1

# Question 1    6

a) $$\sum_{k=1}^{n} \rho_k W_Q^k = \frac{R\rho}{1-\rho} = \frac{R \sum_{k=1}^{n} \rho_k}{1 - \sum_{k=1}^{n} \rho_k} \quad \text{(1)}$$

We will use induction to prove this.

For $n = 1$,

$$W_Q^1 = \frac{R}{1-\rho_1} \Rightarrow \rho_1 W_1 = \frac{R\rho_1}{1-\rho_1}$$

Thus the identity holds for $n = 1$

Assuming (1) holds for $n$, we will show that it holds for $n+1$.

For priority class $n+1$,

$$W_Q^{n+1} = \frac{R}{(1 - \sum_{k=1}^{n} \rho_k)(1 - \sum_{k=1}^{n+1} \rho_k)} \quad \text{(2)}$$

$$\sum_{k=1}^{n+1} \rho_k W_q^k = \sum_{k=1}^{n} \rho_k W_q^k + \rho_{n+1} W_{n+1}$$

$$= \frac{R \sum_{k=1}^{n} \rho_k}{1 - \sum_{k=1}^{n} \rho_k} + \frac{R \cdot \rho_{n+1}}{\left(1 - \sum_{k=1}^{n} \rho_k\right)\left(1 - \sum_{k=1}^{n+1} \rho_k\right)} \qquad \text{(from 1 \& 2)}$$

$$= \frac{R}{1 - \sum_{k=1}^{n} \rho_k} \left[ \sum_{k=1}^{n} \rho_k + \frac{\rho_{n+1}}{1 - \sum_{k=1}^{n} \rho_k - \rho_{n+1}} \right]$$

$$= \frac{R}{\cancel{\left(1 - \sum_{k=1}^{n} \rho_k\right)}} \left[ \frac{\cancel{\left(1 - \sum_{k=1}^{n} \rho_k\right)} \left( \sum_{k=1}^{n} \rho_k + \rho_{n+1} \right)}{1 - \sum_{k=1}^{n} \rho_k - \rho_{n+1}} \right]$$

$$= \frac{R \sum_{k=1}^{n+1} \rho_k}{1 - \sum_{k=1}^{n+1} \rho_k}$$

$\therefore$ Eq 1 holds.

2

b) Let the classes be labelled such that

$$\frac{\overline{X_1}}{C_1} \le \frac{\overline{X_2}}{C_2} \cdots \le \frac{\overline{X_n}}{C_n}$$

Priorities are assigned according to the class labels, with class 1 - highest priority

let $$C = \sum_{k=1}^{n} C_k N_Q^k = \sum_{k=1}^{n} C_k \lambda_k W_Q^k$$

$$= \sum_{k=1}^{n} \frac{C_k}{\overline{X_k}} \rho_k W_Q^k .$$

we have $W_1 \le W_2 \le \cdots \le W_n$

we will show that the cost is minimized by the above ordering by showing that exchanging the priority of two neighbouring classes $i$ & $j = i+1$ will result in a higher cost $C'$. ~~ele~~ $(C' \ge C)$

For any priority class m,

$$W_Q^m = \frac{R}{(1-\sum_{k=1}^{m-1} \rho_k)(1-\sum_{k=1}^{m} \rho_k)}$$

Consider the classes i & j = i+1. After exchanging their priorities, let the resulting waiting times be represented by $\overline{W_Q^m}$.

For m < i, $W_Q^m$ does not depend on $\rho_i$ or $\rho_j$

For m > j, $W_Q^m$ depends on the sum $\rho_i + \rho_j$

∴ The exchange of priorities of i & j does not affect the waiting time of the remaining classes

$$W_Q^m = \overline{W_Q^m} \quad , m \neq i, j$$

∴ Average number in system

$$= 0 \cdot P_0 + 1 \cdot P_1 = \frac{\lambda}{\lambda + \mu}$$

## Question 2

a) $W_Q^{(K)}$ is same as the average waiting time in queue of an M/M/1 system with arrival rate $(\lambda_1 + \cdots + \lambda_K)$ because of 2 reasons

1) The waiting time of classes $1, \ldots K$ is not influenced by the presence of classes $(K+1) \cdots n$.

2✓

2) Since all priority classes have the same service time distribution, interchanging of order of service does not change average waiting time.

b) Avg. # in queue of class K

= Avg # in queue of classes 1 to K

– Avg # in queues of classes 1 to K−1

3

From Little's Theorem,

$$\lambda_K \cdot W_Q^{K} = \left[ W_Q^{(K)} \sum_{i=1}^{K} \lambda_i - W_Q^{(K-1)} \sum_{i=1}^{K-1} \lambda_i \right]$$

$K = 2, 3, \ldots n$.

$W_1 = W_{(1)}$ , $K = 1$

Question 3 :

Let $n_i$ be the # of customers left behind in the system when the $i^{th}$ customer departs.

Let $a_{i+1}$ be the number of arrivals in the $(i+1)^{th}$ service time.

Let $j$ be the # of customers waiting for service when a busy period begins, $j \geq 1$

& $P_j = Pr\{ j$ customers starting the busy period $\}$

$$n_{i+1} = a_{i+1} + j - 1 \quad \text{for } n_i = 0$$
$$= n_i + a_{i+1} - 1 \quad \text{for } n_i \geq 1$$

Above equations can also be written as
$$n_{i+1} = n_i + a_{i+1} - 1 + j\,[1 - U(n_i)] \quad \text{for } i = 1, 2, 3 \ldots$$

Since $a_{i+1}$ is independent of $n_i$ & its previous states

& $j$ (which is a function of # of arrivals during the vacation interval) is also independent of $n_i$ & its previous states.

We have an imbedded Markov Chain