LINEAR DYNAMICS

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Abstract. These are the lecture notes prepared for AIS Geometric Methods in Complex Analysis (2012) to be held at IISc Bangalore. We discuss the basics of the theory of dynamics of linear operators.

1. Topological Transitivity and Hypercyclicity

Let V be a topological space. We say that a continuous transformation $T : V \to V$ is *topologically transitive* if for each pair of non-empty open sets $U, V \subseteq X$ there exists a non-negative integer n such $T^n(U) \cap V \neq \emptyset$.

The Tent Map 1.1. *Consider the function* $T : [0, 1] \to [0, 1]$ *defined by*

$$\begin{aligned} T(x) &= 2x \text{ if } 0 \leq x < 1/2 \\ &= -2x + 2 \text{ if } 1/2 \leq x \leq 1. \end{aligned}$$

We contend that T *is topologically transitive. Let us examine the action of* T^2. *A simple calculation shows that*

$$\begin{aligned} T^2(x) &= 2(2x) \text{ if } 0 \leq x < 1/4 \\ &= -2(2x) + 2 \text{ if } 1/4 \leq x < 1/2 \\ &= -2(-2x + 2) + 2 \text{ if } 1/2 \leq x < 3/4 \\ &= 2(-2x + 2) \text{ if } 3/4 \leq x \leq 1. \end{aligned}$$

Thus T^2 *maps any interval of the form* $[k/4, (k + 1)/4]$ *for* $k = 0, 1, 2, 3$ *onto the interval* $[0, 1]$. *More generally,* T^n *maps any interval of the form* $I_{k,n} := [k/2^n, (k+1)/2^n]$ *for* $k = 0, 1, \cdots, 2^n-1$ *onto the interval* $[0, 1]$. *Given an open subset* U *of* $[0, 1]$, *we may find a positive integer* n *sufficiently large so that* $I_{k,n} \subseteq U$ *for some* $0 \leq k \leq 2^n - 1$. *But then*

$$[0, 1] = T^n(I_{k,n}) \subseteq T^n(U) \subseteq [0, 1]$$

shows that $T^n(U) = [0, 1]$. *In particular,* T *is topologically transitive.* ∎

We say that a continuous transformation $T : V \to V$ is *hypercyclic* if there exists $f \in V$ (to be referred to as a *hypercyclic vector*) such that $O(f, T)$ is dense in V, where

$$O(f, T) := \{T^n f : n \text{ is a non-negative integer}\}.$$

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Remark 1.2. *If a topological space supports a hypercyclic transformation then it is necessarily separable.*

In these notes, the main object of study is linear hypercyclic transformations on a topological vector space. Recall that a *topological vector space* is a vector space together with a topology such that with respect to this topology the addition and scalar multiplication are continuous operations.

Examples 1.1. *Topological Vector Spaces:*

1. *A normed linear space.*
2. *The vector space $\mathcal{O}(\Omega)$ of complex-valued holomorphic functions f defined on the open set $\Omega \subset \mathbb{C}$ endowed with the topology of uniform convergence on compact subsets.* ■

Before we see some examples of linear hypercyclic transformations on a topological vector space (can you think of a simple one?), here are some examples of non-hypercyclic transformations:

Example 1.1. *Let V be a normed linear space and let $T : V \to V$ be a continuous transformation. If $R := \sup_{n \geq 1} \|T^n\| < \infty$ then T is not hypercyclic. Indeed, $O(T, x)$ is contained in the closed ball centered at the origin and of radius R, and hence can not be dense in V.*

Let us see a particular instance when this happens. Assume that T is a bounded linear operator on a Banach space (that is, a complete normed linear space) X. Suppose the spectrum

$$\sigma(T) := \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bijective } \}$$

of T is contained in the open unit disc. By the Spectral Radius Formula,

$$\sup_{\lambda \in \sigma(T)} |\lambda| = \lim_{n \to \infty} \|T^n\|^{\frac{1}{n}}.$$

Since $\sigma(T)$ is a compact subset of the open unit disc, one may find a positive number $K < 1$ and positive integer N such that $\|T^n\| \leq K^n$ for all $n \geq N$. It is obvious now that R is at most $\max\{1, \|T\|, \cdots, \|T^{N-1}\|\}$. ■

Birkhoff's Transitivity Theorem 1.1. *Let X be a separable topological space and let $T : X \to X$ be given. Then the following are true:*

1. *Suppose X is a complete metric space. If T is topologically transitive then T is hypercyclic.*
2. *Suppose X is a topological vector space. If T is hypercyclic then T is topologically transitive.*

In both the cases, the set H of hypercyclic vectors is dense in X.

