Driven quantum coarsening
& symmetries of generating functionals

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Summary

1. A very quick review of **classical coarsening**;
   
ex. the paramagnetic - ferromagnetic case.

2. **Quantum dissipative relaxation**.

3. **Driven quantum coarsening**.
   
   Dynamic phase diagram & phase transitions.
   
   Non-equilibrium steady state.
   
   Non-stationary coarsening regime;
   
   decoherence and super-universality.

4. **Symmetry** of quantum dynamic generating functional.
Classical coarsening 2dIM

A rapid quench below $T_c$

Curvature driven coarsening with non-conserved order parameter.

Competition between the growth of two equilibrium states
Coarse-grained description

- Non-conserved order parameter (e.g. $+ - \rightarrow + +$ possible).
- Typical domain radius $R(t) \sim \lambda(T) t^{1/2}$
- Field-theoretic coarse-grained description.

Coarse-grained magnetization

$$\phi(\vec{r}, t) = \frac{1}{V_r} \sum_{i \in V_r} s_i(t)$$

**Time-dependent Ginzburg-Landau equation**

$$m \dddot{\phi}(\vec{r}, t) + \gamma \dddot{\phi}(\vec{r}, t) = \nabla^2 \phi(\vec{r}, t) - \frac{\delta V[\phi]}{\delta \phi(\vec{r}, t)} + \xi(\vec{r}, t)$$

Negligible in Friction Elasticity Double-well Noise $(T)$

overdamped limit

- Mean-field solvable case: $O(N)$ model in the large $N$ limit, $\phi \rightarrow \vec{\phi}$. 
Dynamic scaling

At late times there is a single length-scale, the typical radius of the domains $\mathcal{R}(T, t)$, such that the domain structure is (in statistical sense) independent of time when lengths are scaled by $\mathcal{R}(T, t)$, e.g.

\[
C(r, t) \equiv \langle s_i(t) s_j(t) \rangle|_{\bar{x}_i - \bar{x}_j = r} \sim m_{eq}^2(T) f \left( \frac{r}{\mathcal{R}(T, t)} \right),
\]

\[
C(t, t_w) \equiv \langle s_i(t) s_i(t_w) \rangle \sim m_{eq}^2(T) f_c \left( \frac{\mathcal{R}(T, t)}{\mathcal{R}(T, t_w)} \right),
\]

etc. when $r \gg \xi(T)$, $t, t_w \gg t_0$ and $C < m_{eq}^2(T)$.

Suggested by experiments and numerical simulations. Proved for

- Ising chain with Glauber dynamics.
- Langevin dynamics of the $O(N)$ model with $N \to \infty$, and the spherical ferromagnet. \(\text{Review Bray, 1994.}\)
- Distribution of hull-enclosed areas in $2d$ curvature driven coarsening.
Take the scalar Ginzburg-Landau model:

\[ H = \int d^d r \left[ \frac{m}{2} \Pi^2 + (\vec{\nabla} \phi)^2 + V(\phi) \right] \]

with the commutation rule

\[ [\Pi(\vec{r}), \phi(\vec{r}')] = -i\hbar \delta^d(\vec{r} - \vec{r}') \]

and its large \( N \) extension.

Similar to

- the quantum \( p = 2 \) spherical disordered spin model
  
  Shukla & Singh 80s, LFC & Lozano 98-99, Rokni & Chandra 04

- and to the rotors model:
Equivalent model

The **fully connected rotor model**

\[
H = \frac{\Gamma}{2M} \sum_{i=1}^{N} \vec{L}_i^2 + \frac{M}{2\sqrt{N}} \sum_{i \neq j} J_{ij} \vec{n}_i \vec{n}_j
\]

with \( L_{i}^{\mu\nu} = n_{i}^{\mu} p_{i}^{\nu} - n_{i}^{\nu} p_{i}^{\mu} \) and \( \vec{L}_i^2 = \sum_{\mu<\nu} (L_{i}^{\mu\nu})^2 \).

The \( p_{i}^{\mu} = -i\hbar \partial / \partial n_{i}^{\mu} \) are momentum operators canonically conjugate to \( n_{i}^{\mu} \) satisfying \([p_{i}^{\mu}, n_{j}^{\nu}] = -i\hbar \delta_{ij} \delta_{\mu\nu} \).

