Practice Problems

Q1:

$\lambda = 5/\text{min}$; $W_q = 5$ min ; $1/\mu = 20$ min

$L_q = \lambda W_q = 25$ customers

Expected # of customers in service $= \rho = \frac{\lambda}{\mu} = 100$

Since with prob 0.5 these customers leave,

actual # of customers in restaurant

$= \#$ in queue (L_q) $+ 0.5 \cdot \rho$

$= 25 + \frac{1}{2} \cdot 100 = \underline{\underline{75}}$

Q2: When all courts are busy, expected time between two departures is $40/4 = 10$ min

If a pair sees K pairs waiting in the queue, there must be exactly K+1 departures from the system before they get the court.

Since all courts would be busy during this whole time, the avg waiting time is

$\underline{(K+1) \times 10}$ minutes

Q3: we have $1/\mu = b$

Let $Y_n = \#$ of arrivals during $[(n-1)b, nb]$

$\therefore X_{n+1} = X_n - U(X_n) + Y_{n+1}$; $U(X_n) = \begin{cases} 1 & X_n > 0 \\ 0 & X_n = 0 \end{cases}$

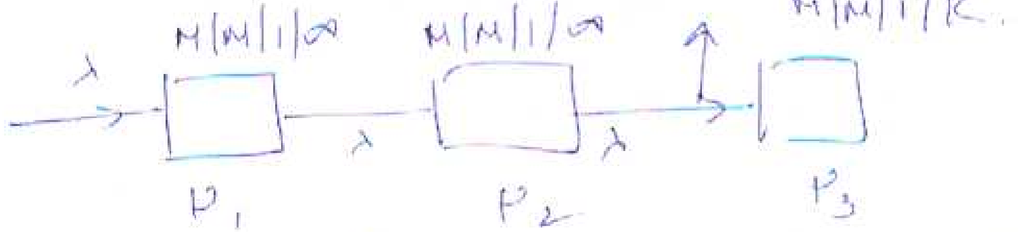
The transition matrix is

$$\begin{bmatrix} K_0 & K_1 & K_2 & \cdots \\ K_0 & K_1 & K_2 & \cdots \\ 0 & K_0 & K_1 & \cdots \\ 0 & 0 & K_0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $k_j = \Pr\{ j \text{ arrivals during } ((n-1)b, nb] \}$

$$= \frac{e^{-\lambda b}(\lambda b)^j}{j!}$$

Q4)



$$L_1 = \frac{\rho_1}{1-\rho_1} \text{ where } \left(\rho_1 = \frac{\lambda}{\mu_1}\right) \quad ; \quad L_2 = \frac{\rho_2}{1-\rho_2} \text{ where } \left(\rho_2 = \frac{\lambda}{\mu_2}\right)$$

$$L_3 = \frac{\rho_3 \left[1 - (K+1)\rho_3^K + K\rho_3^{K+1}\right]}{(1-\rho_3^{K+1})(1-\rho_3)} \quad (\text{For } \rho_3 \neq 1)$$

$$L = L_1 + L_2 + L_3$$

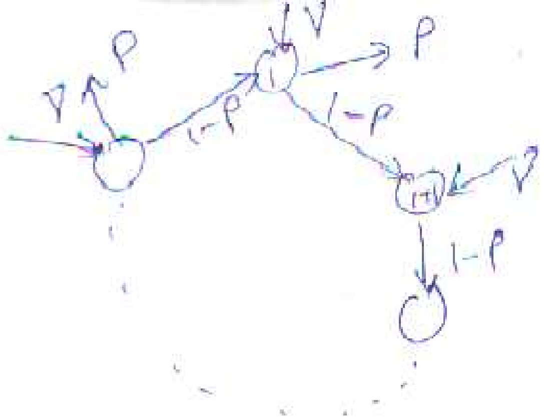
$$W = \frac{L_1 + L_2}{\lambda} + \frac{L_3}{\lambda(1-P_K)} \quad \text{where } P_K = \frac{(1-\rho_3)\rho_3^K}{1-\rho_3^{K+1}}$$