Proof. (1): Let H denote the set of all hypercyclic vectors of T and let $\{V_j\}_{j \in \mathbb{N}}$ be a countable basis for X. Note that a sequence $\{x_n\}_{n \in N}$ is dense in X if and only if for each $j \in \mathbb{N}$ there exists $n_j \in \mathbb{N}$ such that $x_{n_j} \in V_j$. Applying this fact to the sequence $x_n := T^n x$, one obtains

$$H := \bigcap_j \bigcup_{n \geq 0} (T^n)^{-1}(V_j).$$

By hypothesis, for each non-empty open set $U \subseteq X$ there exists a non-negative integer n such $T^n(U) \cap V_j \neq \emptyset$. It follows that $\bigcup_{n \geq 0} T^{-n}(V_j)$ is dense in X for every j. By the Baire Category Theorem, H is dense in X.

(2): If $x \in X$ then since scalar multiplication is continuous, every neighborhood of x contains points from $X \setminus \{x\}$. Thus X has no isolated points. Now it follows that if T is hypercyclic with hypercyclic vector x, then for every non-negative integer n, $T^n x$ is a hypercyclic vector for T. In particular, H is dense in X. Now if $T^N x \in V$ for some non-negative integer N, then for some $n \geq N$, $T^n x \in U$. Thus $T^{n-N}(U) \cap V$ is non-empty. □

Let $U \subset \mathbb{C}$ be open and let $\{K_n\}_{n \geq 1}$ denote a compact exhaustion of U. If $U = \mathbb{C}$ then the collection closed discs centered at the origin and of radius n forms a compact exhaustion of U. Define a metric d on the vector space $\mathcal{C}(U)$ of continuous functions on U as follows:

$$d(f, g) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|f - g\|_{K_n, \infty}}{1 + \|f - g\|_{K_n, \infty}} \ (f, g \in \mathcal{O}(U)), \tag{1.1}$$

where $\|f\|_{K,\infty} = \sup_{z \in K} |f(z)|$. Note that $\mathcal{C}(U)$ is endowed with the topology of uniform convergence on compact sets. Recall that $\mathcal{C}(U)$ is a complete metric space. So is the subspace $\mathcal{O}(U)$ of all holomorphic functions on U (see Exercise 4.1).

The next result relies on the Runge's Polynomial Approximation. This approximation theorem is included in almost every text on complex analysis but very few texts exhibit its utility (see Exercise 4.3 below, and refer to [6] for many such).

Corollary 1.3. *For a non-zero complex number a, let $T_a : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ denote the translation operator defined by $T_a(f)(z) = f(z + a)$. Then T_a is hypercyclic.*

Proof. We apply the Birkhoff's Transivity Theorem to $X := \mathcal{O}(\mathbb{C})$ and $T := T_a$. Let U, V be two non-empty open subsets of $\mathcal{O}(\mathbb{C})$. One may find $\epsilon > 0$, and $f, g \in \mathcal{O}(\mathbb{C})$ such that

$$\{h \in \mathcal{O}(\mathbb{C}) : d(f, h) < 3\epsilon\} \subseteq U, \ \{h \in \mathcal{O}(\mathbb{C}) : d(g, h) < 3\epsilon\} \subseteq V,$$

where d is as given in (1.1). Now it is easy to see that there exists a closed disc $K \subset \mathbb{C}$ such that

$$\{h \in \mathcal{O}(\mathbb{C}) : \|h - f\|_{\infty, K} < \epsilon\} \subseteq U, \ \{h \in \mathcal{O}(\mathbb{C}) : \|h - g\|_{\infty, K} < \epsilon\} \subseteq V$$

(see Exercise 4.2). Let n be a positive integer such that $K \cap (K + an) = \emptyset$. Since the compact set $K \cup (K + an)$ has connected complement in $\mathbb{C}$, Runge's Theorem is applicable to the holomorphic function taking value $f(z)$ in the vicinity of K, and $T^n_{-a}(g)(z) = g(z - na)$ in the vicinity of $K + an$. This gives us an analytic polynomial p such that

$$\|p - f\|_{\infty,K} < \epsilon \text{ and } \|p - T^n_{-a}(g)\|_{\infty,K+an} = \|T^n_a(p) - g\|_{\infty,K} < \epsilon.$$

