Descendant correlators on the torus

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Abstract: Only using simple results about elliptic functions, I develop recursive relations that allow to relate arbitrary torus correlators of descendants to the respective primary correlators.

Contents

1 Introduction

We use some simple results from the theory of elliptic functions to compute Ward identities on the torus and in particular give an algorithm on how to compute correlation functions of generic descendants.

2 Results from the theory of elliptic functions

2.1 Two important theorems

We first want to repeat two theorems for general compact Riemann surfaces.

Theorem 1 *Let S be a compact Riemann surface. Any meromorphic differential ω on S satisfies*

$$\sum_{p\in P} Res_p w = 0\,, \tag{2.1}$$

where P is the finite set of poles of ω.

Two immediate consequences are that no meromorphic function on S can have a single simple pole or a single simple zero.

Theorem 2 *Any holomorphic function on a compact Riemann surface S is necessarily constant.*

This follows from the maximum modulus principle or Liouville's theorem.

2.2 Some results on the torus

Now, I want to specify S to be the torus T^2 which can be given as the quotient

$$T^2 = \mathbb{C}/\Gamma\,, \tag{2.2}$$

where $\Gamma = \{n\omega_1 + m\omega_2 \,|\, n, m \in \mathbb{Z};\}$ is a lattice with primitive periods $\{\omega_1, \omega_2\}$. Meromorphic functions on the torus must be periodic in the directions of the periods. Such functions are called *elliptic*.

2.2.1 Weierstrass functions

Elliptic

The elliptic Weierstrass $\wp$ function is constructed as

$$\wp(z;\Gamma) = \frac{1}{z^2} + \sum_{w\in\Gamma\setminus\{0\}} \left(\frac{1}{(z-w)^2} - \frac{1}{w^2} \right). \tag{2.3}$$

In a neighborhood of the the origin, the Laurent series expansion of $\wp$ is

$$\wp(z;\Gamma) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + \mathcal{O}(z^6) \tag{2.4}$$

with the so-called *invariants*

$$g_2 = 60 \sum_{w\in\Gamma\setminus\{0\}} \frac{1}{w^4} \tag{2.5}$$

$$g_3 = 140 \sum_{w\in\Gamma\setminus\{0\}} \frac{1}{w^6}\,. \tag{2.6}$$

Note that any higher expansion coefficient is given in rather simple expressions of these two numbers.

Quasi-periodic

The Weierstrass ζ function is defined by

$$\frac{d\zeta(z)}{dz} = -\wp(z)\,, \quad \lim_{z\to 0} \left(\zeta(z) - \frac{1}{z} \right) = 0\,. \tag{2.7}$$

Integrating (2.3) term by term gives

$$\zeta(z;\Gamma) = \frac{1}{z} + \sum_{w\in\Gamma\setminus\{0\}} \left(\frac{1}{z-w} + \frac{1}{w} + \frac{z}{w^2} \right). \tag{2.8}$$

The function has a single first order pole on the torus and, hence, cannot be elliptic. However, it is quasi-periodic and one finds

$$\zeta(z+\omega_i) = \zeta(z) + 2\zeta(\omega_i/2)\,. \tag{2.9}$$

Next, we have the Weierstrass σ function which is defined by

$$\frac{\sigma'(z)}{\sigma(z)} = \frac{d\log\sigma}{dz} = \zeta(z)\,, \quad \lim_{z\to 0}\frac{\sigma(z)}{z} = 1\,. \tag{2.10}$$

Integrating (2.8) term by term gives

$$\sigma(z;\Gamma) = z\prod_{w\in\Gamma\setminus\{0\}}\left(1-\frac{z}{w}\right)e^{\frac{z}{w}+\frac{z^2}{2w^2}}\,. \tag{2.11}$$

As a function on the complex plane it is analytic with quasi-periodicity

$$\sigma(z+\omega_i) = -e^{2\zeta(\omega_i/2)(z+\omega_i)}\sigma(z)\,. \tag{2.12}$$

