SPECTRAL THEOREM FOR SYMMETRIC OPERATORS WITH COMPACT RESOLVENT

Abstract. These are the letcure notes prepared for the workshop on "Functional Analysis and Operator Algebras" to be held at NIT-Karnataka, during 2-7 June 2014.

1. Compact Operators

Let X, Y be Banach spaces. A linear operator T from X into Y is said to be *compact* if for every bounded sequence $\{x_n\}$ in X, $\{Tx_n\}$ has a convergent subsequence.

Lemma 1.1. *Compact operators form a subspace of $B(X, Y)$.*

Proof. Clearly, a scalar multiple of a compact operator is compact. Suppose S, T are compact operators and let $\{x_n\}$ be a bounded sequence. Since S is compact, there exists a convergent subsequence $\{Sx_{n_k}\}$ of $\{Sx_n\}$. Since $\{x_{n_k}\}$ is bounded, by the compactness of T, we get a convergent subsequence $\{Tx_{n_{k_l}}\}$ of $\{Tx_n\}$. Thus we obtain a convergent subsequence $\{(S+T)x_{n_{k_l}}\}$ of $\{(S+T)x_n\}$, and hence $S+T$ is compact. □

Remark 1.2 : If S is compact and T is bounded then ST, TS are compact. Thus compact operators actually form an ideal of $B(X, Y)$.

Exercise 1.3 : Show that every finite-rank bounded linear map $T : X \to Y$ is compact.

Hint. Note that there exist vectors $y_1, \cdots, y_k \in Y$ such that $Tx = \sum_{i=1}^{k} \lambda_i(x) y_i$, where $\lambda_1, \cdots, \lambda_k$ are linear functionals on X. If $\{x_n\}$ is bounded then so is each $\{\lambda_i(x_n)\}$. Now apply Heine-Borel Theorem.

The following proposition provides a way to generate examples of compact operators which are not necessarily of finite rank.

Proposition 1.4. *Compact operators form a closed subspace of $B(X, Y)$.*

Proof. In view of Lemma 1.1, it suffices to check that the space of compact operators is sequentially closed. Given $\epsilon > 0$, find an integer $N \geq 1$ such that $\|T - T_N\| < \epsilon$. Now find a convergent sequence $\{T_N x_{n_k}\}$ using compactness of T_N. Check that $\{Tx_{n_k}\}$ is a Cauchy sequence. Since $B(X, Y)$ is complete, $\{Tx_{n_k}\}$ is convergent as required. □

Exercise 1.5 : Let $a := \{a_n\}$ be a bounded sequence. Show that the diagonal operator D_a on l^2 with diagonal entries $a_1, a_2, \cdots$ is compact iff $\{a_n\} \in c_0$.

Hint. If D_b is a diagonal operator with diagonal entries $b_1, b_2, \cdots$, then $\|D_b\| = \sup_n |b_n|$. The sufficient part follows from $\lim_{n\to\infty} \|D_{a,n} - D_a\| = 0$, where $D_{a,n}$ is the diagonal operator with diagonal entries $a_1, \cdots, a_n, 0, 0, \cdots$. To see the necessary part, WLOG, assume that $\{a_n\}$ is bounded from below, and note that $\|D_a e_n - D_a e_m\|^2 = a_n^2 + a_m^2 > c > 0$ for a scalar c.

Example 1.6 : Let $a := \{a_n\} \in c_0$ and let e_n denote the sequence in l^2 with nth entry 1 and all other entries equal to 0. Define $U_a e_n = a_n e_{n+1}$, and extend U_a linearly and continuously to l^2. Then U_a is compact. This follows from Remark 1.2 and the preceding exercise in view of the decomposition $U_a = UD_a$, where $U \in B(l^2)$ is governed by $Ue_n = e_{n+1}$.

The integral operators provide the most interesting examples of compact operators, which are of great importance as far as applications are concerned.

Exercise 1.7 : Consider $(Tf)(x) = \int_0^x f(y)dy$ $(x \in [0,1])$ and

$$(T_n f)(x) = \sum_{k=0}^{n-1} \chi_{[k/n,(k+1)/n)}(x) \int_0^{k/n} f(y)dy \ (x \in [0,1]).$$

Verify that $T \in B(L^2(0,1), L^2(0,1))$ satisfies $\|T - T_n\| \leq n^{-1/2}$. Conclude that T is compact.

