TOPOLOGY: NOTES AND PROBLEMS

Abstract. These are the notes prepared for the course MTH 304 to be offered to undergraduate students at IIT Kanpur.

Contents

1. Topology of Metric Spaces

A function $d : X \times X \to \mathbb{R}_+$ is a *metric* if for any $x, y, z \in X$,

(1) $d(x, y) = 0$ iff $x = y$.
(2) $d(x, y) = d(y, x)$.
(3) $d(x, y) \leq d(x, z) + d(z, y)$.

We refer to (X, d) as a *metric space*.

Exercise 1.1 : Give five of your favourite metrics on $\mathbb{R}^2$.

Exercise 1.2 : Show that $C[0, 1]$ is a metric space with metric $d_\infty(f, g) := \|f - g\|_\infty$.

An *open ball* in a metric space (X, d) is given by

$$B_d(x, R) := \{y \in X : d(y, x) < R\}.$$

Exercise 1.3 : Let (X, d) be your favourite metric (X, d). How does open ball in (X, d) look like ?

Exercise 1.4 : Visualize the open ball $B(f, R)$ in $(C[0, 1], d_\infty)$, where f is the identity function.

We say that $Y \subseteq X$ is *open* in X if for every $y \in Y$, there exists $r > 0$ such that $B(y, r) \subseteq Y$, that is,

$$\{z \in X : d(z, y) < r\} \subseteq Y.$$

Exercise 1.5 : Give five of your favourite open subsets of $\mathbb{R}^2$ endowed with any of your favourite metrics.

Exercise 1.6 : Give five of your favourite non-open subsets of $\mathbb{R}^2$.

Exercise 1.7 : Let $B[0, 1]$ denote the set of all bounded functions $f : [0, 1] \to \mathbb{R}$ endowed with the metric d_∞. Show that $C[0, 1]$ can not be open in $B[0, 1]$.

Hint. Any neighbourhood of 0 in $B[0, 1]$ contains discontinuous functions.

Exercise 1.8 : Show that the open unit ball in $(C[0, 1], d_\infty)$ can not be open in $(C[0, 1], d_1)$, where $d_1(f, g) = \int_{[0,1]} |f(t) - g(t)|dt$.

Hint. Construct a function of maximum equal to $1 + r$ at 0 with area covered less than r.

Exercise 1.9 : Show that the open unit ball in $(C[0, 1], d_1)$ is open in $(C[0, 1], d_\infty)$.

Example 1.10 : Consider the first quadrant of the plane with usual metric. Note that the open unit disc there is given by

$$\{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y^2 < 1\}.$$

We say that a sequence $\{x_n\}$ in a metric space X with metric d converges to x if $d(x_n, x) \to 0$ as $n \to \infty$.

Exercise 1.11 : Discuss the convergence of $f_n(t) = t^n$ in $(C[0, 1], d_1)$ and $(C[0, 1], d_\infty)$.

Exercise 1.12 : Every metric space (X, d) is *Hausdorff*: For distinct $x, y \in X$, there exists $r > 0$ such that $B_d(x, r) \cap B_d(y, r) = \emptyset$. In particular, limit of a convergent sequence is unique.

Exercise 1.13 : (Co-finite Topology) We declare that a subset U of $\mathbb{R}$ is open iff either $U = \emptyset$ or $\mathbb{R} \setminus U$ is finite. Show that $\mathbb{R}$ with this "topology" is not Hausdorff.

A subset U of a metric space X is *closed* if the complement $X \setminus U$ is open. By a *neighbourhood of a point*, we mean an open set containing that point. A point $x \in X$ is a *limit point of* U if every non-empty neighbourhood of x contains a point of U. (This definition *differs* from that given in Munkres). The set $\overline{U}$ is the collection of all limit points of U.

Exercise 1.14 : What are the limit points of bidisc in $\mathbb{C}^2$?

Exercise 1.15 : Let (X, d) be a metric space and let U be a subset of X. Show that $x \in \overline{U}$ iff for every $x \in \overline{U}$, there exists a convergent sequence $\{x_n\} \subseteq U$ such that $\lim_{n\to\infty} x_n = x$.

2. Topological Spaces

Let X be a set with a collection Ω of subsets of X. If Ω contains $\emptyset$ and X, and if Ω is closed under arbitrary union and finite intersection then we say that Ω is a *topology on* X. The pair (X, Ω) will be referred to as the topological space X with topology Ω. An *open set* is a member of Ω.

Exercise 2.1 : Describe all topologies on a 2-point set. Give five topologies on a 3-point set.

Exercise 2.2 : Let (X, Ω) be a topological space and let U be a subset of X. Suppose for every $x \in U$ there exists $U_x \in \Omega$ such that $x \in U_x \subseteq U$. Show that U belongs to Ω.

Exercise 2.3 : (Co-countable Topology) For a set X, define Ω to be the collection of subsets U of X such that either $U = \emptyset$ or $X \setminus U$ is countable. Show that Ω is a topology on X.

Exercise 2.4 : Let Ω be the collection of subsets U of $X := \mathbb{R}$ such that either $X \setminus U = \emptyset$ or $X \setminus U$ is infinite. Show that Ω is not a topology on X.

Hint. The union of $(-\infty, 0)$ and $(0, \infty)$ does not belong to Ω.

Let X be a topological space with topologies Ω_1 and Ω_2. We say that Ω_1 is *finer* than Ω_2 if $\Omega_2 \subseteq \Omega_1$. We say that Ω_1 and Ω_2 are *comparable* if either Ω_1 is finer than Ω_2 or Ω_2 is finer than Ω_1.

Exercise 2.5 : Show that the usual topology is finer than the co-finite topology on $\mathbb{R}$.

Exercise 2.6 : Show that the usual topology and co-countable topology on $\mathbb{R}$ are not comparable.

Remark 2.7 : Note that the co-countable topology is finer than the co-finite topology.

3. Basis for a Topology

Let X be a set. A *basis* $\mathbb{B}$ for a topology on X is a collection of subsets of X such that

(1) For each $x \in X$, there exists $B \in \mathbb{B}$ such that $x \in B$.
(2) If $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathbb{B}$ then there exists $B \in \mathbb{B}$ such that $x \in B \subseteq B_1 \cap B_2$.

Example 3.1 : The collection $\{(a,b) \subseteq \mathbb{R} : a, b \in \mathbb{Q}\}$ is a basis for a topology on $\mathbb{R}$.

Exercise 3.2 : Show that collection of balls (with rational radii) in a metric space forms a basis.

Example 3.3 : (Arithmetic Progression Basis) Let X be the set of positive integers and consider the collection $\mathbb{B}$ of all arithmetic progressions of positive integers. Then $\mathbb{B}$ is a basis. If $m \in X$ then $B := \{m+(n-1)p\}$ contains m. Next consider two arithmetic progressions $B_1 = \{a_1 + (n-1)p_1\}$ and $B_2 = \{a_2+(n-1)p_2\}$ containing an integer m. Then $B := \{m+(n-1)(p)\}$ does the job for $p := \text{lcm}\{p_1, p_2\}$.

4. Topology Generated by a Basis

Let $\mathbb{B}$ be a basis for a topology on X. The *topology* $\Omega_{\mathbb{B}}$ *generated by* $\mathbb{B}$ is defined as

$$\Omega_{\mathbb{B}} := \{U \subseteq X : \text{For each } x \in U, \text{ there exists } B \in \mathbb{B} \text{ such that } x \in B \subseteq U\}.$$

We will see in the class that $\Omega_{\mathbb{B}}$ is indeed a topology that contains $\mathbb{B}$.

Exercise 4.1 : Show that the topology Ω_B generated by the basis $\mathbb{B} := \{(a,b) \subseteq \mathbb{R} : a, b \in \mathbb{Q}\}$ is the usual topology on $\mathbb{R}$.

Example 4.2 : The collection $\{[a,b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}$. The topology generated by it is known as *lower limit topology* on $\mathbb{R}$.

Example 4.3 : Note that $\mathbb{B} := \{p\} \bigcup \{\{p,q\} : q \in X, q \neq p\}$ is a basis. We check that the topology Ω_B generated by $\mathbb{B}$ is the VIP topology on X. Let U be a subset of X containing p. If $x \in U$ then choose $B = \{p\}$ if $x = p$, and $B = \{p, x\}$ otherwise. Note further that if $p \notin U$ then there is no $B \in \mathbb{B}$ such that $B \subseteq U$. This shows that $\Omega_{\mathbb{B}}$ is precisely the VIP topology on X.

Exercise 4.4 : Show that the topology generated by the basis $\mathbb{B} := \{X\} \cup \{\{q\} : q \in X, q \neq p\}$ is the outcast topology.

Exercise 4.5 : Show that the topological space $\mathbb{N}$ of positive numbers with topology generated by arithmetic progression basis is Hausdorff.

Hint. If $m_1 > m_2$ then consider open sets $\{m_1 + (n-1)(m_1 + m_2 + 1)\}$ and $\{m_2 + (n-1)(m_1 + m_2 + 1)\}$.

The following observation justifies the terminology basis:

Proposition 4.6. *If $\mathbb{B}$ is a basis for a topology on X, then $\Omega_{\mathbb{B}}$ is the collection Ω of all union of elements of $\mathbb{B}$.*

Proof. Since $\mathbb{B} \subseteq \Omega_{\mathbb{B}}$, by the very definition of topology, $\Omega \subseteq \Omega_{\mathbb{B}}$. Let $U \in \Omega_{\mathbb{B}}$. Then for each $x \in U$, there exists $B_x \in \mathbb{B}$ such that $x \in B_x \subseteq U$. It follows that $U = \cup_x B_x$, that is, $U \in \Omega$. □

Remark 4.7 : If $\mathbb{B}_1$ and $\mathbb{B}_2$ are bases for topologies on X such that $\mathbb{B}_2 \subseteq \mathbb{B}_1$ then $\Omega_{\mathbb{B}_1}$ is finer than $\Omega_{\mathbb{B}_2}$.

Proposition 4.8. *For $i = 1, 2$ consider the basis $\mathbb{B}_i \subseteq X$ and the topology $\Omega_{\mathbb{B}_i}$ it generates. TFAE:*

(1) *$\Omega_{\mathbb{B}_1}$ is finer than $\Omega_{\mathbb{B}_2}$.*
(2) *For each $x \in X$ and each basis element $B_2 \in \mathbb{B}_2$ containing x, there is a basis element $B_1 \in \mathbb{B}_1$ such that $x \in B_1 \subseteq B_2$.*

Exercise 4.9 : Let $X := \mathbb{R}$. Consider the pairs of bases:

(1) $\mathbb{B}_1 := \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ and $\mathbb{B}_2 := \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{Q}\}$.
(2) $\mathbb{B}_1 := \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ and $\mathbb{B}_2 := \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{Q}\}$.
(3) $\mathbb{B}_1 := \{[a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$ and $\mathbb{B}_2 := \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$.

Do they generate comparable topologies ? If so then do they generate the same topology ?

Example 4.10 : Consider the subset

$$(a, b) \setminus K := \{x \in (a, b) : x \neq 1/n \text{ for any integer } n \geq 1\}$$

of the open interval (a, b). The collection

$$\mathbb{B}_1 := \{(a, b) \subseteq \mathbb{R} : a, b \in \mathbb{R}\} \cup \{(a, b) \setminus K \subseteq \mathbb{R} : a, b \in \mathbb{R}\}$$

is a basis for a topology on $\mathbb{R}$. The topology it generates is known as the *K-topology* on $\mathbb{R}$. Clearly, K-topology is finer than the usual topology. Note that there is no neighbourhood of 0 in the usual topology which is contained in $(-1, 1) \setminus K \in \mathbb{B}_1$. This shows that the usual topology is not finer than K-topology. The same argument shows that the lower limit topology is not finer than K-topology. Consider next the neighbourhood $[2, 3)$ of 2 in the lower limit topology. Then there is no neighbourhood of 2 in the K-topology which is contained in $[2, 3)$. We conclude that the K-topology and the lower limit topology are not comparable.

4.1. **Infinitude of Prime Numbers.** Let (X, Ω) be a topological space with topology Ω. A subset V of X is said to be *closed* if $X \setminus V$ belongs to Ω.

Exercise 4.11 : ([1, H. Fürstenberg]) Consider $\mathbb{N}$ with the arithmetic progression topology. Verify the following:

(1) For a prime number p, the basis element $\{np : n \geq 1\}$ is closed.
(2) There are infinitely prime numbers.

Hint. For (i), note that $\{np\} = \mathbb{N} \setminus \cup_{i=1}^{p-1}\{i + np\}$. For (ii), note that $\mathbb{N} \setminus \{1\} = \cup_p \{np\}$, where union is over all prime numbers. Now note that no finite set is open.

5. Product Topology

Proposition 5.1. *Let (X_1, Ω_1) and (X_2, Ω_2) be topological spaces with bases $\mathbb{B}_1$ and $\mathbb{B}_2$ respectively. Then $\mathbb{B} := \{B_1 \times B_2 : B_1 \in \mathbb{B}_1, B_2 \in \mathbb{B}_2\}$ forms a basis for a topology on $X_1 \times X_2$.*

Proof. We note the following:

(1) Suppose $(x_1, x_2) \in X_1 \times X_2$. Then for $i = 1, 2$, $x_i \in X_i$, and hence there exists $B_i \in \mathbb{B}_i$ such that $x_i \in B_i$. Thus $(x_1, x_2) \in B_1 \times B_2 \in \mathbb{B}$.
(2) Let $(x_1, x_2) \in (B_1 \times B_2) \cap (B_1' \times B_2')$ for some $B_i, B_i' \in \mathbb{B}_i$ $(i = 1, 2)$. Note that $(B_1 \times B_2) \cap (B_1' \times B_2') = (B_1 \cap B_1') \times (B_2 \cap B_2')$. Then there exists $B_i'' \in \mathbb{B}_i$ such that $x_i \in B_i'' \subseteq B_i \cap B_i'$. Thus $B := B_1'' \times B_2'' \in \mathbb{B}$ satisfies $(x_1, x_2) \in B \subseteq (B_1 \times B_2) \cap (B_1' \times B_2')$.

This completes the proof. □

The *product topology* on $X_1 \times X_2$ is the topology generated by the basis $\mathbb{B}$ as given above. For example, the product topology on $\mathbb{R} \times \mathbb{R}$ coincides with usual topology on $\mathbb{R}^2$.

Exercise 5.2 : Show that the open unit disc is open in the product topology on $\mathbb{R}^2$. Show further that it is not of form $U \times V$ for any open subsets U and V of $\mathbb{R}$.

Example 5.3 : Consider the space $\mathbb{R}_l \times \mathbb{R}$ with product topology, where $\mathbb{R}_l$ denotes the real line with lower limit topology. The basis for the product topology consists of $\{(x, y) \in \mathbb{R}^2 : a \leq x < b, c < y < d\}$.