2.2.2 Express any elliptic function in terms of Weierstrass functions

Consider a generic elliptic function f with poles at $\{z_i\}$ and respective Laurent series'

$$f(z) = \frac{c_{i,r_i}}{(z-z_i)^{r_i}} + \cdots + \frac{c_{i,2}}{(z-z_i)^2} + \frac{c_{i,1}}{z-z_i} + \mathcal{O}\Big((z-z_i)^0\Big)\,. \tag{2.13}$$

From theorem 1 we know that $\sum_i c_{i,1} = 0$. We will write it in terms if the Weierstrass ζ functions and its derivatives

$$\frac{d^n\zeta(z)}{dz^n} \equiv \zeta^{(n)}(z) = (-1)^n\frac{n!}{z^{n+1}} + \mathcal{O}(z^0)\,. \tag{2.14}$$

Therefore, let us define the function

$$g(z) := f(z) - \sum_i\left(c_{i,1}\zeta(z-z_i) + c_{i,2}\wp(z-z_i) + \sum_{m=2}^{r_i-1}c_{i,m+1}\frac{\zeta^{(m)}(z-z_i)}{(-1)^m(m)!}\right)\,. \tag{2.15}$$

It is elliptic because $g(z+\omega_i) - g(z) = -2\zeta(\omega_i/2)\sum_i c_i = 0$. Additionaly it has no poles and, hence, has to be constant due to theorem 2. This means that any elliptic function can be expressed as

$$f(z) = C + \sum_i\left(c_{i,1}\zeta(z-z_i) + c_{i,2}\wp(z-z_i) + \sum_{m=2}^{r_i-1}c_{i,m+1}\frac{\zeta^{(m)}(z-z_i)}{(-1)^m(m)!}\right) \tag{2.16}$$

with some constant C. The latter can be fixed if we know the integral of f along one of the periods. I.e. if we assume that

$$\int_z^{z+\omega_k} f(z) \equiv I_k \tag{2.17}$$

is known and use

$$\int_z^{z+\omega_k}\zeta(z-z_i) = \log\sigma(\omega_k - z_i) - \log\sigma(-z_i) \tag{2.18}$$

$$= -2\zeta(\omega_k/2)z_i + 2\omega_i\zeta(\omega_i/2) + i\pi\,, \tag{2.19}$$

$$\int_z^{z+\omega_k}\wp(z-z_i) = -\zeta(\omega_k - z_i) + \zeta(-z_i) = -2\zeta(\omega_k/2)\,, \tag{2.20}$$

$$\int_z^{z+\omega_k}\zeta^{(n)}(z-z_i) = 0\,, \quad \text{for } n>1\,, \tag{2.21}$$

we can write

$$\omega_i C = I_i + \sum_i 2\zeta(\omega_k/2)\,(c_{i,2} + z_i c_{i,1})\ , \tag{2.22}$$

where we used $\sum_i c_{i,1} = 0$ again.

3 Application in CFT

3.1 Warm up: Ward identity on the torus

The result of Eguchi and Ooguri for the Ward identity follows straight forward from above results on elliptic functions. Most importantly we use that the correlator

$$f(z) = \langle T(z)\phi_1(z_1)\dots\phi_N(z_N)\rangle \tag{3.1}$$

is an elliptic function in z. Its poles follow from the OPE

$$T(z)\phi_i(z_i) = \sum_{k=-\infty}^{\infty} \frac{\hat{L}_{-k}\phi_i(z_i)}{(z-z_i)^{2-k}} \tag{3.2}$$

$$= \frac{h_i\phi_i(z_i)}{(z-z_i)^2} + \frac{\partial_{z_i}\phi_i(z_i)}{z-z_i} + \mathcal{O}\big((z-z_i)^0\big)\ , \tag{3.3}$$

where the first line is true for generic fields and the second line holds for primary fields only. Using the notation (2.13) we can write

$$c_{i,1} = \partial_{z_i}\langle\phi_1(z_1)\dots\phi_N(z_N)\rangle\ , \tag{3.4}$$

$$c_{i,2} = h_i\langle\phi_1(z_1)\dots\phi_N(z_N)\rangle\ . \tag{3.5}$$

Now using (2.16) we can write

$$f(z) = C + \sum_{i=1}^{N}\left(\zeta(z-z_i)\partial_{z_i} + \wp(z-z_i)h_i\right)\langle\phi_1(z_1)\dots\phi_N(z_N)\rangle\ . \tag{3.6}$$