Exercise 1.8 : For $t \in C_0(\mathbb{R}^{n+m})$ (continuous with compact support), define a linear operator $T_t : L^2(\mathbb{R}^m) \to L^2(\mathbb{R}^n)$ by

$$(T_t u)(x) = \int_{\mathbb{R}^m} t(x,y)u(y)dy, \ u \in L^2(\mathbb{R}^m), x \in \mathbb{R}^n.$$

Verify the following:

(1) T_t is a bounded, linear operator.

(2) Let $\{u_k\}$ be a bounded sequence with bound M. Then

$$|(T_t u_k)(x)| \leq \|t\|_\infty (2c)^{m/2} M,$$

where c is chosen so that $t(x,y) = 0$ if $\|x\|_2 > c$ or $\|y\|_2 > c$.

(3) Given $\epsilon > 0$, there exists $\delta \in (0,1)$ such that

$$|(T_t u_k)(x) - (T_t u_k)(x')| \leq \epsilon (2c)^{m/2} M$$

whenever $\|x - x'\|_2 < \delta$.

(4) T_t is a compact operator.

Hint. Recall the Arzela-Ascoli Theorem.

Exercise 1.9 : The conclusion of the last exercise holds true for any measurable function t in $L^2(\mathbb{R}^{n+m})$.

Hint. Since $C_0(\mathbb{R}^{n+m})$ is dense in $L^2(\mathbb{R}^{n+m})$, there exists $t_k \in C_0(\mathbb{R}^{n+m})$ such that $\|t_k - t\|_{L^2(\mathbb{R}^{n+m})} \to 0$ as $k \to \infty$. If T_{t_k} is the integral operator associated with the kernel t_k then $\|T_{t_k} - T_t\| \leq \|t_k - t\|_{L^2(\mathbb{R}^{n+m})}$.

Exercise 1.10 : Consider the linear operator $A : L^2(0,\infty) \to L^2(0,\infty)$ given by

$$(Af)(x) = \frac{1}{x}\int_0^x f(t)dt \ (x \in (0,\infty)).$$

Show that A is bounded linear but not compact.

Hint. Consider $f_n(t) = n$ if $0 < t \leq 1/n^2$, and 0 otherwise. Note that $\|Af_n\| \geq 1$ for all n. However, $\langle Af_n, g\rangle = \langle f_n, A^*g\rangle \to 0$ for all $g \in L^2(0,\infty)$.

One of the objectives of these notes is to give an elementary proof of spectral theorem for compact normal operators. This result is usually attributed to Hilbert and Schmidt. Although our treatment is along the lines of [1, Chapter 2], for simplicity, we will work only with complex infinite-dimensional separable Hilbert spaces. The separability assumption makes life simple in many instances (see, for example, Exercise 2.6).

Exercise 1.11 : Show that $L^p(0,1)$ is separable for $1 \leq p < \infty$.

Recall that $C[0,1]$ is dense in $L^p(0,1)$ for $1 \leq p < \infty$. By the Weierstrass Approximation Theorem, continuous functions can be approximated in $L^p(0,1)$ by polynomials, and hence by polynomials with rational coefficients.

2. Spectral Theorem for Compact Normal Operators

We begin with some elementary properties of compact operators.

Proposition 2.1. *Let T be a compact operator with closed range $ran(T)$. Then the closed unit ball in $ran(T)$ is sequentially compact. In particular, $ran(T)$ is finite-dimensional.*

Proof. Let $y_n \in \text{ran}(T)$ be such that $\|y_n\| \leq 1$. Consider the bounded linear surjection $S : X \to \text{ran}(T)$ given by $Sx = Tx$ for $x \in X$. By the Open Mapping Theorem, there exists $x_n \in X$ such that $y_n = Tx_n$ for some bounded sequence $\{x_n\}$. Since T is compact, $\{y_n\}$ has a convergent subsequence, and the unit ball in $\text{ran}(T)$ is sequentially compact.

The remaining part follows from the fact that a Banach space X is finite-dimensional iff the unit ball in X is sequentially compact. □

Corollary 2.2. *Let μ be a non-zero complex number and $T : X \to Y$ be compact. Then* $\ker(T - \mu)$ *is finite-dimensional.*

Proof. If M is a closed subspace of X such that $TM \subseteq M$ then $T|_M$ is compact. Clearly, $T|_{\ker(T-\mu)}$ is a compact operator with closed range $\ker(T-\mu)$. Now apply the last result. □

Exercise 2.3 : Let $T : X \to X$ be a compact operator. Show that either X is finite dimensional or T is non-surjective.