Example 5.4 : Let X_1 denote the topological space $\mathbb{R}$ with discrete topology and let X_2 be $\mathbb{R}$ with usual topology. Then the product topology Ω on $\mathbb{R} \times \mathbb{R}$ is nothing but the dictionary order topology on $\mathbb{R}^2$. Since the basis for the product topology on $\mathbb{R} \times \mathbb{R}$ is given by $\{\{x_1\} \times (a, b) : x_1, a, b \in \mathbb{R}\}$, any open set in the dictionary order topology is union of open sets in the product topology.

We also note that the product topology Ω is finer than the usual topology on $\mathbb{R}^2$. In fact, any basis element $(a, b) \times (c, d)$ of the usual topology can be expressed as the union $\cup_{a<x<b}\{x\} \times (c,d)$ of open sets $\{x\} \times (c,d)$ in the product topology Ω.

6. Subspace Topology

Let (X,Ω) be a topological space and let Y be a subset of X. Then $\Omega_Y := \{U \cap Y : U \in \Omega\}$ is a topology on Y, called the *subspace topology.*

Remark 6.1 : If $U \in \Omega$ and $U \subseteq Y$ then U is open in the subspace topology.

Remark 6.2 : If $V \in \Omega_Y$ and $Y \in \Omega$ then $V \in \Omega$.

Example 6.3 : Consider the real numbers $\mathbb{R}$ with usual topology and let $Y := [-1,1]$. Then

(1) $(1/2,1)$ is open in the subspace topology: $(1/2,1) \subseteq Y$.
(2) $(1/2,1]$ is open in the subspace topology: $(1/2,1] = (1/2,2) \cap Y$.
(3) $[1/2,1)$ is not open in the subspace topology: If $[1/2,1) = U \cap Y$ for some open subset U of $\mathbb{R}$ then $1/2 \in U$. Thus $(1/2-\epsilon, 1/2+\epsilon)$ is contained in $U \cap Y = [1/2,1)$ for some $\epsilon > 0$, which is not possible.
(4) $\{x \in (0,1) : 1/x \neq 1, 2, \cdots\}$ is open in the subspace topology: This set is open in $\mathbb{R}$. Now apply Remark 6.1.

Remark 6.4 : Let (X,Ω) be a topological space with subsets Y, Z such that $Z \subseteq Y \subseteq X$. Then (Y,Ω_Y) is a subspace of (X,Ω) and (Z,Ω_{Y_Z}) is a subspace of (Y,Ω_Y). Also, (Z,Ω_Z) is a subspace of (X,Ω). Then $\Omega_Z = \Omega_{Y_Z}$.

Proposition 6.5. *If $\mathbb{B}$ is basis for the topological space X, and Y is a subspace of X then the basis $\mathbb{B}_Y$ for the subspace topology on Y is given by*

$$\mathbb{B}_Y = \{B \cap Y : B \in \mathbb{B}\}.$$

Example 6.6 : Consider the topological space $\mathbb{R}_l \times \mathbb{R}$ and let L denote a straight line in the plane. Then the basis elements for the subspace topology on L are of the form

$$\{(x,y) \in L : a \leq x < b, c < y < d\} \text{ or } \{(x,y) \in L : a < x < b, c < y < d\}$$

provided L is neither X-axis nor Y-axis.

Let X be a simply ordered set with order topology Ω_o and let Y be a subset of X. Then the order relation $<$ on X makes Y into an ordered set. This makes Y a topological space with order topology $\Omega_{Y,o}$. Also, Y has the subspace topology Ω_Y. We will see in class an example in which $\Omega_{Y,o}$ and Ω_Y are not same [4, Pg 90, Example 3].

Lemma 6.7. $\Omega_{Y,o} \subseteq \Omega_Y$.

Proof. The basis elements for the order topology on Y are of the form $B_1 := \{x \in Y : a < x < b\}, B_2 := \{x \in Y : \tilde{a}_0 \leq x < b\}, B_3 := \{x \in Y : a < x \leq$

$\tilde{b}_0\}$, where $\tilde{a}_0, \tilde{b}_0$ denote the minimal and maximal elements of Y (if exists) respectively. Clearly, $a_0 \leq \tilde{a}_0, \tilde{b}_0 \leq b_0$. Note the following:

(1) Clearly, $B_1 = (a, b) \cap Y$.
(2) If $a_0 = \tilde{a}_0$ then $B_2 = [a_0, b) \cap Y$.
(3) If $a_0 < \tilde{a}_0$ then $B_2 = (a_0, b) \cap Y$.
(4) If $b_0 = \tilde{b}_0$ then $B_3 = (a, b_0] \cap Y$.
(5) If $b_0 < \tilde{b}_0$ then $B_3 = (a, b_0) \cap Y$.

Thus B_1, B_2, B_3 belong to the basis for the subspace topology. That is, $\mathbb{B}_{Y,o} \subseteq \mathbb{B}_Y$, and hence $\Omega_{Y,o} \subseteq \Omega_Y$. □

Let X be a simply ordered set and let Y be a subset of X. We say that Y is *convex* if for any pair $a, b \in Y$, the interval (a, b) is contained in Y.

Example 6.8 : Any interval (x, y) in a simply ordered set is convex: If $a, b \in (x, y)$ then $x < a < b < y$ and hence $(a, b) \subseteq (x, y)$.

Example 6.9 : Consider the subset $[0, 1] \times [0, 1]$ of $\mathbb{R}^2$ with dictionary order. Then $0 \times 0, 1 \times 1$ belong to $[0, 1] \times [0, 1]$, however, $1/2 \times 2 \in (0 \times 0, 1 \times 1)$ does not belong to $[0, 1] \times [0, 1]$.

The first quadrant is not a convex subset of $\mathbb{R}^2$. However, the right half plane is convex. ■

Lemma 6.10. *Let Y be a convex subset of topological space X with order topology. For any open ray R in X, $R \cap Y$ is open in order topology on Y.*

Proof. We prove the lemma only for the ray (a, ∞). Consider the cases:

(1) $a < y$ for all $y \in Y$: Then $Z = Y$.
(2) $y < a$ for all $y \in Y$: Then $Z = \emptyset$.
(3) There exists $y_1, y_2 \in Y$ such that $y_1 < a < y_2$: Then $a \in Y$. The convexity of Y implies that $a \in (y_1, y_2) \subseteq Y$.

If $a \notin Y$ then $Z = Y$ or $Z = \emptyset$. If $a \in Y$ then the open set $Z := (a, \infty) \cap Y$ in the subspace topology is actually an open ray in Y. □

Theorem 6.11. *Let Y be a convex subset of the topological space X with order topology. Then the order topology on Y is same as subspace topology on Y, that is, $\Omega_{Y,o} = \Omega_Y$.*

Proof. By the preceding lemma, for any open ray R in X, $R \cap Y$ is open in order topology on Y. Note the following:

(1) If $a < b$ then $(a, b) \cap Y = ((-\infty, b) \cap Y) \cap ((a, \infty) \cap Y)$.
(2) If X has minimal elment a_0 then $[a_0, b) \cap Y = (-\infty, b) \cap Y$.
(3) If X has maximal element b_0 then $(a, b_0] \cap Y = (a, \infty) \cap Y$.

Thus every basis element for the subspace topology is open in order topology on Y. The fact that every basis element for the order topology is open in the subspace topology is established in Lemma 6.7. □

Remark 6.12 : If may happen that for a non-convex subset Y of X, $\Omega_{Y,o} = \Omega_Y$. Consider for example $X := \mathbb{R}$ and $Y := \mathbb{R} \setminus \{0\}$.

7. Closed Sets, Hausdorff Spaces, and Closure of a Set

A topological space (X, Ω) is *Hausdorff* if for any pair $x, y \in X$ with $x \neq y$, there exist neighbourhoods N_x and N_y of x and y respectively such that $N_x \cap N_y = \emptyset$.

Any metric space is Hausdorff. In particular, the real line $\mathbb{R}$ with usual metric topology is Hausdorff.

Exercise 7.1 : If X is Hausdorff then show that the complement of any finite set is open.

Hint. Let $x_0 \in X$ then let $x \in X \setminus \{x_0\}$ and apply the definition of Hausdorff set to the pair x, x_0.

Recall that a subset of X is closed if its complement in X is open in X.

Remark 7.2 : Arbitrary intersection of closed sets is closed. Finite union of closed sets is closed.

Example 7.3 : Consider the real line $\mathbb{R}$ with usual topology. Then the set $\mathbb{Q}$ of rationals is not closed. This follows since any neighbourhood of an irrational number contains rationals. If we enumerate $\mathbb{Q}$ as a sequence $\{r_n\}$ then $\mathbb{Q} = \cup_n \{r_n\}$, which shows that countable union of closed sets need not be closed.

Exercise 7.4 : Show that X is Hausdorff iff the diagonal $\Delta := \{(x, x) \in X \times X\}$ is closed in $X \times X$.

Hint. Note that $x \neq y$ iff (x, y) belongs to the complement of Δ in $X \times X$. Check that X is Hausdorff iff $X \times X \setminus \Delta$ is open in $X \times X$.

Exercise 7.5 : Show that every topological space with order topology is Hausdorff.

Hint. Given $x_1 \neq x_2$, choose $x_i \in (a_i, b_i)$ possibly $(a_1, b_1) \cap (a_2, b_2) \neq \emptyset$. We may assume that $a_2 < b_1$. If there is $c \in (a_1, b_1) \cap (a_2, b_2)$ then try (a_1, c) and (c, b_2). Otherwise try (a_1, b_1) and (a_2, b_2).

Exercise 7.6 : What are all closed subsets of $\mathbb{R}$ with VIP topology (resp. outcast topology) ?

Example 7.7 : Consider the subset $Y := (-\infty, 0) \cup [1, \infty)$ of $\mathbb{R}$. Clearly, Y is not closed in the usual topology on $\mathbb{R}$ as $\mathbb{R} \setminus Y = [0, 1)$ is not open in $\mathbb{R}$. However, Y is closed in $\mathbb{R}_l$.

Exercise 7.8 : Show that the only non-empty subset of $\mathbb{R}$ which is open as well as closed is $\mathbb{R}$.

Hint. Write $\mathbb{R} = A \cup B$ for open and disjoint sets A and B. If $a \in A$ and $b \in B$ with $a < b$ then $[a, b] = A_0 \cup B_0$, where $A_0 = A \cap [a, b]$ and

$B_0 = B \cap [a, b]$ are disjoint. Let $c := \sup A_0 \in [a, b]$. Then either $c \in A_0$ or $c \in B_0$. If $c \in A_0$ then either $c = a$ or $a < c < b$. Since A_0 is open in the subspace topology on $[a, b]$, there is d such that $[c, d) \subseteq A_0$. This contradicts that $c = \sup A_0$. Similarly, prove that c can not belong to B_0.

Exercise 7.9 : Given an example of a non-empty subset of $Y := \mathbb{R} \setminus \{0\}$ which is open as well as closed in the subspace topology on Y.

Exercise 7.10 : Let $f : \mathbb{R}^n \to \mathbb{R}$ be a continuous function in the variables $x_1, \cdots, x_n$. Show that the zero set $Z(f) := \{x \in \mathbb{R}^n : f(x) = 0\}$ is a closed subset of $\mathbb{R}^n$.

Exercise 7.11 : Identify $M_n(\mathbb{C})$ with $\mathbb{C}^{n^2}$ with usual topology. Show that $GL_n(\mathbb{C})$ is an open subset of $M_n(\mathbb{C})$.

Let (X, Ω) be a topological space with topology Ω. If A is a subset of X then the *closure* $\bar{A}$ *of* A *in* X is defined as the intersection of all closed sets containing A:

$$\bar{A} = \bigcap_{\text{closed } B \supseteq A} B.$$

Remark 7.12 : Clearly, $A \subseteq \bar{A}$. Also, $\bar{A} \subseteq A$ iff A is closed.

Example 7.13 : The closure of rationals in the usual topology on $\mathbb{R}$ is $\mathbb{R}$. To see this, let B be a closed subset of $\mathbb{R}$ such that $\mathbb{Q} \subseteq B$. If $B \neq \mathbb{R}$ then the complement of B in $\mathbb{R}$ is non-empty, and hence contains an interval. But then $\mathbb{R} \setminus B$ contains a rational, which is not possible since $\mathbb{Q} \subseteq B$.

Example 7.14 : Let X be an ordered set with order topology. Note that the complement of $[a, b]$ in X is open. Thus $\overline{(a, b)} \subseteq [a, b]$.

Example 7.15 : Consider the topological space $\mathbb{R}_l$. Then the closure of (non-closed set) $(0, \sqrt{2})$ equals $[0, \sqrt{2})$. This follows from the observation that $\mathbb{R} \setminus [0, \sqrt{2}) = (-\infty, 0) \cup [\sqrt{2}, \infty)$ is open in $\mathbb{R}_l$. If one replaces the lower limit topology by the topology Ω generated by the basis $\{[a, b) : a, b \in \mathbb{Q}\}$ then the closure of $(0, \sqrt{2})$ in Ω equals $[0, \sqrt{2}]$. This provides another verification of the fact that Ω and the lower limit topology are different.

Example 7.16 : Let $X = \mathbb{R}$ with usual topology. Then the closure of $(0, 1)$ in $\mathbb{R}$ equals $[0, 1]$ as $[0, 1]$ is the smallest closed set containing $(0, 1)$. However, the closure of $(0, 1)$ in the subspace topology on $[0, 1)$ equals $[0, 1)$ as $[0, 1)$ is the smallest closed in the subspace topology that contains $(0, 1)$.

Theorem 7.17. *Let* (X, Ω) *be a topological space with topology* Ω. *Let* A, Y *be subsets of* X *such that* $A \subseteq Y$. *Then*

(1) *A is closed in the subspace topology on Y iff $A = B \cap Y$ for some closed subset B of X.*
(2) *The closure of A in the subspace topology on Y equals $\bar{A} \cap Y$.*

Proof. (1) If $A = B \cap Y$ for some closed subset B of X then $(X \setminus B) \cap Y = Y \setminus A$, and hence A is closed in the subspace topology on Y. If A is closed in the subspace topology on Y then $Y \setminus A = U \cap Y$ for some open set U in X. Then $A = Y \setminus (U \cap Y) = B \cap Y$ for closed set $B := X \setminus U$ in X.