To obtain the constant C which is given by (2.22), we need to integrate the correlator along one of the periods. The CFT results cannot depend on absolute orientation of the lattice. Hence, we can choose $\omega_1 = |\omega_1| \equiv L \in \mathbb{R}$. We want to map the integral along this period on (part of) the complex plane via the exponential map $u = e^{\frac{2\pi i}{L}z}$, s.t.

$$I_1 = \oint_1 dz\,\langle T(z)\dots\rangle = -\frac{4\pi^2}{L}\frac{1}{2\pi i}\oint du\,\Big\langle\Big(u\,T^{\text{plane}}(u) - \frac{c}{24u}\Big)\dots\Big\rangle \tag{3.7}$$

$$= -\frac{4\pi^2}{L}\Big\langle\Big(L_0 - \frac{c}{24}\Big)\dots\Big\rangle \tag{3.8}$$

To rewrite this further we use the definitions

$$\langle\dots\rangle = \frac{1}{Z(\tau)}\,\mathrm{Tr}\left[\dots q^{L_0 - \frac{c}{24}}\right]\ , \tag{3.9}$$

where $Z(\tau) = \mathrm{Tr}\left[q^{L_0 - \frac{c}{24}}\right]$ with $q = e^{2\pi i\tau}$ and the modular parameter $\tau = \frac{\omega_2}{\omega_1}$.

Now let's take derivatives with respect to ω_2, s.t. we can write

$$\frac{d}{d\omega_2}\langle\ldots\rangle + \left(\frac{d}{d\omega_2}\log Z(\tau)\right)\langle\ldots\rangle = \frac{2\pi i}{L}\left\langle\left(L_0 - \frac{c}{24}\right)\ldots\right\rangle \tag{3.10}$$

s.t.

$$I_1 = 2\pi i\left(\frac{d}{d\omega_2}\langle\ldots\rangle + \left(\frac{d}{d\omega_2}\log Z(\tau)\right)\langle\ldots\rangle\right). \tag{3.11}$$

and

$$\langle T(z)\phi_1(z_1)\ldots\phi_N(z_N)\rangle = \left(\frac{2\pi i}{L}\frac{d}{d\omega_2} + \left(\frac{2\pi i}{L}\frac{d}{d\omega_2}\log Z(\tau)\right)\right)\langle\phi_1(z_1)\ldots\phi_N(z_N)\rangle \tag{3.12}$$

$$+\sum_{i=1}^{N}\left(\zeta(z-z_i) + \frac{2\zeta(\omega_1/2)z_i}{L}\right)\partial_{z_i}\langle\phi_1(z_1)\ldots\phi_N(z_N)\rangle \tag{3.13}$$

$$+\sum_{i=1}^{N}\left(\wp(z-z_i) + \frac{2\zeta(\omega_1/2)}{L}\right)h_i\langle\phi_1(z_1)\ldots\phi_N(z_N)\rangle. \tag{3.14}$$

3.2 Generic descendant correlators on the torus

Consider some fields Φ_i of a CFT that are not necessary primary. As before their OPE is given by

$$T(z)\Phi_i(z_i) = \sum_{k=-\infty}^{\infty}\frac{\hat{L}_{-k}\Phi_i(z_i)}{(z-z_i)^{2-k}}, \tag{3.15}$$

and the correlators

$$\langle T(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle \tag{3.16}$$

are elliptic functions in z. Now consider some elliptic function $g(z)$ with possibly poles at the insertion points, i.e. it can be expanded in Laurent series' around these points,

$$g(z) = \sum_{m=0}^{\infty} g^{(i)}_{-n_i+m}(z-z_i)^{-n_i+m}. \tag{3.17}$$

Then, the product

$$F(z) := g(z)\langle T(z)\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle \tag{3.18}$$

has Laurent series'

$$\sum_{k=-\infty}^{\infty}\sum_{m=0}^{\infty} g^{(i)}_{-n_i+m}\left\langle\hat{L}_{-k}\Phi_i(z_i)\ldots\right\rangle(z-z_i)^{m+k-n_i-2} \tag{3.19}$$

with residues

$$\mathrm{Res}_{z_i} F(z) = \sum_{m=0}^{\infty} g^{(i)}_{-n_i+m}\left\langle\hat{L}_{m-n_i-1}\Phi_i(z_i)\ldots\right\rangle. \tag{3.20}$$