Definition 2.4 : A bounded linear operator on a Hilbert space H is *normal* if $T^*T = TT^*$. We say that T is *self-adjoint* if $T^* = T$.

The operator of multiplication by $\phi \in L^\infty(0,1)$ on $L^2(0,1)$ is normal. In fact, $M_\phi^* = M_{\bar{\phi}}^*$, where $\bar{\phi}(z) = \overline{\phi(z)}$. In particular, M_ϕ is self-adjoint iff ϕ is real-valued.

Exercise 2.5 : If N is normal then show that so is $N - \lambda$ for any scalar λ. Use this to deduce that the eigenspaces corresponding to distinct eigenvalues are orthogonal.

Hint. First part follows from the definition. Check that M is normal iff $\|Mh\| = \|M^*h\|$ for every $h \in H$. Letting $M := N - \lambda$, we obtain $Nx = \lambda x$ iff $N^*x = \overline{\lambda}x$.Finally, if $Nx = \lambda x$ and $Ny = \mu y$ then $\lambda\langle x, y\rangle = \langle Nx, y\rangle = \langle x, N^*y\rangle = \mu\langle x, y\rangle$.

Exercise 2.6 : Let X be a separable inner-product space. Show that any orthonormal subset of X is countable. Conclude that the point-spectrum (that is, the set of eigenvalues) of any normal operator on X is countable.

Hint. Suppose $\{x_\alpha\}$ is orthonormal. Then $\{\mathbb{B}(x_\alpha, 1/\sqrt{2})\}$ is a collection of disjoint open balls in H. Now if $\{y_n\}$ is countable dense subset of H then each $B(x_\alpha, 1/\sqrt{2})$ must contain at least one y_n.

Exercise 2.7 : Consider the linear operator T on $L^2[0,1]$:

$$(Tf)(x) = (1-x)\int_0^x yf(y)dy + x\int_x^1 (1-y)f(y)dy \ (x \in [0,1]).$$

Verify the following:

(1) $T \in B(L^2(0,1), L^2(0,1))$ is compact and self-adjoint.
(2) If $Tf = \lambda f$ then for some integer $n \geq 1$, $f(x) = c\sin(n\pi x)$ for some scalar c and $\lambda = 1/n^2\pi^2$.

Hint. For (1), use Exercise 1.7 and Remark 1.2. For (2), note that $(Tf)(0) = 0 = (Tf)(1)$, and $\lambda f'' = (Tf)'' = -f$ a.e.

Exercise 2.8 : Show that for any normal operator N on a Hilbert space, norm $\|N\|$ of N is equal to the spectral radius $r(N)$ of N.

Hint. Note that $\|N\|^2 = \|N^*N\| = \|(N^*N)^2\|^{1/2} = \|N^2\|$. Since any positive integral power of a normal operator is normal, one may check by induction that $\|N^{2^k}\| = \|N\|^{2^k}$ for every positive integer k. Use now the formula: $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$.

Exercise 2.9 : A normal operator $N \in B(H)$ is invertible iff there exists a positive scalar α such that $\|Nx\| \geq \alpha\|x\|$ for every $x \in H$.

Hint. Suppose $\|Nx\| \geq \alpha\|x\|$ for every H and for some $\alpha > 0$. Then $\ker(N) = \{0\}$. If $y \in \overline{\text{ran}(N)}$ then for some sequence $\{x_n\}$, $Tx_n \to y$. Check that $\{x_n\}$ is Cauchy, and hence converges to some x. This shows that $y = Tx$, so that $\operatorname{ran}(T)$ is closed. On the other hand, since $\ker(N) = \ker(N^*)$, $\operatorname{ran}(N)$ is dense in H.

