] - T_{F_{[a,x]}}$ for $x \in [a, b]$. In view of the last remark, it suffices to check that $x \to F(x) + T_{F_{[a,x]}}$ is increasing. For $a \leq u < v \leq b$, by (4.1), we obtain

$$T_{F|_{[a,v]}} - T_{F|_{[a,u]}} = T_{F|_{[u,v]}} \geq V(F|_{[u,v]}, \{u, v\}) = |F(v) - F(u)| \geq F(u) - F(v),$$

and hence the desired conclusion. □

Problem 4.10. *Show that the set of discontinuities of a monotone function on* $[a, b]$ *is countable, and hence of measure* 0.

Hint. Suppose F is bounded and increasing. If x is a discontinuity of F, then there exists a rational number r_x such that $\lim\limits_{t \to x, t<x} F(t) < r_x < \lim\limits_{t \to x, t>x} F(t)$. Note that if $x < y$ then $r_x < r_y$, and hence $x \to r_x$ is one-to-one.

In view of the last exercise, it is interesting to know the set of points at which a bounded, monotone function is non-differentiable. A deep result of Lebesgue says that every bounded, monotone function on $[a, b]$ is differentiable almost everywhere with integrable derivative (refer to [4, Chapter 3] for a proof). As a consequence of this and Jordan's theorem, we obtain the following:

Corollary 4.11. *If* $F : [a, b] \to \mathbb{C}$ *is of bounded variation then* F *is differentiable almost everywhere. Moreover,* F' *belongs to* $L^1[a, b]$.

5. Riesz Representation Theorems

In this section, we will see the Riesz representation theorem for $C[a, b]$ and $L^1[a, b]$. We start recalling some preliminaries from functional analysis. Let X be a normed linear space. The *dual space* X' of X is defined as the normed linear space of all bounded linear functionals $f : X \to \mathbb{C}$.

Problem 5.1. *Show that* X' *is a Banach space with norm*

$$\|f\| := \sup\{|f(x)| : \|x\| \leq 1\}.$$

Hint. Note that $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|\|x\|$, and if $\{f_n\}$ is Cauchy then so is $\{f_n(x)\}$. In this case, there exists a linear functional $f : X \to \mathbb{C}$ such that $f_n(x) \to f(x)$ for every x. Show that for $\epsilon > 0$, there exists $N \geq 1$ such that $|f_n(x) - f(x)| \leq \epsilon\|x\|$ for all $n \geq N$.

Problem 5.2. $X \neq \{0\}$ *if and only if* $X' \neq \{0\}$.

Hint. Let $x \in X$ be non-zero. Then $f(\alpha x) = \alpha\|x\|$ is a bounded linear functional on $\mathbb{C}x$. Now apply Hanh-Banach extension theorem.

The Riesz representation theorem for $C[a, b]$ identifies the dual space of $C[a, b]$ with a subspace of functions of bounded variation. To establish it, we need some definitions and observations.

Consider the vector space $BV[a, b]$ of functions $F : [a, b]$ of bounded variation. Note that if $F \in BV[a, b]$ then its total variation T_F is 0 if and only if F is constant. This allows us to make $BV[a, b]$ into a normed linear space with norm

$$\|F\| := T_F + |F(a)|.$$

Problem 5.3. *If $F \in BV[a,b]$ then $\psi_F : C[a,b] \to \mathbb{C}$ given by*

$$\psi_F(f) := \int_a^b f(t)dF(t)$$

is a bounded linear functional with norm $\|\psi_F\|$ at most the total variation T_F of F, where $\int_a^b f(t)dF(t)$ is the limit of the Riemann-Stieljes sum

$$\sum_{j=1}^{n} f(s_j)[F(t_j) - F(t_{j-1}],$$

as the mesh of the partition $\{a = t_0 < t_1 < \cdots < t_n = b]$ tends to 0 and s_j in $[t_{j-1}, t_j]$ for $j = 1, \cdots, n$.

Remark 5.4 : Note that the above limit exists in view of Jordan decomposition theorem. The mapping $F \to \psi_F$ is not injective. Indeed, $\psi_1 = 0 = \psi_2$.

Let $B[a,b]$ denote the normed linear space of bounded functions on $[a,b]$ with norm $\| \cdot \|_\infty$.

