PROBLEMS AND NOTES: UNIFORM CONVERGENCE AND POLYNOMIAL APPROXIMATION

SAMEER CHAVAN

Abstract. These are the lecture notes prepared for the participants of IST to be conducted at BP, Pune from 3rd to 15th November, 2014.

Contents

1. Pointwise and Uniform Convergence 1
2. Lebesgue's Proof of Weierstrass' Theorem 2
3. Bernstein's Theorem 3
4. Applications of Weierstrass' Theorem 4
5. Stone's Theorem and its Consequences 5
6. A Proof of Stone's Theorem 7
7. Müntz-Szász Theorem 8
8. Nowhere Differentiable Continuous Function 9
References 10

1. Pointwise and Uniform Convergence

Exercise 1.1 : Consider the function $f_n(x) = x^n$ for $x \in [0, 1]$. Check that $\{f_n\}$ converges pointwise to f, where $f(x) = 0$ for $x \in [0, 1)$ and $f(1) = 1$.

Exercise 1.2 : Consider the function $f_m(x) = \lim_{n\to\infty}(\cos(m!\pi x))^n$ for $x \in \mathbb{R}$. Verify the following:

(1) $\{f_m\}$ converges pointwise to f, where $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$, and $f(x) = 1$ for $x \in \mathbb{Q}$.
(2) f is discontinuous everywhere, and hence non-integrable.

Remark 1.3 : If f is pointwise limit of a sequence of continuous functions then the set of continuties of f is everywhere dense [1, Pg 115].

Consider the vector space $B[a, b]$ of bounded function from $[a, b]$ into $\mathbb{C}$. Let f_n, f be such that $f_n - f \in B[a, b]$. A sequence $\{f_n\}$ converges uniformly to f if

$$\|f_n - f\|_\infty := \sup_{x\in[a,b]} |f_n(x) - f(x)| \to 0 \text{ as } n \to \infty.$$

Theorem 1.4. *Let $\{f_n\}$ be a sequence of continuous functions. If $\{f_n\}$ converges uniformly to f on $[a,b]$ then f is continuous. Moreover,*

$$\lim_{n\to\infty}\int_a^b f_n(x)dx = \int_a^b f(x)dx.$$

Proof. Let $N \geq 1$ be such that $\|f_N - f\|_\infty < \epsilon/3$. Recall that f_N is uniformly continuous on $[a,b]$, that is, for some δ, $|f_N(x) - f_N(y)| < \epsilon/3$ whenever $|x-y| < \delta$. Finally, for all x, y such that $|x-y| < \delta$,

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f(y)| \\ &\leq \|f_N - f\|_\infty + \epsilon/3 + \|f_N - f\|_\infty \\ &\leq \epsilon/3 + \epsilon/3 + \epsilon/3. \end{aligned}$$

The remaining part follows from

$$|\int_a^b f_n(x)dx - \int_a^b f(x)dx| \leq \|f_n - f\|_\infty (b-a).$$

That's the end of the proof. □

Corollary 1.5. *Let $\overline{\mathcal{P}} = \{f \in B[a,b] : \exists\ \{p_n\}$ such that $\|p_n - f\|_\infty \to 0\}$. Then $\overline{\mathcal{P}}$ is contained in $C[a,b]$.*

A remarkable result of Weierstrass asserts indeed that $\overline{\mathcal{P}} = C[a,b]$. In particular, any continuous function on $[a,b]$ can be approximated uniformly by a sequence of polynomials.

Theorem 1.6. *For any $f \in C[a,b]$, there exists a sequence $\{p_n\}$ of polynomials such that $\|f - p_n\|_\infty \to 0$ as $n \to \infty$.*

There are several proofs of different flavors of this great theorem (apart from Weierstrass' original proof). Perhaps the most elementary proof is due to Lebesgue (which relies on the conclusion of Weierstrass' theorem for the function $|x|$). A constructive proof is due to Bernstein (which relies on a Chebyshev's version of Bernoulli's law of large numbers). We will also discuss two remarkable generalizations of Weierstrass' Theorem: Stone's Theorem and Müntz-Szász's Theorem.

Exercise 1.7 : Let $\{p_n\}$ be a sequence of polynomials of degree d_n. Suppose that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$ for some continuous function $f \in C[a,b]$. If f is not a polynomial then $d_n \to \infty$ as $n \to \infty$.