Lemma 2.10. *Let λ be a non-zero complex number and let T be a compact normal operator on a separable Hilbert space. If $T - \lambda$ is not invertible then $T - \lambda$ is not one-to-one. In particular, $\sigma(T) \setminus \{0\} = \sigma_p(T) \setminus \{0\}$.*

Proof. Since $T-\lambda$ is not invertible, by the preceding exercise, there exists $x_n \in \mathcal{H}$ such that $\|x_n\| = 1$ and $\|(T-\lambda)x_n\| < 1/n$. Since T is a compact operator, there exists a subsequence $\{x_{n_l}\}$ such that $Tx_{n_l} \to y$ for some $y \in H$. It follows that $\lambda x_{n_l} \to y$, and hence $Ty = \lambda y$ with $\|y\| = |\lambda| > 0$. It follows that $\sigma(T) \setminus \{0\} \subseteq \sigma_p(T) \setminus \{0\}$. The reverse inclusion holds for any bounded linear operator. □

Remark 2.11 : The conclusion of the lemma holds for any compact operator.

Now we state and proof the main result of this section:

Theorem 2.12. *Let T be a non-zero normal operator on a complex Hilbert space H. If T is compact then there exists an orthonormal basis $\{e_n\}$ of $\ker(T)^{\perp}$ and a sequence $\{\lambda_n\}$ of non-zero complex numbers (possibly repeated) such that $Te_n = \lambda_n e_n$. Moreover, the following statements hold:*

1. *For each $n \geq 1$, $\ker(T - \lambda_n)$ is finite-dimensional.*
2. $\inf_n |\lambda_n| = 0$.
3. *$\{\lambda_n\}$ has no accumulation point except 0.*
4. *For any $\epsilon > 0$, the annulus $A(0, \epsilon, r(T)) := \{z \in \mathbb{C} : \epsilon \leq |z| \leq r(T)\}$ intersects $\sigma(T)$ in finitely many points. In particular, $\Delta_\epsilon := \{n \in \mathbb{N} : |\lambda_n| \geq \epsilon\}$ contains λ_n for finitely many values of n.*
5. *If $x \in H$, then $Tx = \sum_{n \in \sigma_p(T)\setminus\{0\}} \lambda_n \langle x, e_n\rangle e_n$ in the following sense: For any $\epsilon > 0$,*

$$\left\| Tx - \sum_{n\in\Delta_\epsilon} \lambda_n \langle x, e_n\rangle e_n \right\| < \epsilon.$$

Proof. We will prove the result for a separable Hilbert space H. In view of Exercise 2.6, we may enumerate all the eigenvalues of T as $\{\lambda_n\}$. We may choose an orthonormal set $\{e_n\}$ of corresponding eigenvectors: $Te_n = \lambda_n e_n$. Note that (1) is already obtained in Corollary 2.2. Let us see (2). Suppose $\{\lambda_n\}$ has a subsequence $\{\lambda_{n_k}\}$ which is bounded from below in modulus by $c > 0$. Then

$$\|Te_{n_k} - Te_{n_l}\|^2 = |\lambda_{n_k}|^2 + |\lambda_{n_l}|^2 \geq 2c^2.$$

Thus we have the bounded sequence $\{e_{n_k}\}$ for which $\{Te_{n_k}\}$ has no convergent subsequence. Since T is compact, this is not possible unless $\inf_n |\lambda_n| = 0$. This also shows that $\{\lambda_n\}$ has no non-zero accumulation point. Further, (4) follows from (3), Lemma 2.10, and Bolzano-Weierstrass Theorem. Indeed, since any infinite subset of the compact set $A(0, \epsilon, r(T))$ has a limit point, $A(0, \epsilon, r(T))$ contains λ_n for finitely many values of n.

To see (5), let $\epsilon > 0$. By (4), Δ_ϵ is a finite set. Consider the sequence $\{T_k\}$ of non-zero compact normal operators T_k defined by

$$T_k x := Tx - \sum_{n\in\Delta_\epsilon} \lambda_n \langle x, e_n\rangle e_n \ (x \in H).$$

We check that $\|T_k\| < \epsilon$. By Exercise 2.8, there exists $\lambda \in \sigma(T_k)$ such that $\|T_k\| = |\lambda|$. By the preceding lemma, λ must belong to $\sigma_p(T_k)$. Thus, for some $y \neq 0$, $T_k y = \lambda y$ and $T_k^* y = \overline{\lambda} y$. We now complete the proof of (5). Note that

$$\lambda y = Ty - \sum_{n\in\Delta_\epsilon} \lambda_n \langle y, e_n\rangle e_n. \tag{2.1}$$