(2) Let B denote the closure of A in the subspace topology. Since $\bar{A} \cap Y$ is a closed set containing A, and B is the smallest closed set in the subspace topology that contains A, we have $B \subseteq \bar{A} \cap Y$. Also, since $B = C \cap Y$ for some closed set C in X that contains A, $\bar{A} \subseteq C$. But then $\bar{A} \cap Y \subseteq C \cap Y = B$. □

Corollary 7.18. *Let Y be a subspace of X. If A is closed in Y and Y is closed in X, then A is closed in X.*

Proof. Let $\bar{A}$ and $\bar{A}^Y$ denote the closures of A in X and Y respectively. By the last theorem, $\bar{A}^Y = \bar{A} \cap Y$. Suppose that A is closed in Y, that is, $\bar{A}^Y = A$. But then $A = \bar{A}^Y = \bar{A} \cap Y$, which is the intersection of closed sets $\bar{A}$ and Y in X. Hence A is closed in X. □

Example 7.19 : Consider the topological space $\mathbb{R} \times \mathbb{R}$ with dictionary order topology and consider the unit square $[0, 1] \times [0, 1]$ with subspace topology. Note that $A := \{(1/n) \times 0 : n \in \mathbb{N}\}$ is closed in $\mathbb{R} \times \mathbb{R}$, and hence in the subspace topology on $[0, 1] \times [0, 1]$. Note that the closure of $B := \{0 \times (1/n) : n \in \mathbb{N}\}$ in $\mathbb{R} \times \mathbb{R}$ equals $B \cup \{0 \times 0\}$. This also shows that the closure of B in the subspace topology equals $B \cup \{0 \times 0\}$.

By a *neighbourhood of a point*, we mean an open set containing that point. By a *deleted neighbourhood of a point x*, we mean $U \setminus \{x\}$, where U is a neighbourhood of x. If A is a subset of the topological space X and if $x \in X$ then we say that x is a *cluster point* of A if every deleted neighbourhood of x intersects A.

Example 7.20 : Consider the subset $A := [0, 1) \cup \{2\}$ of the real line with usual topology. Note that 1 is a cluster point of A but 2 is not a cluster point of A.

Exercise 7.21 : What are all cluster points of $\{q\}$ in VIP topology with base point p on $\mathbb{R}$?

Lemma 7.22. *Every cluster point of A belongs to $\bar{A}$.*

Proof. If possible, suppose that $x \notin \bar{A}$. But then $U := X \setminus \bar{A}$ is a neighbourhood of x. Since x is a cluster point of x, the deleted neighbourhood $U \setminus \{x\}$ must intersect A. This is not possible since U is contained in $X \setminus A$. □

Lemma 7.23. *If $x \in \bar{A} \cap (X \setminus A)$ then x is a cluster point of A.*

Proof. If x is not a cluster point of A then there is a neighbourhood U of x such that $(U \setminus \{x\}) \cap A = \emptyset$. Since $x \notin A$, we have $U \cap A = \emptyset$, and hence A is contained in the closed set $X \setminus U$. By the definition of closure of a set, $\bar{A} \subseteq X \setminus U$. Since $x \in \bar{A}$, we arrive at the contradiction that $x \notin U$. □

We combine the preceding two lemmas to get the following:

Theorem 7.24. *$x \in \bar{A}$ iff either $x \in A$ or x is a cluster point of A.*

Theorem 7.25. *If X is a Hausdorff space then every neighbourhood of a cluster point of A contains infinitely many points from A.*

Proof. Suppose that X is Hausdorff and let x be a cluster point of a subset A of X. Suppose, if possible, there is a neighbourhood U of x such that $U \cap A = F$ is a finite set. Since X is Hausdorff, F is closed in X. But then the deleted neighbourhood $U \cap (X \setminus F)$ of x does not intersect at all. That contradicts that x is a cluster point. □

8. Continuous Functions

Let (X_i, Ω_i) $(i = 1, 2)$ be two topological spaces and let $f : X_1 \to X_2$ be a function. We say that f is *continuous* if for any $V \in \Omega_2$, the pre-image $f^{-1}(V)$ of V under f belongs to Ω_1.

Remark 8.1 : Note that f is continuous iff for any closed set B in X_2, the pre-image $f^{-1}(B)$ of B under f is closed in X_1. This follows from the observation $f^{-1}(X_2 \setminus B) = X_1 \setminus f^{-1}(B)$.

Exercise 8.2 : Let (X, Ω_i) $(i = 1, 2)$ be two topological spaces and consider the identity function id from X onto itself. Show that Ω_1 is finer than Ω_2 iff the identity function id is continuous.

Example 8.3 : Let X_1, X_2, X_3 denote the set $\mathbb{R}$ with usual topology, lower limit topology, K-topology respectively. The identity mapping id from X_3 onto X_1 is continuous. However, neither the identity mapping id from X_2 onto X_3 nor the identity mapping id from X_3 onto X_2 is continuous.

Exercise 8.4 : Let X be a topological space with discrete topology and Y be any topological space. Show that any function $f : X \to Y$ is continuous.

Exercise 8.5 : Let (X_i, Ω_i) be two topological spaces with topology Ω_i $(i = 1, 2)$ and let $x_0 \in X_2$. If X_2 is Hausdorff then show that for any continuous function $f : X_1 \to X_2$, the set $C := \{x \in X_1 : f(x) = x_0\}$ is closed in X_1.

Hint. Note that $f^{-1}(X_2 \setminus \{x_0\}) = X_1 \setminus C$.

Example 8.6 : Consider the metric space $C[0, 1]$ with sup metric and $\mathbb{C}$ with usual topology. For fixed $t_0 \in [0, 1]$, consider the evaluation functional $E_{t_0} : C[0, 1] \to \mathbb{C}$ given by $E_{t_0}(f) = f(t_0)$. Since $|f(t_0) - g(t_0)| \leq d_\infty(f, g)$, the convergence in sup metric implies point-wise convergence. It follows that E_{t_0} is continuous. Now $\{f \in C[0,1] : f(t_0) \neq 0\}$ is the complement of the null-space of E_{t_0}, and hence by the last exercise, it is open in $C[0,1]$.

Exercise 8.7 : If $f : X \to Y$ is continuous and a sequence $\{x_n\}$ in X converges to $x \in X$, show that the sequence $\{f(x_n)\}$ in Y converges to $f(x)$.

Proposition 8.8. *Let (X_i, Ω_i) $(i = 1, 2)$ be two topological spaces and let $\mathbb{B}_2$ be a basis for Ω_2. A function $f : X_1 \to X_2$ is continuous iff for any $B_2 \in \mathbb{B}_2$, the pre-image $f^{-1}(B_2)$ of V under f belongs to Ω_1.*

Proof. Suppose that for any $B_2 \in \mathbb{B}_2$, the pre-image $f^{-1}(B_2)$ of V under f belongs to Ω_1. Let V be an open subset of Ω_2. Then $V = \cup_\alpha B_\alpha$. Since $f^{-1}(V) = \cup_\alpha f^{-1}(B_\alpha)$, $f^{-1}(V)$ belongs to Ω_1. □

Example 8.9 : Consider the topological space $X_1 := (0, \infty)$ with the subspace topology inherited from $\mathbb{R}$. Then the function $f(x) = x^2$ from X_1 onto itself is continuous. In view of the last proposition, we need to check that the pre-image of any open interval in $(0, \infty)$ is open. However, note that for $0 < a < b < \infty$, $f^{-1}(a, b) = (\sqrt{a}, \sqrt{b})$. If X_2 denotes the topological space $(0, \infty)$ with lower limit topology then by the same argument, the function $f : X_1 \to X_2$ given by $f(x) = x^2$ is not continuous. Finally, note that the function $f : X_2 \to X_1$ given by $f(x) = x^2$ is continuous.

Exercise 8.10 : Let E be a non-empty proper subset of the topological space X. Consider the characteristic function χ_E of E. Show that χ_E is continuous iff E is a closed and open subset of X.

Exercise 8.11 : Consider topological spaces $X_i = (X, \Omega_i)$, i=1, 2 such that Ω_1 is finer than Ω_2. Show that $f : X_1 \to Y$ is continuous if so is $f : X_2 \to Y$.

The next result says that continuity is a *local* property.

Proposition 8.12. *Let (X_i, Ω_i) $(i = 1, 2)$ be two topological spaces. A function $f : X_1 \to X_2$ is continuous iff for each $x \in X$ and each neighbourhood V_x of $f(x)$, there is a neighbourhood U_x of x such that $f(U_x) \subseteq V_x$.*

Proof. If f is continuous then $U_x := f^{-1}(V_x)$ satisfies $f(U_x) \subseteq V_x$.

Conversely, suppose that for each $x \in X$ and each neighbourhood V_x of $f(x)$, there is a neighbourhood U_x of x such that $f(U_x) \subseteq V_x$. Let V be an open subset of X_2 and let $x \in f^{-1}(V)$. Then $f(x) \in V$, and hence by hypothesis, there is a neighbourhood U_x of x such that $f(U_x) \subseteq V$. But then $U_x \subseteq f^{-1}(V)$, and hence $f^{-1}(V)$ is open. □

We say that $f : X_1 \to X_2$ is *continuous at* $x \in X_1$ if for each neighbourhood V_x of $f(x)$, there is a neighbourhood U_x of x such that $f(U_x) \subseteq V_x$. By the last result, f is continuous iff f is continuous at every point.

Example 8.13 : Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = -x$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. We check that f is continuous only at 0.

To see that, first note that any neighbourhood V_0 of $f(0) = 0$ contains an interval $(-\epsilon, \epsilon)$ for some $\epsilon > 0$. Then $U_0 := (-\epsilon, \epsilon)$ satisfies $f(U_0) \subseteq (-\epsilon, \epsilon) \subseteq V_0$. Thus f is continuous at 0. If $x > 0$ then for $V_x := (x/2, 3x/2) \subseteq (0, \infty)$, there is no neighbourhood U_x of x such that $f(U_x) \subseteq V_x$ (since $f(U_x)$ always contains negative numbers). Similarly, one can see that f is not continuous at $x < 0$.

Exercise 8.14 : Let X be a topological space and Y be a topological space with ordered topology. Let $f, g : X \to Y$ be continuous functions. Show that the set $U := \{x \in X : f(x) > g(x)\}$ is open in X.

Hint. Let $x_0 \in U$. If $(g(x_0), f(x_0)) = \emptyset$ then $g^{-1}(-\infty, f(x_0)) \cap f^{-1}(g(x_0), \infty)$ is a neighbourhood of x_0 contained in U. Otherwise, for any $y \in (g(x_0), f(x_0))$, $g^{-1}(-\infty, y) \cap f^{-1}(y, \infty)$ is a neighbourhood of x_0 contained in U.

Exercise 8.15 : Prove: Composition of continuous functions is continuous.

Hint. Use Proposition 8.12.

Corollary 8.16. *Let (X_i, Ω_i) $(i = 1, 2)$ be topological spaces and let A be a subspace of X_1. If $f : X_1 \to X_2$ is continuous then so is map $f|_A : A \to X_2$.*

Proof. We apply the last proposition. Let $a \in A$ and let V_a be a neighbourhood of $f(a)$ in X_2. Since $A \subseteq X$, $a \in X$. Since the f is continuous, there is a neighbourhood U_a of a such that $f(U_a) \subseteq V_a$. But then $f(A \cap U_a) \subseteq f(U_a) \subseteq V_a$, where $A \cap U_a$ is a neighbourhood of a in the subspace topology. □

Remark 8.17 : The inclusion map $i : A \hookrightarrow X$ is always continuous.

One may wish to know whether or not the converse of Corollary 8.16 is true: For $X = \cup_\alpha A_\alpha$ and $f : X \to A$, if each $f|_{A_\alpha}$ is continuous then whether f is continuous ? (*Restriction Problem*). The answer is No even if each A_α is closed.

Example 8.18 : Let $X := [0, 1]$ and $A_0 := \{0\}$, $A_n := [1/n, 1]$. Then the function $f : [0, 1] \to \mathbb{R}$ given by $f(0) = 1, f(x) = 1/x$ $(0 < x \leq 1)$ is discontinuous at 0. However, $f|_{A_n}$ is continuous for every $n \geq 0$. ■

The answer to the Restriction Problem is yes if each A_α is open.

Corollary 8.19. *For $X = \cup_\alpha A_\alpha$ with open A_α and $f : X \to A$, if each $f|_{A_\alpha}$ is continuous then f is continuous.*

Proof. We apply Proposition 8.12. Let $x \in X$ and let V_x be a neighbourhood of x. Then $x \in A_\alpha$ for some α. Since $f|_{A_\alpha}$ is continuous, there exists a neighbourhood $U_{x,\alpha}$ of x such that $f(U_{x,\alpha} \cap A_\alpha) \subseteq V_x$, where $U_x := U_{x,\alpha} \cap A_\alpha$ is the desired neighbourhood of x. □

An indexed family $\{A_\alpha\}$ of subsets of X is *locally finite* if each $x \in X$ has a neighbourhood that intersects with finitely many A_α's. The family

$\{A_n\}$ as discussed in Example 8.18 is not locally finite. Any finite family is trivially locally finite (choose the neighbourhood to be the entire space).

The answer to the Restriction Problem is yes for any locally finite family of closed sets.

Proposition 8.20. *Let $\{A_\alpha$ be a locally finite family of closed sets such that $X = \cup_\alpha A_\alpha$ and $f : X \to Y$, if each $f|_{A_\alpha}$ is continuous then f is continuous.*

Proof. Let $x \in X$ and let U_x be a neighbourhood of x such that U_x intersects only with $A_{\alpha_1}, \cdots, A_{\alpha_k}$. It suffices to check that $g := f|_{U_x}$ is continuous. Note that $U_x = \cup_{i=1}^k B_i$, where each $B_i := A_{\alpha_i} \cap U_x$ is closed in the subspace topology on U_x and each $g|_{B_i}$ is continuous.

In view of Remark 8.1, it suffices to check that $g^{-1}(C)$ is closed for any closed subset C of Y. Note however that $g^{-1}(C) = \cup_{i=1}^k (g|_{B_i})^{-1}(C)$, where $(g|_{B_i})^{-1}(C)$ is closed in B_i and hence in U_x. □

We immediately obtain the following:

Corollary 8.21. *(Pasting Lemma) Let $X = A \cup B$, where A and B are closed subsets of X. Let $f : A \to Y$ and $g : B \to Y$ be continuous. If f and g agree on $A \cap B$ then the function $h : X \to Y$ defined by*

$$\begin{aligned} h(x) &= f(x) \text{ if } x \in A \\ &= g(x) \text{ if } x \in B \end{aligned}$$

is continuous.

Example 8.22 : Let X be a topological space and Y be a topological space with ordered topology. Let $f, g : X \to Y$ be continuous functions. Consider the function $m_{f,g}(x) := \max\{f(x), g(x)\}$ from X into Y. Then $m_{f,g}$ is continuous. To see that, consider the subsets $A_1 := \{x \in X : f(x) \leq g(x)\}$ and $A_2 := \{x \in X : g(x) \leq f(x)\}$ of X, and note that $m_{f,g}|_{A_1} = g|_{A_1}$ and $m_{f,g}|_{A_2} = f|_{A_2}$ are continuous. Also, A_1 and A_2 are closed subsets of X (Exercise 8.14). Since f and g agree on $A_1 \cap A_2$, the desired conclusion follows from the Pasting Lemma.