\( i \) is the rotor index; \( \mu \) is the component index; \( J_{ij} \) are Gaussian i.i.d. random variables.

The long-range nature of the interactions is in \( \sum_{i \neq j} \).

Statics of the isolated model in the limit \( M \to \infty \) Ye, Read & Sachdev 93
Dissipation and drive

The system is not isolated

Quantum mechanically the effect of an environment can be highly non-trivial leading to decoherence and localization in a two-level system

Bray & Moore 84, Chakravarty 84, Leggett et al. 80s

At the initial time $t = 0$ we couple each variable to two (source and drain) electron baths. In rotor’s language

$$H_{int} = \hbar \omega_c \sum_{i\mu kll'} n_i^\nu \left[ \bar{\psi}_{iklL} \sigma_{ll'}^\nu \psi_{ikl'R} + L \leftrightarrow R \right]$$

We treat the coupling in perturbation theory.

Similar to what done by Millis & Mitra 05-08

The system’s energy density is not conserved: $\mathcal{E}(t) \neq ct$
• **Path-integral Schwinger-Keldysh** formalism.

\[ T \]

• Connect system to reservoirs at time \( t = 0 \), factorized density matrix:

\[
\rho(0) = \rho_S(0) \otimes \rho_B(0)
\]

and take each reservoir in equilibrium at their own \( \beta \) and \( \mu_{L,R} \).

• Integrate out each fermion bath – assumed to be in equilibrium – using 2nd order perturbation theory.

• Obtain an effective action

\[
S = S_S + S_{SB} + S_\lambda
\]
Quadratic action

In rotor's language

\[ S_S = \sum_{a=\pm} \int dt \left[ \frac{\hbar^2}{2\Gamma} \sum_i (\dot{n}_{ia}(t))^2 + \sum_{i<j} J_{ij} n_{ia}(t) n_{ja}(t) \right] . \]

\[ S_{SB} = -\frac{1}{2} \sum_{ab=\pm} \int dt dt' \Sigma^B_{ab}(t, t') \sum_i n_{ia}(t) n_{ib}(t') , \]

\[ S_{\lambda} = \sum_{a=\pm} \frac{a}{2} \int dt \sum_i \lambda_{ia}(t) (n_{ia}^2(t) - M) \]

with the bath induced kernels :

\[ \Sigma^B_{ab}(t, t') = -iab\hbar\omega_c^2 \left[ G^R_{ab}(t, t')G^L_{ba}(t', t) + L \leftrightarrow R \right] \]

and \( G_{ab}(t, t') \equiv -i\langle T \psi_a(t)\psi_b^\dagger(t') \rangle \) with \( \psi_a(t), \psi_a^\dagger(t) \) the fermionic fields and \( T \) the time-ordering operator on the closed contour.
The reservoirs

- The fermions are in **equilibrium** \( \Rightarrow G_{ab}(t, t') = G_{ab}(t - t') \).
- We use **free fermions** so that \( G_{ab}(t - t') \) are known exactly.
- The ‘physical’ components \( \Sigma^K_B \) and \( \Sigma^R_B \) are linear combinations of \( \Sigma^B_{ab} \).
- In the \( \hbar \omega \ll T \) limit one can define an **equivalent temperature**

\[
T^* \equiv \lim_{\hbar \omega \ll T} \frac{\Sigma^K_B(\omega)}{2\partial \omega \text{Im} \Sigma^R_B(\omega)}
\]

that would be the one of a **classical thermal bath in equilibrium**.
- We choose the same symmetric DOS for left and right reservoirs with
  width \( \epsilon_F \) and finite cut-off \( \epsilon_{cut} \), e.g. a semi-circle, though \( \mu_L \) and \( \mu_R \)
- In the large band-width limit \( \epsilon_F \to \infty \) the resulting bath becomes

  ‘almost’ Ohmic : \( \Sigma^R_B \propto i\gamma \omega \) although \( \Sigma^K_B \) remains non-trivial if \( \omega \)
  finite
Real-time dynamics