Moreover, for any $m \in \Delta_\epsilon$, one has

$$\begin{aligned}\lambda\langle y, e_m\rangle &= \langle Ty - \sum_{n\in\Delta_\epsilon} \lambda_n\langle y, e_n\rangle e_n, e_m\rangle \\ &= \langle y, T^* e_m\rangle - \sum_{n\in\Delta_\epsilon}\langle \lambda_n\langle y, e_n\rangle e_n, e_m\rangle, \\ &= \langle y, \overline{\lambda}_m e_m\rangle - \lambda_m\langle y, e_m\rangle = 0.\end{aligned}$$

Thus y is orthogonal to $e_n \in \ker(T-\lambda_n)$ provided $n \in \Delta_\epsilon$. By (2.1), $\lambda y = Ty$, and hence $\lambda = \lambda_m$ and $y \in \ker(T - \lambda_m)$ for some $m \geq 1$. Since $y \neq 0$, $m \notin \Delta$. That is, $\|T_k\| = |\lambda| = |\lambda_m| < \epsilon$.

Finally, we check that $\{e_n\}$ of is an orthonormal basis of $\ker(T)^\perp$. Let $x \in \ker(T)^\perp$ be such that $\langle x, e_n\rangle = 0$ for all integers $n \geq 1$. But then by (5), for any $\epsilon > 0$, $\|Tx\| < \epsilon$, that is, $Tx = 0$. It follows that $x \in \ker(T)$, and hence it must be 0. □

Remark 2.13 : Let $z = x + y$, where $x \in \ker(T)^\perp$ and $y \in \ker(T)$. Since $\{e_n\}$ of is an orthonormal basis of $\ker(T)^\perp$, we can write $x = \sum_{n=1}^\infty \langle x, e_n\rangle e_n$. By continuity of T,

$$Tz = Tx = \sum_{n=1}^\infty \lambda_n\langle x, e_n\rangle e_n = \sum_{n=1}^\infty \lambda_n\langle z, e_n\rangle e_n.$$

3. Unbounded Operators with Compact Resolvent

Let X be a complex normed linear space. By a *linear operator S in X with domain $\mathcal{D}(S)$*, we mean a linear transformation S defined on some linear subspace $\mathcal{D}(S)$ of X.

If S is a linear operator in X with domain $\mathcal{D}(S)$, then the *resolvent set* of S, denoted by $\rho(S)$, is defined to be the set of all scalars $\mu \in \mathbb{C}$ for which there exists a bounded linear operator $R(\mu)$ on X such that

(R1) for every $y \in X$ we have that $R(\mu)y \in \mathcal{D}(S)$ and $(S-\mu)R(\mu)y = y$,
(R2) $R(\mu)(S-\mu)x = x$ for all $x \in \mathcal{D}(S)$.

Let $R(\mu) := (S-\mu)^{-1}$. Thus

$$\rho(S) := \left\{\mu \in \mathbb{C} \ : \ (S-\mu)^{-1} \text{ exists as a bounded linear operator on } X\right\}.$$

Define the spectrum $\sigma(S) := \rho(S)^c$, and the point spectrum

$$\sigma_p(S) := \{\mu \in \mathbb{C} \ : \ Sx = \mu x \text{ for some non-zero } x \in \mathcal{D}(S)\}.$$

Remark 3.1 : We note the following:

(1) $\sigma_p(T) \subseteq \sigma(T)$.
(2) If (R1) holds and $\mu \notin \sigma_p(T)$ then (R2) is satisfied.

Definition 3.2 : A linear operator S in X is *closed* if for every sequence $\{x_n\} \subset \mathcal{D}(S)$ such that $x_n \to x, Sx_n \to y$ for some $y \in X$, one has $x \in \mathcal{D}(S)$ and $Sx = y$.

Exercise 3.3 : Let T be a closed linear operator in a Banach space X. Then $\mu \in \rho(T)$ iff $T - \mu$ is bijective.

Hint. Use the Closed Graph Theorem.

Exercise 3.4 : If T is one-one and closed then T^{-1} is also closed.

Hint. Let $\{x_n\}$ be such that $x_n \to x$ and $T^{-1}x_n \to y$. Then $y_n := T^{-1}x_n \in \mathcal{D}(T)$, so that $y_n \to y$ and $Ty_n = x_n \to x$.

Exercise 3.5 : If $\rho(T) \neq \emptyset$ then show that T is closed.

Hint. If $\mu \in \rho(T)$ then $(T-\mu)^{-1}$ is closed; so is $T - \mu$ and hence T.