Hint. The subspace of polynomials of degree at most d is finite-dimensional, and hence closed in $C[a,b]$.

2. Lebesgue's Proof of Weierstrass' Theorem

Exercise 2.1 : Consider the function $f_n(x) = nx(1-x^2)^n$ for $x \in [0,1]$. Verify:

(1) $\{f_n\}$ converges pointwise to f, where $f(x) = 0$ for all $x \in [0,1]$.
(2) $\lim_{n\to\infty}\int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$, and hence $\{f_n\}$ can not converge uniformly to f.

Here is a partial converse to Theorem 1.4.

Theorem 2.2. *Let $\{f_n\}$ be a sequence in $C[a,b]$ converging pointwise to a continuous function f. If $\{f_n(x)\}$ decreasing for all $x \in [a,b]$ then $\{f_n\}$ converges uniformly to f.*

Proof. Let $g_n := f_n - f \geq 0$. For $\epsilon > 0$, consider the closed subset $K_n := \{x \in [a,b] : g_n(x) \geq \epsilon\}$ of $[a,b]$. Since $g_n \geq g_{n+1}$, $K_{n+1} \subseteq K_n$. In particular, finite intersection of K_n's is non-empty if every K_n is non-empty. If $x \in [a,b]$ then since $g_n(x) \to 0$, $x \notin K_n$ for sufficiently large n. If each K_n is non-empty then we must have $\cap_{n=1}^{\infty} K_n \neq \emptyset$ (Exercise), and hence K_N is empty for some N. That is, $0 \leq g_n(x) < \epsilon$ for $n \geq N$. □

Here is an important special case of Weierstrass' Theorem.

Corollary 2.3. *Define a sequence $\{p_n\}$ of polynomials by $p_0(x) = 0$, and*

$$p_{n+1}(x) := p_n(x) + (x - p_n(x)^2)/2 \ (n \geq 0).$$

If $q_n(x) := p_n(x^2)$ then the $\{q_n\}$ converges uniformly to $f(x) = |x|$ on $[-1,1]$.

Proof. Let $y = 1 - x^2$ ($x \in [-1,1]$) then $f(x) = \sqrt{1-y}$ for $y \in [0,1]$. Thus it suffices to check that $\{p_n\}$ converges uniformly to $\sqrt{x}$ on $[0,1]$. One may verify inductively that $0 \leq p_n(x) \leq \sqrt{x}$ for $x \in [0,1]$. It follows that $p_n(x) \leq p_{n+1}(x)$ for all $n \geq 0$ and all $x \in [0,1]$. In particular, $\{p_n(x)\}$ converges pointwise to $\sqrt{x}$. Now apply Theorem 2.2 to $f_n := -p_n$. □

Exercise 2.4 : Show that the function $g : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} g(x) &= 0 \text{ for } x \leq 0, \\ &= x \text{ for } x \geq 0. \end{aligned}$$

Show that for any positive number α, g can be uniformly approximated by polynomials on $[-\alpha, \alpha]$.

Hint. Note that $g(x) = \frac{1}{2}(x + |x|)$.

Lebesgue's Proof of Weierstrass' Theorem. Since f is uniformly continuous on $[a,b]$, there exists a positive integer N such that $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < 1/N$. For $x_i := a + (b-a)(i/N)$ $(i = 0, \cdots, N)$, consider the function $h(x)$ with graph a polygon of vertices at

$$(a, f(a)), (x_1, f(x_1)), \cdots, (x_{n-1}, f(x_{n-1})), (b, f(b)).$$

Check that $\|f - g\|_\infty \leq \epsilon/2$. On the other hand,

$$h(x) = f(a) + \sum_{i=0}^{N-1} c_i g(x - x_i) \ (x \in [a,b])$$

for some scalars $c_0, \cdots, c_{N-1}$. The result now follows from Exercise 3.2. □

3. Bernstein's Theorem

For $f \in C[0,1]$, define the nth Bernstein polynomial $B_n(f)$ by

$$B_n(f)(x) := \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \ (x \in [0,1], n \geq 1).$$

Remark 3.1 : $B_n(1) = 1$ (Probability distribution) and $B_n(nt) = nx$ (Mean of the distribution), and $B_n(n^2t^2) = n(n-1)x^2 + nx$ (Variance of the distribution).