Q 5) 1) $\lambda_1 = \nu + (1-p)\lambda_K$

$\lambda_2 = \nu + (1-p)\lambda_1$

$\vdots$

$\lambda_K = \nu + (1-p)\lambda_{K-1}$



Since all nodes are symmetric

$\lambda_1 = \lambda_2 = \dots = \lambda_K = \lambda$

$\therefore \lambda = \nu + (1-p)\lambda \Rightarrow \lambda = \nu/p$

2) Stationary distribution exists if for every node

$\nu/p < \mu$

stationary distribution (from Jackson's Theorem)

$$p(n_1, \dots n_K) = \prod_{i=1}^{K} (1-\rho)\rho^{n_i} = (1-\rho)^K \rho^{n_1 + \dots + n_K}$$

where $\rho = \dfrac{\nu}{p\mu}$

3) Avg # of customers at each queue

$$N_i = \frac{\rho}{1-\rho}$$

Avg # of customers in the network

$$N = \frac{K\rho}{1-\rho}$$

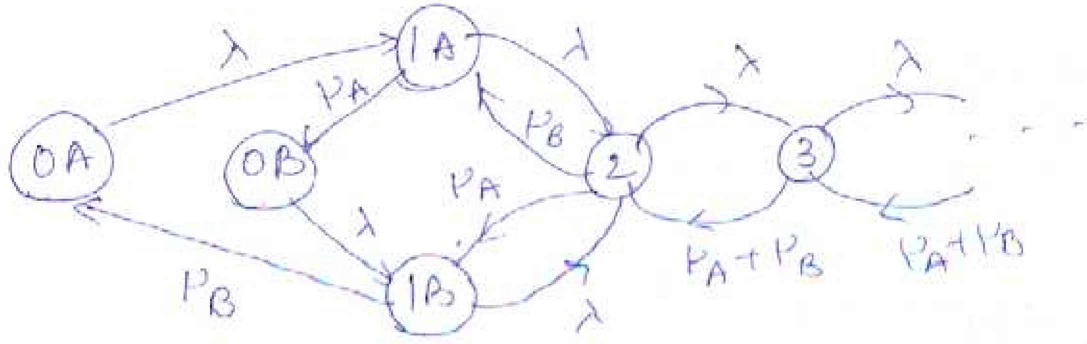
Total arrival rate to the network $K\nu$

From little theorem, avg time a customer spends in network is $\dfrac{N}{K\nu} = \dfrac{1}{p\mu - \nu}$

4) The internal flows are not poisson since the same customer can traverse the same link multiple times before he leaves the network [feed back exists] (4)

Q6)

1)



(0A) → 0 customers in system; server A idle for the longest time

(0B) → " ; server B "

(1A) → 1 customer in system served at server A

(1B) → " " at server B.

$n = 2, 3 \ldots$ → n customers in the system

2) Global balance equations:

$$\lambda P_{0,A} = \mu_B P_{1,B}$$

$$\lambda P_{0,B} = \mu_A P_{1,A}$$

$$\lambda P_{0,A} + \mu_B P_2 = (\lambda + \mu_A) P_{1,A}$$

$$\lambda P_{0,B} + \mu_A P_2 = (\lambda + \mu_B) P_{1,B}$$

$$(\mu_A + \mu_B) P_3 + \lambda P_{1,A} + \lambda P_{1,B} = (\mu_A + \mu_B + \lambda) P_2$$

$$P_{n-1} \cdot \lambda + P_{n+1} (\mu_A + \mu_B) = (\lambda + \mu_A + \mu_B) P_n \quad , n = 3, 4 \ldots$$

Expressing everything as a function of P_2

$$P_{0,A} = P_{0,B} = \frac{\mu_A \mu_B}{\lambda^2} P_2$$

$$P_{1,A} = \frac{\mu_B}{\lambda} \cdot P_2 \; ; \quad P_{1B} = \frac{\mu_A}{\lambda} P_2$$