Exercise 3.6 : If A and B are linear operators such that $A = B$ on $\mathcal{D}(A) \subseteq \mathcal{D}(B)$. If $\rho(A) \cap \rho(B) \neq \emptyset$ then $\mathcal{D}(A) = \mathcal{D}(B)$, and hence $A = B$.

Hint. If $\mu \in \rho(A) \cap \rho(B)$ and $x \in \mathcal{D}(B)$ then $(B-\mu)x \in H = \text{ran}(A-\mu)$, and hence for some $y \in \mathcal{D}(A)$, $(B-\mu)x = (A-\mu)y$.

Exercise 3.7 : Let T be the linear operator of differentiation with domain $\mathcal{D}(T)$ in $L^p(0,1)$ for $p \in [1, \infty]$. Verify:

(1) If $\mathcal{D}(T) = \{f \in C^1[0,1] : f(0) = 0\}$ then $\sigma(T) = \mathbb{C}$.
(2) If $\mathcal{D}(T) = \{f \in AC[0,1] : f' \in L^p(0,1),\ f(0) = 0\}$ then $\sigma(T) = \emptyset$.
(3) If $\mathcal{D}(T) = \{f \in AC[0,1] : f' \in L^p(0,1),\ f(0) = f(1)\}$ then

$$\sigma(T) = \sigma_p(T) = \{2\pi i n : n \in \mathbb{Z}\}.$$

(4) For the operator T in part (3), $(T-1)^{-1}$ is compact.

Show that all the operators T are densely defined.

Hint. In (1), $T-\lambda$ is never surjective. For (2), consider $(R(\lambda)g)(x) = \int_{(0,x)} e^{\lambda(x-s)} g(s)ds$ for $g \in L^p(0,1)$ and $\lambda \in \mathbb{C}$. To see (3), note first that for $f \in \mathcal{D}(T)$, $Tf = \lambda f$ iff $f' = \lambda f$ a.e. iff $(e^{-\lambda t} f)' = 0$ a.e. To see that $\sigma_p(T) = \{2\pi i n : n \in \mathbb{Z}\}$, use now the fundamental theorem of calculus [2, Theorem 7.18], and the boundary conditions $f(0) = f(1)$. If $\lambda \notin \sigma_p(T)$ then $R(\lambda)$ is given by

$$(R(\lambda)g)(x) = \int_{(0,1)} \frac{e^{\lambda(x-y)}}{1-e^{\lambda}} t(x,y) g(y) dy \ (g \in L^p(0,1)),$$

where $t(x,y) = e^{\lambda}$ if $x \leq y$; and $t(x,y) = 1$ if $x > y$. The last part may be concluded from the Arzela-Ascoli Theorem.

Definition 3.8 : A linear operator T in a Hilbert space is said to have *compact resolvent* if there exists λ_0 in the resolvent $\rho(T)$ of T such that $R(\lambda_0) := (T-\lambda_0)^{-1}$ is compact.

Exercise 3.9 : If T has compact resolvent then show that $R(\lambda)$ is compact for every $\lambda \in \rho(T)$.

Hint. Derive the resolvent identity: $R(\lambda) - R(\mu) = (\lambda - \mu)R(\lambda)R(\mu)$ for $\lambda, \mu \in \rho(T)$. If $R(\lambda_0)$ is compact then so is $R(\lambda) = R(\lambda_0) + (\lambda - \lambda_0)R(\lambda)R(\lambda_0)$ for any $\lambda \in \rho(T)$.

Exercise 3.10 : Suppose that T has compact resolvent in X. For every $\lambda \neq 0$, show that $\ker(T-\lambda)$ is a closed subspace of X.

Hint. Suppose $(T-\lambda_0)^{-1}$ is compact for some $\lambda_0 \in \mathbb{C}$. Observe that $\ker(T-\lambda) = \ker((\lambda-\lambda_0)R(\lambda_0) - 1)$, and apply Corollary 2.2.

4. Spectral Theorem for Symmetric Operators with Compact Resolvent

A densely defined linear operator S in a Hilbert space H is said to be *symmetric* if

$$\langle Sx, y\rangle = \langle x, Sy\rangle \text{ for all } x, y \in \mathcal{D}(S).$$

Exercise 4.1 : Let $\lambda \in \rho(T) \cap \mathbb{R}$. Show that T is symmetric iff $R(\lambda) := (T-\lambda)^{-1}$ is self-adjoint.