In particular, $p \in U$ and $T^n_a(p) \in V$. □

Corollary 1.4. *Let X be a separable, complete metric space and a topological vector space. Suppose T is invertible with a continuous inverse $T^{-1} : X \to X$. Then T is hypercyclic if and only if so is T^{-1}.*

Proof. Note that $T^n(U) \cap V \neq \emptyset$ if and only if $T^{-n}(V) \cap U \neq \emptyset$. Now appeal to the Birkhoff's Transitivity Theorem. □

In the remaining part of these notes, we confine ourselves to the study of hypercyclicity of bounded, linear operators. We use $L(X)$ to denote the algebra of continuous linear operators on X.

Kitai's Criterion 1.1. *Let X be a complete metric space and a separable topological vector space. Let $T \in L(X)$. If there exist a dense subset $\mathcal{D}$ of X and a sequence $\{S_n\}$ of transformations $S_n : X \to X$ such that*

(1) $T^n(x) \to 0$ *for any* $x \in \mathcal{D}$,
(2) $S_n(x) \to 0$ *for any* $x \in \mathcal{D}$, *and*
(3) $T^n S_n(x) \to x$ *for any* $x \in \mathcal{D}$.

Then T is hypercyclic with a dense set of hypercyclic vectors.

Proof. It suffices to verify the hypothesis of the Birkhoff's Transivity Theorem. To see that, let U, V be two non-empty open subsets of X and pick $x \in D \cap U$, $y \in D \cap V$. Then $x + S_n(y) \to x \in U$ as $k \to \infty$. Choose a positive integer k_1 so that $x + S_n(y) \in U$ for $n \geq k_1$. Since T is linear, $T^n(x + S_n(y)) = T^n(x) + T^n S_n(y) \to y \in V$. Choose a positive integer k_2 so that $T^n(x + S_n(y)) \in V$ for $n \geq k_2$. It follows that $T^n(U) \cap V$ is non-empty for every $n \geq \max\{k_1, k_2\}$. □

The following result is due to G. R. MacLane.

Theorem 1.5. *The derivative operator $\frac{d}{dz} : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ is hypercyclic.*

Proof. One may conclude from the Weierstrass' Convergence Theorem that $\frac{d}{dz} \in L(\mathcal{O}(\mathbb{C}))$. Set $\mathcal{D} :=$ the vector space of analytic polynomials. Since every entire function can be globally represented as a power series (convergent uniformly on compact sets), $\mathcal{D}$ is dense in $\mathcal{O}(\mathbb{C})$. Consider the transformation $S : \mathcal{O}(\mathbb{C}) \to \mathcal{O}(\mathbb{C})$ given by

$$S(f)(z) = \int_{[0,z]} f(w)dw,$$

where $[0, z]$ denotes the line segment joining the origin and z. Set $S_n = S^n$. We now apply the Kitai's Criterion to $X := \mathcal{O}(\mathbb{C}), T := \frac{d}{dz}$. Fix positive integers k and l. Since $T^k(z^l) = 0$ if $k \geq l$; $S_k(z^l) = \frac{l!}{(l+k)!} z^{l+k} \to 0$ on any compact subset of $\mathbb{C}$, $TS(z^l) = z^l$, conditions (1), (2), (3) of the Kitai's Criterion are satisfied. □

Remark 1.6. *Note that the set of hypercyclic vectors of* $\frac{d}{dz}$ *is dense in* $\mathcal{O}(\mathbb{C})$.

To see an interesting application of the preceding result to the complex function theory, we need a definition.

Definition 1.7. *The final set* $L(f)$ *of a function* f *meromorphic in the complex plane* $\mathbb{C}$ *is the set of points* z_0 *of the Riemann sphere* $\mathbb{C} \cup \{\infty\}$ *for which the following holds true: Each neighbourhood of* z_0 *contains zeros of infinitely many derivatives of* f.

A result of Pólya, which says that the final set of any non-entire, meromorphic function consists of a union of rays, lines, and line segments. In view of this, the following is quite striking.

Corollary 1.8. *There exists a dense subset* H *of* $\mathcal{O}(\mathbb{C})$ *such that* $L(f) = \mathbb{C} \cup \{\infty\}$ *for all* $f \in H$.