Exercise 3.2 : If $g_n(x) := \sum_{k=0}^{n}(x-k/n)^2\binom{n}{k}x^k(1-x)^{n-k}$ $(x \in [0,1])$ then verify that $\|g_n\|_\infty \leq 1/4n$.

Hint. Use Remark 3.1.

Theorem 3.3. *If $f \in C[0,1]$ then $\|B_n f - f\|_\infty \to 0$ as $n \to \infty$.*

Remark 3.4 : Since $\phi(x) = (1-x)a + xb$ is a homeomorphism from $[0,1]$ onto $[a,b]$, Weierstrass' Theorem follows from Bernstein's theorem.

The proof of Bernstein's Theorem depends on the following variant of Bernoulli's law of large numbers due to Chebyshev.

Lemma 3.5. *Given $\delta > 0$ and $x \in [0,1]$, let F denote the set*

$$F := \{k \in \mathbb{N} : 0 \leq k \leq n \text{ such that } |k/n - x| \geq \delta\}.$$

Then, for every $x \in [0,1]$,

$$\sum_{k \in F}\binom{n}{k}x^k(1-x)^{n-k} \leq \frac{1}{4n\delta^2}.$$

Proof. Note that for any $x \in [0,1]$ and $k \in F$, $\frac{|k/n-x|^2}{\delta^2} \geq 1$. It follows that

$$\begin{aligned}\sum_{k \in F}\binom{n}{k}x^k(1-x)^{n-k} &\leq \frac{1}{\delta^2}\sum_{k \in F}(k/n-x)^2\binom{n}{k}x^k(1-x)^{n-k} \\ &\leq \frac{1}{\delta^2}\sum_{k=0}^{n}(k/n-x)^2\binom{n}{k}x^k(1-x)^{n-k}.\end{aligned}$$

The desired estimate follows from Exercise 3.2. □

Proof of Bernstein's Theorem. Since f is uniformly continuous, there exists an integer $N \geq 1$ such that $|f(x) - f(y)| < \epsilon/2$ whenever $|x - y| < \delta = 1/N$. Let us estimate $\|B_n(f) - f\|_\infty$. Let F be as in Lemma 3.5, and note that

$$\begin{aligned}|B_n(f)(x) - f(x)| &= \left|\sum_{k=0}^{n}(f(k/n) - f(x))\binom{n}{k}x^k(1-x)^{n-k}\right| \\ &\leq \sum_{k \in F}|f(k/n) - f(x)|\binom{n}{k}x^k(1-x)^{n-k} \\ &+ \sum_{k \notin F}|f(k/n) - f(x)|\binom{n}{k}x^k(1-x)^{n-k} \\ &\leq \frac{2\|f\|_\infty}{4n\delta^2} + \epsilon/2,\end{aligned}$$

where we used Remark 3.1. Choose now an integer $n \geq 1$ so that $\frac{\|f\|_\infty}{n\delta^2} < \epsilon$, and note that $\|B_n(f) - f\|_\infty < \epsilon$. □

4. Applications of Weierstrass' Theorem

Exercise 4.1 : Use Weierstrass' Theorem to show that $C[a,b]$ is separable.

Exercise 4.2 : For $0 \leq x < y \leq 1$, consider the indicator function $\chi_{[x,y]} : [0,1] \to \mathbb{R}$. Show that there exists a sequence $\{p_n\}$ of polynomials such that

$$\int_0^1 |p_n(x) - \chi_{[x,y]}(x)| dx \to 0 \text{ as } n \to \infty.$$

Conclude that finite linear combination of indicator functions of subintervals of $[0,1]$ can be approximated uniformly by polynomials.

Hint. Approximate $\chi_{[x,y]}$ by continuous functions in the L^1 norm.

Exercise 4.3 : Let $f \in C[a,b]$ be such that $\int_a^b t^n f(t) dt = 0$ for all non-negative integers n. Show that $f(t) = 0$ for every $t \in [a,b]$.

Hint. Get a sequence $\{p_n\}$ such that $\|p_n - f\|_\infty \to 0$, and apply Theorem 1.4.

Exercise 4.4 : Let $f : [a,b] \to \mathbb{R}$ be a continuouly differentiable function. Show that there exists a sequence $\{r_n\}$ of polynomials such that

$$\|r_n - f\|_\infty \to 0 \text{ and } \|r_n' - f'\|_\infty \to 0 \text{ as } n \to \infty.$$

Conclude that $C^1[a,b]$ is separable with norm $\|f\| := \|f\|_\infty + \|f'\|_\infty$.