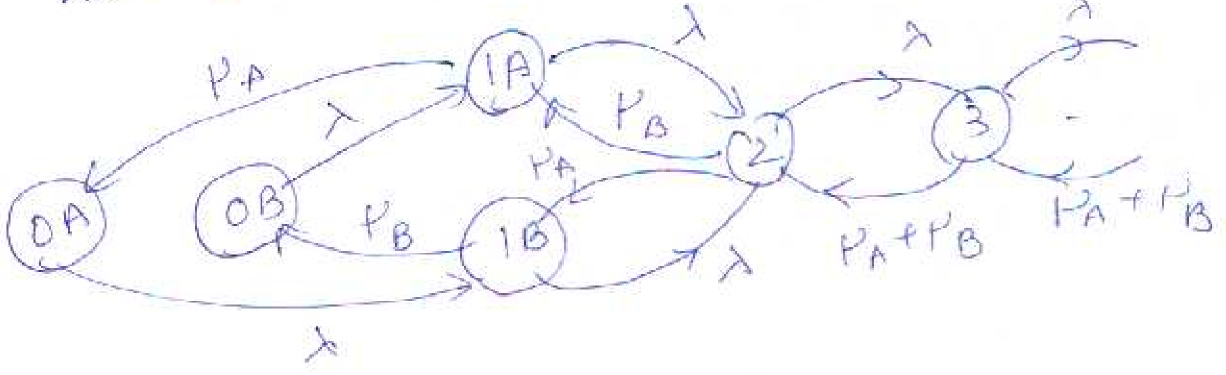
$$P_n = \left(\frac{\lambda}{\mu_A + \mu_B}\right)^{n-2} P_2 \, , \quad n = 2, 3, \ldots$$

Normalizing. i.e $\Sigma P_n = 1$

$$P_2 = \left[\frac{2\mu_A \mu_B}{\lambda^2} + \frac{\mu_B}{\lambda} + \frac{\mu_A}{\lambda} + \sum_{n=2}^{\infty}\left(\frac{\lambda}{\mu_A + \mu_B}\right)^n\right]^{-1}$$

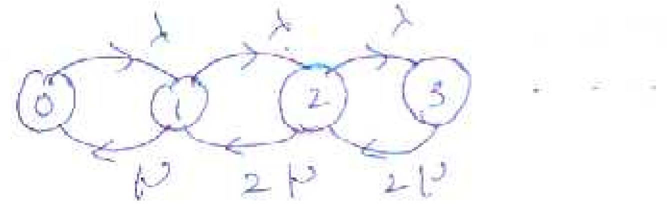
3) The system is not reversible.
Consider the loop $0A \to 1A \to 0B \to 1B \to 0A$
In the forward direction, the product of the transition rates along the loop is $\lambda^2 \mu_A \mu_B$. In reverse direction, it is 0.

4) A bit tedious, but working out the reverse transition rates gives rise to the following reversed M.C



Q7)

1) This is an M/M/2 system.



$P_1 = \frac{\lambda}{\mu} P_0 = 2\rho P_0$ $\quad\left(\rho = \frac{\lambda}{2\mu}\right)$

$P_n = \left(\frac{\lambda}{2\mu}\right)^{n-1} P_1 = 2\rho^n P_0 \quad , n \geq 1$

P_0 (from $\sum P_n = 1$) turns out to be $\frac{1-\rho}{1+\rho}$

$L_1 = \frac{2\rho}{1-\rho^2}$; $W_1 = \frac{L_1}{\lambda} = \frac{1/\mu}{1-\rho^2}$ (subscript 1 refers to case 1)

$L_Q^1 = \frac{2\rho^3}{1-\rho^2}$

2) This is a case of M/M/1 queue with service rate of 2μ.

$N_2 = \frac{\rho}{1-\rho}$ $\quad\left(\rho = \frac{\lambda}{2\mu}\right)$

$W_2 = \frac{N_2}{\lambda} = \frac{1/2\mu}{1-\rho}$

3) Each queue is an M/M/1 queue with an arrival rate of $\lambda/2$ & service rate μ.

$N_3 = \frac{2\rho}{1-\rho}$ $\quad\left(\rho = \frac{\lambda}{2\mu}\right)$

$W_3 = \frac{1/\mu}{1-\rho}$.

4) We have $T_2 < T_1 < T_3$

In terms of delay, it is better to have a faster server

than two slower ones. ~~Re difference~~
When there are two servers, delay is less if
they share a common queue.

Q8:

1) Prob server is busy $= \rho$
Prob server is idle $= 1-\rho = 1-\lambda\bar{X}$

2) Length of an idle period is the inter arrival time between a customer arrivals & a customer departure. This is exponential with parameter λ. Hence average length $= 1/\lambda$

3) Let B be the avg length of a busy period
Let I " " idle period.
Portion of time server is busy $= \dfrac{B}{I+B} = \lambda\bar{X}$

$$\Rightarrow B = \frac{\bar{X}}{1-\lambda\bar{X}}$$

4) Average # of customers served in a busy period $= \dfrac{B}{\bar{X}} = \dfrac{1}{1-\lambda\bar{X}}$