8.1. **A Theorem of Volterra Vito.**

Exercise 8.23 : Show that the function $g : (0, 1) \to \mathbb{R}$ given below is continuous on irrationals and discontinuous on rationals:

$$\begin{aligned} g(x) &= \frac{1}{q} \text{ if } x \in \mathbb{Q} \cap (0, 1) \text{ and } x = \frac{p}{q} \text{ in reduced form} \\ &= 0 \text{ otherwise.} \end{aligned}$$

We say that a set is a G_δ set if it is countable intersection of open sets.

Remark 8.24 : The irrationals $\mathbb{R}\backslash\mathbb{Q}$ form a G_δ set for $\mathbb{R}\backslash\mathbb{Q} = \bigcap_{r \in \mathbb{Q}} \mathbb{R}\backslash\{r\}$.

Exercise 8.25 : The rationals do not form a G_δ set.

Hint. Recall the Baire Category Theorem: *A countable intersection of open dense sets in $\mathbb{R}$ is again dense.*

Exercise 8.26 : Let f be a function defined on an open subset U. The set of points in U where a function $f : U \to \mathbb{R}$ is continuous is a G_δ set.

Hint. For positive integer n, consider the set A_n given by

$\{x_0 \in U : \exists$ an open nbhd V_n of x_0 such that $|f(x) - f(y)| < 1/n \ \forall \ x, y \in V_n\}$.

Theorem 8.27. *(Volterra Vito) There is no function $g : (0, 1) \to \mathbb{R}$ which is continuous on rationals and discontinuous on irrationals.*

9. Homeomorphisms

A function $f : X \to Y$ is a *homeomorphism* if f is continuous, one-one, and onto with continuous inverse. We say that X and Y are homeomorphic if there exists a homeomorphism $f : X \to Y$.

Remark 9.1 : Let $f : X \to Y$ be a homeomorphism. Here are some key observations to decide whether or not a given function is homeomorphism.

(1) If f is a homeomorphism then so is f^{-1}.
(2) If U is open in X then $f(U)$ is open in Y.
(3) If A is a subset of X then A and $f(A)$ are homeomorphic.
(4) Composition of homeomorphisms is again a homeomorphism. In particular, if X is homeomorphic to Y, and Y is homeomorphic to Z then X is homeomorphic to Z.

Exercise 9.2 : Show that a bijective continuous map from a compact metric space into a metric space sends closed sets to closed sets, and hence it is a homeomorphism.

Example 9.3 : Define $f : \mathbb{R} \to [0, 1]$ by $f(x) = x$ if $|x| \leq 1$, and $f(x) = 1/|x|$ if $|x| \geq 1$. Then f is continuous on $\mathbb{R}$. Note that f is onto but not one-one.

Exercise 9.4 : Show that the interval $(a, b) \subseteq \mathbb{R}$ is homeomorphic to any other interval $(c, d) \subseteq \mathbb{R}$.

Hint. Try $\alpha(t - b) + \beta(t - a)$ for appropriate scalars α and β.

Exercise 9.5 : Show that e^{-x} is a homeomorphism from $(0, \infty)$ onto $(0, 1)$.

Example 9.6 : We check that $X = \mathbb{R}$ with VIP topology and $Y = \mathbb{R}$ with outcast topology (with base point 0) are not homeomorphic. In fact, if there exists a homeomorphism ϕ between X and Y, then $\phi([-n, n])$ is an open set in the outcast topology, and hence $0 \notin \phi([-n, n])$ for any $n \geq 1$. However, $\cup_n \phi([-n, n]) = \phi(X) = Y$, which is impossible.

A relatively simpler argument I learnt from one of the students. Suppose 0 is mapped to y. Consider an open set V in the outcast topology which excludes y. Then inverse image U of V can not contain 0, and hence U is not open in the VIP topology, a contradiction.

Exercise 9.7 : Which of spaces X and Y are homeomorphic:

(1) $X = \mathbb{R}$ and $Y = [0, 1)$
(2) $X = \mathbb{R}$ and $Y = [0, 1]$
(3) $X = [1, \infty)$ and $Y = (0, 1]$
(4) $X = \mathbb{R}$ and $Y = (0, 1)$
(5) $X = \mathbb{Q}$ and $Y = \mathbb{Z}$
(6) $X = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z, r < z < R\}$ and $Y = A(0, r, R)$
(7) $X = \{x \in \mathbb{R}^{n+1} : x_{n+1} = 0\}$ and $Y = \mathbb{R}^n$

Hint. Use Remark 9.1 to prove that two spaces are not homeomorphic. For (1), (2), use Intermediate Value Theorem while for (5), choose a neighbourhood of an integer which contains finitely many elements, and analyze its image in $\mathbb{Q}$. For (4), write $X = (-\infty, 1) \cup [1, \infty)$, and note that by (3), $[1, \infty) \cong (0, 1]$. Also, $(-\infty, 1) \cong (-\infty, -1) \cong (-1, 0)$.

Exercise 9.8 : Verify that $\phi(t) = e^{it}$ is a homeomorphism between $(0, 2\pi)$ and $\mathbb{T} \setminus \{1\}$, where $\mathbb{T}$ denotes the unit circle in the complex plane. Conclude that unit circle minus a point is homeomorphic to the real line.

Proposition 9.9. *$\mathbb{R}^n$ is homeomorphic to $\mathbb{R}$ iff $n = 1$.*

Proof. Suppose that for $n > 1$, there is a continuous bijection $f : \mathbb{R}^n \to \mathbb{R}$. By Remark 9.1, $g = f|_X$ is a homeomorphism from $X := \mathbb{R}^n \setminus \{0\}$ onto $Y := \mathbb{R} \setminus \{y_0\}$ for some $y_0 \in \mathbb{R}$. Choose $y_1, y_2 \in Y$ such that $y_1 < y_0 < y_2$ and let $x_1, x_2 \in X$ be such that $f(x_1) = y_1$ and $f(x_2) = y_2$.

Let L denote the line segment connecting x_1 and x_2. If L does not pass through 0 then let $\gamma(t) = (1 - t)x_1 + tx_2$. If L passes through 0 then choose any point $x_3 \notin L$ and let $\gamma(t)$ be given by

$$\begin{aligned} \gamma(t) &= (1 - 2t)x_1 + 2tx_3 \text{ if } 0 \leq t \leq 1/2 \\ &= 2(1 - t)x_3 + tx_2 \text{ if } 1/2 \leq t \leq 1. \end{aligned}$$

Thus $\gamma : [0, 1] \to \mathbb{R}$ is a continuous function such that $\gamma(0) = x_1$ and $\gamma(1) = x_2$. We consider the continuous function $h : [0, 1] \to [y_1, y_2]$ by $h(t) = g(\gamma(t))$. By the Intermediate Value Theorem, there exists $t_0 \in [0, 1]$ such that $h(t_0) = y_0$, which then implies that y_0 belongs to the range of g, a contradiction. □

Remark 9.10 : It is highly non-trivial fact that $\mathbb{R}^n$ is homeomorphic to $\mathbb{R}^m$ iff $m = n$.

Exercise 9.11 : Show that $X = A(0, r, R)$ (open annulus) and $Y = \mathbb{T}$ (unit circle) are not homeomorphic.

Hint. If X and Y are homeomorphic with homeomorphism $\phi : X \to Y$ then $X \setminus \{x_1\}$ is homeomorphic to $\mathbb{T} \setminus \{\phi(x_1)\} \cong \mathbb{R}$. Now argue as in the preceding proposition.

Exercise 9.12 : Let $X = S^n \setminus \{(0, \cdots, 1)\} \subseteq \mathbb{R}^{n+1}$ and $Y = \mathbb{R}^n$, and define $f : X \to Y$ and $g : Y \to X$ by

$$\begin{aligned} f(x_1, \cdots, x_{n+1}) &= \left(\frac{x_1}{1 - x_{n+1}}, \cdots, \frac{x_n}{1 - x_{n+1}}\right), \\ g(x_1, \cdots, x_n) &= \left(\frac{2x_1}{\|x\|_2^2 + 1}, \cdots, \frac{2x_n}{\|x\|_2^2 + 1}, \frac{\|x\|_2^2 - 1}{\|x\|_2^2 + 1}\right) \end{aligned}$$

Verify that f and g are homeomorphisms such that $f^{-1} = g$.

Exercise 9.13 : Show that the graph G of a continuous function $f : X \to Y$ (with the subspace topology inherited from $X \times Y$) is homeomorphic to X.

Hint. Define $\phi : G \to X$ by $\phi(x, f(x)) = x$. Note that if U is an open subset of X then $\phi^{-1}(U) = \{(x, f(x)) : x \in U\} = (U \times Y) \cap G$.

Example 9.14 : Consider the continuous function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ by $f(x) = 1/x$. The graph of f is the hyperbola $xy = 1$. One may conclude from the last exercise that $\mathbb{R} \setminus \{0\}$ is homeomorphic to $xy = 1$.

10. Product, Box, and Uniform Topologies

By $\prod_{i=1}^m X_i$, we mean the cartesian product $X_1 \times \cdots X_m$ of $X_1, \cdots, X_m$.

Example 10.1 : Consider the metric spaces (X_i, d_i), $i = 1, \cdots, m$. Check that $d(x, y) = \max_i d_i(x_i, y_i)$ defined on $X = \prod_{i=1}^m X_i$ is a metric. The metric topology coincides with the product topology on X. This may be concluded from the fact that for $x = (x_1, \cdots, x_m)$, $\mathbb{B}_d(x, r) = \prod_{i=1}^m \mathbb{B}_{d_i}(x_i, r)$.

Exercise 10.2 : Show that projections $\pi_j : \prod_{i=1}^m X_i \to X_j$ is continuous.

Hint. $(\pi_j)^{-1}(U) = X_1 \times X_{j-1} \times U \times X_{j+1} \times X_m$.

Proposition 10.3. *Let $f : A \to \prod_{i=1}^m X_i$ be given by $f(a) = (f_1(a), \cdots, f_m(a))$, where the functions $f_i : A \to X_i$ $(i = 1, \cdot, m)$ are given. Then f is continuous iff each f_i is continuous.*

Proof. If each f is continuous then so is $P_i \circ f = f_i$. If each f_i is continuous then f is continuous since $f^{-1}(\prod_{i=1}^m U_i) = \cap_{i=1}^m (f_i)^{-1}(U_i)$. □

We wish to have an analog of the last result for arbitrary family $\{X_\alpha\}$ of topological spaces. For that, it is necessary to understand the topology on $\prod_{\alpha \in I} X_\alpha$. We expect that such a topology should be consistent with the product topology on finite products, and also it should yield the continuity of projections. There are however two natural candidates for a topology on $\prod_{\alpha \in I} X_\alpha$.

Exercise 10.4 : Verify that $\mathbb{B}_b$ and $\mathbb{B}_p$ are basis for a topology on $\prod_{\alpha}^{m} X_\alpha$:

(1) $\mathbb{B}_b := \{\prod_{\alpha\in I} U_\alpha : U_\alpha \text{ is open in } X_\alpha\}$.
(2) $\mathbb{B}_p := \{\prod_{\alpha\in I} U_\alpha \in \mathbb{B}_b : U_\alpha = X_\alpha \text{ for finitely many values of } \alpha\}$.

Definition 10.5 : The *box topology* is the topology generated by the basis $\mathbb{B}_b$, and the *product topology* is the topology generated by the basis $\mathbb{B}_p$.

Remark 10.6 : Since $\mathbb{B}_p \subseteq \mathbb{B}_b$, we have $\Omega_{\mathbb{B}_p} \subseteq \Omega_{\mathbb{B}_b}$, that is, the box topology is finer than the product topology.

Exercise 10.7 : Suppose that each X_α contains a non-empty proper open subset U_α. Show that $\Omega_{\mathbb{B}_p} = \Omega_{\mathbb{B}_b}$ iff I has finite cardinality.

Hint. If $|I|$ is infinite then $\prod_{\alpha\in I} U_\alpha$ does not belong to $\Omega_{\mathbb{B}_p}$.

Remark 10.8 : The projection $\pi_\alpha(x_\alpha) = x_\alpha$ is continuous in box and product topologies.

Exercise 10.9 : Show that the product topology is the smallest topology which makes all projections π_α continuous.

Exercise 10.10 : Let $f : A \to \prod_\alpha X_\alpha$ be given by $f(a) = (f_\alpha(a))$, where the functions $f_\alpha : A \to X_\alpha$ are given. Suppose $\prod_\alpha X_\alpha$ has either box topology or product topology. Show that if f is continuous then each f_α is continuous.

Example 2 on page 117 of [4] shows that it may happen that each f_α is continuous but (f_α) is discontinuous at a point in case the product space $\prod_{\alpha\in I} X_\alpha$ carries box topology. This is not possible if the product space $\prod_{\alpha\in I} X_\alpha$ carries product topology.

Proposition 10.11. *Let $f : A \to \prod_\alpha X_\alpha$ be given by $f(a) = (f_\alpha(a))$, where the functions $f_\alpha : A \to X_\alpha$ are given. Suppose $\prod_\alpha X_\alpha$ has product topology. Show that f is continuous iff each f_α is continuous.*

Proof. Suppose $U_\alpha = X_\alpha$ for finitely many values of α, say $\alpha_1, \cdots, \alpha_m$. Then

$$\begin{aligned} f^{-1}\Big(\prod_{\alpha\in I} U_\alpha\Big) &= \{a \in A : f(a) \in \prod_{\alpha\in I} U_\alpha\} \\ &= \{a \in A : f_{\alpha_i}(a) \in U_\alpha \text{ for } i = 1, \cdots, m\} = \cap_{i=1}^{m} (f_\alpha)^{-1}(U_{\alpha_i}). \end{aligned}$$

The desired conclusion follows from Proposition 8.8. □

Recall that if any continuous function sends convergent sequences to convergent sequences. In particular, if $\{x_n = (x_{n\alpha})\}$ is a convergent sequence in $\prod_\alpha X_\alpha$ then $\{\pi_\alpha(x_n) = x_{n\alpha}\}$ is also convergent in X_α. The converse is not true if $\prod_{\alpha\in I} X_\alpha$ carries box topology.

Example 10.12 : Consider the product space $\mathbb{R}^\omega$ (that is, countably infinite product of $\mathbb{R}$ with itself) with box topology. Consider the sequence

$\{x_n = (1/(n+1), 1/(n+2), \cdots,)\}$ in $\mathbb{R}^\omega$. Note that for any integer $m \geq 1$, $\{\pi_m(x_n) = 1/(n+m)\}$ converges to 0. However, the sequence $\{x_n\}$ does not converge to $(0, 0, \cdots,)$ in the box topology. Indeed, if we consider the neighbourhood $N_0 := \prod_n(-1/n^2, 1/n^2)$ of $(0, 0, \cdots,)$ then there is no positive integer N such that $x_N \in N_0$.