Hint. Let $g(x) = f(x) - f(a)$ and note that $g' = f'$. Find a sequence $\{q_n\}$ of polynomials such that $\|q_n - g'\|_\infty \to 0$. Set $p_n(x) := \int_a^x q_n(t)dt$. Note that $p_n' = q_n$, and hence $\|p_n' - g'\|_\infty \to 0$. Also,

$$|p_n(x) - g(x)| = \left| \int_a^x q_n(t)dt - \int_a^x g'(t)dt \right| \leq (b-a)\|q_n - g'\|_\infty.$$

Let $r_n(x) := p_n(x) + f(a)$.

Exercise 4.5 : Let $f : [a,b] \to \mathbb{R}$ be a Riemann-integrable function. Show that there exists a sequence $\{p_n\}$ of polynomials such that

$$\int_a^b |p_n(x) - f(x)|^2 dx \to 0 \text{ as } n \to \infty.$$

Hint. Approximate any Riemann-integrable function by a continuous function, and then apply Weierstrass Theorem.

Exercise 4.6 : Let K be a compact subset of $\mathbb{R}^n$ and let $\mathscr{A}$ be a subalgebra of $C(K)$. If $f \in \overline{\mathscr{A}}$ then show that $|f| \in \overline{\mathscr{A}}$, where $|f|(x) = |f(x)|$.

Hint. Let $a := \sup_{x \in K} |f(x)|$ and let $\epsilon > 0$. Let $p : [-a,a] \to \mathbb{R}$ be a polynomial such that $\|p - |t|\|_\infty < \epsilon$. Since $p(f) \in \overline{\mathscr{A}}$, we must have

$$\sup_{x \in K} |p(f)(x) - |f(x)|| < \epsilon.$$

5. Stone's Theorem and its Consequences

Recall that the Weierstrass' Theorem says that $\overline{\mathcal{P}} = C[a,b]$. It is interesting to note that $\mathcal{P}$ is an algebra which enjoys the following properties: If $x \neq y \in [a,b]$ then trivially $id(x) \neq id(y)$, and if $x \in [a,b]$ then $1(x) \neq 0$, where $id(x) = x$ and $1(x) = 1$. It turns out these properties of subalgebras $\mathscr{A}$ of $C[a,b]$ ensure that

$\overline{\mathscr{A}} = C[a,b]$. This remarkable fact was first discovered by Stone in much more generality.

Theorem 5.1. *Let K be a compact subset of $\mathbb{K}^n$, where $\mathbb{K}$ is either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathscr{A}$ be an algebra of continuous functions $f : K \to \mathbb{R}$ with the following properties:*

(1) *If $x \neq y \in K$, then there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$.*
(2) *For every $x \in K$, there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$.*

Then $\overline{\mathscr{A}} = C_{\mathbb{R}}(K)$.

Before we present a proof of Stone's Theorem, it is advisable to understand it through its wide range of applications/consequences.

Exercise 5.2 : For positive integer n, consider the polynomial subalgebra $\mathscr{A}$ of $C_{\mathbb{R}}[0,1]$ generated by 1 and x^n. Show that $\overline{\mathscr{A}} = C_{\mathbb{R}}[0,1]$.

Exercise 5.3 : Show that the algebra $\mathcal{P}$ of analytic polynomials on the closed unit $\overline{\mathbb{D}}$ satisfies assumptions of Theorem 5.1, however, $\overline{\mathcal{P}} \subsetneq C(\overline{\mathbb{D}})$. What goes wrong ?

Hint. $\overline{z}$ belongs to $C(\mathbb{D})$ but $\overline{z} \notin \overline{\mathcal{P}}$.

Corollary 5.4. *Let K be a compact subset of $\mathbb{K}^n$. Let $\mathscr{A}$ be an algebra of continuous functions $f : K \to \mathbb{C}$ with the following properties:*

(1) *If $x \neq y \in K$, then there exists $f \in \mathscr{A}$ such that $f(x) \neq f(y)$.*
(2) *For every $x \in K$, there exists $f \in \mathscr{A}$ such that $f(x) \neq 0$.*
(3) *For every $f \in \mathscr{A}$, $\overline{f} \in \mathscr{A}$, where $\overline{f}(x) = \overline{f(x)}$.*

Then $\overline{\mathscr{A}} = C(K)$.