Proof. We take H to be the set of hypercyclic vectors of $\frac{d}{dz}$. By Remark 1.6, H is dense in $\mathcal{O}(\mathbb{C})$. Fix $f \in H$ and $z_0 \in \mathbb{C}$. One may now approximate $z - z_0$ by some subsequence of $\{\frac{d^n f}{dz^n}\}_{n \geq 0}$, uniformly on compact subsets of $\mathbb{C}$. By the Hurwitz's Theorem, each disc centered at z_0 contains zeros of all but finitely many members of this subsequence. Hence $z_0 \in L(f)$. To see that $\infty \in L(f)$, note simply that any neighbourhood of ∞ contains an open subset of $\mathbb{C}$. This shows that $L(f) = \mathbb{C} \cup \{\infty\}$ for all $f \in H$. □

Recall that the Hardy space H^2 of the unit disc consists of complex-valued functions f holomorphic on the unit disc $\mathbb{D}_1$ for which

$$\|f\|^2_{H^2} := \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{d\theta}{2\pi} < \infty.$$

Since the norm on H^2 satisfies the Paralleogram Law, H^2 is an inner-product space endowed with the inner-product

$$\langle f, g \rangle_{H^2} := \frac{1}{4} \left(\|f+g\|^2_{H^2} - \|f-g\|^2_{H^2} + i\|f+ig\|^2_{H^2} - i\|f-ig\|^2_{H^2} \right)$$

for all $f, g \in H^2$ (see Exercise 4.10). What is not so obvious is that H^2 is complete in this inner-product. To see that, let $\{f_n\}$ be a Cauchy sequence in H^2. By the Cauchy Integral Formula,

$$f_n(z) - f_m(z) = \int_{|z|=R} \frac{f(w)}{w-z} dw \ (|z| < R < 1),$$

which leads to

$$\begin{aligned} |f_n(z) - f_m(z)| &\leq \frac{1}{R-r} \int_0^{2\pi} |f(Re^{i\theta})| d\theta \\ &\leq \frac{2\pi}{R-r} \int_0^{2\pi} |f(Re^{i\theta})|^2 d\theta \\ &\leq \frac{\|f_n - f_m\|_{H^2}}{R-r} \end{aligned}$$

for any $|z| \leq r < R < 1$. This shows that $f_n - f_m$ converges compactly to 0. It follows that $\{f_n\}$ converges compactly to some holomorphic function f. Now, it is easy to see that $\|f_n - f\|_{H^2}$ converges to 0.

The following result provides an example of a hypercyclic linear operator on a Hilbert space and it is due to S. Rolewicz.

Corollary 1.9. *αS^* is hypercyclic for $|\alpha| > 1$.*

Proof. Consider the function $\kappa : \mathbb{D}_1 \to H^2$ defined by

$$\kappa(\lambda) \equiv \sum_{n=0}^{\infty} \lambda^n S^n(1).$$

Since $\|S\| \leq 1$, κ is well-defined on the open unit disc. Also, $S^*(\kappa(\lambda)) = \lambda\kappa(\lambda)$ in view of $S^*(1) = 0$. Set

$$\mathcal{D}_r := \text{linspan}\{\kappa(\lambda) : \lambda \in \mathbb{D}_r\}$$

for a real $r > 0$. We check that $\mathcal{D}_r$ is dense in H^2. Let $f(z) := \sum_{n=0}^{\infty} a_n z^n \in H^2$ be such that $\langle f, \kappa(\lambda)\rangle = 0$ for every $\lambda \in \mathbb{D}_r$. Since $\{z^n\}_{n\geq 0}$ forms an othonormal set in H^2, $\langle f, \kappa(\lambda)\rangle = f(\overline{\lambda})$. Thus $f(z)$ is zero on $\mathbb{D}_r$, and hence by the Identity Theorem, $f(z)$ is identically zero. Thus the claim stands verified. Now one may apply the Kitai's Criterion with $T := \alpha S^*$ and $S_n := \alpha^n S^n$ for $|\alpha| > 1$ to obtain the desired result. □

2. Spectral Properties

For simplicity, in the remaining part of these lecture notes, all the topological spaces are assumed to be complex, separable Banach spaces. We use $B(X)$ to denote the algebra of bounded linear operators on X. We use X' to denote the dual space of X.