Proposition 10.13. *Let $\{x_n = (x_{n1}, x_{n2}, \cdots,)\}$ be a sequence in the product space $\prod_{\alpha\in I} X_\alpha$ with product topology. Then the sequence $\{x_n\}$ converges to $x = (x_1, x_2, \cdots,) \in \prod_\alpha X_\alpha$ iff for every positive integer m, $\{\pi_m(x_n) = x_{nm}\}$ converges to x_m.*

Proof. Suppose that for every positive integer m, $\{\pi_m(x_n) = x_{nm}\}$ converges to x_m. Let $\prod_{\alpha\in I} U_\alpha$ be an open neighbourhood of x in the product topology. Thus there exist $\alpha_1, \cdots, \alpha_k \in I$ such that $U_\alpha = X_\alpha$ for every $\alpha \neq \alpha_1, \cdots, \alpha_k$. Fix $i = 1, \cdots, k$. Since $\{\pi_i(x_n) = x_{ni}\}$ converges to x_i, there exists positive integer N_i such that $x_{ni} \in U_{\alpha_i}$ for all $n \geq N_i$. Check that $x_n \in \prod_{\alpha\in I} U_\alpha$ for all $n \geq \max\{N_1, \cdots, N_k\}$. $\square$

Exercise 10.14 : Show that $d : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ given by $d(x, y) = \min\{|x-y|, 1\}$ defines a metric on $\mathbb{R}$. Verify further that the metric topology induced by d coincide with the usual topology on $\mathbb{R}$.

Given a non-empty index set I, consider the product space $\mathbb{R}^I := \prod_{\alpha\in I} \mathbb{R}$ of the product of $|I|$ number of copies of $\mathbb{R}$. Define $d_u : \mathbb{R}^I \times \mathbb{R}^I \to \mathbb{R}$ by

$$d_u((x_\alpha), (y_\alpha)) = \sup\{\min\{|x_\alpha - y_\alpha|, 1\} : \alpha \in I\}.$$

It is easy to see that d_u defines a metric on $\mathbb{R}^I$. Check that $\mathbb{B}_u := \{\mathbb{B}_{d_u}(x, r) : x \in \mathbb{R}^I, r > 0\}$ forms a basis for a topology on $\mathbb{R}^I$. Let $\Omega_{\mathbb{B}_u}$ be the metric topology generated by $\mathbb{B}_u$.

Theorem 10.15. $\Omega_{\mathbb{B}_p} \subseteq \Omega_{\mathbb{B}_u} \subseteq \Omega_{\mathbb{B}_b}$ *with strict inclusions if I is infinite.*

Proof. We first note that any open subset of $\mathbb{R}$ in the usual topology is open in the metric topology induced by $d(x, y) = \min\{|x - y|, 1\}$ (If $d(x, y) < 1$ then $|x - y| < 1$). In particular, if $\prod_{\alpha\in I} U_\alpha$ is open in the product topology then it is open in the uniform topology. Also, $\prod_{\alpha\in I}(x_\alpha - r/2, x_\alpha + r/2) \subseteq B_{d_u}(x, r)$ for any positive real number $r < 1$, so the box topology is finer than the uniform topology.

Suppose that I is infinite. Let us see that both the inclusions are strict. Let $x_0 = (0, 0, \cdots,)$. Since I is infinite, there exists a sequence $\{i_k\}$ contained in I. Consider $U := \prod_{\alpha\in I} U_\alpha$, where $U_\alpha = (-1/k, 1/k)$ if $\alpha = i_k$, and $U_\alpha = \mathbb{R}$ otherwise. Note that there is no $0 < r < 1$ such that $\mathbb{B}_{d_u}(x_0, r) \subseteq U \in \Omega_{\mathbb{B}_b}$. In fact, $(r/2, r/2, \cdots,) \in \mathbb{B}_{d_u}(x_0, r)$ but it does not belong to U. This shows that $U \notin \Omega_{\mathbb{B}_u}$. Also, since $\mathbb{B}_{d_u}(x_0, 1) \subseteq \prod_n(-1, 1)$, there is no open set U in the product topology which is contained in $\mathbb{B}_{d_u}(x_0, 1)$. That is, $\mathbb{B}_{d_u}(x_0, 1) \in \Omega_{\mathbb{B}_u}$ but $\mathbb{B}_{d_u}(x_0, 1) \notin \Omega_{\mathbb{B}_p}$. $\square$

A topological space (X, Ω) is said to be *metrizable* if there exists a metric $d : X \times X \to \mathbb{R}$ such that Ω coincides with the topology generated by the basis $\{\mathbb{B}_d(x, r) : x \in X, r > 0\}$.

Example 10.16 : Define $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ by

$$\begin{aligned} d((x_1, y_1), (x_2, y_2)) &= \max\{1, |y_1 - y_2|\} \text{ if } x_1 \neq x_2 \\ &= |y_1 - y_2| \text{ otherwise.} \end{aligned}$$

Check that d is a metric. Moreover, for $r \leq 1$,

$$\mathbb{B}_d((x_1, y_1), r) = \{(x_1, y) \in \mathbb{R}^2 : |y - y_1| < r\},$$

and for $r > 1$, $\mathbb{B}_d((x_1, y_1), r) = \mathbb{R}^2$. We note that the metric topology coincides with the dictionary order topology on the plane. Hence the plane $\mathbb{R}^2$ with dictionary order topology is metrizable.

Theorem 10.17. *The product space $\mathbb{R}^\omega$ with box topology is not metrizable.*

Proof. We use the following property of metric spaces. If (X, d) is a metric space then for any cluster point x of $A \subseteq X$, there exists a sequence $\{x_n\} \subseteq A$ such that $d(x_n, x) \to 0$. Indeed, for $n \geq 1$, let $x_n \in (\mathbb{B}_d(x, 1/n) \setminus \{x\}) \cap A$.

We claim that there is no sequence in $\mathbb{R}^\omega$ which converges to the cluster point $x_0 := (0, 0, \cdots,)$ in the box topology of $A := \prod_n \mathbb{R} \setminus \{0\}$. Contrary to this, suppose there exists a sequence $\{x_m := (x_{nm})_{n \geq 1}\} \subseteq A$ such that x_n converges to x_0. Consider the neighbourhood $\prod_n \mathbb{R} \setminus \{x_{nn}\}$ of x_0, and note that there is no positive integer $N \geq 1$ such that $x_m \in \prod_n \mathbb{R} \setminus \{x_{nn}\}$ for every $m \geq N$. Hence we arrive at a contradiction to the assumption that $\{x_m\}$ converges to x_0. □

Exercise 10.18 : Show that $D : \mathbb{R}^\omega \times \mathbb{R}^\omega \to \mathbb{R}$ given by

$$D((x_n), (y_n)) := \sup_n \frac{\min\{|x_n - y_n|, 1\}}{n}$$

defines a metric on $\mathbb{R}^\omega$. Verify further that $D \leq d_u$, where d_u is the uniform metric.

Remark 10.19 : Note that $\mathbb{B}_{d_u}(x, r) \subseteq \mathbb{B}_D(x, r)$. Thus uniform topology $\Omega_{\mathbb{B}_u}$ is finer than the topology generated by the metric D.

We refer the reader to [4] for the proof of the following:

Theorem 10.20. *The product space $\mathbb{R}^\omega$ with product topology is metrizable with metric D.*

11. Compact Spaces

Let X be a topological space X with topology Ω. By *open cover* $\{U_\alpha\}$ of X, we mean a collection $\{U_\alpha\} \subseteq \Omega$ such that $X \subseteq \cup_\alpha U_\alpha$. By a *finite subcover* of $\{U_\alpha\}$, we mean finite subset $\{U_{\alpha_1}, \cdots, U_{\alpha_k}\}$ of $\{U_\alpha\}$ such that $X \subseteq U_{\alpha_1} \cup \cdots \cup U_{\alpha_k}$.

We say that X is *compact* if any open cover of X admits a finite subcover.

Exercise 11.1 : Show that any finite set is compact.

A subspace Y of X is said to be *compact in* X if Y is compact in the subspace topology.

Example 11.2 : The real line with usual topology is not compact: The open cover $\{(-n, n)\}$ has no finite subcover.

The set of rationals with subspace topology is not compact: The open cover $\{(-n, n) \cap \mathbb{Q}\}$ has no finite subcover.

The set of integers with discrete topology is not compact: The open cover $\{\{n\}\}$ has no finite subcover.

The open ball $\mathbb{B}_d(x, r)$ in $\mathbb{R}^n$ is not compact: If $N > 1/r$ then the open cover $\{\mathbb{B}_d(x, r - 1/n\}_{n \geq N}$ has no finite subcover.

The set $\{1/n\}_{n\geq 1} \cup \{0\}$ is compact.

Exercise 11.3 : Let $K \subseteq Y \subseteq X$. Show that K is compact in X iff K is compact in Y.

We invoke a classical result from Analysis, which characterizes all compact subsets of $\mathbb{R}^n$.

Theorem 11.4. *A subset A of $\mathbb{R}^n$ is compact iff it is closed and A is contained some ball of finite radius.*

Example 11.5 : Let p be a polynomial in the real variables x, y. Then the zero set $Z(p)$ of p is always closed. However, $Z(p)$ may or may not be compact. For instance, if $p(x, y) = x^2 + y^2 - 1$ then $Z(p)$ being the unit circle is compact. If $p(x, y) = x$ then $Z(p)$ is the Y-axis, which is certainly non-compact.

Exercise 11.6 : Let p be a non-constant polynomial in the complex variables $z_1, \cdots, z_m$. Show that the zero set $Z(p)$ of p is compact iff $m = 1$.

Hint. Suppose $m = 2$. We check that $Z(p)$ is unbounded. To see that, fix a positive integer n. Write $p(z, w) = \sum_{i=0}^{d} p_i(z)w^i$ for some polynomials $p_0, p_1, \cdots, p_d$ in the complex variable z. One may now choose z_n such that $|z_n| \geq n$ and $z_n \in \mathbb{C} \setminus Z(p_1)$. But then $p(z_n, w)$ is a non-constant polynomial in the complex variable w. By the Fundamental Theorem of Algebra, there exists w_n such that $p(z_n, w_n) = 0$. We thus obtain $(z_n, w_n) \in Z(p)$ with $\|(z_n, w_n)\| \geq n$, that is, $Z(p)$ is unbounded.

Exercise 11.7 : Show that every subset of $\mathbb{R}$ with co-finite topology is compact.

Theorem 11.8. *Let X be a compact topological space and let Y be a subspace of X. Then the following are true:*

(1) *If Y is closed then Y is compact.*
(2) *If X is Hausdorff then Y is closed iff Y is compact.*

Proof. (1) If $\{U_\alpha \cap Y\}$ is an open covering of Y (with open subsets U_α of X) then $\{U_\alpha\} \cup \{X \setminus Y\}$ is an open cover of X. Now use compactness of X.

(2) We check that $X \setminus Y$ is open. Let $x_0 \in X \setminus Y$. Since X is Hausdorff, one can choose neighbourhoods U_y and V_y of x_0 and y such that $U_y \cap V_y = \emptyset$. Now $Y \subseteq \cup_{y \in Y} V_y$ and Y is compact. So there exists a finite subcover $\{V_{y_1}, \cdots, V_{y_k}\}$ of Y. But then $U_{y_1} \cap \cdots \cap U_{y_k}$ is a neighbourhood of x_0 which is disjoint from Y. □

Remark 11.9 : The above result is not true in case the condition that X is Hausdorff is relaxed. In fact, every subset (closed or non-closed) of $\mathbb{R}$ with co-finite topology is compact.

Exercise 11.10 : Show that the continuous image of a compact space is again compact.

Hint. If f is continuous then $\{V_\alpha\}$ is an open cover of $f(X)$ iff $\{f^{-1}(V_\alpha)\}$ is an open cover of X.

Remark 11.11 : Compactness is a topological property: If two spaces X and Y are homeomorphic then X is compact iff Y is compact.

Theorem 11.12. *Let $f : X \to Y$ be a continuous bijection. If X is compact and Y is Hausdorff then f is a homeomorphism.*

Proof. To see that f^{-1} is continuous, it suffices to check that $f(U)$ is open in Y for any open subset U of X. Note that $X \setminus U$ being closed in X is actually compact. By the last exercise, $f(X \setminus U)$ is compact, and since Y is Hausdorff, $f(X \setminus U)$ is closed in Y. But $Y \setminus f(U) = f(X \setminus U)$, so $f(U)$ is open in Y as desired. □

12. Quotient Topology

Let X be a set with an equivalence relation $\sim$. Given $x \in X$, let $[x] := \{y \in X : y \sim x\}$ be an equivalence class containing x. Let $X/\sim$ denote the set of all equivalence classes of elements in X.

Example 12.1 : Let $X := [0,1]$ and define an equivalence relation $\sim$ on X as follows: If $x, y \in (0,1)$ then $x \sim y$ iff $x = y$. If $x, y \in \{0,1\}$ then $x \sim y$. Note that $X/\sim = \{[x] = \{x\} : 0 < x < 1\} \cup \{[0] = \{0,1\}\}$.

Our primary aim in this section is to introduce a "best possible" topology (of course inherited from X) on $X/\sim$, which will make $X/\sim$ a topological space. To do that, suppose (X, Ω) is a topological space, and consider the *quotient map* $q : X \to X/\sim$ given by $q(x) = [x]$. Since q is surjective, we endow with $X/\sim$, the *quotient topology* Ω_q:

$$\Omega_q := \{U \subseteq X/\sim : q^{-1}(U) \in \Omega\}.$$

Let us check that Ω_q is indeed a topology. Clearly, $q^{-1}(\emptyset) = \emptyset \in \Omega$ and $q^{-1}(X/\sim) = X \in \Omega$, so that $\emptyset, X/\sim$ belong to Ω_q. If $\{U_\alpha\} \subseteq \Omega_q$ then $q^{-1}(U_\alpha) \in \Omega$, and hence $q^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha q^{-1}(U_\alpha) \in \Omega$. It follows that Ω_q is closed under arbitrary union. Along similar lines, we may check that Ω_q is closed under finite intersection.

Remark 12.2 : Note that the quotient map $q : (X, \Omega) \to (X/\sim, \Omega_q)$ is a continuous surjection. In particular, if X is compact then so is $X/\sim$.

Example 12.3 : Let X and $\sim$ be as in Example 12.1. Let us understand q and Ω_q in this case. Note that $q(x) = \{x\}$ for $0 < x < 1$, and $q(0) = \{0, 1\} = q(1)$. Let $U \in \Omega_q$. Then there are two possibilities: either $U \subseteq q((0, 1))$, or U contains $q(0) = q(1)$. Consequently, $q^{-1}(U)$ is an open subset of $(0, 1)$ (containing precisely those points x for which $\{x\} \in U$) or $q^{-1}(U)$ is an open subset, which contains two open sets V_1 and V_2 in the subspace topology containing points 0 and 1 respectively. Note that $\{[x] : x \in (1/2, 1]\}$ is not open in the quotient topology on $X/\sim$. However, $\{[x] : 0 \leq x < 1/2\} \cup \{[x] : 1/2 < x \leq 1\}$ belongs to Ω_q.