Proof. Let $\mathscr{A}_{\mathbb{R}}$ denote the algebra functions in $\mathscr{A}$ which are real-valued, and let $\Re\mathscr{A}$ be the set of real parts of functions in $\mathscr{A}$. Since $\Re f := (f + \overline{f})/2 \in \mathscr{A}$ for every $f \in \mathscr{A}$, $\Re\mathscr{A} = \mathscr{A}_{\mathbb{R}}$. Check that $\mathscr{A}_{\mathbb{R}}$ satisfies (1) and (2) of Theorem 5.1. Thus every real-valued continuous function on K can be uniformly approximated by polynomials. Since $\overline{\mathscr{A}}$ is an algebra, $\overline{\mathscr{A}} = C(K)$. □

Exercise 5.5 : Show that the trigonometric polynomials are dense in $C(\mathbb{T})$, where $\mathbb{T}$ stands for the unit circle.

Exercise 5.6 : Let $f : \mathbb{T} \to \mathbb{R}$ be a continuous function such that $\hat{f}(n) := \int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta = 0$ for all $n \in \mathbb{Z}$. Show that f is identically 0.

Hint. Use the last exercise.

Exercise 5.7 : For an integer $n \geq 1$, consider the polynomial subalgebra $\mathscr{A}$ of $C[a,b]$ generated by x. If $0 \notin [a,b]$ then show that $\overline{\mathscr{A}} = C[a,b]$.

Corollary 5.8. *Let $\mathscr{A}$ be as in Theorem 5.1 except that (2) is not given. If $\overline{f} \in \mathscr{A}$ whenever $f \in \mathscr{A}$, then there exists $a \in K$ such that $\overline{\mathscr{A}} = \{f \in C(K) : f(a) = 0\}$.*

Proof. If (2) does not hold true then there exists $a \in K$ such that $f(a) = 0$ for every $f \in \mathscr{A}$, that is, $\mathscr{A} \subseteq \{f \in C(K) : f(a) = 0\}$. Since $\{f \in C(K) : f(a) = 0\}$ is closed in $C(K)$, we must have $\overline{\mathscr{A}} \subseteq \{f \in C(K) : f(a) = 0\}$. Now let $f \in C(K)$ be such that $f(a) = 0$. By Stone's Theorem, the algebra $\mathscr{B}$ generated by $\mathscr{A}$ and 1 is dense in $C(K)$. Thus there exists a sequence $\{f_n\}$ in $\mathscr{B}$ such that $\|f_n - f\|_\infty \to 0$

as $n \to \infty$. Note that $f_n = g_n + \alpha_n$ for $g_n \in \mathscr{A}$. Also, $f_n(a) \to f(a) = 0$, and hence $\alpha_n \to 0$. It follows that $\|g_n - f\|_\infty \leq \|f_n - f\|_\infty + |\alpha_n| \to 0$ as $n \to \infty$. □

Exercise 5.9 : Let $\mathscr{A} := \{x^2 p(x) : p \text{ is a polynomial}\}$. Show that

$$\overline{\mathscr{A}} = \{f \in C[0,1] : f(0) = 0\}.$$

Exercise 5.10 : Show that the subalgebra $\mathscr{A} := \{p(x^2) : p \text{ is a polynomial}\}$ is not dense in $C[-1,1]$.

Hint. If possible then there exists sequence $\{p_n\}$ of polynomials such that $\|p_n(x^2) - x\|_\infty \to 0$ as $n \to \infty$. In particular,

$$2\int_0^1 p_n(x^2)dx = \int_{-1}^1 p_n(x^2)dx \to \int_{-1}^1 xdx = 0.$$

However, $\int_0^1 p_n(x^2)dx \to \int_0^1 xdx = 1$, which is absurd.

Exercise 5.11 : Show that for any $a > 0$, $\mathscr{A} := \{p(|x|) : p \text{ is a polynomial}\}$ is not dense in $C[-a,a]$.