If we, for now, assume that g has no poles other than the insertion points, then we can write

$$0 = \sum_{i=1}^{N} \mathrm{Res}_{z_i} F(z) \tag{3.21}$$

$$= \sum_{i=1}^{N} \sum_{m=0}^{\infty} g^{(i)}_{-n_i+m} \left\langle \hat{L}_{m-n_i-1} \Phi_i(z_i) \dots \right\rangle . \tag{3.22}$$

The latter equation relates the correlation function of different descendants with each other. These relations obviously depend on the choice of g.

As a first choice we want to use

$$g(z) = \wp^{(n)}(z - z_i)\,, \quad n \geq 0\,. \tag{3.23}$$

With $\wp^{(n)}(z) \equiv \frac{(-1)^n (n+1!)}{z^{2+n}} + \sum_{k=0}^{\infty} a^{(n)}_k z^{2k+\epsilon}$, where $\epsilon = 1$ for odd n and $\epsilon = 0$ for even n, and using (3.21) we can write

$$\frac{(n+1)!}{(-1)^{n+1}} \left\langle \hat{L}_{-3-n} \Phi_i(z_i) \dots \right\rangle = \sum_{k=0}^{\infty} a^{(n)}_k \left\langle \hat{L}_{2k+\epsilon-1} \Phi_i(z_i) \dots \right\rangle \tag{3.24}$$

$$+ \sum_{j\neq i} \sum_{k=0}^{\infty} \frac{\wp^{(n+k)}(z_j - z_i)}{k!} \left\langle \hat{L}_{-1+k} \Phi_j(z_j) \dots \right\rangle . \tag{3.25}$$

The total level of each term on the r.h.s. is lower than the one on the l.h.s. of the latter equation. We, hence, can use it recursively to express a correlator of descendants with L_{-3-n}, $n \geq 0$, in terms of correlators of lower level that only consist of descendants that are built from L_{-2} and L_{-1}.

As a second choice we want to use

$$g(z) = \zeta(z - z_0) - \zeta(z - z_i)\,, \tag{3.26}$$

where z_0 is not an insertion point. Its expansion around z_i is

$$g(z) = -\frac{1}{z - z_i} + \sum_{n=0} \left(\frac{\zeta^{(n)}(z_i - z_0)}{n!} - \zeta_n \right) (z - z_i)^n\,, \tag{3.27}$$

where ζ_n are the Laurent coefficients of $\zeta(z)$ around $z = 0$. At the other insertion points z_j, $j \neq i$, we get

$$g(z) = \sum_{n=0} \frac{1}{n!} \left(\zeta^{(n)}(z_j - z_0) - \zeta^{(n)}(z_j - z_i) \right) (z - z_i)^n\,. \tag{3.28}$$

Now using again (3.21) but with the extra point z_0 we can write

$$\left\langle \hat{L}_{-2} \Phi_i(z_i) \dots \right\rangle = \langle T(z_0) \dots \rangle + \sum_{n=0}^{\infty} \left(\frac{1}{n!} \zeta^{(n)}(z_i - z_0) - \zeta_n \right) \left\langle \hat{L}_{-1+n} \Phi_i(z_i) \dots \right\rangle \tag{3.29}$$

$$+ \sum_{j\neq i} \sum_{n=0}^{\infty} \frac{1}{n!} \left(\zeta^{(n)}(z_j - z_0) - \zeta^{(n)}(z_j - z_i) \right) \left\langle \hat{L}_{-1+n} \Phi_j(z_j) \dots \right\rangle . \tag{3.30}$$