Proposition 2.1. *Let $T \in B(X)$ be hypercyclic. Then the adjoint operator T^* has no eigenvalue.*

Proof. Suppose $T^*f = \lambda f$ for some non-zero $f \in X'$ and complex number λ. Then, for $g \in X$ and for all integers $n \geq 0$,

$$\langle f, T^n g\rangle = \langle T^{n*} f, g\rangle = \langle T^{*n} f, g\rangle = \lambda^n \langle f, g\rangle.$$

Clearly, for any g, the set $\{\lambda^n \langle f, g\rangle : n \geq 0\}$ is not dense in $\mathbb{C}$ (Why?) On the other hand, if g is a hypercyclic vector for T, then αf can be approximated by a subsequence of $\{T^g\}_{n\geq 0}$ for any complex number α, and hence in that case $\{\langle f, T^n g\rangle : n \geq 0\}$ is dense in $\mathbb{C}$. Consequently, no g can be a hypercyclic vector for T. □

Corollary 2.2. *A finite-dimensional complex vector space does not support hypercyclic operators.*

Remark 2.3. *By a more direct argument, this result can be proved even for real vector spaces (see Exercise 4.6).*

The conclusion of Corollary 2.2 is not actually satisfactory as it just concludes that the orbit of an $n \times n$ complex (or real) matrix can not be dense. We claim that it can not even be somewhere dense. To see that, let us examine the orbit of a complex $k \times k$ matrix T. By the Jordan Decomposition, it suffices to examine the orbit of an $n \times n$ Jordan block $J = \lambda I + N$, where $\lambda \in \mathbb{C}$, and N is a nilpotent operator with the superdiagonal with all entries equal to 1. Since $N^n = 0$ and λI commutes with N, one obtains

$$T^m = \begin{bmatrix} \lambda^m & \binom{m}{1}\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} & \binom{m}{3}\lambda^{m-3} & \cdots \\ 0 & \lambda^m & \binom{m}{1}\lambda^{m-1} & \binom{m}{2}\lambda^{m-2} & \cdots \\ 0 & 0 & \lambda^m & \binom{m}{1}\lambda^{m-1} & \cdots \\ 0 & 0 & 0 & \lambda^m & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

It is not now difficult to see that $O(x, J)$ is a discrete subset of $\mathbb{C}^m$.

For a positive real number r, let

$$\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\},\ \partial\mathbb{D}_r := \{z \in \mathbb{C} : |z| = r\}, \overline{\mathbb{D}_r} := \{z \in \mathbb{C} : |z| \leq r\}.$$

Theorem 2.4. *Let $T \in B(X)$ be such that*

$$\sigma(T) \cap \partial\mathbb{D}_r = \emptyset,\ \ \sigma(T) \cap \mathbb{D}_r \neq \emptyset$$

for some positive real number $r \leq 1$. Then T is not hypercyclic.

Proof. Note that $\sigma(T)$ is a union of disjoint closed sets σ_1 and σ_2, where

$$\sigma_1 = \sigma(T) \cap \overline{\mathbb{D}_r},\ \ \sigma_2 = \sigma(T) \cap (\mathbb{C} \setminus \mathbb{D}_r).$$

By the Riesz Decomposition Theorem ([1], Appendix D), there exist bounded linear operators T_1 and T_2 such that $T = T_1 \oplus T_2$ and $\sigma(T_1) = \sigma_1, \sigma(T_2) = \sigma_2$. If T is hypercyclic then so is T_1 (Exercise 4.8). Since $\sigma(T_1) = \sigma_1 \subseteq \mathbb{D}_r \subseteq \mathbb{D}_1$, by Example 1.1, T_1 and hence T is not hypercyclic. □

Corollary 2.5. *Let $T \in B(X)$ be hypercyclic. Then the spectrum $\sigma(T)$ of T intersects the unit circle $\partial\mathbb{D}_1$.*

Proof. Suppose $\sigma(T) \cap \partial\mathbb{D}_1 = \emptyset$. By the preceding theorem (with $r = 1$), we must have $\sigma(T) \cap \mathbb{D}_1 = \emptyset$. It follows that $\sigma(T) \subseteq (\mathbb{C} \setminus \overline{\mathbb{D}_1})$. But then T is invertible, and hence by Corollary 1.3, T^{-1} is hypercyclic. Since $\sigma(T^{-1}) = \{\lambda^{-1} : \lambda \in \sigma(T)\} \subseteq \mathbb{D}_1$, we arrive at a contradiction. □

Remark 2.6. *Further, it can be proved that every connected component of a hypercyclic operator intersects the unit circle* [1].