Corollary 5.12. *Let $f \in C(K)$ be such that $f(x) > 0$ for all $x \in K$. Let $\mathscr{A}$ be a subalgebra of $C(K)$ that contains f. If f is injective then $\overline{\mathscr{A}} = C(K)$.*

Proof. Note that $f \in \mathscr{A}$ satisfies (1) and (2) of Stone's Theorem. □

Remark 5.13 : Let $X = [a,b]$. Consider the algebra $\mathscr{A}$ of functions of the form $p(f)$, where p is a polynomial such that $p(0) = 0$. The last corollary is applicable to $f(x) = x^\alpha$ for $\alpha > 0$, $f(x) = e^x$, $f(x) = 1/x$ if $0 \notin [a,b]$.

6. A Proof of Stone's Theorem

We start with some elementary properties of subalgebras of $C_{\mathbb{R}}(K)$.

Exercise 6.1 : Let $\mathscr{A}$ be a subalgebra of $C_{\mathbb{R}}(K)$. If $f_1, \cdots, f_k \in \overline{\mathscr{A}}$ then so are $\max\{f_1, \cdots, f_k\}$ and $\min\{f_1, \cdots, f_k\}$.

Hint. $\max\{f_1, f_2\} = \frac{f_1+f_2}{2} + \frac{|f_1-f_2|}{2} \in \overline{\mathscr{A}}$ by Exercise 4.6. Now apply finite induction.

Lemma 6.2. *Under the assumptions of Theorem 5.1, for distinct points x_1, x_2 of K and (real) scalars c_1, c_2, there exists $f \in \mathscr{A}$ such that $f(x_1) = c_1$ and $f(x_2) = c_2$.*

Proof. By assumptions there exist $g, h, k \in \mathscr{A}$ such that $g(x_1) \neq g(x_2)$ and $h(x_1) \neq 0 \neq h(x_2)$. Then $u := (g - g(x_1))k$ and $v := (g - g(x_2)h$ belong to $\mathscr{A}$, and $f := \frac{c_1 v}{v(x_1)} + \frac{c_2 u}{u(x_2)}$ does the job. □

Proof of Stone's Theorem. Fix $x \in K$ and $\epsilon > 0$. For x and every $y \in K$, by the preceding lemma, there exists $f_y \mathscr{A}$ such that $f_y(x) = f(x)$ and $f_y(y) = f(y)$. Note that $f_y - f$ is continuous such that $(f_y - f)(x) = 0$. Thus there exists an open neighbourhood U_y of y such that $f_y(t) - f(t) > -\epsilon$ for every $t \in U_y$. Now $\{U_y\}$ is an open cover of K and K is compact. Thus, for some $y_1, \cdots, y_k \in K$, $K \subseteq \cup_{i=1}^k U_{y_i}$. Also, for $g_x := \max\{f_{y_1}, \cdots, f_{y_k}\} \in \mathscr{A}$ (Exercise 6.1) and

$$g_x(t) > f(t) - \epsilon \text{ for every } t \in K. \tag{6.1}$$

Now we vary x. Note that $g_x(x) = \max\{f_{y_1}(x), \cdots, f_{y_k}(x)\} = f(x)$. Hence, by the continuity of g_x, there exists an open neighbourhood V_x of x such that $g_x(t) - f(t) < \epsilon$ for every $t \in V_x$. Now $\{V_x\}$ is an open cover of K and K is compact. Thus, for some $x_1, \cdots, x_l \in K$, $K \subseteq \cup_{i=1}^{l} V_{x_i}$. Also, for $h := \min\{g_{x_1}, \cdots, g_{x_l}\} \in \mathscr{A}$ and

$$h(t) < f(t) + \epsilon \text{ for every } t \in K.$$

Also, by (6.1), $h(t) > f(t) + \epsilon$ for every $t \in K$. That is, $\|h - f\|_\infty < \epsilon$. □

7. Müntz-Szász Theorem

In this section, we see another remarkable generalization of Weierstrass' Theorem, which relates the topological property of density of polynomials with divergence of certain Harmonic series of positive scalars.

Theorem 7.1. *Suppose $0 < \lambda_1 < \lambda_2 < \cdots$. Then the closed linear span of $\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ equals $C[0,1]$ if and only if $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$.*

Remark 7.2 : The choice $\lambda_k = k$ gives the conclusion of Weierstrass' Theorem. Note that for any integer $l \geq 1$, the closed linear span of $\{1, t^{\lambda_l}, t^{\lambda_{l+1}}, \cdots\}$ equals $C[0,1]$ provided $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$.