Hint. If T is symmetric then

$$\langle R(\lambda)x, y\rangle = \langle R(\lambda)x, (T-\lambda)R(\lambda)y\rangle = \langle (T-\lambda)R(\lambda)x, R(\lambda)y\rangle = \langle x, R(\lambda)y\rangle$$

for any $x, y \in H$. Similarly, one can check that if $R(\lambda)$ is symmetric then $\langle (T-\lambda)x, y\rangle = \langle x, (T-\lambda)y\rangle$ for any $x, y \in \mathcal{D}(T)$.

Exercise 4.2 : Show that T given by $Tf = if'$ $(f \in \mathcal{D}(T))$ is a symmetric operator with compact resolvent, where

$$\mathcal{D}(T) = \{f \in AC[0,1] : f' \in L^2(0,1),\ f(0) = f(1)\}.$$

Hint. Let $f, g \in \mathcal{D}(T)$. Integration by parts gives

$$\langle Tf,\, g\rangle = i\int_{(0,1)} f'(t)\overline{g(t)}dt = -i\int_{(0,1)} f(t)\overline{g'(t)}dt = \langle f,\, Tg\rangle.$$

Also, T has compact resolvent, note that $T - I$ is invertible with inverse $R(1)$ given by

$$(R(1)g)(x) = -i\int_{(0,1)} \frac{e^{x-y}}{1-e} t(x,y)g(y)dy \ (g \in L^p(0,1)),$$

where $t(x,y) = e$ if $x \leq y$; and $t(x,y) = 1$ if $x > y$. The second part may be concluded from Exercise 1.9.

Exercise 4.3 : Show that the point-spectrum $\sigma_p(T)$ of a symmetric operator T is a subset of the real line $\mathbb{R}$.

Exercise 4.4 : Let T be a symmetric operator in a Hilbert space H such that $(T-\lambda_0)^{-1}$ is compact. Show that

$$\sigma(T) = \sigma_p(T) = \{\lambda_0 + \mu^{-1} : \mu \in \sigma_p((T-\lambda_0)^{-1})\}.$$

Hint. Use Lemma 2.10.

Lemma 4.5. *If T is a symmetric operator with compact resolvent then there exists $\lambda_1 \in \mathbb{R} \cap \rho(T)$ such that $(T-\lambda_1)^{-1}$ is a compact self-adjoint operator.*

Proof. Let $\lambda_0 \in \mathbb{C}$ be such that $R(\lambda_0) := (T-\lambda_0)^{-1}$ is a compact operator. We claim that $\sigma(T)$ intersects $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$ in finitely many points. By Exercise 4.4, if $\lambda \in \sigma(T) \cap \mathbb{D}_r$ then $\lambda = \lambda_0 + \mu^{-1}$ for some $\mu \in \sigma_p(R(\lambda_0))$. Also, $|\mu^{-1}| = |\lambda - \lambda_0| < r + |\lambda_0|$, so that $|\mu| > (r + |\lambda_0|)^{-1}$. However, by Theorem 2.12(4), $\sigma_p(R(\lambda_0))$ intersects $\mathbb{C} \setminus \mathbb{D}_r$ in finitely many points. Thus there are finitely many choices for μ, and hence for λ. Hence the claim stands verified.

Choose $\lambda_1 \in (-r, r) \cap \rho(T)$. By Exercises 3.9 and 4.1, $(T-\lambda_1)^{-1}$ is a compact self-adjoint operator as required. □

Theorem 4.6. *Let T be a symmetric operator with compact resolvent in a Hilbert space H. For each $\lambda \in \sigma_p(T)$, define $n(\lambda) = \dim\ker(T-\lambda)$, and choose an orthonormal basis $\{\phi_{\lambda,1}, \cdots, \phi_{\lambda,n(\lambda)}\}$ of $\ker(T-\lambda)$. Then $\{\phi_{\lambda,i} : \lambda \in \sigma_p(T), i = 1, \cdots, n(\lambda)\}$ forms a complete orthonormal set for H. Moreover, we have the following:*

1. *$x \in \mathcal{D}(T)$ iff $\sum_{\lambda\in\sigma_p(T)} \sum_{i=1}^{n(\lambda)} |\lambda|^2 |\langle x,\, \phi_{\lambda,i}\rangle|^2 < \infty$.*
2. *For any $x \in \mathcal{D}(T)$, $Tx = \sum_{\lambda\in\sigma_p(T)} \sum_{i=1}^{n(\lambda)} \lambda\langle x,\, \phi_{\lambda,i}\rangle\phi_{\lambda,i}$.*