Corollary 2.7. *Let $T \in B(X)$ be a compact operator. Then T is not hypercyclic.*

Proof. By Corollary 2.1, we may assume that X is infinite-dimensional. It is known that the spectrum of a compact operator is countable and contains 0. Thus one can find a positive real number $r \leq 1$ such that $\sigma(T) \cap \partial\mathbb{D}_r = \emptyset$. Since $0 \in \sigma(T) \cap \mathbb{D}_r$, by Theorem 2.4, T is not hypercyclic. □

3. The Set of Hypercyclic Vectors

Let $T \in B(X)$ be hypercyclic and let H denote the set of its hypercyclic vectors. Since any dense subset of X remains dense after removal of finitely many points, $T^n x \in H$ for any positive integer whenever $x \in H$. Thus either H is empty or dense subset of X. This topological dichotomy leads to the following rather striking representation theorem.

Proposition 3.1. *Let $T \in B(X)$ be hypercyclic and let H denote the set of its hypercyclic vectors. Then $X = H + H$.*

Proof. Let $x \in X$. As observed in the proof of Birkhoff's Transitivity Theorem, one has

$$H := \bigcap_j \bigcup_{n \geq 0} T^{-n}(V_j),$$

where $\{V_j\}_{j \in \mathbb{N}}$ is a countable basis for X. Thus both H and $x - H$ are dense G_δ subsets of X. By the Baire Category Theorem, H must intersect with $x - H$. Now if $y \in H$ and $y \in x - H$ then $y = x - z$ for some $z \in H$, and hence $x = y + z \in H + H$. □

For $T \in B(X)$ and $p(x) := a_0 + a_1 x + \cdots + a_k x^k$, set

$$p(T) := a_0 I + a_1 T + \cdots + a_k T^k.$$

Theorem 3.2. *Let $T \in B(X)$ be hypercyclic. If $h \in X$ is a hypercyclic vector then so is $p(T)h$ for any non-zero polynomial $p(x)$. In particular, the set of all hypercyclic vectors for T is connected.*

Proof. Let $p(x)$ be a polynomial. If $\lambda_1, \cdots, \lambda_k$ are (possibly repeated) complex roots of $p(x)$ then $p(T) = a(T - \lambda_1 I) \cdots (T - \lambda_k I)$, where $a \neq 0$. By Theorem 2.1, $\ker(T^* - \lambda_i I) = \{0\}$ for each $i = 1, \cdots, k$. It is now easy to see that $\ker(p(T^*)) = \{0\}$. Since $p(T)^* = p(T^*)$, the range of $p(T)$ must be dense in X.

Suppose that h is a hypercyclic vector for T. Consider now

$$\begin{aligned} O(p(T)h, T) &:= \{T^n p(T)h : n \text{ is a non-negative integer}\} \\ &= \{p(T)(T^n h) : n \text{ is a non-negative integer}\}, \end{aligned}$$

which is a dense subset of the range of $p(T)$. Since the range of $p(T)$ itself is dense, so is $O(p(T)h, T)$.

Since for any complex vector space V, the set $V \setminus \{0\}$ is connected, $\{T^n p(T)h : n \text{ is a non-negative integer}\}$ is path-connected. Now to see the remaining part, note that the set of all hypercyclic vectors lies between two connected, dense sets $\{p(T)h : p(x) \text{ is a non-zero polynomial}\}$ and X, and hence connected. □

Corollary 3.3. *If $T \in B(X)$ is hypercyclic then so is T^2.*

Proof. Suppose that T is hypercyclic with a hypercyclic vector h. By the Birkhoff's Transitivity Theorem, it suffices to check that T^2 is topologically transitive. To see that, let U, V be two non-empty open subsets of X. Since T is hypercyclic, $x := T^n h \in U$ for some positive integer n. To complete the proof, we must find a positive integer k such that $T^{2k}x \in V$.

Let H denote the set of all hypercyclic vectors for T. Consider the following subsets of H :

$$E := H \cap \overline{\{T^{2k}x : k \geq 0\}}, \; O := H \cap \overline{\{T^{2k+1}x : k \geq 0\}}.$$

We claim that $E \cap O \neq \emptyset$. Since $H \cap \{T^k x : k \geq 0\} \subseteq E \cup O$, and x is a hypercyclic vector for T, $H = E \cup O$. Clearly, E and O are closed subsets of H. Also, since $x \in E$ and $Tx \in O$, by the connectedness of H, the intersection of E and O is non-empty as desired.