Corollary 7.3. *The closed linear span of the constant function 1 and monomials $\{t^p : p$ is a prime number$\}$ equals $C[0,1]$.*

Proof. This follows from the fact that $\sum_p 1/p = \infty$, where the sum is taken over the set of prime numbers. □

Corollary 7.4. *Suppose $0 < \lambda_1 < \lambda_2 < \cdots$. Then the closed linear span of $\{t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ equals $\{f \in C[0,1] : f(0) = 0\}$ provided $\sum_{k=1}^{\infty} \frac{1}{\lambda_k} = \infty$.*

Proof. Clearly, the closed linear span of $\{t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ is contained in $\{f \in C[0,1] : f(0) = 0\}$. Let $f \in C[0,1]$ be such that $f(0) = 0$. Then, by Müntz-Szász Theorem, there exists a sequence $\{p_n\}$ in the closed linear span of $\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ such that $\|p_n - f\|_\infty \to 0$ as $n \to \infty$. But then $p_n(0) \to f(0) = 0$, and hence $q_n(t) := p_n(t) - p_n(0)$ in the linear span of $\{t^{\lambda_1}, t^{\lambda_2}, \cdots\}$ converges uniformly to f. □

We will only sketch the outline of the proof of the sufficiency part. Let us first collect all the ingredients required for a proof of Müntz-Szász Theorem.

Lemma 7.5. *Let $\mathcal{M}$ be a closed subspace of $C[0,1]$. If $M \neq C[0,1]$ then there exists a non-zero bounded linear map $\phi : C[0,1] \to \mathbb{C}$ such that $\phi(f) = 0$ for every $f \in M$.*

Proof. Let $f \in C[0,1] \setminus M$ and let $Y = M + \{\lambda f : \lambda \in \mathbb{C}\}$. Define $\psi(g + \alpha f) = \alpha d_\infty(f, M)$ for $g \in M$ and $\alpha \in \mathbb{C}$. Clearly, $\psi = 0$ on M and $\phi(f) = d_\infty(f, M) > 0$. The desired conclusion now follows from Hanh-Banach Extension Theorem. □

Lemma 7.6. *Every bounded linear map $\phi : C[0,1] \to \mathbb{C}$ is given by $\phi(f) = \int_{[0,1]} f(t) d\mu(t)$ for a complex Borel measure μ on $[0,1]$.*

Lemma 7.7. *If f is bounded holomorphic function defined on the open unit disc $\mathbb{D}$ with zeros $\alpha_1, \alpha_2, \cdots$ then $\sum_{n=1}^{\infty}(1 - |\alpha_n|) = \infty$ implies that $f(z) = 0$ for all $z \in \mathbb{D}$.*

Proof of Müntz-Szász Theorem. Suppose there exists a bounded linear map $\phi : C[0,1] \to \mathbb{C}$ such that $\phi(f) = 0$ for every $f \in M$. By Lemma 7.6, for some complex Borel measure μ on $[0,1]$, $\phi(t^{\lambda_k}) = \int_{[0,1]} t^{\lambda_k} d\mu(t) = 0$ for every positive integer k. In view of Lemma 7.5, it suffices to check that $\phi = 0$. By Weierstrass' Theorem, it suffices to check that $\phi(t^k) = \int_{[0,1]} t^k d\mu(t) = 0$ for every positive integer k.

We now define a function g on the open right half plane $\mathbb{H}$ by setting

$$g(z) = \int_{(0,1]} e^{z \log t} d\mu(t) \ (z \in \mathbb{H}).$$

Since $|e^{z \log t}| = e^{\log t \Re z} \leq 1$ for all $z \in \mathbb{H}$ and $t \in (0,1]$, by the dominated convergence theorem, g is continuous on $\mathbb{H}$. By theorems of Fubini and Morera, it is easily seen that g is holomorphic in $\mathbb{H}$ such that $g(\lambda_k) = 0$ for every integer $k \geq 1$.