Remark 12.4 : It's clear that the quotient space $X/\sim$ above is obtained by identifying the points 0 and 1 in $[0, 1]$ without disturbing the points in $(0, 1)$. This suggests one to believe that the topological space $X/\sim$ could be identified with the unit circle.

The assertion in the last remark can be made precise.

Theorem 12.5. *Let X and Y be compact spaces. Assume further that Y is Hausdorff. Let $f : X \to Y$ be a continuous surjection. Define the equivalence relation $\sim$ on X by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then $g : X/\sim \to Y$ given by $g([x]) = f(x)$ is a well-defined homeomorphism.*

Proof. By Remark 12.2, $q(X) = X/\sim$ is compact. In view of Theorem 11.12, it suffices to check that $g : X/\sim \to Y$ is a continuous bijection.

g is a bijection: Note that $g \circ q = f$. Since f is a surjection, so is g. Also, if $g([x_1]) = g([x_2])$ then $f(x_1) = f(x_2)$, and hence $x_1 \sim x_2$, so that $[x_1] = [x_2]$. Thus g is injective.

g is continuous: We know that $g \circ q = f$ is continuous. For an open set V in Y, note that $f^{-1}(V) = (g \circ q)^{-1}(V) = q^{-1}(g^{-1}(V))$ is an open subset of X. By the definition of quotient topology, $g^{-1}(V)$ is open in $X/\sim$. □

Remark 12.6 : Note that the level sets $\{x \in X : f(x) = \lambda\}$ of f are precisely the equivalence classes in $X/\sim$

Let $\mathbb{T}$ denote the unit circle in the plane. Consider the function $f : [0, 1] \to \mathbb{T}$ given by $f(t) = e^{2\pi i t}$. Then f is a continuous surjection. Also, for $x_1 < x_2$, $f(x_1) = f(x_2)$ iff $x_1 = 0, x_2 = 1$. By the preceding theorem, $g([t]) = e^{2\pi i t}$ is a homeomorphism from $[0, 1]/\sim$ onto the unit circle $\mathbb{T}$.

Exercise 12.7 : Let $X := \{(u,v) \in \mathbb{R}^2 : -\pi \leq u \leq \pi, -1 \leq v \leq 1\}$ (Rectangle) and let $Y := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1, |z| \leq 1\}$ (Cylinder). Verify the following:

(1) If $u - u' = \pm 2\pi$ then define $(u,v) \sim (u',v')$ iff $v = v'$. Otherwise, define $(u,v) \sim (u',v')$ iff $(u,v) = (u',v')$. Then $\sim$ defines an equivalence relation on X.

(2) $f : X \to Y$, $f(u,v) = (\cos u, \sin u, v)$ is a continuous surjection.

Conclude that $X/\sim$ and Y are homeomorphic.

Exercise 12.8 : Let $X := [0,1] \times [0,1]$ (Unit Square) and let $Y := \mathbb{T} \times \mathbb{T}$ (Torus). Verify the following:

(1) If $s_1 - s_2 = \pm 1$ then define $(s_1,t_1) \sim (s_2,t_2)$ iff $t_1 = t_2$. If $t_1 - t_2 = \pm 1$ then define $(s_1,t_1) \sim (s_2,t_2)$ iff $s_1 = s_2$. If $(s_1,t_1),(s_2,t_2) \in [0,1] \times [0,1]$ then $(s_1,t_1) \sim (s_2,t_2)$ provided $s_1 = s_2$ and $t_1 = t_2$. Then $\sim$ defines an equivalence relation on X.

(2) $f : X \to Y$ given by $f(s,t) = (e^{2\pi is}, e^{2\pi it})$ is a continuous surjection.

Conclude that $X/\sim$ and Y are homeomorphic.

Example 12.9 : Let $X := [0,1] \times [0,1]$. Define the equivalence relation $\sim$ on X as follows: $(x,y) \sim (x',y')$ iff either ($x = 0$, $x' = 1$ and $y = 1 - y'$) or ($x = x'$ and $y = y'$). The quotient space is known as the *Möbius strip.*

Define the equivalence relation $\sim$ on X as follows: $(x,y) \sim (x',y')$ iff either ($x = 0$, $x' = 1$) or ($x = 1 - x'$, $y = 0$, $y' = 1$) or ($x = x'$ and $y = y'$). The quotient space is known as the *Klein's bottle.*

Exercise 12.10 : Let $X := \overline{\mathbb{D}} \subseteq \mathbb{R}^2$ (Closed Unit Disc) and let $Y := \mathbb{S}^2 \subseteq \mathbb{R}^3$ (Unit Sphere). Verify the following:

(1) If $z, w \in \mathbb{T}$ then set $z \sim w$. If $z, w \in \mathbb{D}$ then $z \sim w$ provided $z = w$. Then $\sim$ defines an equivalence relation on X.

(2) $f : X \to Y$, $f(tx,ty) = (\cos\pi(1-t), x\sin\pi(1-t), y\sin\pi(1-t))$ for $0 \leq t \leq 1, (x,y) \in \mathbb{T}$ is a continuous surjection.

Conclude that $X/\sim$ and Y are homeomorphic.

Let X be a compact Hausdorff space and let A be a closed subset of X. Consider the topological space $Y := (X \setminus A) \cup \{\infty\}$ with the topology Ω_Y

$$\Omega_Y := \{U \subseteq X \setminus A : U \text{ is open}\} \cup \{Y \setminus C : C \text{ is compact in } X \setminus A\}.$$

It is easy to see that Ω_Y defines a topology on Y.

Lemma 12.11. *If X is compact Hausdorff then so is Y.*

Proof. Y is compact: Let $\{U_\alpha\} \cup \{V_\beta\}$ be an open cover of Y, where $U_\alpha \subseteq X \setminus A$ is open is X, and $V_\beta = Y \setminus C_\beta$ for some compact set C_β in $X \setminus A$. Fix β_0. Note that the compact set C_{β_0} has the open cover $\{U_\alpha\} \cup \{V_\beta \cap (X \setminus A)\}$, and hence there exists finite subcover $\{U_{\alpha_i}\} \cup \{V_{\beta_j} \cap (X \setminus A)\}$ of C_{β_0}. It follows that Y has the finite subcover $\{U_{\alpha_i}\} \cup \{V_{\beta_j}\} \cup \{V_{\beta_0}\}$.

Y is Hausdorff: Let $x, y \in Y$ be two distinct points. Since X is Hausdorff, we may assume without loss of generality that $x \in X \setminus A$ and $y = \infty$. One may choose a neighbourhood U of x such that $\overline{U} \subseteq X \setminus A$ (see the proof of Theorem 11.8(2)). Since $\overline{U}$ is a subset of $X \setminus A$ and since it is compact in X, it is also compact in $X \setminus A$. Then U and $Y \setminus \overline{U}$ are disjoint neighbourhoods of x and y. $\square$

Although, $X \setminus A$ may not be compact, we see that Y is always compact. Any space homeomorphic to Y is known as the *one-point compactification* of $X \setminus A$.

Corollary 12.12. *(One-point Compactification) Let X be a compact Hausdorff space and let A be a closed subset of X. Let $\sim$ be the equivalence relation defined on X as follows: If $x_1, x_2 \in A$ then $x_1 \sim x_2$. If $x_1, x_2 \in X \setminus A$ then $x_1 \sim x_2$ iff $x_1 = x_2$. Consider the topological space $Y := (X \setminus A) \cup \{\infty\}$ with topology Ω_Y as discussed above. Then Y is homeomorphic to $X/\sim$.*

Proof. Define $f : X \to Y$ by $f(x) = x$ if $x \in X \setminus A$, and $f(a) = \infty$ if $a \in A$. Clearly, f is a surjection with level sets precisely the equivalence classes in $X/\sim$. We check that f is continuous. Let $U \in \Omega_Y$. If $U \subseteq X \setminus A$ then $f^{-1}(U) = U$, which is open in X. If $U = Y \setminus C$ for some compact C in $X \setminus A$ then $f^{-1}(U) = f^{-1}(Y) \setminus f^{-1}(C) = X \setminus C$, which is open since C is closed in X. By Theorem 12.5, the space $X/\sim$ is homeomorphic to Y. $\square$

Example 12.13 : One may apply Exercise 12.10 to the last theorem with $X := \overline{\mathbb{D}}$ and $A := \mathbb{T}$ to conclude that the one-point compactification of the open unit disc is the sphere $\mathbb{S}^2$ in $\mathbb{R}^3$.

Exercise 12.14 : Find one-point compactifications of $\mathbb{R}$ and $\mathbb{R}^2$.

Hint. Note that $\mathbb{R}$ is homeomorphic to $\mathbb{T} \setminus \{1\}$. By the last corollary, the one-point compactification of $\mathbb{T} \setminus \{1\}$ equals $(\mathbb{T} \setminus \{1\}) \cup \{\infty\} \cong \mathbb{T}$.

A map $f : X \to Y$ is said to be *open* if f sends open sets in X to open sets in Y, that is, $f(U)$ is open in Y for every open set U in X.

Remark 12.15 : Note that composition of open maps is open.

Exercise 12.16 : Let $\mathbb{R}^{n+1} \setminus \{0\}$ be the punctured Euclidean space and let $\mathbb{S}^n$ denote the unit sphere in $\mathbb{R}^{n+1}$. Define $g : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{S}^n$ by $g(x) = x/\|x\|_2$. Show that g is an open map.

Hint. It suffices to check that g maps open unit ball in $\mathbb{R}^{n+1} \setminus \{0\}$ to an open set in the unit sphere. This follows from $\mathbb{B}(g(x), \delta/\|x\|) \cap \mathbb{S}^n \subseteq g(\mathbb{B}(x, \delta))$: Suppose $\|y - x/\|x\|\| < \delta/\|x\|$ and $\|y\| = 1$. Then $\|y\|x\| - x\| < \delta$, and hence $y = y/\|y\| = z/\|z\| = g(z)$ with $z = \|x\|y \in \mathbb{B}(x, \delta)$.

The following variant of Theorem 12.5 is useful in instances in which X is non-compact.

Theorem 12.17. *Let $f : X \to Y$ be an open continuous surjection. Define the equivalence relation $\sim$ on X by $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Then $g : X/\sim \to Y$ given by $g([x]) = f(x)$ is a well-defined homeomorphism.*

Proof. As in the proof of Theorem 12.5, one may check that g is a continuous bijection. We check that g^{-1} is continuous. Let $q : X \to X/\sim$ be the quotient map. For an open set U in $X/\sim$, note that $g(U) = g \circ q(q^{-1}(U)) = f(q^{-1}(U))$, which is open since $q^{-1}(U)$ is open in X and f sends open sets to open sets. □

Example 12.18 : Consider the real line $\mathbb{R}$ with the usual topology. Define the equivalence relation $\sim$ on $\mathbb{R}$ by $x \sim y$ if $x - y$ is an integer. Let us determine the quotient space $\mathbb{R}/\sim$. Consider $f : \mathbb{R} \to \mathbb{T}$ by $f(x) = e^{2\pi i x}$, and note that f is a continuous surjection such that $x_1 \sim x_2$ iff $f(x_1) = f(x_2)$. Since f maps open intervals in $\mathbb{R}$ to open arcs in $\mathbb{T}$, f is open. By the last theorem, $\mathbb{R}/\sim$ is homeomorphic to $\mathbb{T}$. The space $\mathbb{R}/\sim$ is usually denoted by $\mathbb{R}/\mathbb{Z}$.

Example 12.19 : Let $X := \mathbb{R}^{n+1} \setminus \{0\}$. Consider the equivalence relation $\sim$ on X given by $x \sim y$ iff $x = ty$ for some non-zero $t \in \mathbb{R}$. The quotient space $X/\sim$ is known as the *n-dimensional projective space* $\mathbb{P}^n(\mathbb{R})$ *over* $\mathbb{R}$.

Let $Y := \mathbb{S}^n/\sim$ with equivalence relation given by $x \sim y$ iff $x = \pm y$. Consider the continuous surjection $f : X \to Y$ given by $f(x) = [x/\|x\|_2]$. Note that $f = q \circ g$ with $g(x) = x/\|x\|_2$ is continuous and $q(x) = [x]$ is the quotient map. It follows that f is continuous. We claim that f is open. In view of Exercise 12.16, it suffices to check that q is an open map from $\mathbb{S}^n$ onto Y. To see that, let U be an open subset of $\mathbb{S}^n$ in the subspace topology. We must check that $q(U)$ is open in the quotient topology. By the definition of quotient topology, $q(U)$ is open provided $q^{-1}(q(U))$ is open in $\mathbb{S}^n$. However, if $V := \{x \in \mathbb{S}^n : -x \in U\}$ then $q^{-1}(q(U)) = U \cup V$, which is open in $\mathbb{S}^n$.

By the last theorem, $\mathbb{P}^n(\mathbb{R})$ is homeomorphic to Y. In particular, $\mathbb{P}^n(\mathbb{R})$ is always compact.

Exercise 12.20 : Show that $\mathbb{P}^1(\mathbb{R})$ is homeomorphic to the unit circle $\mathbb{T}$.

13. Connected and Path-connected Spaces

Let (X, Ω) be a topological space. We say that X is *connected* if the only subsets of X, which are both open and closed are $\emptyset$ and X.

Example 13.1 : We already recorded that $\mathbb{R}$ is connected (see Exercise 7.8). Along similar lines, one can prove that any interval is connected.

Example 13.2 : The space $\mathbb{Q}$ is not connected. In fact, for any irrational $x \in \mathbb{R}$, the set $[-x, x] \cap \mathbb{Q} = (-x, x) \cap \mathbb{Q}$ is both open and closed in the subspace topology on $\mathbb{Q}$.

Exercise 13.3 : Show that $GL_n(\mathbb{R})$ is not connected.

Example 13.4 : If X is an infinite set with co-finite topology then X can not contain a proper set which is both closed and open, and hence X is connected.

The following proposition gives a way to generate connected spaces out of known ones.

Proposition 13.5. *The continuous image of a connected space is connected.*

Proof. Let $f : X \to Y$ is a continuous surjection. Suppose Y has a proper simultaneously open and closed subset. Then $f^{-1}(U)$ is a proper simultanously open and closed subset of X. Thus disconnectedness of Y implies disconnectedness of X. □

Remark 13.6 : Connectedness is a topological property.

Corollary 13.7. *Any path is connected.*

Proof. Any path connecting points x, y is a continuous function f on $[0, 1]$ such that $f(0) = x$ and $f(1) = y$. Since $[0, 1]$ is connected, so is $f([0, 1])$. □

Example 13.8 : Let us see that $\mathbb{R}^n \setminus \{0\}$ is connected. Suppose there exists a proper set U which is both open and closed in $\mathbb{R}^n \setminus \{0\}$. Let $x \in U$ and $y \in (\mathbb{R}^n \setminus \{0\}) \setminus U$. It is easy to see that x and y can be connected by union L of line segments. But then L contains simultaneously open and closed proper subset $L \cap U$. However, L being path is connected, which is contrary to our assumption.