Let $y \in E \cap O$. Since $y \in H$, one can find a positive integer m such that $T^m y \in V$. Thus y belongs to the open set $T^{-m}V$. If, for some positive integer k, $m = 2k$ (resp. $m = 2k + 1$) then since $y \in E$ (resp. $y \in O$), there exists a positive integer l such that $T^{2l}x \in T^{-2k}V$, and hence $T^{2(l+k)}x \in V$ (resp. $T^{2l+1}x \in T^{-2k-1}V$, and hence $T^{2(l+k+1)}x \in V$). □

Remark 3.4. *The proof of Corollary 3.2 is borrowed from ([1], Chapter 3). Actually, Corollary 3.2 is a special case of a result of S. A. Ansari, which says that T^n is hypercyclic for any positive integer n if so is T (see Corollary 3.6 below).*

Example 3.1. *One may conclude from the Maclane's Theorem and the Corollary 3.3 that there exists an entire function $f \in \mathcal{O}(\mathbb{C})$ such that the sequence*

$$\left\{\frac{d^{2n} f}{dz^{2n}}\right\}_{n\geq 0}$$

is dense in $\mathcal{O}(\mathbb{C})$. ■

In the remaining part of this section, we discuss implications of the following beautiful result of Bourdon and Feldman: *Either $O(x, T)$ is dense or nowhere dense.* We refer the reader to ([3], Chapter 6) for a proof.

The following result, conjectured by Herrero, is due to Costakis and Peris.

Theorem 3.5. *Let $T \in B(X)$. If, for some $x_1, \cdots, x_n \in X$,*

$$\bigcup_{j=1}^{n} O(x_j, T) \text{ is dense in } X \tag{3.2}$$

then T is hypercyclic.

Proof. Since finite union of closed sets is closed, the union of closure of $O(x_1, T), \cdots, O(x_n, T)$ is closure of union of $O(x_1, T), \cdots, O(x_n, T)$. Now if (3.2) holds true then

$$\bigcup_{j=1}^{n} \overline{O(x_j, T)} = X,$$

and since finite union of nowhere dense sets is nowhere dense, by the Bourdon-Feldman Theorem, for some j, $O(x_j, T)$ is dense in X. □

Corollary 3.6. *If $T \in B(X)$ is hypercyclic then so is T^n for any positive integer n. Moreover, if $x \in X$ is a hypercyclic vector for T then x is also a hypercyclic vector for T^n for any positive integer n.*

Proof. Note that

$$O(x, T) = \bigcup_{j=0}^{n-1} O(T^j x, T^n).$$

Now if T is hypercyclic then by the preceding theorem, so is T^n. In particular, $O(T^j x, T^n)$ is dense in X for some $1 \leq j < n$. Since

$$T^{n-j}(O(T^j x, T^n)) \subset O(x, T^n)$$

and since the range of T^{n-j} is dense (Proposition 2.1), $O(x, T^n)$ must be dense in X. □

4. Exercises

Exercise 4.1. *Show that the subspace $\mathcal{O}(U)$ of $\mathcal{C}(U)$ is closed.*

Exercise 4.2. *Let d be as given by (1.1). Show that for any $\epsilon > 0$ and $f \in \mathcal{O}(U)$, there exists a compact subset K of U such that*

$$\{g \in \mathcal{O}(U) : \|f - g\|_{K,\infty} < \epsilon\} \subset \{g \in \mathcal{O}(U) : d(f, g) < 3\epsilon\}.$$

Exercise 4.3. *Show that there exists a sequence of complex polynomials $\{p_n\}_{n\geq 1}$ such that*

$$\lim_{n\to\infty} p_n(z) = \left\{ \begin{array}{rl} 1 & (Im(z) > 0) \\ 0 & (Im(z) = 0) \\ -1 & (Im(z) < 0) \end{array} \right\}$$

Hint. Let K_n denote the union of line segment $[-n, n]$ and rectangles $[-n+i/n, n+i/n, n+in, -n+in]$, $[-n-i/n, n-i/n, n-in, -n-in]$, where $[a, b, c, d]$ denotes the rectangle with vertices a, b, c, d. Apply the Runge's Theorem to an appropriate holomorphic function on an open set containing the compact set K_n.