Q9)

$$0.2\lambda_2 + 0.2\lambda_3 + \lambda = \lambda_1$$

$$\lambda_2 = \lambda_3 = 0.4\lambda_1$$

$$\lambda_4 = 0.2\lambda_1 + 0.6\lambda_2 + 0.6\lambda_3$$

$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1.1905\lambda, 0.4762\lambda, 0.4762\lambda, 0.8095\lambda)$

$(\rho_1, \rho_2, \rho_3, \rho_4) = (1.1905\rho_0, 0.9524\rho_0, 0.9524\rho_0, 1.6190\rho_0)$ $\left(\rho_0 = \frac{\lambda}{\mu}\right)$

$N_1 = \frac{\rho_1}{1-\rho_1}$, $N_2 = N_3 = \frac{\rho_2}{1-\rho_2}$, $N_4 = \frac{\rho_4}{1-\rho_4}$

Q_4 has the highest utilization. Hence as the traffic arrival rate λ increases, this queue will become unstable first.

Hence we require $\rho_4 < 1$

$\Rightarrow \lambda < 0.6176\mu$

Q10) class 1 - control packets (λ_1, μ_1)
class 2 - data packets (λ_2, μ_2)

For Control packets $1/\mu_1 = \frac{48}{9600} = 0.005$ sec ; $\frac{1}{\mu_1^2}$ is second moment

data Packets $\frac{1}{\mu_2} = \frac{960}{9600} = 0.1$ sec. $\frac{2}{\mu_2^2}$ is the second moment

we have $\lambda_1 = 0.2(\lambda_1 + \lambda_2)$.

1) M/G/1 system

$\rho = \rho_1 + \rho_2 = \frac{\lambda_1}{\mu_1} + \frac{\lambda_2}{\mu_2} = 0.5$ (given)

$\lambda_1 = 0.2(\lambda_1 + \lambda_2)$ given.

Solving for λ_1 & λ_2, $\lambda_1 = 1.235$, $\lambda_2 = 4.94$

$$W_q = \frac{\lambda_1\left(\frac{1}{\mu_1}\right)^2 + \lambda_2\left(\frac{2}{\mu_2}\right)^2}{2(1-\rho)} = 0.099$$

Q 10)

2) $$W_q^1 = \frac{\frac{\lambda_1}{(\mu_1)^2} + \lambda_2 \cdot \frac{2}{(\mu_2)^2}}{2\left(1 - \frac{\lambda_1}{\mu_1}\right)} = 0.05$$

$$W_q^2 = \frac{\frac{\lambda_1}{(\mu_1)^2} + \lambda_2 \cdot \frac{2}{(\mu_2)^2}}{2\left(1 - \frac{\lambda_1}{\mu_1}\right)(1-\rho)} = 0.1$$

when using non-premptive priority, avg waiting time for the control packet dropped almost two times from 99 ms to 50 ms while the avg data packet delay slightly increased from 99 ms to 100 ms.

Q 11) Batch service time is either (X) or $(2X + \Delta)$ with probability 0.5.

The Laplace transform of the pdf is given by

$$L_B^*(s) = 0.5\, L_B(s) + 0.5\, L_B^2(s)\, e^{-s\Delta}$$

where $L_B(s)$ is Laplace transform of exponential dist.

Moments are $\overline{X^*} = 1.5\overline{X} + 0.5\Delta$; $\overline{X^{*2}} = 2.5\overline{X^2} + 2\overline{X}\Delta + 0.5\Delta^2$

Now viewing this as an M/G/1 System ; (where the service times are general drawn according to above distribution) mean queueing delay (W_{q_b}) before service can start of a batch is

$$W_{q_b} = \frac{\lambda \overline{X^{*2}}}{2(1-\rho)}, \quad \rho = \lambda \overline{X^*}$$

Mean queueing delay (W_2) within a batch

$$= \frac{\Pr\{\text{batch size} = 2\}}{\text{Mean batch size}} \cdot \overline{X} = \frac{0.5}{1.5} \cdot \overline{X} = \frac{1}{3}\overline{X}$$

Total queueing delay

$$W_q = W_{q_b} + W_2$$

$$= \frac{\lambda(2.5\,\overline{X^2} + 2\overline{X}\Delta + 0.5\Delta^2)}{2(1 - 1.5\lambda\overline{X} - 0.5\lambda\Delta)} + \frac{1}{3}\overline{X}$$