Two-time dependence

\[ C(t + t_w, t_w) = \langle [\hat{O}(t + t_w), \hat{O}(t_w)]_+ \rangle \]

Correlation

Linear response

\[ R(t + t_w, t_w) = \frac{\delta \langle \hat{O}(t + t_w) \rangle}{\delta h(t_w)} \bigg|_{h=0} = \langle [\hat{O}(t + t_w), \hat{O}(t_w)]_- \rangle \]
Phase diagram

2nd order dynamic phase transitions between
a quantum non-equilibrium steady state (QNESS)
and
a driven coarsening regime

Obtained from a destabilization of the QNESS: $t - t_w$ lost
No drive: $eV = 0$.

In the $\simeq$ Ohmic limit $\epsilon_F \rightarrow \infty$

Critical line in

Rokni & Chandra 04 recovered model coupled to oscillators’ bath
Classical system coupled to two reservoirs

$$T^*(T_c, eV_c) = J$$

$$\overline{T}_c = J :$$

same as for Langevin dynamics

Shukla & Singh JPA 80s
LFC & Dean JPA 95
Zero temperature plane
The harder to study.
No full analytic form but only numerical.
The effect of a stronger coupling to the bath, as measured by

\[ g \equiv \hbar \omega_c / \epsilon_F \]

is to increase the extent of the coarsening phase.

For increasing \( g \) the full ‘tent’ is pulled up while keeping its ‘base’ fixed.

Extension of the ‘localization transition’ à la Caldeira-Leggett to the driven coarsening problem.

Similar results in LFC, Grempel, Lozano & da Silva Santos 00 for a glassy model & Rokni & Chandra 04 for oscillator baths.
Transformation

Effectively quadratic model

After a transformation of the variables $n^+(t), n^-(t)$ into

$$n(t) = (n^+ + n^-)/2 \quad \text{and} \quad i\hbar \hat{n} = (n^+ - n^-)/\hbar$$

the quantum Schwinger-Keldysh generating functional takes the form of the classical Martin-Siggia-Rose generating functional of a Langevin process although with a weird noise represented by $\Sigma^K_B$ and $\Sigma^R_B$.

This is thanks to the fact that the action is quadratic in $n^+, n^-$!

At not too large $\Gamma$ one can prove analytically that the dynamics at sufficiently long times and time-differences is the same as in the classical limit with $T$ replaced by $T^*$. This is consistent with the phase diagram found exactly.
Effective Langevin eq.

Consequences

We proved that the growth law is

\[ \mathcal{R}(t) \simeq t^{1/2} \]

and also the super-universal character of the quantum dynamics in the late stages of the coarsening regime:

the scaling functions, e.g. \( f_c(x) \) in

\[ C(t, t_w) \simeq f_c \left( \frac{\mathcal{R}(t)}{\mathcal{R}(t_w)} \right) \]

are the classical ones when \( \mathcal{R}(t)/\mathcal{R}(t_w) \sim 1 \).

cfr. classical coarsening with weak disorder Fisher & Huse 87

At large values of \( \Gamma \) we expect the same. Shown numerically and to a certain extent, that we need to improve, also analytically.
Fluctuation-dissipation theorem

Classical dynamics in equilibrium

The classical FDT for a stationary system with $\tau \equiv t - t_w$ reads

$$\chi(\tau) = \int_0^\tau dt' R(t') = -\beta [C(\tau) - C(0)] = \beta [1 - C(\tau)]$$

choosing $C(0) = 1$.

Linear relation between $\chi$ and $C$

Quantum dynamics in equilibrium

The quantum FDT reads

$$\chi(\tau) = \int_0^\tau dt' R(t') = \int_0^\tau dt' \int_{-\infty}^{\infty} \frac{i d\omega}{\pi \hbar} e^{-i\omega t'} \tanh \left( \frac{\beta \hbar \omega}{2} \right) C(\omega)$$

Complicated relation between $\chi$ and $C$
FDT & effective temperatures

Slow quantum out of equilibrium dynamics

If one measures $\chi(t, t_w)$ and $C(t, t_w)$ and finds

$$\chi(t, t_w) \approx ct - \beta_{\text{eff}} C(t, t_w)$$

in some times regime

It is tempting to interpret $\beta_{\text{eff}}$ as an effective temperature.