Remark 5.5 : If $\phi \in (C[a,b])'$, then there exists a bounded linear functional $\psi \in (B[0,1])'$ such that $\psi(f) = \phi(f)$ for all $f \in C[a,b]$ and $\|\psi\| = \|\phi\|$. This is a consequence of Hanh-Banach extension theorem.

Problem 5.6. *Let $\psi \in (B[a,b])'$. Define $F : [0,1] \to \mathbb{K}$ by $F(0) = 0$ and $F(t) = \psi(\chi_{(0,t]})$. Show that $T_F \leq \|\psi\|$.*

Hint. There exists a $\theta \in \mathbb{R}$ such that $|F(t_i) - F(t_{i-1})| = e^{i\theta}(F(t_i) - G(t_{i-1})) = \psi(e^{i\theta}\chi_{(t_{i-1},t_i]})$.

A function $F \in BV[a,b]$ is said to be *normalized* if $F(a) = 0$ and F is right continuous on (a,b). Let $NBV[a,b]$ denote the space of normalized functions of bounded variation. Then $NBV[a,b]$ is a normed linear space with norm as the total variation.

Lemma 5.7. *If $F \in BV[a,b]$ then there exists a $G \in NBV[a,b]$ such that*

$$\int_a^b f(t)dF(t) = \int_a^b f(t)dG(t)$$

for every $f \in C[a,b]$. Moreover, $T_G \leq T_F$.

Proof. Define G by setting $G(a) = 0$ and

$$\begin{aligned} G(t) &= F(t^+) - F(a) \text{ if } t \in (a,b) \\ &= F(b) - F(a) \text{ if } t = b. \end{aligned}$$

Clearly, G is right continuous. We claim that $T_G \leq T_F + \epsilon$ for any $\epsilon > 0$. Given a partition $P : \{t_0 = a, \cdots, t_n = b\}$ of $[a,b]$, find a partition $Q : \{s_0 = a, s_1, \cdots, s_{n-1}, s_n = b\}$ such that $t_j < s_j$ and

$$|F(t_j^+) - F(s_j)| < \epsilon/2n \text{ for } j = 1, \cdots, n-1.$$

It is easy to see that $V(P,G) \leq V(Q,F) + \epsilon$, and hence the claim stands verified. However, since $\epsilon > 0$ is arbitrary, we have $T_G \leq T_F$. This shows that $G \in NBV[a,b]$.

Since $F - G$ is a constant almost everywhere, both the integrals given in the statement agree. $\square$

Problem 5.8. *The G in the preceding lemma is unique.*

Hint. Suppose that $\int_a^b f(t)dG(t)$ for every $f \in C[a, b]$. For any $c \in (a, b)$, show that $|G(c)| \leq T_{G|_{[c,c+h]}}$ for sufficiently small h. Now note that the function $T_{G|_{[a,t]}}$ is right continuous if so is G.

Theorem 5.9. *If $\phi \in (C[a, b])'$, then $\phi = \phi_F$ for some function F of bounded variation. Moreover, the mapping $F \to \phi_F$ from $NBV[a, b]$ onto $(C[a, b])'$ is an isometric isomorphism.*

Proof. For simplicity, we assume that $a = 0$ and $b = 1$. For $f \in C[0, 1]$, consider $s_n := \sum_{r=1}^n f(r/n)\chi_{((r-1)/n,r/n]}$. Given $\epsilon > 0$, choose $n \geq 1$ such that $|f(s) - f(t)| < \epsilon$ for all x, y such that $|x - y| < 1/n$. Note that

$$s_n(t) - f(t) = \sum_{r=1}^{n} (f(r/n) - f(t))\chi_{((r-1)/n,r/n]}(t).$$

It is now easy to see that s_n converges uniformly to f. Let $\psi \in (B[0, 1])'$ be the extension of ϕ an ensured by Remark 5.5. We claim that $\phi = \phi_F$, where F is the function of bounded variation given by Problem 5.6. By the continuity of ψ, $\psi(s_n)$ converges to $\psi(f) = \phi(f)$, and hence it suffices to check that $\psi(s_n) = \int_0^1 s_n(t)dF(t)$. This follows from $\psi(\chi_{((r-1)/n,r/n]}) = \int_0^1 \chi_{((r-1)/n,r/n]}(t)dF(t)$. The equality $T_F = \|\phi\|$ follows from Problems 5.3 and 5.6. The remaining part follows from Problems 5.3 and 5.8 and Lemma 5.7. $\square$

Remark 5.10 : The normed linear space $NBV[a, b]$ with norm as total variation is complete.