The unit sphere $\mathbb{S}^n$ in $\mathbb{R}^n$ is connected since $\mathbb{R}^n \setminus \{0\}$ is connected and $g : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^n$ given by $g(x) = x/\|x\|_2$ is a continuous surjection.

Exercise 13.9 : If A is connected and B is a set such that $A \subseteq B \subseteq \overline{A}$ then B is also connected.

Example 13.10 : (Topologist's Sine Curve) Consider the graph S of the function $s : (0, 1] \to [0, 1]$, $s(x) = \sin(1/x)$. Note that $S = f((0, 1])$, where $f(x) = (x, \sin(1/x))$ is a continuous function. It follows that S is connnected. By Exercise 13.9, $\overline{S}$ is also connnected. The topological space $\overline{S}$ is commonly known as the *topologist's sine curve.*

Let (X, Ω) be a topological space. We say that X is *path-connected* if for any two points $x, y \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$.

Remark 13.11 : Every path-connected space is connected. In fact, if U is path-connected then for open sets V, W of U such that $U = V \cup W$ and for any path f in U, the range of f being the continuous image of the connected set $[0, 1]$ is connected, and hence lies entirely either in V or W. This shows that one of V, W must be empty, that is, U is connected.

Sometimes, the path-connectedness can be checked more easily than the connectedness.

Exercise 13.12 : Show that the complement of any countable subset $\mathcal{C}$ in $\mathbb{R}^2$ is path-connected.

Hint. Given $p, q \in \mathbb{R}^2 \setminus \mathcal{C}$, consider

$$\mathcal{F} := \{f : f \text{ is a path connecting } p \text{ and } q\}.$$

Note that $\mathcal{F}$ consists of uncountable disjoint paths. Thus the countable set $\mathcal{C}$ can intersect only countable paths from $\mathcal{F}$. Thus there exists $f \in \mathcal{F}$ such that $f[0,1] \subseteq \mathbb{R}^2 \setminus \mathcal{C}$, and f path-connects p and q.

Example 13.13 : (Comb Space) Let K denote the set $\{1/n : n \geq 1\}$ and consider $E := ([0,1] \times \{0\}) \bigcup (K \times [0,1])$. It is easy to see that E is path-connected. The *comb space* $\mathcal{C}$ is defined to be the space $E \cup (\{0\} \times [0,1])$. Then $\mathcal{C}$ is also connected since E is connected and $\overline{E} = \mathcal{C}$. The *deleted comb space* $\mathcal{C}_0$ is defined as $E \cup \{0 \times 1\}$. Since $E \subseteq \mathcal{C}_0 \subseteq \overline{E}$, by Exercise 13.9, $\mathcal{C}_0$ is connected. However, $\mathcal{C}_0$ is not path-connected as there is no path which connects the points $p = 0 \times 1$ and $q = 1 \times 0$.

Proposition 13.14. *The deleted comb space is not path-connected.*

Proof. Suppose, contrary to this, that there is a path $\gamma : [0,1] \to \mathcal{C}_0$ such that $\gamma(0) = p$ and $\gamma(1) = q$. Then the set $\gamma^{-1}(\{p\})$ is a closed subset of $[0,1]$, and hence it has a maximum t_0 in $[0,1]$. Consider the projection P_1 of $\mathbb{R}^2$ onto the X-axis. We claim that there exists $t_1 \in (t_0, 1]$ such that $(P_1 \circ \gamma)(t_0, t_1) \subseteq K$. Assume that the claim is false and let $\{t_n\} \subseteq (t_0, 1]$ be a sequence converging to t_0. By assumption, there exists $s_n \in (t_0, t_n)$ such that $\gamma(s_n) = x_n \times 0$ for some $x_n \in [0,1] \setminus K$. Note that $\{s_n\}$ converges to t_0. By continuity, $x_n \times 0 = \gamma(s_n) \to \gamma(t_0) = p = (0,1)$, which is absurd. Thus the claim stands verified.

Thus there exists $t_1 \in (t_0, 1]$ such that $1 \in (P_1 \circ \gamma)(t_0, t_1)$ is a connected subset of K. It follows that $(P_1 \circ \gamma)(t_0, t_1) = \{1\}$. Again, by continuity, $(P_1 \circ \gamma)[t_0, t_1) = \{1\}$, which is impossible since $(P_1 \circ \gamma)(t_0) = 0$. $\square$

Exercise 13.15 : Show that the continuous image of a path-connected space is path-connected.

Proposition 13.16. *Let p denote an analytic polynomial in the complex variables $z_1, \cdots, z_n$, and let $Z(p)$ denote the zero set of p. Then $\mathbb{C}^n \setminus Z(p)$ is path-connected.*

Proof. Let $z, w \in \mathbb{C}^n \setminus Z(p)$. Consider the straight-line path

$$\gamma(t) = (1-t)z + tw \ (t \in \mathbb{C}).$$

Note that $\{t \in \mathbb{C} : \gamma(t) \in Z(p)\}$ is precisely the zero set $Z(p \circ \gamma) := Z$. Since $p \circ \gamma$ is a polynomial in one variable, Z is a finite subset of $\mathbb{C}$. Thus γ maps the path-connected set $\mathbb{C} \setminus Z$ continuously into $\mathbb{C}^n \setminus Z(p)$. In particular, z and w belong to the path-connected subset $\gamma(\mathbb{C} \setminus Z)$ of $\mathbb{C}^n \setminus Z(p)$. □

Example 13.17 : The general linear group $GL_n(\mathbb{C})$ is path-connected. In fact, if one identifies in a natural way the space of complex $n \times n$ matrices with $\mathbb{C}^{n^2}$ then $GL_n(\mathbb{C})$ can be seen as $\mathbb{C}^{n^2} \setminus Z(\det)$, where det is the analytic polynomial which sends a matrix to its determinant. Now the desired conclusion is immediate from the last proposition.

Exercise 13.18 : For open subset U of $\mathbb{R}^n$, show that U is connected if and only if U is path-connected.

Hint. It suffices to check that for any $p \in U$, the set S of points in U which can be path-connected to p is the whole of U. This can be obtained by simply showing that the set S above is non-empty, closed, and open.

Finally, we record that the topologist's sine curve $\overline{S}$ is connected but not path-connected (for details, refer to [4]).

14. Compactness Revisited

Exercise 14.1 : Let X be a compact space with a nested sequence $\{C_n\}$ of non-empty closed subsets: $C_1 \supseteq C_2 \supseteq C_3 \cdots$. Show that the intersection $\cap_n C_n$ is non-empty.

Hint. Suppose that $\cap_n C_n = \emptyset$. Note that $X = \cup_n (X \setminus C_n)$, and then use compactness of X.

Remark 14.2 : The conclusion of the last exercise is no more true for non-compact spaces. In fact, for $X = \mathbb{R}$ and $C_n = [n, \infty)$ then $C_1 \supseteq C_2 \supseteq C_3 \cdots$. However, $\cap_n C_n$ is empty.

A point $x_0 \in X$ is said to be *isolated* if $\{x_0\}$ is open in X.

Every point in $\mathbb{Z}$ is isolated point. No point of $\mathbb{Q}$ is isolated. The space $\mathbb{R}$ with co-finite topology has no isolated points. The space $[0, 1] \cup \{2\}$ has 2 as the only isolated point.

Lemma 14.3. *Let X be a Hausdorff space without an isolated point. Let U be a non-empty open subset of X and let $x \in X$. Then there exists a non-empty open set V contained in U such that $x \notin \overline{V}$.*

Proof. Since U is open and X has no isolated points, there exists a point $y \in U \setminus \{x\}$. Since X is Hausdorff, there exist open disjoint neighbourhoods U_x and U_y of x and y respectively. Note that $V := U_y \cap U$ is a neighbourhood of y. Clearly, $\overline{V} \subseteq \overline{U_y}$. Since $x \in U_x$ and $U_x \cap U_y = \emptyset$, $x \notin \overline{V}$. □

Theorem 14.4. *Let X be a non-empty compact Hausdorff space without any isolated points. Then no function $f : \mathbb{N} \to X$ can be surjective.*

Proof. We show that there exists $x \in X \setminus f(\mathbb{N})$. Apply the last lemma to $U_1 := X$ and $x_1 := f(1)$ to find a non-empty open set $V_1 \subseteq X$ such that

$x_1 \notin \overline{V_1}$. Apply again the last lemma to $U_2 := V_1$ and $x_2 := f(2)$ to find a non-empty open set $V_2 \subseteq V_1$ such that $x_2 \notin \overline{V_2}$. By induction, we get a non-empty open set $V_n \subseteq V_{n-1}$ such that $x_n \notin \overline{V_n}$. Thus we have nested sequence $C_n := \overline{V}_n$ of closed sets: $C_1 \supseteq C_2 \supseteq C_3 \cdots$. Since X is compact, $\cap_n C_n$ is non-empty. Let $x \in \cap_n C_n$ and note that $x \neq f(n)$ for any n. □

One may use the last theorem to give a topological proof of the set-theoretic fact that $[0, 1]$ is uncountable. The conclusion of last theorem may fail for compact Hausdorff spaces: $\{1/n\} \cup \{0\}$ is compact Hausdorff but not uncountable.

We now discuss a property of compact spaces, which willl be required in the proof of Tychonoff Theorem. For that, we need some terminology.

A collection $\mathcal{C}$ of closed subsets of X is said to have *finite intersection property* if for every finite subcollection $\{C_1, \cdots, C_n\}$ of $\mathcal{C}$, $C_1 \cap \cdots \cap C_n$ is non-empty.

Exercise 14.5 : A topological space X is compact if and only if for every collection $\mathcal{C}$ with finite intersection property, $\cap_{C \in \mathcal{C}} C$ is non-empty.

Hint. Suppose $\cap_{C \in \mathcal{C}} C = \emptyset$. Then $X = \cup_{C \in \mathcal{C}} X \setminus C$, and hence by compactness of X, there exists $C_1, \cdots, C_n$ such that $X = \cup_{i=1}^n X \setminus C_i$, which violates finite intersection property of $\mathcal{C}$.

15. Countability Axioms

Let (X, Ω) be a topological space and let $x \in X$. We say that X has *countable basis at x* if there exists a countable collection $\{U_n\}$ of neighbourhoods of x such that for each neighbourhood U of x there exists n_0 such that $U_{n_0} \subseteq U$. A space X that has a countable basis at each of its points is said to satisfy the *first countability axiom*. Sometimes, we say that X is *first countable*.

Example 15.1 : Consider the metric space (X, d). Given $x \in X$, the collection $\{\mathbb{B}_d(x, 1/n)\}$ forms a countable basis at x. Thus any metric space satisfies the first countability axiom.

Example 15.2 : The real line with co-finite topology does not satisfy the first countability axiom: If $\{U_n\}$ is a countable collection of neighbourhoods of 0 then there is no n_0 such that $U_{n_0} \subseteq \mathbb{R} \setminus \{x_0\}$, where x_0 is a non-zero real number in the complement of the countable subset $\cup_n(\mathbb{R} \setminus U_n)$ of $\mathbb{R}$.

Theorem 15.3. *Let X be a first countable space and let A be a subset of X. Then the following are true:*

1. *(Sequence Lemma) A point $x \in \overline{A}$ if and only if there is a sequence of points of A converging to x.*
2. *(Continuity Versus Sequential Continuity) Let $f : X \to Y$. Then f is continuous if and only if f is sequentially continuous.*

Proof. Suppose x is a cluster point of A. Let $\{U_n\}$ be a countable basis at x. Since x is a cluster point, for each n, the deleted neighbourhood $\cap_{k=1}^n U_k \setminus \{x\}$ of x intersects with A. Let $a_n \in (\cap_{k=1}^n U_k \setminus \{x\}) \cap A$, and note that $\{a_n\}$ converges to x. The converse is true without the first countability axiom.

Again, continuity always implies sequential continuity. Suppose f is sequentially continuous. Let B be a closed subset of Y and let a be a cluster point of $f^{-1}(B)$. By the previous part, there exists a sequence $\{a_n\}$ in $f^{-1}(B)$ such that a_n converges to a. Thus $f(a_n) \in B$. By sequentially continuity, $f(a_n)$ converges to $f(a)$. Since B is closed, $f(a) \in B$, that is, $a \in f^{-1}(B)$. This shows that $f^{-1}(B)$ is closed in X. □

Remark 15.4 : It is observed in the proof of Theorem 10.17 that the conclusion of Sequence Lemma does not hold for the product space $\mathbb{R}^\omega$ with box topology, and hence we may conclude that it is not first countable.

We now discuss a special class of first countable spaces. A space X is said to be *second countable* if it has countable basis.

Remark 15.5 : Let $\mathbb{B}$ be a countable basis $\{B_n\}$ for X, and let $x_n \in B_n$ be given. If $x \in X$ then any neighbourhood of x contains some basis element, and hence it intersects with $\{x_n\}$. This shows that $\overline{\{x_n\}} = X$, that is, $\{x_n\}$ is *dense* in X.

Example 15.6 : Consider the space $\mathbb{R}_l$ with lower limit topology. Given $x \in \mathbb{R}$, the collection $\{[x, x + 1/n)\}$ forms a countable basis at x. Thus $\mathbb{R}_l$ is first countable. We claim that $\mathbb{R}_l$ is not second countable. Let $\mathbb{B}$ be any basis for $\mathbb{R}_l$. For each $x \in \mathbb{R}$, there exists a basis element such that $x \in B_x \subseteq [x, x+1)$. If $x \neq y$ then $\inf B_x = x \neq y = \inf B_y$, so that $B_x \neq B_y$. Thus $\mathbb{B}$ is uncountable, and hence the claim stands verified.

Proposition 15.7. *A metric space (X, d) is second countable if any one of the following holds true:*

1. (1) *X has a countable dense subset,*
2. (2) *X is compact.*

Proof. (1) Let $\{x_n\}$ be a countable dense subset of X. We check that the countable collection $\{\mathbb{B}_d(x_n, 1/m) : n \geq 1, m \geq 1\}$ forms a basis:

1. (1) Let $x \in X$. Then there exists an integer $n \geq 1$ such that $x_n \in \mathbb{B}_d(x, 1/2)$. Then $x \in \mathbb{B}_d(x_n, 1/2)$.
2. (2) If $x \in U := \mathbb{B}_d(x_{n_1}, 1/m_1) \cap \mathbb{B}_d(x_{n_2}, 1/m_2)$ then there exists integer $r \geq 1$ such that $\mathbb{B}_d(x, 1/r) \subseteq U$. By density of $\{x_n\}$, there exists $x_l \in \mathbb{B}_d(x, 1/4r)$. Then $x \in \mathbb{B}_d(x_l, 1/2r) \subseteq \mathbb{B}_d(x, 1/r) \subseteq U$.