Exercise 4.4. *For $\lambda \in \mathbb{C}$, consider the set $S := \{\lambda^n : n \geq 0\}$. If $z \in \mathbb{C} \setminus \{0\}$ is such that $|z| \notin \{|\lambda|^n : n \geq 0\}$ then z does not belong to the closure of S. Conclude that S is nowhere-dense in the complex plane.*

Exercise 4.5. *Let $S, T \in B(X)$. If there exists an invertible operator $U \in B(X)$ such that $SU = UT$ then S is hypercyclic if and only if so is T.*

Exercise 4.6. *Show that a linear operator on a real, finite-dimensional vector space can not be hypercyclic.*

Hint. The idea of the proof is due to Rolewicz. If x is a hypercyclic vector for T then $x, Tx, \cdots, T^{n-1}x$ form a basis for $\mathbb{R}^n$. If $T^{n_k}x \to \alpha x$ then verify that $T^{n_k}y \to \alpha y$ for every $y \in \mathbb{R}^n$. Conclude that $T^{n_k} \to \alpha I$ and then use the continuity of the determinant.

Exercise 4.7. *Suppose $X = X_1 \oplus X_2$ for closed subspaces X_1 and X_2 of X. Show that $P(x \oplus y) = x$ $(x \in X_1, y \in X_2)$ is continuous.*

Hint. Use the Closed Graph Theorem.

Exercise 4.8. *Prove: If $T = T_1 \oplus T_2 \in B(X_1 \oplus X_2)$ is hypercyclic then so are $T_1 \in B(X_1)$ and $T_2 \in B(X_2)$.*

Hint. Use the last exercise.

Exercise 4.9. *Suppose N is bounded linear operator on a Hilbert space. If $N^*N = NN^*$ then show that N is never hypercyclic.*

Hint. There are elementary ways to obtain far generalizations of this exercise [3]. However, to illustrate the power of the functional calculus of a normal operator, we sketch an alternative proof.

Suppose N is hypercyclic with a hypercyclic vector h. Consider $\mathcal{K} = \chi(N)\mathcal{H}$, where χ is the characteristic function of the $\overline{\mathbb{D}} \cap \sigma(N)$, and $\chi(N)$ is given by the functional calculus of N. Note that $N = N_1 \oplus N_2$ on $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}^{\perp}$. By Exercise 4.8, both N_1 and N_2 are hypercyclic. The norm of N_1 is at most 1. Also, if $\mathcal{K} \subsetneq \mathcal{H}$ then N_2 is invertible with the norm of N_2^{-1} at least 1.

Exercise 4.10. *Let X denote a normed linear space with the norm $\|\cdot\|$. If X satisfies the Parallelogram Law*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \ (x, y \in X),$$

then the function

$$\langle x,\, y\rangle := \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2\right) \ (x, y \in X)$$

satisfies $\sqrt{\langle\cdot,\,\cdot\rangle} = \|\cdot\|$ and defines an inner-product on X.

In other words, norm on any normed linear space is induced by an inner-product if and only if it satisfies the Parallelogram Law.

Hint. We divide the verification into four steps:

(1) $\langle x,\, y\rangle = \overline{\langle y,\, x\rangle}$ $(x, y \in X)$.

(2) $\langle x/2,\, y\rangle = 1/2\langle x,\, y\rangle$ $(x, y \in X)$.

(3) $\langle x+y,\, z\rangle = \langle x,\, z\rangle + \langle y,\, z\rangle =$ $(x, y \in X)$.

Note that $x + y + z = x + y/2 + y/2 + z$ etc...

(4) $\langle \alpha x,\, y\rangle = \alpha\langle x,\, y\rangle$ for any $\alpha \in \mathbb{C}$.

Use density of $\{m/2^n : m \in \mathbb{Z},\ n \in \mathbb{N} \cup \{0\}\}$ in $\mathbb{R}$ to conclude that $\langle \alpha x,\, y\rangle = \alpha\langle x,\, y\rangle$ for any real α.

Exercise 4.11. *Show that if $T \in B(X)$ is hypercyclic and non-invertible then its spectrum is uncountable.*

Hint. Use Theorem 2.4.

5. Comments

The basic purpose for writing these notes is to give a brief introduction to this rapidly evolving branch of functional analysis in a short time-span. Undoubtedly, our main source is the masterful exposition [1]. However, there are a few exceptions.

1. Example 1.1 of Tent Map is from [4].
2. Corollary 1.8 is from [2].
3. Although the proof of Theorem 2.4 is an adaptation of that of Theorem 1.18 from [1], the statement is apparently new.

4. Theorem 3.5 and Corollary 3.6 are from [3].

Finally, for beginners, we recommend the excellent introductory text [3].

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