Recall that the zeros of any non-zero holomorphic function on a connected domain are isolated. If $\{\lambda_k\}$ is a bounded sequence then by the Identity Theorem, $g = 0$. So suppose that $\lambda_k \to \infty$ as $k \to \infty$. Define $h : \mathbb{D} \to \mathbb{C}$ by $h(z) = g\left(\frac{1+z}{1-z}\right)$, and note that h is a bounded holomorphic function such that $g\left(\frac{\lambda_k - 1}{\lambda_k + 1}\right) = 0$ for every integer $k \geq 1$. Also, since $\sum_{k=0}^{\infty} \frac{1}{\lambda_k} = \infty$, $\sum_{n=1}^{\infty} \left(1 - \frac{\lambda_k - 1}{\lambda_k + 1}\right) = 2 \sum_{n=1}^{\infty} \frac{1}{\lambda_k + 1} = \infty$. By Lemma 7.7, we must have $h = 0$, and hence $g = 0$. It follows that $g(k) = 0$, and therefore $\phi(t^k) = 0$ for every integer $k \geq 1$. □

We separate out one important technique employed in the preceding proof, which can be used to prove many approximation results (e.g. Stone's Theorem).

Exercise 7.8 : Let S be a subset of $C(K)$. Show that $\overline{S} = C(K)$ if and only if for any complex Borel measure, $\int_K f(t) d\mu(t) = 0$ for every $f \in S$ implies $\int_K f(t) d\mu(t) = 0$ for every $f \in C(K)$.

Hint. Use Lemmas 7.5 and 7.6.

8. Nowhere Differentiable Continuous Function

By Weierstrass' Theorem, any continuous function on $[0,1]$ is a uniform limit of infinitely differentiable functions. It is quite striking that there exists a nowhere differentiable continuous function on $[0,1]$. In particular, uniform limit of infinitely real differentiable functions could be nowhere differentiable.

Theorem 8.1. *Let $\phi(x) = |x|$ for $x \in [-1,1]$, which is extended periodically (with period 2) to $\mathbb{R}$ by setting $\phi(x+2) = \phi(x)$ $(x \in \mathbb{R})$. Then the function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x)$$

is continuous. However, for every $x \in \mathbb{R}$, there exists a sequence $\{\delta_m\}$ converging to 0 such that

$$|(f(x+\delta_m) - f(x))/\delta_m| \to \infty \text{ as } m \to \infty.$$

Proof. Clearly, ϕ is a continuous function such that $0 \leq \phi(x) \leq 1$ $(x \in \mathbb{R})$. Also,

$$\left| f(x) - \sum_{n=0}^{k} \left(\frac{3}{4}\right)^n \phi(4^n x) \right| \leq \sum_{n=k+1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x) \leq \sum_{n=k+1}^{\infty} \left(\frac{3}{4}\right)^n \to 0.$$

Thus, $\sum_{n=0}^{k}\left(\frac{3}{4}\right)^n\phi(4^n x)$ converges uniformly to f on $\mathbb{R}$, and hence f is continuous.

Let $x \in \mathbb{R}$ and let m be a positive integer. Set $\delta_m = \pm\frac{1}{2}4^{-m}$, where the sign is so chosen that no integer lies between $4^m x$ and $4^m x + \delta_m$. Define

$$\gamma_n := (\phi(4^n(x+\delta_m)) - \phi(4^n x))/\delta_m.$$

If $n > m$ then $\phi(4^n(x+\delta_m)) = \phi(4^n x + 4^{n-m}/2) = \phi(4^n x)$, and hence $\gamma_n = 0$. Note that $|\gamma_m| = 4^m$. When $0 \leq n < m$,

$$|\gamma_n| = \frac{||4^n(x+\delta_m)| - |4^n x||}{|\delta_m|} \leq \frac{|4^n\delta_m|}{|\delta_m|} = 4^n.$$

Now we complete the argument. Note that

$$\begin{aligned} |(f(x+\delta_m) - f(x))/\delta_m| &= \left|\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\gamma_n\right| = \left|\sum_{n=0}^{m}\left(\frac{3}{4}\right)^n\gamma_n\right| = \left|\sum_{n=0}^{m-1}\left(\frac{3}{4}\right)^n\gamma_n + 3^m\right| \\ &\geq 3^m - \sum_{n=0}^{m-1} 3^n = \frac{1}{2}(3^m+1), \end{aligned}$$

which blows up as $m \to \infty$. □

References

[1] R. Boas, *A Primer of Real Functions*, The mathematical association of America, 1981.
[2] J. Burkill, *Lectures on Approximation by Polynomials*, Lecture Notes, Tata Institute of Fundamental Research, Bombay, 1959.
[3] N. Carothers, *Real Analysis*, Cambridge Univ. Press, 2000.
[4] W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, 1976.
[5] W. Rudin, *Real and Complex Analysis*, McGraw-Hill Book Co. New York, 1987.