To get rid of the supposedly z_0 dependence on the r.h.s. of this equation we integrate over the ω_1 cycle. Using (2.18) – (2.21) and (3.11) we obtain

$$\left\langle \hat{L}_{-2}\Phi_i(z_i)\ldots \right\rangle = \frac{2\pi}{L}\left(i\frac{d}{d\omega_2}\langle\ldots\rangle + \left(i\frac{d}{d\omega_2}\log Z(\tau)\right)\langle\ldots\rangle\right) \tag{3.31}$$

$$- \sum_{n=0}^{\infty}\zeta_n \left\langle \hat{L}_{-1+n}\Phi_i(z_i)\ldots \right\rangle \tag{3.32}$$

$$- \sum_{j\neq i}\sum_{n=0}^{\infty}\frac{1}{n!}\zeta^{(n)}(z_j - z_i)\left\langle \hat{L}_{-1+n}\Phi_j(z_j)\ldots \right\rangle \tag{3.33}$$

$$+ \sum_{j}\left(\frac{2\zeta(\omega_1/2)}{L}h_{\Phi_j} + \frac{2\zeta(\omega_1/2)}{L}z_j\partial_{z_j}\right)\langle\Phi_1(z_1)\ldots\Phi_N(z_N)\rangle\,. \tag{3.34}$$

The first term on the r.h.s. of this equation shows that we might need to know the ω_2 derivatives of the Weierstrass functions. We found the identities

$$\frac{d\zeta(z)}{d\omega_2} = -\frac{i}{2\pi}\left(\frac{\omega_1}{2}\left(\wp'(z) + 2\zeta(z)\wp(z) - \frac{g_2 z}{6}\right) + 2\eta_1\left(\zeta(z) - z\wp(z)\right)\right), \tag{3.35}$$

$$\frac{d\wp(z)}{d\omega_2} = \frac{i}{\pi}\left(\frac{\omega_1}{2}\left(2\wp(z)^2 + \zeta(z)\wp'(z) - \frac{g_2}{3}\right) - \eta_1\left(2\wp(z) + z\wp'(z)\right)\right), \tag{3.36}$$

$$\frac{d\wp'(z)}{d\omega_2} = \frac{i}{2\pi}\left(\frac{\omega_1}{2}\left(6\wp(z)\wp'(z) + 12\zeta(z)\wp(z)^2 - g_2\zeta(z)\right)\right. \tag{3.37}$$

$$\left. - \eta_1\left(6\wp'(z) + 12z\wp(z)^2 - g_2 z\right)\right), \tag{3.38}$$

where $\eta_i = \zeta(\omega_i/2)$, and g_i are as before the Weierstrass invariants. To compute the correlator recursively we also define

$$e_i := \wp(\omega_i/2) \tag{3.39}$$

$$f_i := \partial_z\wp(\omega_i/2) \tag{3.40}$$

and use the identities

$$\frac{d\eta_1}{d\omega_2} = -\frac{i}{2\pi}\left(\frac{\omega_1}{2}\left(f_1 + 2\eta_1 e_1 - \frac{g_2\omega_1}{12}\right) + 2\eta_1\left(\eta_1 - \frac{\omega_1 e_1}{2}\right)\right), \tag{3.41}$$

$$\frac{de_1}{d\omega_2} = \frac{i}{\pi}\left(\frac{\omega_1}{2}\left(2e_1^2 + \eta_1 f_1 - \frac{g_2}{3}\right) - \eta_1\left(2e_1 + \frac{\omega_1 f_1}{2}\right)\right), \tag{3.42}$$

$$\frac{df_1}{d\omega_2} = \frac{i}{2\pi}\left(\frac{\omega_1}{2}\left(6e_1 f_1 + 12\eta_1 e_1^2 - g_2\eta_1\right) - \eta_1\left(6f_1 + 6\omega_1 e_1^2 - \frac{\omega_1 g_2}{2}\right)\right). \tag{3.43}$$

Finally we also need the the derivative of the Weierstrass invariants which is given by

$$\frac{d\{g_2, g_3\}}{d\omega_2} = \frac{i}{2\pi}\left\{6g_3\omega_1 - 8g_2\eta_1, \frac{g_2^2\omega_1}{3} - 12g_3\eta_1\right\}. \tag{3.44}$$