(2) Suppose X is compact. For each integer $n \geq 1$, consider the open cover $\{\mathbb{B}(x, 1/n)\}$ of X. Since X is compact, X has a finite subcover $\{\mathbb{B}(x_{n_i}, 1/n) : i = 1, \cdots, k_n\}$ of X. Check that $\cup_{n=1}^\infty \{\mathbb{B}(x_{n_i}, 1/n) : i = 1, \cdots, k_n\}$ forms a countable basis for X. □

Exercise 15.8 : Show that $\mathbb{R}_l$ is not metrizable.

Hint. $\mathbb{Q}$ is dense in $\mathbb{R}_l$.

Theorem 15.9. *If X is second countable then every open covering of X contains a countable subcover.*

Proof. Let $\mathbb{B}$ be a countable basis $\{B_n\}$ for X, and let $\{U_\alpha\}$ be an open cover of X. Let J denote the set of positive integers n for which it is possible to find some U_{α_n} which contains B_n. We check that the countable collection $\{U_{\alpha_n} : n \in J\}$ covers X (there may be many such U_α's; we pick up only one such set denoted by U_{α_n}). In fact, if $x \in X$ then $x \in U_\alpha$ for some α, and hence there exists $n \geq 1$ such that $x \in B_n \subseteq U_\alpha$. But then $n \in J$, and hence $x \in B_n \subseteq U_{\alpha_n}$. That is, $\{U_{\alpha_n} : n \in J\}$ covers X. □

A deep theorem of Urysohn ensures that any second countable space which separates a point and a closed set by open neighbourhoods is necesarily metrizable.

16. Separation Axioms

Exercise 16.1 : Let (X, d) be a metric space with metric. Let A be a subset of X, and for $x \in X$, let $d(x, A) := \inf\{d(x, a) : a \in A\}$. Show that $d(x, A)$ is a continuous function of x.

Hint. For $x, y \in X$ and $a, b \in A$, $d(x, a) \leq d(x, y) + d(y, b) + d(b, a)$.

Let (X, d) be a metric space with metric d. Let A and B be disjoint non-empty closed subsets of X. For $x \in X$, define

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Clearly, $f : X \to [0, 1]$ is a continuous function. Note that $f(a) = 0$ and $f(b) = 1$ for every $a \in A$ and $b \in B$. In particular, the disjoint non-empty closed subsets A and B of X are separated by the continuous function f.

Exercise 16.2 : (Functional Separation implies Topological Separation) Let (X, Ω) be a topological space with the property that any two disjoint non-empty closed subsets of X can be separated by a continuous function. Show that for any disjoint non-empty closed subsets A and B of X, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$.

Let (X, Ω) be a topological space with the property that for any disjoint non-empty open subsets A and B of X, there exist disjoint open sets U and V such that $A \subseteq U$ and $B \subseteq V$. Such a space is known as *normal space.*

Remark 16.3 : Clearly, a normal space is Hausdorff. By the last exercise, every metric space is normal.

Exercise 16.4 : Consider the topological space $\mathbb{R}_K$. Show that there are no disjoint open sets U and V of $\mathbb{R}_K$ such that $\{0\} \subseteq U$ and $K \subseteq V$. In particular, $\mathbb{R}_K$ is Hausdorff but not normal.

Hint. Note that U must contain a basis element of the form $(-a, a) \setminus K$. Thus for large n, any deleted neighbourhood of $1/n$ (and hence, in particular, $V \setminus \{1/n\}$) intersects with $(-a, a) \setminus K$. It follows that U and V are not disjoint.

Remark 16.5 : The topological space $\mathbb{R}_K$ is not metrizable.

Proposition 16.6. *Every compact Hausdorff space is normal.*

Proof. Let A and B be two closed subsets of a compact Hausdorff space X. It is observed in the proof of Theorem 11.8(2) that any closed set and a point in X can be separated by open sets. For each $a \in A$, there exists disjoint open sets U_a and V_a such that $a \in U_a$ and $B \subseteq V_a$. Clearly, $A \subseteq \cup_{a \in A} U_a$. Since A is compact, there exist $a_1, \cdots, a_k \in A$ such that $U := U_{a_1} \cap \cdots \cap U_{a_k}$ is an open set containing A. Then $V := V_{a_1} \cap \cdots \cap V_{a_k}$ is an open set containing B, and $U \cap V = \emptyset$. □

Problem 16.7. *Whether or not any two disjoint non-empty closed subsets of a normal space X can be separated by a continuous function ?*

The question has affirmative answer, and a solution is provided by so-called Urysohn's Lemma. This result ensures that normal Hausdorff spaces admit enormous number of continuous functions.

We find it convenient to introduce the following definition:

Let (X, Ω) be a topological space and let A, B be closed subsets of X. We say that X *admits a family* $\{U_r\}_{r \in \mathbb{Q}}$ *of nested neighbourhoods* if

1. $A \subseteq U_0$ and $U_1 = X \setminus B$,
2. $\overline{U}_r \subseteq U_s$ for any $r < s$,
3. $U_r = \emptyset$ for $r < 0$, and $U_r = X$ for $r > 1$.

Remark 16.8 : Given a point $x \in X$, define

$$\mathbb{Q}(x) := \{r \in \mathbb{Q} : x \in U_r\}.$$

For $r < 0$, $U_r = \emptyset$, so that $\inf \mathbb{Q}(x) \geq 0$. For $r > 1$, $U_r = X$, so that $\inf \mathbb{Q}(x) \leq 1$. Thus $0 \leq \inf \mathbb{Q}(x) \leq 1$.

Exercise 16.9 : Suppose X admits a family $\{U_r\}_{r \in \mathbb{Q}}$ of nested neighbourhoods. Define $f : X \to [0, 1]$ by $f(x) = \inf \mathbb{Q}(x)$. Verify the following:

1. $f(a) = 0$ for every $a \in A$.
2. $f(b) = 1$ for every $b \in B$.
3. $f(x) \leq r$ for any $x \in \overline{U}_r$,
4. $f(x) \geq r$ for any $x \notin U_r$.

Hint. If $x \in \overline{U}_r$ then $x \in U_s$ for any $s > r$. Thus $r + 1/n \in \mathbb{Q}(x)$ for every integer $n \geq 1$. Thus $f(x) \leq r + 1/n$ for all $n \geq 1$, and hence $f(x) \leq r$.

Lemma 16.10. *Suppose X admits a family $\{U_r\}_{r\in\mathbb{Q}}$ of nested neighbourhoods. Then $f : X \to [0,1]$ given by $f(x) = \inf \mathbb{Q}(x)$ is continuous.*

Proof. Let $x_0 \in X$ and let (c,d) be an open interval containing $f(x_0)$. Choose rational numbers p and q such that $c < p < f(x_0) < q < d$. By the preceding exercise, x_0 belongs to U_q since $f(x_0) < q$ and $x_0 \notin \overline{U}_p$ since $f(x_0) > p$. Thus $x_0 \in U := U_q \setminus \overline{U}_p \subseteq \overline{U_q} \setminus U_p$. We check that $f(U) \subseteq [p,q] \subseteq (c,d)$. In fact, if $x \in U$ then $x \in \overline{U}_q$, so that $f(x) \leq q$, and $x \notin U_p$, so that $f(x) \geq p$. $\square$

Exercise 16.11 : Let X be a normal space and let A be a closed subset of X contained in some open subset U of X. Show that there exists an open set V such that $A \subseteq V \subseteq \overline{V} \subseteq U$.

Hint. Apply the definition of normal spaces to closed sets A and $X \setminus U_1$.

Theorem 16.12. *(Urysohn's Lemma) Given closed non-empty disjoint subsets A and B of a normal space X, there exists a continuous function $f : X \to [0,1]$ such that $f|_A = 0$ and $f|_B = 1$.*

Proof. In view of the preceding observations, it suffices to check that any normal space admits a family $\{U_r\}_{r\in\mathbb{Q}}$ of nested neighbourhoods. Rewrite the countable set $P := \mathbb{Q} \cap [0,1]$ as $\{r_0, r_1, \cdots,\}$, where $r_0 = 1$ and $r_1 = 0$. Let $P_n = \{r_0, r_1, \cdots, r_{n-1}\}$. We will use induction on $n \geq 2$ to prove that there exists a family $\{U_r\}_{r\in P}$ of nested neighbourhoods.

Let $U_1 = X\setminus B$, and note that $A \subseteq U_1$. By the last exercise, there exists an open set U_0 such that $A \subseteq U_0 \subseteq \overline{U}_0 \subseteq U_1$. Thus we have a family $\{U_r\}_{r\in P_1}$ of nested neighbourhoods.

Now suppose that we have a family $\{U_r\}_{r\in P_n}$ of nested neighbourhoods for some $n \geq 2$. Note that $P_{n+1} = P_n \cup \{r_n\}$. Let $p, q \in P_n$ such that $p < r_n < q$ and $P_{n+1} \cap (p,q) = \{r_n\}$. By induction hypothesis, we have open sets U_p and U_q such that $\overline{U}_p \subseteq U_q$. Again by the preceding exercise, there exists an open set U_{r_n} such that $\overline{U}_p \subseteq U_{r_n} \subseteq \overline{U}_{r_n} \subseteq U_q$. It is easy to see that $\{U_r\}_{r\in P_n} \cup \{U_{r_n}\}$ is a family of nested neighbourhoods.

If $r \in \mathbb{Q}$ then define $U_r = \emptyset$ for $r < 0$, and $U_r = X$ for $r > 1$. Note that $\{U_r\}_{r\in\mathbb{Q}}$ is the desired family of nested neighbourhoods. $\square$

Here is an immediate application of the Urysohn's Lemma.

Corollary 16.13. *Let X be a connected normal space, which is also Hausdorff and which contains at least two points. Then there exists a continuous surjection $f : X \to [0,1]$. In particular, X is uncountable.*

Proof. Fix distinct points a, b in X. By Urysohn's Lemma, there exists a continuous function $f : X \to [0,1]$ such that $f(a) = 0$ and $f(b) = 1$. Suppose there is $r \in [0,1]$ such that r has no preimage. Then $0 \in f(X) \cap [0,r)$ is open and closed subset of $f(X)$, which does not contain 1. This is not possible since $f(X)$ is a connected set containing 1, so that f is surjective. $\square$

A particular case of the last corollary is worth-notable: Any connected metric space with at least two points contains uncountably many points.

Exercise 16.14 : Let X be a compact Hausdorff space without any isolated points. Consider the vector space $C(X)$ of continuous functions $f : X \to \mathbb{R}$. Show that $C(X)$ is infinite-dimensional.

Hint. X is uncountable, so that X contains infinitely many single-ton disjoint closed subsets. Now apply Urysohn's Lemma.

17. Tychonoff's Theorem

Exercise 17.1 : Let $X = \prod_{n=1}^{\infty}\{0, 1\}$. Show that X is not compact in the box topology. Whether X is compact in the product topology ?

Theorem 17.2. *(Tychonoff's Theorem) An arbitrary product of compact spaces is compact in the product topology.*

Let $X = \prod_{\alpha \in J} X_\alpha$ be a product space, where the index set J is non-empty. If each X_α is non-empty then by axiom of choice, we may pick up some x_α from X_α, and hence it follows that $(x_\alpha) \in X$, so that X is non-empty. We use the following criterion to prove Tychonoff's Theorem: Recall that a space is compact iff for every collection $\mathcal{C}$ of closed sets with finite intersection property, $\cap_{C \in \mathcal{C}} C$ is non-empty.

Let $\mathcal{C}$ be a family of closed sets in X with the finite intersection property. We must check that $\cap_{C \in \mathcal{C}} C$ is non-empty. The first step is to get a maximal family $\mathcal{F} \supseteq \mathcal{C}$ with finite intersection property. This is achieved by an application of Zorn's Lemma:

Lemma 17.3. *There exists a maximal family $\mathcal{F} \supseteq \mathcal{C}$ with the finite intersection property.*

Proof. Consider the collection $\mathbb{F}$ of families $\mathcal{F} \supseteq \mathcal{C}$ of subsets of X with the finite intersection property with strictly partial order given by $\mathcal{F}_1 < \mathcal{F}_2$ iff $\mathcal{F}_1 \subsetneq \mathcal{F}_2$. Let $\mathbb{F}_0$ be a simply ordered subcollection of $\mathbb{F}$ (in particular, if $\mathcal{F}_1 \neq \mathcal{F}_2 \in \mathbb{F}_0$ then either $\mathcal{F}_1 \subseteq \mathcal{F}_2$ or $\mathcal{F}_2 \subseteq \mathcal{F}_1$). Then $\mathbb{F}_0$ has the upper bound $\mathcal{U} = \cup_{\mathcal{F} \in \mathbb{F}_0} \mathcal{F}$ in $\mathbb{F}$. To see this, we must check that $\mathcal{U}$ belongs to $\mathcal{F}$, that is, $\mathcal{U}$ has finite intersection property. If $A_1, \cdots, A_n \in \mathcal{U}$ then $A_i \in \mathcal{F}_i$ for some i, and hence $A_1, \cdots, A_n \in \mathcal{F}_j$ for some j (since $\mathbb{F}_0$ is simply ordered). Since $\mathcal{F}_j$ has finite intersection property, $A_1 \cap \cdots \cap A_n$ is non-empty. Thus Zorn's lemma is applicable, which ensures existence of maximal family $\mathcal{F} \supseteq \mathcal{C}$ with the finite intersection property. $\square$

For each $\alpha \in J$, consider the projection $\pi_\alpha : X \to X_\alpha$ from the product space X onto X_α.

Exercise 17.4 : If a family $\mathcal{F}$ of subsets of X has finite intersection property then for each $\alpha \in J$, $\cap_{A \in \mathcal{F}} \overline{\pi_\alpha(A)}$ is non-empty.

Hint. $\{\pi_\alpha(A) : A \in \mathcal{F}\}$ has finite intersection property for each $\alpha \in J$.

By the preceding exercise, there exists $x_\alpha \in \cap_{A \in \mathcal{F}} \overline{\pi_\alpha(A)}$. If $x = (x_\alpha)$ then it can be seen that x belongs to $\cap_{A \in \mathcal{F}} \overline{A}$. Here we need the definition of product topology.

Lemma 17.5. *x belongs to $\cap_{A \in \mathcal{F}} \overline{A}$.*

Since $\cap_{A \in \mathcal{F}} \overline{A} \subseteq \cap_{C \in \mathcal{C}} C$, we obtain $\cap_{C \in \mathcal{C}} C$ is also non-empty. For details, the reader is referred to [4].

References

[1] H. Fürstenberg, *On the infinitude of primes*, Amer. Math. Monthly **62**, (1955), 353.
[2] S. Kumaresan, *Outline of a Topology Course*, private communication.
[3] B. Mendelson, *Introduction to topology*, Dover Publications, New York, 1990.
[4] J. Munkres, *Topology*, Prentice-Hall, Inc. Englewood Cliffs, N.J. 1975.