Proof. By the last lemma, there exists $\lambda_0 \in \mathbb{R} \cap \rho(T)$ such that $R(\lambda) := (T-\lambda_0)^{-1}$ is a compact self-adjoint operator. By Theorem 2.12,

$$R(\lambda_0)x = \sum_{\mu\in\sigma_p((T-\lambda_0)^{-1})} \sum_{i=1}^{n(\mu)} \mu\langle x,\, \phi_{\mu,i}\rangle\phi_{\mu,i} \ (x \in H). \tag{4.2}$$

Also, by Exercise 4.4, $\sigma(T) = \sigma_p(T) = \{\lambda_0 + \mu^{-1} : \mu \in \sigma_p((T-\lambda_0)^{-1})\}$. Thus (4.2) can be rewritten as

$$R(\lambda_0)x = \sum_{\lambda\in\sigma_p(T)} \sum_{i=1}^{m(\lambda)} \frac{\langle x,\, \psi_{\lambda,i}\rangle}{\lambda-\lambda_0}\psi_{\lambda,i} \ (x \in H),$$

where $m(\lambda) := n(\mu)$ and $\psi_{\lambda,i} := \phi_{\mu,i}$. Since $\ker(R(\lambda_0)) = \{0\}$, it follows from the completeness part of Theorem 2.12 that $\{\phi_{\lambda,i} : \lambda \in \sigma_p(T), i = 1, \cdots, n(\lambda)\}$ is a complete orthonormal set for H.

Let $x \in \mathcal{D}(T)$. Since T is symmetric, $\langle Tx,\, \psi_{\lambda,i}\rangle = \langle x,\, T\psi_{\lambda,i}\rangle = \lambda\langle x,\, \psi_{\lambda,i}\rangle$. It is easy to see from the Parseval's identity that $\sum_{\lambda\in\sigma_p(T)} \sum_{i=1}^{n(\lambda)} |\lambda|^2 \langle x,\, \phi_{\lambda,i}\rangle|^2 = \|Tx\|^2 < \infty$. Conversely, suppose that $\sum_{\lambda\in\sigma_p(T)} \sum_{i=1}^{n(\lambda)} |\lambda|^2 |\langle x,\, \phi_{\lambda,i}\rangle|^2 < \infty$, and set $y = \sum_{\lambda\in\sigma_p(T)} \sum_{i=1}^{n(\lambda)} \lambda\langle x,\, \phi_{\lambda,i}\rangle\phi_{\lambda,i}$.

It is easy to see that $R(\lambda_0)(y-\lambda_0 x) = x$. In particular, $x \in \mathcal{D}(T)$ and $(T-\lambda_0)x = y - \lambda_0 x$. Finally, note that $Tx = y = \sum_{\lambda \in \sigma_p(T)} \sum_{i=1}^{n(\lambda)} \lambda \langle x, \phi_{\lambda,i} \rangle \phi_{\lambda,i}$. $\square$

Remark 4.7 : As a part of the definition of symmetric operators, we assumed that T is densely defined. The conclusion of the theorem holds without this assumption.

Example 4.8 : Recall that the operator T as defined in Exercise 4.2 is a symmetric operator with compact resolvent. It may be concluded from the last theorem that the eigenvectors $1, e^{2\pi ix}, e^{-2\pi ix}, e^{4\pi ix}, e^{-4\pi ix}, \cdots$ of T form a complete orthonormal set in $L^2[0,1]$ (for details, see Exercise 3.7). This provides (possibly the most elegant) proof of the completeness of the Fourier series in $L^2(-\pi,\pi)$: For any $f \in L^2(-\pi,\pi)$,

$$\sum_{n\in\mathbb{Z}} \hat{f}(n) e^{n\pi ix} \text{ converges to } f \text{ in } L^2(-\pi,\pi),$$

where $\hat{f}(n) = \frac{1}{2\pi}\int_{(-\pi,\pi)} f(x) e^{n\pi ix} dx$ for $n \in \mathbb{Z}$.

There are numerous applications of Theorem 4.6 to the theory of PDE's including Sturm-Liouville problems (refer, for instance, to [1, Section 2.7]).

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