In the remaining part of this section, we present a proof of Riesz representation theorem for $L^1[a, b]$. To do that, we state (a special case of) the Radon-Nikodym theorem (without proof).

Theorem 5.11. *Let μ be a positive σ-finite measure defined on the σ-algebra Σ and let λ be a complex measure on Σ with the property that $\lambda(E) = 0$ for every $E \in \Sigma$ for which $\mu(E) = 0$. Then there exists a unique $f \in L^1(\mu)$ such that*

$$\lambda(E) = \int_E f(t)d\mu(t)$$

for every $E \in \Sigma$.

Remark 5.12 : The Radon-Nikodym theorem may be viewed as a generalization of the fundamental theorem of calculus:

$$F(b) - F(a) = \int_{[a,b]} F'(t)dt.$$

For instance, in view of Corollary 4.11, if F is a function of bounded variation, then we have a complex measure λ obtained by setting

$$\lambda(E) = \int_E F'(t)dt$$

for every Lebesgue measurable set E of $[a, b]$. In this case, $F(b) - F(a) = \lambda([a, b])$. We wish to recall here that the conclusion of Fundamental theorem of calculus may fail even for a function of bounded variation (see Remark 4.7).

A complex-valued function f is said to be *essentially bounded* if $\|\phi\|_{m,\infty}$ is finite, where $\|\phi\|_{m,\infty} \equiv \inf\{M \in \mathbb{R}_+ : |\phi(z)| \leq M$ outside set of Lebesgue measure $0\}$.

Remark 5.13 : If $f \sim g$ then $\|f\|_\infty$ may be not be equal to $\|g\|_\infty$. However, we always have $\|f\|_{m,\infty} = \|g\|_{m,\infty}$.

Problem 5.14. *For a measurable set X, let $L^\infty(X)$ denote the set of all (equivalence classes of) measurable functions f for which $\|f\|_{m,\infty} < \infty$. Show that $L^\infty(X)$ is a normed linear space with norm $\|f\|_{m,\infty}$.*

Problem 5.15. *Let $g \in L^\infty[a, b]$. Define the linear functional $\phi_g : L^1[a, b] \to \mathbb{C}$ by $\phi_g(f) := \int_{[a,b]} f(t)g(t)dt$. Show that $\phi_g \in (L^1[a, b])'$.*

Theorem 5.16. *For $\phi \in (L^1[a, b])'$, there exists a unique $g \in L^\infty[a, b]$ such that $\phi = \phi_g$, where ϕ_g is as defined in the preceding problem.*

Proof. For a Lebesgue measurable subset Δ of $[a, b]$, define $\lambda(\Delta) = \phi(\chi_\Delta)$. Then λ is a countably additive measure. If $m(\Delta) = 0$ then $\chi_\Delta = 0$ almost everywhere, and hence by linearity of ϕ, $\lambda(\Delta) = \phi(0) = 0$. By Radon-Nikodym theorem, there exists $g \in L^1[a, b]$ such that $\lambda(\Delta) = \int_\Delta g(t)dt$ for every Lebesgue measurable subset Δ of $[a, b]$. For $\epsilon > 0$, let

$$A := \{x \in [a, b] : |g(x)| > \|\phi\| + \epsilon\}$$

and let $f = \chi_A(\bar{g}/g)$. Calculate $\|f\|_1$ and examine $\phi_g(f)$ to see that $\|\phi_g\| = \|g\|_\infty$. This also shows that $g \in L^\infty[a, b]$. To see that $\phi = \phi_g$, we check that $\phi(s) = \phi_g(s)$ for any simple measurable function s. By a simple application of dominated convergence theorem, we get $\phi(f) = \phi_g(f)$ for any $f \in L^1[a, b]$. We leave the uniqueness of g to the reader. $\square$

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