Meaning of $\beta_{\text{eff}}$ for classical out of equilibrium problems discussed in LFC, Kurchan & Peliti 97
Quantum coarsening

\[ C(t, t_w) = \langle [s_i(t), s_i(t_w)]_+ \rangle \]

\[ \chi(t, t_w) = \int_{t_w}^{t} dt' \langle [s_i(t), s_i(t')]_- \rangle \]

- Quantum stationary regime, with oscillations, quantum FDT if \( eV = 0 \), etc. (short length-scales).

- Classical aging regime, two-time decoherence and super-universality (long length-scales).
Summary of results

- Quantum critical point ($\Gamma_c$) as for equilibrium oscillator bath (if $eV = 0$).
- Second order critical lines (smooth close to $\Gamma_c$ and $T_c$, and non-analytic close to $V_c$).
- Coarsening survives a finite drive ($eV \neq 0$).
- Localization: ordering phase is enlarged by an increasing coupling to the electron reservoirs (as for the oscillator bath).
- Two-time decoherence: large structures behave classically.
- Super-universality: the scaling laws coincide with the classical ones.
- The current depends only on the fast regime (since the linear response decays too fast, $\beta_{\text{eff}} \to \infty$).
Classical path integrals

The **Langevin equation** with **additive coloured noise** for a particle in 1d is

$$m\dddot{x}(t) + \int_{-T}^{T} dt' \Gamma(t - t') \theta(t - t') \dot{x}(t') = -\frac{\delta V(x)}{dx(t)} + \xi(t)$$

with the coloured noise

$$\langle \xi(t) \xi(t') \rangle = \Gamma(t - t')$$

The initial time is $-T$.

(The extension of what follows to multiplicative coloured noise is easy.)

The dynamic generating functional is a path-integral

$$Z_{\text{dyn}}[\eta] = \int dx_0 d\dot{x}_0 \int Dx Di\dot{x} Dc D\bar{c} e^{-S[x, i\dot{x}, c, \bar{c}; \eta]}$$

with $i\dot{x}(t)$ the ‘response’ variable and $c$ and $\bar{c}$ two fermions used to exponentiate a determinant arising from a change of variables. $x_0$ and $\dot{x}_0$ are the initial conditions.

**Martin-Siggia-Rose-Jenssen-deDominicis formalism**
Classical path integral

The action has a deterministic part that includes the initial condition and a dissipative part (dependent on $\gamma$) : $S = S_{det} + S_{diss}$.

The transformation

\[
\begin{align*}
x(t) & \rightarrow x(-t) \\
i\dot{x}(t) & \rightarrow i\dot{x}(-t) + \beta \dot{x}(-t)
\end{align*}
\]

leaves $S_{det}$ invariant if $P_{eq}(x_0, \dot{x}_0)$, and $S_{diss}$ independently, and the path-integral measure as well.

A consequence of this symmetry are the FDTs valid for Newton or Langevin dynamics

\[
R_{AB}(t, t') + R_{AB}(t', t) = \beta \partial_{t'} C_{AB}(t, t')
\]

with $R_{AB}(t, t') = \delta \langle A[x(t)] / \delta f_B(t') \rangle |_{f=0}$ the linear response and $C_{AB}(t, t') = \langle A[x(t)] B[x(t')] \rangle$ the correlation. And all the possible higher-order extensions. Also the fluctuation theorems are proven this way.
Quantum path integral

Path integral representation

\[ C_{AB}(t, t') = \text{Tr} \ A_H(t) B_H(t') \rho(0) = \text{Tr} \ e^{i \frac{Ht}{\hbar}} A e^{-i \frac{Ht}{\hbar}} e^{i \frac{Ht'}{\hbar}} B e^{-i \frac{Ht'}{\hbar}} \rho(0) \]

with the initial density matrix

\[ \rho(0) \propto e^{-\beta H} \]
Quantum path integral

Equivalent but more symmetric representation

The integration over the imaginary time runs from $-\beta \hbar / 2$ to $\beta \hbar / 2$.

Playing with transformations of variables and contours we can prove the FDTs and fluctuation theorems (by using symmetries or their violations).